LATTICE WALKS AND PRIMARY DECOMPOSITION

by PERSI DIACONIS, DAVID EISENBUD,
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Introduction

This paper shows how primary decompositions of an ideal can give useful descriptions of components of a graph arising in problems from Combinatorics, Statistics, and Operations Research. We begin this introduction with the general formulation. Then we give the simplest interesting example of our theory, followed by a statistical example similar to that which provided our original motivation. In the body of the paper we study the primary decompositions corresponding to some natural combinatorial problems.

Let $B$ be a set of vectors in $\mathbb{Z}^n$. Define a graph $G_B$ on whose vertices are the non-negative $n$-tuples $\mathbb{N}^n$ as follows: $u, v \in \mathbb{N}^n$ are connected by an edge of $G_B$ if and only if $u - v$ is in $\pm B$. We say $u$ and $v$ in $\mathbb{N}^n$ can be connected via $B$ if they are in the same connected component of $G_B$. We shall consider the problem of characterizing the components of $G_B$.

The simplest sort of characterization we know is by linear functionals. For example, if $n = 2$ and $B = \{(1, -1)\}$ then two vectors $u = (u_1, u_2)$ and $u' = (u'_1, u'_2)$ are in the same component of $G_B$ if and only if $u_1 + u_2 = u'_1 + u'_2$. However in the case where $B = \{(2, -2), (3, -3)\}$, there is no such characterization.

To treat such cases, we connect our problem with commutative algebra. For any vector $u = (u_1, \ldots, u_n)$ of positive integers we define a monomial $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n} \in k[z_1, \ldots, z_n]$. Every vector $u \in \mathbb{Z}^n$ can be written uniquely as $u = u_+ - u_-$ where $u_+, u_-$ are non-negative vectors with disjoint support. For example $(1, -2) = (1, 0) - (0, 2)$. To a vector $u \in \mathbb{Z}^n$ we associate the binomial difference $x^u_+ - x^u_-$. Let $k$ be any field. To the subset $B \subset \mathbb{Z}^n$ we associate the ideal generated by the $x^u_+ - x^u_-:

I_B = \langle x^u_+ - x^u_- : u \in B \rangle \subset k[z_1, \ldots, z_n].$

The following theorem shows that this is a good encoding scheme:

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Theorem 1.1. Two vectors \( u, v \in \mathbb{N}^n \) are in the same component of \( G_B \) if and only if \( x^u - x^v \in I_B \).

Theorem 1.1 has been rediscovered many times. One early reference is [MM]. See also [Stu, §5]. Applications of the graphs \( G_C \) to integer programming can be found in [Tho].

As explained further in Section 2, there are various decompositions of the form

\[
I_B = J_1 \cap J_2 \cap \ldots \cap J_r,
\]

where the \( J_i \) correspond to other — and in some ways simpler — combinatorial problems. For example, we might take primary decomposition, and this is the case mainly studied in this paper. Since \( u \) and \( v \) are in the same component of \( G_B \) if and only if \( x^u - x^v \in J_i \), \( 1 \leq i \leq r \), such a decomposition of ideals allows a decomposition of the original problem. The following examples suggest how the theory will go:

Example. 1.2

The simplest typical example is provided by the set

\[
B = \{(2, -2), (3, -3)\} \subset \mathbb{Z}^2.
\]

If \( u, v \in \mathbb{N}^2 \) are connected via \( B \) then clearly \( u_1 + u_2 = v_1 + v_2 \). We shall see that the converse is true provided that \( u_1 + u_2 \geq 3 \). When this inequality is not satisfied, the situation is more delicate: Of the vectors \( (2, 0), (1, 1), (0, 2) \) with sum 2, only \((0, 2)\) and \((2, 0)\) are connected. Each of the vectors \((1, 0), (0, 1), \) and \((0, 0)\) is isolated.

These statements are all easy, but we will now derive them using the general method of this paper. We first compute the primary decomposition

\[
I_B = \langle x^2 - y^2, x^3 - y^3 \rangle = \langle x - y \rangle \cap \langle x + y, x^3, x^2y, xy^2, y^3 \rangle.
\]

From this we see that \((u_1, u_2)\) is connected to \((v_1, v_2)\) via \( B \) if and only if it satisfies two conditions, corresponding to the two ideals on the right hand side of the equation. The first condition is that \( x^{u_1}y^{u_2} - x^{v_1}y^{v_2} \in \langle x - y \rangle \), that is, \( x - y \) divides \( x^{u_1}y^{u_2} - x^{v_1}y^{v_2} \), or equivalently \( u_1 + u_2 = v_1 + v_2 \).

The second condition is harder to interpret combinatorially. Note that \( \langle x + y, x^3, x^2y, xy^2, y^3 \rangle \) contains all monomials of degree \( \geq 3 \). Thus if \( u_1 + u_2 = v_1 + v_2 \geq 3 \), then \( u \) and \( v \) are connected via \( B \). Since \( x^2 - y^2 \) is divisible by \( x + y \) it is also in the second ideal, and \((2, 0)\) is connected with \((0, 2)\). It is not hard to show that no other difference of monomials is in \( I_B \), completing the proof.
**Example.** 1.3 (Poisson regression)

Here is a small example of a more realistic kind, exhibiting the behavior that led us to consider such problems. In it we are interested in a graph $G$ defined by the simple conditions given by linear functionals. We wish to perform a random walk on this graph, and there are certain moves in the walk that are particularly easy to describe; these are the elements of $B$. Our primary decomposition technique shows that the condition for two vectors to be connected via $B$ is stronger than the desired condition, but suggests that it is nevertheless a good approximation. In this example we show experimentally that the approximation is good in a sense useful for applications.

Suppose that a chemical to control insects is sprayed on successive equally infested plots in increasing concentrations 0,1,2,3,4 (in some units). After the spraying the numbers of insects left alive on the plots are 44,25,21,19,11. Roughly: Greater concentration leads to fewer insects.

To extrapolate we need a model. One standard model postulates that the number of insects at concentration $i$ has a Poisson distribution with mean parameter $e^{a+bi}$ where $a$ and $b$ are parameters to be fitted from the data. If $\hat{a}$ and $\hat{b}$ are estimates of $a$ and $b$, and $\hat{\lambda}_i = e^{\hat{a}+i\hat{b}}$, then a test for goodness of fit of the Poisson model can be based on the chi-square statistic

$$\sum_{i=1}^{5} \frac{(\hat{\lambda}_i - N_i)^2}{\hat{\lambda}_i}.$$ 

Asymptotic theory says this should have an approximate chi-square distribution on 3 degrees of freedom. In this example, $\hat{a} = 3.707, \hat{b} = -.3125$, the 5 fitted values are $\hat{\lambda}_i = (40.8,29.8,21.8,16.0,11.7)$ and the chi-square statistic is 1.7.

Does this value of chi-square show that the data were well fitted? Poorly fitted? Calibrating the chi-square test leads to a combinatorial problem of the type considered above. Let $\mathcal{X}$ be the set of all non-negative $n$-tuples $x = (x_0, \ldots, x_4)$ with $S(x) = x_0 + \ldots + x_4 = 120$ and $T(x) = 0x_0 + x_1 + 2x_2 + 3x_3 + 4x_4 = 168$ matching the data above. $\mathcal{X}$ is a finite set, a component of the graph $G$ defined by the linear functionals $S$ and $T$. We want to know what proportion of the $n$-tuples in $\mathcal{X}$ have chi-square greater than 1.7. We do not want to enumerate all $n$-tuples to find out (indeed, in realistic problems of this kind there are simply too many to enumerate) and we know no general theory that will solve this problem accurately. Thus we will approximate a solution by choosing examples uniformly from the set, and seeing what proportion of the examples chosen have chi-square greater than 1.7.

To make these choices, we might run a random walk starting at the original data vector $x^* = (44,25,21,19,11)$. It seems reasonable to take for a set
of elements that span the sublattice of $\mathbb{Z}^n$ defined by the vanishing of the functionals $S$ and $T$. For example, we might take

$$B = \{(1, -1, -1, 1, 0), (1, -1, 0, -1, 1), (0, 1, -1, -1, 1)\}.$$ 

At each step of our walk we randomly choose $\pm$ one of the vectors in $B$ and then add it to the current vector in $X$. If the entries of the result is positive, we step to the sum, which is again a vector in $X$. Otherwise we discard it. Thus, the walk might go

$$(44, 25, 21, 19, 11) \rightarrow (45, 24, 20, 20, 11) \rightarrow (45, 23, 21, 10) \rightarrow \ldots.$$ 

This walk generates a symmetric process which leads to the uniform distribution on the component of $G_B$ containing $x^*$. If we actually do the computation, we find that the observed chi-square represents a rather good fit.

Unfortunately, not every pair of vectors in $X$ can be connected by steps in $B$, so the set of 5-tuples over which we are averaging is not quite the same as the set we want! For example, the vector $(36, 0, 84, 0, 0)$ cannot be connected to $x^*$ by steps in $B$ keeping all entries positive. The primary decomposition exhibited below shows that two non-negative integer vectors $(i_1, j_1, h_1, l_1, m_1)$ and $(i_2, j_2, h_2, l_2, m_2)$ in $X$ are connected via $B$ if and only if

\begin{align*}
i_r + j_r + h_r &\geq 1 \text{ and } \\
i_r + j_r + l_r &\geq 1 \text{ and } \\
j_r + l_r + m_r &\geq 1 \text{ and } \\
h_r + l_r + n_r &\geq 1. \tag{1.1}
\end{align*}

The set $B$ can be enlarged to

$$B' = B \cup \{(1, -2, 1, 0, 0), (0, 1, -1, 1, 0), (0, 0, 1, -2, 1)\},$$

and we shall see that this enlargement is sufficient. However, the distribution observed for chi-square coming from a random walk based on $B$ is close to one based on $B'$. In particular, using $B$, the proportion of samples with chi-squared $< 1.7$ is .0031, whereas using $B'$ the proportion is .0046. See the histograms below, each of which is based on a walk of 90,000 steps, an initial 10,000 having been discarded.
Walk with 3 moves

To understand the situation we turn to primary decomposition. We work in the polynomial ring \( k[x_1, x_2, x_3, x_4, x_5] \). The set \( \mathcal{B} \) is encoded by the binomial ideal

\[
\mathcal{I}_B = (x_2x_3 - x_1x_4, x_2x_4 - x_1x_5, x_3x_4 - x_2x_5)
\]

Two 5-tuples are connected via \( \mathcal{B} \) if and only if \( \mathcal{I}_B \) contains

\[
x_1^{i_1}x_2^{j_1}x_3^{k_1}x_4^{l_1}x_5^{m_1} - x_1^{i_2}x_2^{j_2}x_3^{k_2}x_4^{l_2}x_5^{m_2}
\] (1.2)

The primary decomposition of \( \mathcal{I}_B \) is found to be

\[
\mathcal{I}_B = I \cap (x_1, x_2, x_3) \cap (x_1, x_2, x_4) \cap (x_2, x_4, x_5) \cap (x_3, x_4, x_5)
\] (1.3)

where \( I \) is the prime ideal generated by the \( 2 \times 2 \)-minors of the matrix as is

\[
\begin{pmatrix}
  x_1 & x_2 & x_3 & x_4 \\
  x_2 & x_3 & x_4 & x_5
\end{pmatrix}
\]

(The ideal \( I \) appears in Algebraic Geometry as the defining ideal of the rational normal curve in \( \mathbb{P}^4 \).) To translate this into combinatorics, note that a binomial difference (1.2) is in \( \mathcal{I}_B \) if and only if it is in each ideal on the right of (1.3). The difference is in \( I \) if and only if \( i_1 + j_1 + k_1 + l_1 + m_1 = i_2 + j_2 + k_2 + l_2 + m_2 \) and \( 0 \cdot i_1 + j_1 + 2k_1 + 3l_1 + 4m_1 = 0 \cdot i_1 + j_2 + 2k_2 + 3l_2 + 4m_2 \). The remaining ideals on the right side are generated by monomials. A polynomial is in a monomial ideal \( I \) if and only if each term in \( I \). This gives the remaining characterizing relations claimed in (1.1).

The ideal \( I \), which corresponds to the moves in \( \mathcal{B}' \), encodes precisely a lattice walk that connects all of \( \mathcal{X} \). The reader may wonder why we didn’t simply begin with \( \mathcal{B}' \) in constructing the random walk above. In larger problems, it is computationally quite taxing to find connecting sets of moves. It
is natural to take a smaller set of moves such as a lattice basis and "hope for
the best"; indeed, this approach is taken in several published studies. From
the algebraic theory described here it *appears* that the random walk generated
by a set of moves \( B \) should be a good approximation (in some useful sense)
to the one generated by the set of moves \( B' \) whenever the two bases generate
the same sublattice of \( \mathbb{Z}^n \). Algebraically this can be expressed by the fact
that the ideals corresponding to these sets of moves in our dictionary have the
same intersection with the Laurent polynomial ring [ES]; combinatorically the
they correspond to the solution of combinatorial problems that differ only for
\( n \)-tuples with a priori bounds on some of the components.

The rest of this paper is laid out as follows. In Section two we set up
the algebraic technique which comes from the work on binomial ideals [ES].
Section three treats contingency tables; \( a \times b \) arrays of non-negative integers
with given row and column sums. Section 4 describes a different basis \( B \) for
the problem of contingency tables, the "adjacent minors". In the final section
we discuss a systematic way of making a relatively small choice of basis for any
lattice walk problem: we call these circuit walks.

## 2. Lattice ideals

With notation as above let \( B \) be a set of vectors in \( \mathbb{Z}^n \). Let \( \mathcal{L} \) be the subgroup
of \( \mathbb{Z}^n \) generated by \( B \). Call \( u, v \in \mathbb{N}^n \) *equivalent* if \( u - v \in \mathcal{L} \). In Example
1.2 \( \mathcal{L} \) is the set \((u_1, u_2) \in \mathbb{Z}^2 \) with \( u_1 + u_2 = 0 \) and \((v_1, v_2), (v_1, v_2) \in \mathbb{N}^2 \) are
equivalent if and only if \( u_1 + u_2 = v_1 + v_2 \). In Example 1.3 the equivalence
classes generated by \( \mathcal{L} \) are the set of all \( u \in \mathbb{N} \) with \( u_1 + u_2 + u_3 + u_4 + u_5, 0 \cdot u_1 + u_2 + u_3 + u_4 + u_5 \) having fixed values. In applied problems
the equivalence classes are often the basic objects of interest. One wants to
construct a set of edges (that is a choice of \( B \) that connects elements within
an equivalence class with all components staying non-negative. In [DS] it is
shown how to find a finite set \( B \) by finding a Gröbner basis for \( I_{\mathcal{L}} \). The division
algorithm gives an effective algorithm for finding a connecting path.

If \( u \) and \( v \) lie in the same connected component of \( G_B \) they are equivalent
but not conversely. It can be shown that two vectors \( u, v \in \mathbb{N}^n \) are equivalent
if and only if \( u + w \) can be connected to \( B \) for some (sufficiently large) \( w \in \mathbb{N}^n \).
It is an interesting open problem to give useful bounds on \( w \). Theorem 1.1
gives the following condition for equivalence.

**Corollary 2.1.** Let \( B \subseteq \mathbb{Z}^n \) generate \( \mathcal{L} = \mathbb{Z}B \). Every pair of \( \mathcal{L} \) equivalent
vectors is connected via \( B \) if and only if \( I_{\mathcal{L}} = I_B \).

The ideals considered here are all generated by binomial \( x^u + x^v \). There
is a recently developed theory of binomial ideals [ES]. We briefly review what
is needed here. General references for commutative algebra with emphasis
on computational aspects are [CLO], [Stu]. Thorough treatments of primary decomposition can be found in [AM], [ES].

Let \( \mathcal{L} \) be a lattice in \( \mathbb{Z}^n \). Call \( \mathcal{L} \) saturated if for each \( r \in \mathbb{Z}, u \in \mathbb{Z}^n, r \cdot u \in \mathcal{L} \) implies \( u \in \mathcal{L} \). Equivalently, \( \mathcal{L} \) is saturated if and only if the quotient group \( \mathbb{Z}^n/\mathcal{L} \) is free abelian. Saturated lattices are discussed in [ES] where it is proved that \( I_{\mathcal{L}} \) is prime if and only if \( \mathcal{L} \) is saturated. All of the lattices that appear in the examples of this paper are saturated.

A binomial in \( k[z_1, \ldots, z_n] \) is a polynomial with at most two terms. A binomial ideal is an ideal generated by binomials. Thus, monomial ideals are also binomial ideals.

The following theorem is proved in [ES].

**Theorem 2.2.** Every binomial ideal possesses a binomial primary decomposition.

In [ES, §9] there is an explicit algorithm which expresses a given binomial ideal as an intersection of primary binomial ideals. A primary decomposition algorithm for general polynomial ideals is given in [BW §8]. Specializing to the situation of the paper we get

\[
I_B = J_1 \cap J_2 \cap \ldots \cap J_r
\]

where each \( J_i \) is primary and generated by binomials \( \alpha x^u - \beta x^v \in I_B \). If \( \mathcal{L} = \mathbb{Z} B \) is saturated then [ES] show that the prime \( I_B \) appears among the \( J_i \). Otherwise, \( I_{\mathcal{L}} \) equals the intersection of some of the \( J_i \). All other \( J_i \)'s must contain monomials by [ES §2].

Theorem 2.2 shows that \( u \) and \( v \) are connected via \( B \) if and only if \( x^u - x^v \) lies in \( J_i \) for all \( i \). If \( J_i \) is a monomial ideal, the corresponding combinatorial condition is easy: suppose

\[
J_i = \langle x^a, x^b, \ldots, x^c \rangle.
\]

then \( x^u - x^v \in J_i \) if and only if \( x^u, x^v \in J_i \). Further, \( x^u \in J_i \) if and only if \( u \geq a \) or \( u \geq b \) or \( \ldots \) or \( u \geq c \). Even if the \( J_i \)'s are not monomial ideals, artful choices may allow neat necessary and sufficient conditions or neat necessary conditions. Examples appear in Section 3 and 4 below. It is often convenient to combine the \( J_i \) in groups, so that the intersection of each group is generated by monomials \( x^u \) and pure binomials \( x^u - x^v \). Such a regrouping facilitates the combinatorial translation of containment in \( J_i \). See [ES, Cor. 8.2] for further details.
3. Corner minors

The prototype of the problems considered here is the problem of generating random "contingency tables" — tables of positive integers of given size with fixed row and column sums. The statisticians J. Darroch and G. Glonek (see [GL]) introduced a random walk technique: Start at a given table and take steps that do not change the positivity or the row and column sums. For example, consider the following procedure: at each step, a position \( l \) in the first row and \( m \) in the first column are chosen randomly. The current table is changed to a new table by altering the four entries in positions \((1,1), (l,1), (1,m), (l,m)\) by adding or subtracting 1 following either the pattern of signs \(\begin{pmatrix} + & - \\ - & + \end{pmatrix}\) or \(\begin{pmatrix} - & + \\ + & - \end{pmatrix}\), the choice being random as well. For instance, for \(a = b = 3\) this basis is

\[
B_{\text{cor}} = \begin{Bmatrix}
\begin{pmatrix} +1 & -1 & 0 \\ -1 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} +1 & 0 & -1 \\ -1 & 0 & +1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} +1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & +1 & 0 \end{pmatrix}, \begin{pmatrix} +1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & +1 \end{pmatrix}
\end{Bmatrix}.
\]

A change is suppressed if it would lead to a table with negative entries. This process defines a symmetric random walk with a uniform stationary distribution on the set of tables connected to the starting table by the given moves. It turns out, however, that these moves may, not connect all the non-negative tables with the given row and column sums. Glonek showed that they do suffice to connect these tables when the row and column sums are all \(\geq 2\). In this section we derive a strengthening of Glonek's result by describing the primary decomposition of the ideal corresponding to the set of chosen moves.

More formally, we are concerned with the lattice \(\mathbb{Z}^{a\times b}\) of \(a \times b\)-integer matrices and \(L\) is the sublattice of matrices with zero row sums and zero column sums. We begin by considering a random walk with a larger set \(B_{\text{all}}\) of possible moves: Again, the walk is over all non-negative integer \(a \times b\)-matrices with fixed row and column sums. To describe a move in \(B_{\text{all}}\) we select the positions in a \(2 \times 2\)-submatrix and alter them by adding

\[
\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}. \tag{3.1}
\]

**Lemma 3.1.** The moves (3.1) are necessary and sufficient to connect any pair of non-negative integer \(a \times b\)-matrices with the same row and column sums.

**Proof.** It is well-known (see e.g. [Stu, Proposition 5.4]) that \(I_L\) is a prime ideal which is minimally generated by the \(2 \times 2\)-minors of an \(a \times b\)-matrix of
Lemma 3.1 now follows from Corollary 2.2.

The number of moves (3.1) is \(\binom{a}{2} \binom{b}{2}\), which is much larger than \(\text{rank}(\mathcal{L}) = (a - 1)(b - 1)\), the size of the collection of moves described at the beginning of this section — quartic rather than quadratic in the size of the tables. It turns out that though these moves do not connect all possible tables with the given row and column sums, they come rather close. The following primary decomposition result is the basis for analyzing this.

**Theorem 3.2.** Let \(I_\mathcal{L}\) be the prime ideal generated by all \(2 \times 2\)-minors of (3.2), where \(a, b \geq 2\). Let \(R := (x_{11}, \ldots, x_{1b})\) and \(C := (x_{11}, \ldots, x_{a1})\). The ideal of "corner minors" \(I_{B_{\text{cor}}} := (x_{1i}x_{ij} - x_{1j}x_{i1} : 2 \leq i \leq a, 2 \leq j \leq b)\) has the primary decomposition

\[
I_{B_{\text{cor}}} = I_\mathcal{L} \cap R \cap C \cap (I_{B_{\text{cor}}} + R^2 + C^2). \tag{3.3}
\]

If \(a, b > 2\) then this primary decomposition is minimal, whereas if \(b = 2\) the last two terms can be dropped, and similarly if \(a = 2\).

From (3.3) we can read off the connectivity properties of the lattice basis \(B_{\text{cor}}\).

**Corollary 3.3.** Two non-negative integer \(a \times b\)-matrices \(U, V\) are connected via \(B_{\text{cor}}\) if

(a) \(U\) and \(V\) have the same row and column sums, and
(b) \(U, V\) have positive first row sum,
(c) \(U, V\) have positive first column sum,
(d) \(U, V\) have either row sum \(\geq 2\) or column sum \(\geq 2\).

**Proof.** The primary decomposition (3.3) implies

\[
I_{B_{\text{cor}}} \supseteq I_\mathcal{L} \cap R \cap C \cap (R^2 + C^2).
\]

This inclusion of ideals is equivalent to the assertion of Corollary 3.3.

**Proof of Theorem 3.2.** We will deal only with the case \(a, b > 2\), leaving the (easy) contrary cases to the reader. The left hand side of (3.3) is clearly
contained in the right hand side of (3.3). Order the variables row-wise \( x_{11} < x_{12} < \ldots < x_{1b} < x_{21} < \ldots < x_{ab} \) and let \( \prec \) denote the resulting reverse lexicographic term order. We shall prove the equality

\[
in_{\prec}(I_c) \cap in_{\prec}(R) \cap in_{\prec}(C) \cap in_{\prec}(I_{B_{cor}} + R^2 + C^2) = in_{\prec}(I_{B_{cor}}) \tag{3.4}
\]

This implies (3.3) because the left hand side of (3.4) contains the initial ideal of the right hand side of (3.3); hence both sides of (3.3) have the same initial ideal and are thus equal.

In order to evaluate the constituents in (3.4) we introduce the monomial ideals

\[
R' := \langle x_{12}, x_{13}, \ldots, x_{1b} \rangle \quad \text{and} \quad C' := \langle x_{21}, x_{31}, \ldots, x_{a1} \rangle \quad \text{and}
I_{s,t} := \langle x_{ij}x_{kl} \mid s \leq i < k, j > l \geq t \rangle \quad \text{for} \ 1 \leq s \leq a \ \text{and} \ 1 \leq t \leq b.
\]

Since the \( 2 \times 2 \)-minors are a Gröbner basis (see e.g. [Stu, Proposition 5.4]), we have

\[
in_{\prec}(I_c) = I_{1,1}. \tag{3.5}
\]

Note that \( in_{\prec}(R) = R \) and \( in_{\prec}(C) = C \). We next derive the identity

\[
in_{\prec}(I_{B_{cor}} + R^2 + C^2) = \ (R + C)^2. \tag{3.6}
\]

It is evident that \( R'C' \subseteq in_{\prec}(I_{B_{cor}}) \) and it follows that \((R + C)^2 \subseteq in_{\prec}(I_{B_{cor}} + R^2 + C^2)\).

For the reverse inclusion it suffices to show that the minimal generators of \( I_{B_{cor}} + R^2 + C^2 \) are a Gröbner basis. Using Buchberger's first criterion [BW, Theorem 5.68], the computation reduces to a few easily checked cases, such as

\[
s\prec (x_{k1}x_{1j} - x_{11}x_{kj}, \ x_{k1}x_{1l} - x_{11}x_{kl}) = x_{11}x_{1j}x_{kl} - x_{11}x_{1l}x_{kj} \in R^2 \tag{3.7}
\]

Having thus verified (3.6), we now claim the following more complicated identity:

\[
in_{\prec}(I_{B_{cor}}) = R'C' + x_{11}^2I_{2,2} + \sum_{i \geq 2} x_{11}x_{1i}I_{2,i} + \sum_{s \geq 2} x_{11}x_{s1}I_{s,2} + x_{11} \cdot \langle x_{ij}x_{kl} \mid i < k \ \text{and} \ j > l \ \text{and} \ (i = 1 \ \text{or} \ l = 1) \rangle. \tag{3.8}
\]

Here each summand is a product of ideals. We abbreviate the last summand by \( M \). We first show that \( in_{\prec}(I_{B_{cor}}) \) contains the right hand side of (3.8).

To see that \( in_{\prec}(I_{B_{cor}}) \) contains \( M \), it suffices (by symmetry) to check that
it contains $x_{1j}x_{kl}$ for $1 < k, j > l$. This is clear from (3.7). Next let $s \leq i < k, j \geq l \geq 2$ and consider the following $B_{cor}$-walk:

$$
\begin{align*}
&x_{1s}x_{1j}x_{kl} \to x_{1s}x_{1j}x_{kl}x_{k1} \to x_{s}x_{11}x_{1j}x_{kl} \to x_{s}x_{11}x_{1j}x_{kl}x_{k1} \\
&\to x_{s}x_{11}x_{1j}x_{kl}x_{k1} \to x_{s}x_{11}x_{1j}x_{kl}x_{k1} \to x_{s}x_{11}x_{1j}x_{kl}x_{k1}
\end{align*}
$$

This shows that $x_{1s}x_{1j} \cdot (x_{1j}x_{kl} - x_{1j}x_{k1}) \in I_{B_{cor}}$. Applying (3.5) to $(a - s) \times (b - 1)$-matrices, we conclude that $x_{11}^2I_{2,2} \subset \text{in}_{<}(I_{B_{cor}})$ and $x_{11}x_{s}I_{s,2} \subset \text{in}_{<}(I_{B_{cor}})$ for $s \geq 2$. By symmetry, we also obtain $x_{11}x_{11}I_{2,2} \subset \text{in}_{<}(I_{B_{cor}})$ for $t \geq 2$. It is evident that $R'C' \subset \text{in}_{<}(I_{B_{cor}})$. We have shown that the right hand side of (3.8) lies in the left hand side.

The inclusion $\supset$ in (3.4) is obvious. To prove equality in both (3.4) and (3.8), it suffices to show that the right hand side of (3.8) contains the left hand side in (3.4), i.e.,

$$
I_{1,1} \cap R \cap C \cap (R + C)^2 \subseteq R'C' + x_{11}^2I_{2,2} + \sum_{t \geq 2} x_{11}x_{1t}I_{2,2} + \sum_{t \geq 2} x_{11}x_{s}I_{s,2} + M.
$$

Let $m$ be a monomial in $I_{1,1} \cap R \cap C \cap (R + C)^2$. If $m$ is not divisible by $x_{11}$, then since $m \in R \cap C$ we must have $m \in R'C' = R'C'$. Thus we may suppose that $x_{11}$ divides $m$. If $m$ is not divisible by $x_{11}^2$, then since $m \in (R + C)^2$ we must have $m \in R'$ or $m \in C'$, say $m$ is divisible by $x_{11}x_{1t}$ for some $t$. Since $m \in I_{1,1}$ as well, we see that either:

- $m$ is also divisible by $x_{su}$ for some $s > 1$ and $u < t$, in which case $m \in M$, or else
- $m$ is also divisible by $x_{su}x_{vw}$ with $s > v > 1$, $t \leq u < w$, in which case $m \in x_{11}x_{1t}I_{2,2}$.

In either case $m$ lies in the desired sum. Now consider the case where $x_{11}^2$ divides $m$. Since $m \in I_{1,1}$, $m$ is also divisible by a product of the form $x_{ij}x_{kl}$ with $i < j, k > l$. If $i = l = 1$, then $m \in R'C'$. If exactly one of $i$ or $l$ equals 1, then $m \in M$. If both $i > 1$ and $l > 1$ then $m \in x_{11}^2I_{2,2}$, and we are done.

Finally, we show that the intersection on the right hand side of (3.3) is irredundant. It suffices to show this for $a = b = 3$. In this special case the monomial ideal (3.8) equals

$$
\text{in}_{<}(I_{B_{cor}}) = \langle x_{12}x_{21}, x_{13}x_{31}, x_{12}x_{31}, x_{13}x_{31}, x_{11}^2x_{23}x_{32}, x_{11}x_{12}x_{23}x_{32}, x_{11}x_{21}x_{23}x_{32}, x_{11}x_{22}x_{31}, x_{11}x_{22}x_{31}, x_{11}x_{13}x_{22}, x_{11}x_{13}x_{22}, x_{11}x_{13}x_{32} \rangle.
$$

(3.9)

The following "witnesses" show that each of the four primary components in (3.3) is needed:

$$
x_{11}^2, \quad x_{31} \cdot (x_{21}x_{32} - x_{22}x_{31}), \quad x_{13} \cdot (x_{13}x_{22} - x_{12}x_{23}), \quad x_{11} \cdot (x_{22}x_{33} - x_{23}x_{32}).
$$
The intersection any three of the four primary ideals on the right hand side of (3.3) contains one the four polynomials listed. But none of these four polynomials is in \( I_{B_{cor}} \), because none of their terms lies in (3.9). This completes the proof of Theorem 3.2.

**Remark 3.4.** In general when we have a primary decomposition \( I = \cap_j I_j \) in a polynomial ring with a term order \( < \), then \( \text{in}_<(I) \subseteq \cap_j \text{in}_<(I_j) \), but the two sides will usually not be equal. For a simple example, suppose that \( x < y \) are indeterminates, and consider

\[
\langle x^2 \rangle = \text{in}_<(x^2 - y^2) \neq \langle x \rangle \cap \langle y \rangle = \text{in}_<(x - y) \cap \text{in}_<(x + y).
\]

But in the setting of Theorem 3.1 a small miracle occurs and the corresponding intersections of initial ideals are equal for the right choice of term order. It would be interesting to understand when such things happen in general.

### 4. Adjacent minors

Another natural basis for the lattice \( \mathcal{L} \) in Section 3 is the set \( B_{adj} \) of adjacent \( 2 \times 2 \)-minors. Here the situation is more complicated than before. Let us examine the case \( a = b = 4 \) in detail. The adjacent \( 2 \times 2 \)-moves for \( 4 \times 4 \)-matrices are encoded by the ideal

\[
I := I_{B_{adj}} = \langle x_{12}x_{21} - x_{11}x_{22}, x_{13}x_{22} - x_{12}x_{23}, x_{14}x_{23} - x_{13}x_{24}, \\
x_{22}x_{31} - x_{21}x_{32}, x_{23}x_{32} - x_{22}x_{33}, x_{24}x_{33} - x_{23}x_{34}, \\
x_{32}x_{41} - x_{31}x_{42}, x_{33}x_{42} - x_{32}x_{43}, x_{34}x_{43} - x_{33}x_{44} \rangle.
\]

Two nonnegative integer \( 4 \times 4 \)-matrices \((a_{ij})\) and \((b_{ij})\) with the same row and column sums can be connected by a sequence of adjacent \( 2 \times 2 \)-moves if and only if the binomial

\[
\prod_{1 \leq i,j \leq 4} x_{ij}^{a_{ij}} - \prod_{1 \leq i,j \leq 4} x_{ij}^{b_{ij}}
\]

lies in the ideal \( I \), by Corollary 2.1.

**Proposition 4.1.** Two non-negative integer \( 4 \times 4 \)-matrices with the same row and column sums can be connected by a sequence of adjacent \( 2 \times 2 \)-moves if both of them satisfy the following six inequalities:

(i) \( a_{21} + a_{22} + a_{23} + a_{24} \geq 2 \);
(ii) \( a_{31} + a_{32} + a_{33} + a_{34} \geq 2 \);
(iii) \( a_{12} + a_{22} + a_{32} + a_{42} \geq 2; \)
(iv) \( a_{13} + a_{23} + a_{33} + a_{43} \geq 2; \)
(v) \( a_{12} + a_{22} + a_{23} + a_{32} + a_{33} + a_{42} \geq 1; \)
(vi) \( a_{13} + a_{21} + a_{22} + a_{23} + a_{32} + a_{33} + a_{34} + a_{42} \geq 1. \)

We remark that these sufficient conditions remain valid if (at most) one of the four inequalities \( \geq 2 \) is replaced by \( \geq 1 \). No further relaxation of the conditions (i)-(vi) is possible, as is shown by the following two pairs of matrices, which are disconnected:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The necessity of the conditions (v) and (vi) can be seen from the disconnected matrices

\[
\begin{pmatrix}
n & n & 0 & 0 \\
0 & 0 & 0 & n \\
n & 0 & 0 & 0 \\
n & 0 & n & n
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
n & 0 & n & n \\
n & 0 & 0 & 0 \\
0 & 0 & 0 & n \\
n & n & 0 & n
\end{pmatrix}
\quad \text{for any integer} \quad n \geq 0.
\]

**Proof of Proposition 4.1** Let \( I_\mathcal{L} \) be the prime ideal generated by all 36 \( 2 \times 2 \)-minors of a \( 4 \times 4 \)-matrix \((x_{ij})\) of indeterminates. Define also the prime ideals

\[
C_1 := \langle x_{12}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{43} \rangle \quad \text{and} \quad C_2 := \langle x_{13}, x_{21}, x_{22}, x_{23}, x_{33}, x_{34}, x_{42} \rangle.
\]

Using a computer algebra system – such as MACAULAY – it can be verified easily that

\[
I_\mathcal{L} \cap C_1 \cap C_2 \cap \langle x_{21}, x_{22}, x_{23}, x_{24} \rangle \cap \langle x_{31}, x_{32}, x_{33}, x_{34} \rangle^2 \\
\cap \langle x_{12}, x_{22}, x_{32}, x_{42} \rangle^2 \cap \langle x_{13}, x_{23}, x_{33}, x_{43} \rangle^2 \subseteq I_{B_{adj}}.
\]

This containment of ideals implies Proposition 4.1.

For completeness we describe the primary decomposition of \( I = I_{B_{adj}} \). This is a good test case for implementations of (binomial) primary decomposition. Consider the prime ideals

\[
A := \langle x_{12}x_{21} - x_{11}x_{22}, x_{13}, x_{23}, x_{31}, x_{32}, x_{33}, x_{43} \rangle \quad \text{and}
\]

Proposition 4.1. The minimal associated primes of the binomial ideal $I$ are the 15 primes $A_i$, $B_j$, $C_j$ and $P$. Each of these occurs with multiplicity one in $I$, so that

$$\text{Rad}(I) = A_1 \cap A_2 \cap \cdots \cap A_8 \cap B_1 \cap B_2 \cap B_3 \cap B_4 \cap C_1 \cap C_2 \cap P.$$  

In particular, both $I$ and its radical $\text{Rad}(I)$ have codimension 7 and degree 32.

We next describe the embedded primary components of $I$. Our first primary ideal is

$$D := \left( (I + (x_{13}^2, x_{23}^2, x_{31}^2, x_{32}^2, x_{33}^2, x_{34}^2)) : x_{11}x_{12}x_{14}x_{21}x_{22}x_{24}x_{41}x_{42}x_{44} \right).$$

Its radical $\text{Rad}(D)$ is a prime of codimension 10 and degree 5. (Commutative algebra experts will notice that $\text{Rad}(D)$ is a ladder determinantal ideal.) Up to symmetry, there are four such ideals $D_1, D_2, D_3, D_4$.

Our second type of primary ideal is

$$E := \left( [I + (x_{12}^2, x_{21}^2, x_{22}^2, x_{23}^2, x_{24}^2, x_{32}^2, x_{33}^2, x_{34}^2, x_{42}^2, x_{43}^2)] : (x_{11}x_{13}x_{14}x_{31}x_{41}x_{44})^2 \right).$$

Its radical $\text{Rad}(E)$ is a monomial prime of codimension 1. Up to symmetry, there are eight such primary ideals $E_1, E_2, \ldots, E_8$.

Our third type of primary ideal has codimension 11:

$$F := \left( [I + (x_{12}^3, x_{13}^3, x_{22}^3, x_{23}^3, x_{31}^3, x_{32}^3, x_{33}^3, x_{34}^3, x_{42}^3, x_{43}^3)] : (x_{11}x_{14}x_{21}x_{24}x_{41}x_{44})^2 \right).$$

Its radical is the degree 2 prime

$$\text{Rad}(F) = (x_{11}x_{24} - x_{21}x_{14}, x_{12}, x_{13}, x_{22}, \ldots, x_{43}).$$

Up to symmetry, there are four such primary ideals $F_1, F_2, F_3, F_4$. 
Our last primary ideal has codimension 12, and it is unique up to symmetry:

\[
G := \left( [I + \langle x_{12}^5, x_{13}^5, x_{21}^5, x_{22}^5, x_{23}^5, x_{24}^5, x_{31}^5, x_{32}^5, x_{33}^5, x_{34}^5, x_{42}^5, x_{43}^5 \rangle] \\
:(x_{11} x_{14} x_{41} x_{44})^5 \right).
\]

In summary, we have the following theorem, which can be checked by MACAULAY:

**Theorem 4.2.** The ideal I has precisely 32 associated primes, 15 minimal and 17 embedded. Using the decomposition in Proposition 2, we get the minimal primary decomposition

\[
I = \text{Rad}(I) \cap D_1 \cap D_2 \cap D_3 \cap D_4 \cap E_1 \cap E_2 \cap \cdots \cap E_8 \cap F_1 \cap F_2 \cap F_3 \cap F_4 \cap G.
\]

It remains an open problem to find a primary decomposition for the ideal of adjacent 2 \times 2-minors for larger sizes. We do not even have a reasonable conjecture for generalizing the result in Proposition 4.1.

In the special case \( a = 2 \) the ideal \( I_{B_{adj}} \) is radical and has a nice explicit prime decomposition. We shall present this decomposition using a slightly simplified notation. We write \( I \) as the ideal generated by the following \( n \) binomials in \( 2n + 2 \) variables:

\[
x_{i-1} \cdot y_i - x_i \cdot y_{i-1} \quad (i = 1, 2, \ldots , n).
\]

(4.2)

Let \( f(n) \) denote the \( n \)-th Fibonacci number, which is defined recursively by \( f(0) = f(1) = 1 \) and \( f(n) = f(n - 1) + f(n - 2) \).

**Theorem 4.3.** The ideal I of adjacent 2 \times 2-minors of a generic 2 \times (n + 1)-matrix is the intersection of \( f(n) \) prime ideals; in particular, I is radical.

**Proof.** The ideal I is a complete intersection, which means I has codimension \( n \) and degree \( 2^n \). (To see this note that the left hand terms in (4.2) are pairwise relatively prime. They are the leading terms in the lexicographic order.) By Macaulay's Unmixedness Theorem [Eis, Corollary 18.14], every associated prime of I is minimal and has codimension \( n \).

Let \( D(n) \) denote the set of all subsets of \( \{1, 2, \ldots , n - 1\} \) which do not contain two consecutive integers. The cardinality of \( D(n) \) equals the Fibonacci number \( f(n) \). For instance, \( D(4) = \{ \emptyset , \{1\} , \{2\} , \{3\} , \{4\} , \{1,3\} , \{1,4\} , \{2,4\} \} \).

For each element \( S \) we define a binomial ideal \( I_S \) in \( k[x_0 , \ldots , x_n , y_0 , \ldots , y_n] \). The generators of \( I_S \) are the variables \( x_i \) and \( y_i \) for all \( i \in S \), and the binomials \( x_j y_k - x_k y_j \) for all \( j, k \notin S \) such that no element of \( S \) lies between \( j \) and \( k \). It is easy to see that \( I_S \) is a prime ideal of codimension \( n \). Moreover, \( I_S \) contains
$I$, and therefore $I_S$ is a minimal prime of $I$. We claim that

$$I = \bigcap_{S \in \mathcal{D}(n)} I_S$$  \hspace{1cm} (4.3)

In view of Macaulay's Unmixedness Theorem, it suffices to prove the identity

$$\sum_{S \in \mathcal{D}(n)} \deg(I_S) = 2^n$$  \hspace{1cm} (4.4)

First note that $I_{\emptyset}$ is the determinantal ideal $(x_iy_j - x_jx_i : 0 \leq i < j \leq n)$. It is known (see e.g. [Har, Example 19.10]) that the degree of $I_{\emptyset}$ equals $n + 1$. Using the same fact for matrices of smaller size, we find that, for $S$ non-empty, the degree of $I_S$ equals the product

$$i_1(i_2 - i_1 + 1)(i_3 - i_2 + 1) \cdots (i_r - i_{r-1} + 1)i_r$$ where $S = \{i_1 < i_2 < \cdots < i_r\}$. \hspace{1cm} (4.5)

Consider the surjection $\phi : 2^{\{1, \ldots, n\}} \to \mathcal{D}(n)$ defined by $\phi(\{j_1 < j_2 < \cdots < j_r\}) = \{j_{r-1}, j_{r-3}, j_{r-5}, \ldots\}$. The product in (4.5) is the cardinality of the inverse image $\phi^{-1}(S)$. This proves $\sum_{S \in \mathcal{D}(n)} \#(\phi^{-1}(S)) = 2^n$, which implies (4.4) and hence Theorem 4.3.

5. Circuit walks

Let $A$ be a $d \times n$-integer matrix of rank $d$. The integer kernel of $A$ is a sublattice $L$ in $\mathbb{Z}^n$ of rank $n - d$. In this case $L$ is saturated and hence $I_L$ is a prime ideal. A non-zero vector $u = (u_1, \ldots, u_n)$ in $L$ is called a circuit if its coordinates $u_i$ are relatively prime and its support $\text{supp}(u) = \{i : u_i \neq 0\}$ is minimal with respect to inclusion. In this section we discuss the walk defined by the set $C$ of all circuits in $L$. This makes sense for two reasons:

- The lattice $L$ is generated by the circuits, i.e., $\mathbb{Z}L = L$ (see e.g. [ES, Lemma 8.8]).
- The circuits can be computed easily from the matrix $A$.

Here is a simple algorithm for computing $C$. (See [BM] for a more sophisticated approach.) Initialize $C := \emptyset$. For any $(d + 1)$-subset $\tau = \{\tau_1, \ldots, \tau_{d+1}\}$ of \{1, \ldots, n\} form the vector

$$C_{\tau} = (-1)^i \cdot \det(A_{\tau \setminus \{i\}}) \cdot e_{\tau_i},$$

where $e_j$ denotes the $j$-th unit vector and $A_\sigma$ denote the submatrix of $A$ with column indices $\sigma$. If $C_{\tau}$ is non-zero then remove common factors from its
coordinates. The resulting vector is a circuit and all circuits are obtained in this manner (see e.g. [Stu, §4]).

Example. 5.1

Let \( d - 2, n = 4 \) and \( A = \begin{pmatrix} 0 & 2 & 5 & 7 \\ 7 & 5 & 2 & 0 \end{pmatrix} \). Then the set of circuits equals

\[
C = \pm \{ (3, -5, 2, 0), (5, -7, 0, 2), (2, 0, -7, 5), (0, 2, -5, 3) \}. \tag{5.1}
\]

It is instructive to check that the \( \mathbb{Z} \)-span of \( C \) equals \( L = \ker_{\mathbb{Z}}(A) \). (For instance, try to write \((1, -1, -1, 1) \in L \) as a \( \mathbb{Z} \)-linear combination of \( C \).) We shall derive the following result: Two \( L \)-equivalent non-negative integer vectors \((A, B, C, D)\) and \((A', B', C', D')\) can be connected by the circuits in (5.1) if both of them satisfy the following inequality

\[
\min \left\{ \max \{A, B, C, D\}, \max \left\{ \frac{9}{4} C, \frac{9}{4} D \right\}, \max \left\{ \frac{9}{4} A, \frac{9}{4} B, C \right\} \right\} \geq 9 \tag{5.2}
\]

We remark that the following two \( L \)-equivalent pairs cannot be connected by circuits:

\[
(4, 9, 0, 2) \leftrightarrow (5, 8, 1, 1) \quad \text{and} \quad (1, 6, 6, 1) \leftrightarrow (3, 4, 4, 3) \tag{5.3}
\]

To analyze circuit walks in general we consider the circuit ideal \( I_C \) generated by the binomials \( x^{u^+} - x^{u^-} \) where \( u = u_+ - u_- \) runs over all circuits in \( L \). The primary decomposition of circuit ideals was studied in [ES, §8]. We summarize the relevant results. Let \( \text{pos}(A) \) denote the \( d \)-dimensional convex polyhedral cone in \( \mathbb{R}^d \) spanned by the column vectors of \( A \). Each face of \( \text{pos}(A) \) is identified with the subset \( \sigma \subseteq \{1, \ldots, n\} \) consisting of all indices \( i \) such that the \( i \)-th column of \( A \) lies on that face. If \( \sigma \) is a face of \( \text{pos}(A) \) then the ideal \( I_\sigma := \langle x_i : i \notin \sigma \rangle + I_L \) is prime. Note that \( I_{\{1, \ldots, n\}} = I_L \) and \( I_{\{\}} = \langle x_1, x_2, \ldots, x_n \rangle \).

**Theorem 5.2.** [ES, Theorem 8.3, Example 8.6 and Proposition 8.7]

\[
\text{Rad}(I_C) = I_L \quad \text{and} \quad \text{Ass}(I_C) \subseteq \{ I_\sigma : \sigma \text{ is a face of } \text{pos}(A) \}.
\]

Theorem 8.3 in [ES] gives a procedure for computing a binomial primary decomposition of the circuit ideal \( I_C \). This enables us to analyze the connectivity of the circuit walk in terms of the faces of the polyhedral cone \( \text{pos}(A) \).

Example. 5.1 (continued)
We choose variables \( a, b, c, d \) for the four columns of \( A \). The cone \( \text{pos}(A) = \text{pos}\{(7, 0), (5, 2), (2, 5), (0, 7)\} \) equals the positive orthant in \( \mathbb{R}^2 \). It has one 2-dimensional face, labeled \( \{a, b, c, d\} \), two 1-dimensional faces, labeled \( \{a\} \) and \( \{d\} \) and one 0-dimensional face, labeled \( \{} \). The lattice ideal of \( \mathcal{L} = \ker(z(A)) \) is the prime ideal

\[
I_{\mathcal{L}} = \langle ad - bc, ac^4 - b^3d^2, a^3c^2 - b^5, b^2d^3 - c^5, a^2c^3 - b^4d \rangle.
\]

The circuit ideal equals

\[
I_{\mathcal{C}} = \langle a^3c^2 - b^5, a^5d^2 - b^7, a^2d^5 - c^7, b^2d^3 - c^5 \rangle.
\]

It has the minimal primary decomposition

\[
I_{\mathcal{C}} = I_{\mathcal{L}} \cap \langle b^9, c^4, d^4, b^2b^2, c^2d^2, b^2c^2 - a^2d^2, b^5 - a^3c^2 \rangle \\
\cap \langle a^4, b^4, c^9, a^2b^2, a^2c^2, b^2c^2 - a^2d^2, c^5 - b^2d^3 \rangle \\
\cap \langle (a^9, b^9, c^9, d^9) + I_{\mathcal{C}} \rangle.
\]

Here the second ideal is primary to \( I_{\{a\}} = \langle b, c, d \rangle \) and the third ideal is primary to \( I_{\{d\}} = \langle a, b, c \rangle \). The given primary decomposition implies the condition (5.2) because

\[
\langle a^9, b^9, c^9, d^9 \rangle \cap \langle b^9, c^4, d^4 \rangle \cap \langle a^4, b^4, c^9 \rangle \cap I_{\mathcal{L}} \subseteq I_{\mathcal{C}}. \tag{5.2'}
\]

Returning to our general discussion, Theorem 5.2 implies that for each face \( \sigma \) of the polyhedral cone \( \text{pos}(A) \) there exists a non-negative integer \( M_\sigma \) such that

\[
I_{\mathcal{L}} \cap \bigcap_{\sigma \text{ face of } \text{pos}(A)} \langle x_i : i \notin \sigma \rangle^{M_\sigma} \subseteq I_{\mathcal{C}}. \tag{5.4}
\]

**Corollary 5.3.** For each proper face \( \sigma \) of the cone \( \text{pos}(A) \) there exists an integer \( M_\sigma \) such that any two \( \mathcal{L} \)-equivalent vectors \( (a_1, \ldots, a_n) \) and \( (b_1, \ldots, b_n) \) in \( \mathbb{N}^n \) with the property

\[
\sum_{i \in \sigma} a_i \geq M_\sigma \quad \text{and} \quad \sum_{i \notin \sigma} b_i \geq M_\sigma \quad \text{for all proper faces } \sigma \text{ of } \text{pos}(A)
\]

\[
\sum_{i \notin \sigma} a_i \geq M_\sigma \quad \text{and} \quad \sum_{i \notin \sigma} b_i \geq M_\sigma
\]

\[
\text{can be connected by circuits.}
\]

**Problem 5.4.** Find good bounds for the integers \( M_\sigma \) in terms of the matrix \( A \).
The optimal value of $M_\sigma$ seems to be related to the singularity of the toric variety defined by $I_C$ along the torus orbit labeled $\sigma$: The worse the singularity is the higher the value of $M_\sigma$. It would be very interesting to understand these geometric aspects.

In Example 5.1 we can choose the integers $M_\sigma$ as follows:

$$M_{(i)} = 15 \text{ and } M_{(a)} = 11 \text{ and } M_{(d)} = 11.$$

These choices are optimal; this can be seen from the disconnected pairs in (5.3).

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