THE CONTINUOUS AND DISCRETE BROWNIAN BRIDGES: REPRESENTATIONS AND APPLICATIONS

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Summary

In this paper we give an exposition of Brownian motion and the Brownian bridge, both continuous and discrete. Several examples are given where these processes, or ones closely related to them, are used in statistical applications. Representations of the processes, in terms of weighted standard normal variables, are given, and it is suggested how these might be used in simulation studies.

Keywords: Gaussian process; goodness-of-fit; simulation of Gaussian processes; time series analysis

1 Introduction

The process known as Brownian motion, and the related process called the Brownian bridge, have been studied for almost a century by many authors. The discrete Brownian bridge has a less illustrious history, but it arises in many applications of statistics, for example, in continuous metric scaling (Cuadras and Fortiana, 1993, 1995), Cramér-von Mises statistics for discrete distributions (Choulakian, Lockhart and Stephens, 1994), serial correlation coefficients, and modified Cramér-von Mises statistics for spectral distributions (Anderson and Stephens, 1993). The aim of this paper is to clarify the relationship of Brownian motion and the Brownian bridge to their discrete analogues and to describe some applications. The description of the Brownian motion and Brownian bridge processes is given in the first four
sections, and applications are discussed in Sections 5 to 7. We focus also on representations of the Brownian bridge and in Section 8 we suggest ways in which these might be useful in simulation studies.

2 Brownian Motion and the Brownian Bridge

In this section we survey the continuous Brownian motion and Brownian bridge processes, and the modified Brownian bridge which arises often in statistical work. First we recall that a Gaussian stochastic process \( G(u) \) is a stochastic process with the property that its values \( G(t_i) \) at a finite set of values \( t_i, i = 1, \ldots, n \) are \( n \) multivariate normal variables with a mean \( \mu(t_i) \) and a covariance \( \mathcal{E}(G(t_i) - \mu(t_i))(G(t_j) - \mu(t_j)) = \rho(t_i, t_j) \). In what follows the process \( G(u) \) will be continuous with probability 1 and \( \mu = 0 \); the process will be called a mean-zero Gaussian process and \( \rho(u, v) \) is the covariance function.

The covariance function \( \rho(u, v) \) is symmetric. When \( \rho(u, v) \) is continuous, the function can be represented in terms of the eigenvalues and eigenfunctions of the kernel \( \rho(u, v) \) as

\[
\rho(u, v) = \sum_{j=1}^{\infty} \frac{1}{\theta_j} f_j(u)f_j(v),
\]

where \( \theta_j \) and \( f_j(u) \) are the solutions to the integral equation

\[
f(u) = \theta \int_0^1 \rho(u, v)f(v)dv
\]

normalized by

\[
\int_0^1 f^2(u)dv = 1.
\]

Furthermore, consider the process \( G^*(u) \) defined by

\[
G^*(u) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\theta_j}} f_j(u)X_j,
\]

2
where $X_1, X_2, \ldots$ are independent $N(0, 1)$ variables. This process is Gaussian with mean 0, and its covariance is given by (1). Thus the process $G^*(u)$ has the same probabilistic properties as $G(u)$ and is said to be a representation of $G(u)$. From now on, the same notation will be used for a process and its representation.

Standard Brownian motion (or the Wiener process) is a mean-zero Gaussian process $W(u), 0 \leq u < \infty$, with

$$\mathcal{E}W(u)W(v) = \min(u, v), \quad 0 \leq u, v < \infty.$$  \hfill (5)

The Brownian bridge is a mean-zero Gaussian process $B(u)$ with the covariance function

$$\mathcal{E}B(u)B(v) = h(u, v) = \min(u, v) - uv, \quad 0 \leq u, v \leq 1.$$  \hfill (6)

The Brownian bridge is the residual process after regressing Brownian motion $W(u)$ on $W(1)$; that is, $B(u) = W(u) - uW(1), \quad 0 \leq u \leq 1$.

For both Brownian motion and the Brownian bridge (2) is solved by differentiating twice with respect to $u$ to obtain the differential equation of simple harmonic motion

$$f''(u) = -\theta f(u).$$ \hfill (7)

Let $\theta = k^2$. The general solution of (7) is

$$f(u) = a \cos ku + b \sin ku.$$ \hfill (8)

For Brownian motion we have $\rho(u, v) = \min(u, v)$; thus $\rho(0, v) = 0$ and $f(0) = 0$. Also, the first differentiation gives $f'(1) = 0$. These boundary conditions determine the solutions $f_j(u) = b \sin ku$ with $k = (2j - 1)\pi/2$ for some integer $j$. Hence, the eigenvalues are $\theta_j = (2j - 1)^2\pi^2/4, j = 1, 2, \cdots$. The normalization (3) determines $b = \sqrt{2}$. The eigenfunctions are orthonormal satisfying (3) and

$$\int_0^1 f_i(u)f_j(u)du = 0, \quad i \neq j.$$ \hfill (9)

The representation for the covariance function is therefore

$$\rho(u, v) = \sum_{j=1}^{\infty} \frac{8}{\pi^2(2j - 1)^2} \sin[(2j - 1)\pi u/2] \sin[(2j - 1)\pi v/2]$$
Representation of Brownian motion

The Brownian motion has the representation

\[ W(u) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\theta_j}} f_j(u) X_j = \sum_{j=1}^{\infty} \frac{2\sqrt{2}}{\pi(2j-1)} \sin\{\pi(2j-1)u/2\} X_j, \quad (10) \]

where \( X_1, X_2, \ldots \) are independent \( N(0,1) \) variables. Another representation will be given in (14) below.

From (10) and the orthonormal properties (3) and (9) we obtain

\[ \int_0^1 W^2(u) du = \sum_{j=1}^{\infty} \frac{X_j^2}{\theta_j} = \sum_{j=1}^{\infty} \frac{4X_j^2}{(2j-1)^2\pi^2}. \]

The Brownian Bridge

For the Brownian bridge, with \( \rho(u, v) = h(u, v) \) given in (6), the differential equation (7) holds with general solution (8). Since \( h(0, v) = h(1, v) = 0, \quad 0 \leq v \leq 1 \), the boundary conditions for the solution (8) are \( f(0) = f(1) = 0 \), which imply \( a = 0 \) and \( k = \pi j \) for some \( j \). Hence the eigenvalues are now \( \theta_j = \pi j^2, j = 1, 2, \ldots \), and the eigenfunctions are \( f_j(u) = b \sin \pi j u; \) use of (3) again determines \( b = \sqrt{2} \). The eigenfunctions are orthonormal satisfying (3) and (9).

Thus the representation for the covariance function of the Brownian bridge is

\[ h(u, v) = \sum_{j=1}^{\infty} \frac{1}{\theta_j} f_j(u) f_j(v) = \sum_{j=1}^{\infty} \frac{2}{\pi^2 j^2} \sin j\pi u \sin j\pi v. \]

Representation of the Brownian bridge

The Brownian bridge has the representation

\[ B(u) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\theta_j}} f_j(u) X_j = \sum_{j=1}^{\infty} \frac{\sqrt{2}}{\pi j} \sin \pi j u X_j, \quad (11) \]

where \( X_1, X_2, \ldots \) are independent \( N(0,1) \) variables. From (11) and the orthonormal properties (3) and (9) we obtain

\[ \int_0^1 B^2(u) du = \sum_{j=1}^{\infty} \frac{X_j^2}{\theta_j} = \sum_{j=1}^{\infty} \frac{X_j^2}{\pi^2 j^2}. \quad (12) \]
Note that
\[ \int_0^1 h(u, u)du = \sum_{j=1}^{\infty} \tilde{\theta}_j^{-1} = \sum_{j=1}^{\infty} (\pi^2 j^2)^{-1} = 1/6. \] (13)
Thus 1/6 is the expectation of the left-hand side of equation (12). This integral arises in the theory of goodness-of-fit tests. (See Section 7 below.)

The connection between the Brownian bridge and Brownian motion can be used to give another representation of Brownian motion. Since \( \min(u, v) = h(u, v) + uv, \ 0 \leq u, v < 1, \) we can write Brownian motion for \( 0 \leq u \leq 1 \) as

\[ W(u) = B(u) + uX_0 \]
\[ = \sum_{j=1}^{\infty} \frac{\sqrt{2}}{\pi j} \sin \pi j u \ X_j + uX_0, \quad 0 \leq u \leq 1, \]

where \( X_0 \sim N(0, 1) \) independently of \( X_1, X_2, \ldots. \) This representation is Gaussian with mean zero, and its covariance is \( \min(u, v), \) as required.

Throughout the paper a set \( X_0, X_1, \ldots \) will consist of i.i.d. \( N(0, 1) \) variables. Also, \( 1 \) will be a vector with all components 1, and \( 0 \) will be a vector with all components 0. Matrix \( I \) will be the identity matrix. The length of vectors \( 1 \) and \( 0, \) and the size of \( I \) will be determined by the context.

### 3 Discrete Brownian Motion

The *discrete Brownian motion* on \([0, 1]\) is the Brownian motion process \( W(u) \) sampled at the \( n + 1 \) equally-spaced values \( u = 0, 1/n, \ldots, n/n = 1. \) This sampling produces \( n + 1 \) random variables

\[ W_j = W \left( \frac{j}{n} \right) \]

with \( \mathcal{E}W_j = 0 \) and \( \mathcal{E}W_iW_j = \min(i, j)/n. \) Let \( \sqrt{n}W_j = V_j, \ j = 0, 1, \ldots, n. \) Then \( \mathcal{E}V_j = 0 \) and \( \mathcal{E}V_iV_j = \min(i, j), \ i, j = 0, 1, \ldots, n; \) also \( W_0 = V_0 = 0. \)

Let \( G \) be the \( n + 1 \) by \( n + 1 \) matrix with entries \( g_{ij} = \min(i, j). \) Also, let \( \tilde{G} \) be the matrix \( G \) with the first row and first column (whose entries are all zero) omitted.
To obtain a representation of \( V_j \) corresponding to that of \( W(u) \) in (10), we need the characteristic values \( \nu \) and characteristic vectors \( y = (y_0, \cdots, y_n)' \) of \( G \). They are the solutions of

\[
\nu y_i = \sum_{j=0}^{n} g_{ij} y_j = \sum_{j=0}^{i-1} j y_j + i \sum_{j=i}^{n} y_j, \quad i = 0, 1, \cdots, n, \tag{15}
\]

where the first sum on the right-hand side of (15) is interpreted as 0 for \( i = 0 \). The first difference of \( \nu y_i \) is

\[
\nu (y_{i+1} - y_i) = \sum_{j=i+1}^{n} y_j, \quad i = 0, 1, \cdots, n-1. \tag{16}
\]

The second difference is

\[
\nu (y_{i+1} - y_i) - \nu (y_i - y_{i-1}) = \nu (y_{i+1} - 2y_i + y_{i-1}) = -y_i, \quad i = 1, \cdots, n-1. \tag{17}
\]

The general solution to the difference equation (17) is

\[
y_i = a \cos ki + b \sin ki, \quad i = 0, 1, \cdots, n, \tag{18}
\]

and \( \nu = 1/(4 \sin^2 k/2) \). One boundary condition is \( 0 = y_0 = a \). Then \( y_i = b \sin ki \). Substitution in (16) for \( i = n - 1 \) yields

\[
4 \sin^2 \frac{k}{2} \sin kn = \sin kn - \sin k(n-1) = \sin kn - \sin kn \cos k + \cos kn \sin k = 2 \sin^2 \frac{k}{2} \sin kn + 2 \cos kn \sin \frac{k}{2} \cos \frac{k}{2}
\]

or

\[
0 = \cos kn \cos \frac{k}{2} - \sin kn \sin \frac{k}{2} = \cos k \left( n + \frac{1}{2} \right).
\]

This equation is satisfied by \( k = \frac{2j-1}{2n+1} \pi, \quad j = 1, \cdots, n \). Thus the \( j \)-th characteristic root is

\[
\nu_j = \frac{1}{4 \sin^2 \frac{2j-1}{2(n+1)} \pi}, \quad j = 1, \cdots, n.
\]
The corresponding characteristic vector $y_j, \ j \neq 0$, has as $i$-th component $y_{ji} = b \sin ki$; choosing $b$ to normalize the vector to unit length gives

$$y_{ji} = \frac{2}{\sqrt{2n+1}} \sin \frac{2j - 1}{2n+1} i \pi, \quad j = 1, \ldots, n, \quad i = 0, 1, \ldots, n.$$  

Let $y = (y_0, \tilde{y})'$. The characteristic equation $Gy = \nu y$ in partitioned form is

$$\begin{bmatrix} 0 & 0' \\ 0 & \tilde{G} \end{bmatrix} \begin{bmatrix} y_0 \\ \tilde{y} \end{bmatrix} = \nu \begin{bmatrix} y_0 \\ \tilde{y} \end{bmatrix}. \quad (19)$$

Note that $\nu_0 = 0$ and $y_0 = (y_0, \tilde{y})' = (1, 0')'$ is a solution to (19). If $\nu \neq 0$ (that is, $\nu = \nu_j, j \neq 0$), (19) implies $y_0 = 0$. Define $\tilde{Y}$ to be the matrix whose columns are $\tilde{y}_i, \ i = 1, \ldots, n$, and $\tilde{N}$ to be the diagonal matrix with entries $\nu_i, \ i = 1, \ldots, n$. Further, define

$$\begin{align*}
Y &= (y_0, y_1, \ldots, y_n) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \tilde{y}_1 & \cdots & \tilde{y}_n \end{bmatrix} = \begin{bmatrix} 1 & 0' \\ 0 & \tilde{Y} \end{bmatrix}, \\
N &= \text{diag}(\nu_0, \nu_1, \ldots, \nu_n) = \begin{bmatrix} 0 & 0' \\ 0 & \text{diag}(\nu_1, \ldots, \nu_n) \end{bmatrix} = \begin{bmatrix} 0 & 0' \\ 0 & \tilde{N} \end{bmatrix}.
\end{align*}$$

Then

$$G = YNY' = \begin{bmatrix} 0 & 0' \\ 0 & \tilde{Y} \tilde{N} \tilde{Y}' \end{bmatrix}. \quad (20)$$

The individual entries of $G$ are

$$g_{ij} = \sum_{k=1}^{n} \frac{1}{(2n+1) \sin^2 \frac{2k-1}{2(2n+1)} \pi} \sin \left( \frac{2k-1}{2n+1} i \pi \right) \sin \left( \frac{2k-1}{2n+1} j \pi \right), \quad (21)$$

where $i, j = 1, \ldots, n$.

The differencing of $y_i$ in (16) and (17) shows that

$$\tilde{G}^{-1} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix},$$
whose characteristic roots are \( \tilde{v}_j = 1/\nu_j = 4\sin^2[(2j - 1)\pi/(4n + 2)], \ j = 1, \cdots, n. \)

**Representation of discrete Brownian motion**

Let \( \tilde{\nabla} = (V_1, \cdots, V_n)' \) and \( \nabla = (V_0, \tilde{\nabla})' \). Then the covariance matrix of \( \nabla \) is \( \mathcal{E}\nabla\nabla' = G = \nabla\nabla' \) partitioned as in (20); that is, \( V_0 = 0 \) and \( \mathcal{E}\tilde{\nabla}\nabla' = \tilde{\nabla}\nabla' \). Let \( \tilde{X} = \tilde{\nabla}^{-\frac{1}{2}}\tilde{\nabla}'\tilde{\nabla} = (X_1, \cdots, X_n)' \), say, and \( X = (X_0, \tilde{X}')' \) with \( X_0 = 0 \). Then, since \( \tilde{\nabla} \) is orthogonal, we have \( \tilde{\nabla} = \tilde{\nabla}\tilde{\nabla}' \tilde{\nabla} \); also \( \mathcal{E}\tilde{X}\tilde{X}' = 1 \). Thus \( X_j \) are i.i.d. standard normal random variables, as defined in Section 2. In components \( V_0 = 0 \) and a representation of \( V_j \) is

\[
V_j = \sum_{r=1}^{n} \frac{\sin \frac{2r-1}{2n+1} \pi}{\sqrt{2n+1} \sin \frac{2r-1}{2} \pi} X_r, \quad j = 1, \cdots, n.
\]

An approximation to \( \int_0^1 W^2(u)du \) is

\[
\frac{1}{n} \sum_{j=0}^{n} W^2(\frac{j}{n}) = \frac{1}{n^2} \nabla' \nabla = \frac{1}{n^2} \tilde{X}' \tilde{\nabla}^{-\frac{1}{2}} \tilde{\nabla}' \tilde{\nabla} \tilde{\nabla}' \tilde{\nabla}^{-\frac{1}{2}} \tilde{X} = \frac{1}{n^2} \tilde{X}' \tilde{\nabla} \tilde{X}
\]

\[
= \sum_{j=1}^{n-1} \frac{X_j^2}{4n^2 \sin^2 \frac{2r-1}{2(2n+1)} \pi}.
\]

**4 The Discrete Brownian bridge**

The **discrete Brownian bridge** is the Brownian bridge process \( B(u) \) sampled at the values \( u = 0, 1/n, \cdots, n/n = 1 \), as was done for discrete Brownian motion. This produces \( n+1 \) random variables

\[
B_j = B(\frac{j}{n}) , \quad j = 0, 1, \cdots, n.
\]

These random variables have expected value \( \mathcal{E}B_j = 0 \) and covariance

\[
\mathcal{E}B_iB_j = h\left(\frac{i}{n}, \frac{j}{n}\right) = \min\left(\frac{i}{n}, \frac{j}{n}\right) - \frac{i}{n} \cdot \frac{j}{n} \quad (22)
\]

\[
= \frac{n \min(i, j) - ij}{n^2}, \quad i, j = 0, 1, \cdots, n.
\]
It will be convenient to study \( Z_j = nB_j, j = 0, 1, \ldots, n \), with mean \( \mathbb{E}Z_j = 0 \) and covariance \( \mathbb{E}Z_i Z_j = n^2 h(i/n, j/n) \). Let \( Z = (Z_0, Z_1, \ldots, Z_n)' \), and let \( \tilde{Z} = (Z_1, \ldots, Z_{n-1})' \). The covariance matrix \( \mathbb{E}ZZ' \) is the \( n+1 \) by \( n+1 \) matrix \( K \) defined by

\[
K = (k_{ij}) = (n \min(i, j) - ij) = \left[ n^2 h\left( \frac{i}{n}, \frac{j}{n} \right) \right], \quad i, j = 0, 1, \ldots, n. \tag{23}
\]

As for discrete Brownian motion above, for a representation of \( Z_j \), we need the characteristic roots and vectors of \( K \). The equation for a characteristic vector \( s = (s_0, s_1, \ldots, s_n)' \) and root \( \lambda \) of \( K \) is

\[
\lambda s_i = \sum_{j=0}^{n} k_{ij} s_j = n \sum_{j=0}^{i-1} j s_j + n i \sum_{j=i}^{n} s_j - i \sum_{j=0}^{n} j s_j, \quad i = 0, 1, \ldots, n, \tag{24}
\]

where the first sum on the right-hand side of (24) is interpreted as 0 for \( i = 0 \).

The first difference of \( \lambda s_i \) is

\[
\lambda(s_{i+1} - s_i) = n \sum_{j=i+1}^{n} s_j - \sum_{j=0}^{n} j s_j, \quad i = 0, 1, \ldots, n - 1. \tag{25}
\]

The second difference is

\[
\lambda(s_{i+1} - s_i) - \lambda(s_i - s_{i-1}) = \lambda(s_{i+1} - 2s_i + s_{i-1}) = -n s_i, \tag{26}
\]

which is identical to (17) with \( \nu \) replaced by \( \lambda/n \) and \( y_i \) replaced by \( s_i \). The general solution to the difference equation (26) is (18) with \( y_i \) replaced by \( s_i \).

When we substitute (18) into (26) we obtain

\[2\lambda(\cos k - 1)s_i = -ns_i.\]

So \( \lambda = n/[2(1 - \cos k)] = n/(4 \sin^2 k/2) \). The boundary conditions are

\[0 = s_0 = a,\]

\[0 = s_n = a \cos kn + b \sin kn = b \sin kn,\]
which implies \( kn = j\pi \) for some \( j = 1, \ldots, n - 1 \). Thus the characteristic roots of \( K \) are

\[
\lambda_j = \frac{n}{2(1 - \cos \frac{j\pi}{n})} = \frac{n}{4 \sin^2 \frac{j\pi}{2n}}, \quad j = 1, \ldots, n - 1,
\]  

(27)

and \( \lambda_0 = \lambda_n = 0 \). The corresponding characteristic vector \( s_j \), for \( j \neq 0 \) or \( n \), has as \( i \)-th component

\[
s_{ji} = \sqrt{\frac{2}{n}} \sin \frac{pij}{n}, \quad i = 0, 1, \ldots, n,
\]

(28)

and \( \sum_{i=0}^{n} s_{ji}^2 = 1 \).

Let \( \tilde{s} \) be the vector \((s_1, \ldots, s_{n-1})'\). and let the \( n - 1 \) by \( n - 1 \) matrix \( \tilde{K} \) be the matrix \( K \) with first and last rows and first and last columns omitted; the entries in these rows and columns are zero.

The characteristic equation \( KS = \lambda s \) has the form

\[
\begin{bmatrix}
0 & 0' & 0 \\
0 & \tilde{K} & 0 \\
0 & 0' & 0
\end{bmatrix}
\begin{bmatrix}
s_0 \\
\tilde{s} \\
s_n
\end{bmatrix}
= \lambda
\begin{bmatrix}
s_0 \\
\tilde{s} \\
s_n
\end{bmatrix}.
\]

(29)

If \( \lambda = 0 \), (29) implies \( \tilde{s} = 0 \). The vectors \((1, 0', 0)' = s_0 \) and \((0, 0', 1)' = s_n \) satisfy (29) with \( \lambda_0 = \lambda_n = 0 \). If \( \lambda \neq 0 \), (29) implies \( s_0 = s_n = 0 \). Define \( \tilde{S} \) and \( \tilde{\Lambda} \) by partitioning \( S \) and \( \Lambda \) as follows:

\[
S = (s_0, s_1, \ldots, s_n) = \begin{bmatrix}
1 & 0' & 0 \\
0 & \tilde{S} & 0 \\
0 & 0' & 1
\end{bmatrix},
\]

\[
\tilde{\Lambda} = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_n) = \begin{bmatrix}
0 & 0' & 0 \\
0 & \tilde{\Lambda} & 0 \\
0 & 0' & 0
\end{bmatrix}.
\]

We can write

\[
K = SAS' = \begin{bmatrix}
0 & 0' & 0 \\
0 & \tilde{S} \tilde{\Lambda} \tilde{S}' & 0 \\
0 & 0' & 0
\end{bmatrix}.
\]

(30)
The individual entries of (30) are
\[ k_{ij} = \sum_{r=1}^{n-1} \frac{1}{2 \sin^2 \frac{r \pi}{2n}} \sin \frac{r i \pi}{n} \sin \frac{r j \pi}{n}, \quad i, j = 0, \ldots, n + 1. \] (31)

Note that \( \tilde{S} \tilde{S}' = I = \tilde{S}' \tilde{S} \) and
\[ \text{tr } K = \sum_{i=0}^{n} (ni - i^2) = \frac{n(n^2 - 1)}{6} = \sum_{i=0}^{n} \lambda_i = n \sum_{j=1}^{n-1} \left( 4 \sin^2 \frac{2 \pi j}{n} \right)^{-1}. \]

We have \( Z = (Z_0, \tilde{Z}', Z_n)', \) and \( \mathcal{E} ZZ' = K = SAS' \) partitioned as in (30). Define \( \tilde{X} = (X_1, \ldots, X_{n-1})' = \tilde{A}^{-\frac{1}{2}} \tilde{S}' \tilde{Z}. \) Then \( \tilde{Z} = \tilde{S} \tilde{A}^{\frac{1}{2}} \tilde{X}, \) and \( Z_0 = Z_n = 0. \) Also, since \( \mathcal{E} \tilde{X} \tilde{X}' = I, \) the \( X_j, \quad j = 1, \ldots, n \) are i.i.d. standard normal variables as in Section 2.

**Representation of the discrete Brownian Bridge**

A representation of \( B_i \) is then
\[ B_i = \frac{Z_i}{n} = \sum_{k=1}^{n-1} s_{ki} \sqrt{\lambda_k} X_k = \sum_{k=1}^{n-1} \frac{\sqrt{2 \sin \frac{\pi k}{n^2}}}{2n \sin \frac{\pi k}{2n}} X_k, \quad i = 0, 1, \ldots, n. \]

This is to be compared with (11) for \( u = i/n. \)

An approximation to \( \int_{0}^{1} B^2(u) du \) is
\[ \frac{1}{n} \sum_{j=0}^{n} B_j^2 \left( \frac{j}{n} \right) = \frac{1}{n} \sum_{j=0}^{n} B_j^2 = \frac{1}{n^3} Z' Z \]
\[ = \frac{1}{n^3} \tilde{X} \tilde{A}^{\frac{1}{2}} \tilde{S} \tilde{S}' \tilde{A}^{\frac{1}{2}} \tilde{X} = \frac{1}{n^3} \tilde{X} \tilde{A} \tilde{X} \]
\[ = \frac{1}{n^3} \sum_{j=1}^{n-1} \lambda_j X_j^2 = \sum_{j=1}^{n-1} \frac{1}{(2n \sin \frac{\pi}{2n})^2} X_j^2. \] (32)

This is to be compared with (12). Note that
\[ 2n \sin \frac{j \pi}{2n} = 2n \left[ \frac{j \pi}{2n} - \frac{1}{3!} \left( \frac{j \pi}{2n} \right)^3 + \cdots \right] \]
\[ = j \pi - \frac{(j \pi)^3}{24n^2} + \cdots. \]
The differencing of $s_i$ in (25) and (26) shows that

$$
\tilde{K}^{-1} = \frac{1}{n} \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & 0 \\
0 & -1 & 2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 2
\end{bmatrix}.
$$

Then a generalized inverse of $K$ is

$$
K^- = \begin{bmatrix}
0 & 0' & 0 \\
0 & \tilde{K}^{-1} & 0 \\
0 & 0' & 0
\end{bmatrix}.
$$

The characteristic roots of $\tilde{K}^{-1}$ are $\tilde{\lambda}_j = 1/\lambda_j = 4(\sin^2 \frac{j\pi}{2n})/n$, $j = 1, \ldots, n-1$.

We now relate the characteristic equation (24) to the integral equation (2). The integral equation (2) can be approximated by

$$
f^*(\frac{i}{n}) = \theta^* \sum_{j=0}^{n} h(\frac{i}{n}, \frac{j}{n}) f^*(\frac{j}{n}) \frac{1}{n}, \quad i = 0, 1, \ldots, n,
$$

where $\theta^*$ approximates $\theta$. The normalization (3) is approximated by

$$
\sum_{j=0}^{n} f^{*2}(\frac{j}{n}) \frac{1}{n} = 1.
$$

Since $h(0, \frac{i}{n}) = h(1, \frac{i}{n}) = 0$, $j = 0, 1, \ldots, n$, $f^*(0) = f^*(1) = 0$. Then (33) and (34) can be written equivalently

$$
f^*(\frac{i}{n}) = \theta^* \left[ \frac{1}{2} h(\frac{i}{n}, 0) f^*(0) \frac{1}{n} + \sum_{j=1}^{n-1} h(\frac{i}{n}, \frac{j}{n}) f^*(\frac{j}{n}) \frac{1}{n} \right] + \frac{1}{2} h(\frac{i}{n}, 1) f^*(1) \frac{1}{n},
$$

$$
\frac{1}{2} f^{*2}(0) \frac{1}{n} + \sum_{j=1}^{n-1} f^{*2}(\frac{j}{n}) \frac{1}{n} + \frac{1}{2} f^{*2}(1) \frac{1}{n} = 1.
$$
These show that (35) and (36) are the numerical computation of the integrals (2) and (3) by the trapezoidal rule with \( n \) intervals.

The characteristic roots of \( \left[ \frac{1}{n} \, h \left( \frac{1}{n} \right) \right] \) are those of \( \left( 1/n^3 \right) \hat{K} \), namely \( \lambda_j/n^3 = 1/(4n^2 \sin^2 \frac{j\pi}{2n}) \), \( j = 1, \ldots, n-1 \), and two roots of 0. The \( i \)-th component of the \( j \)-th characteristic vector normalized by (34) is \( f_i^n(j/n) = \sqrt{n}s_{ji} = \sqrt{2}\sin \frac{j\pi}{n}, \quad j = 1, \ldots, n-1, \quad i = 0, 1, \ldots, n \). Note that \( f_i^n(j/n) = f_j(i/n) \) and \( \hat{\theta}_j = \pi^2 j^2 \) is approximated by

\[
\hat{\theta}_j^* = \frac{n^3}{\lambda_j} = \left( \frac{2n \sin \frac{j\pi}{2n}}{2n} \right)^2.
\]

We compare \( \sum_{j=1}^{\infty} \hat{\theta}_j^{-1} = 1/6 \) with \( \sum_{j=1}^{n-1} \hat{\theta}_j^{-1} = \frac{1}{6} \left( 1 - \frac{1}{n^2} \right) \); also \( 1/\hat{\theta}_j < 1/\hat{\theta}_j^* \).

From these facts it follows that \( (1/n) \sum_{j=0}^{n-1} B^2(j/n) \xrightarrow{d} \int_0^1 B^2(u) du \). (One can choose \( N \) so that \( \sum_{j=N+1}^{\infty} \hat{\theta}_j^{-1} \tilde{X}_j^2 < \epsilon \); then \( \sum_{j=N+1}^{n} \hat{\theta}_j^{-1} \tilde{X}_j^2 < \epsilon \).)

Since \( \hat{\theta}_j^* \to \hat{\theta}_j \) as \( n \to \infty \), \( \sum_{j=1}^{N} \hat{\theta}_j^* \tilde{X}_j^2 \xrightarrow{d} \sum_{j=1}^{N} \hat{\theta}_j \tilde{X}_j^2 \).

We can write \( W_i = B_i + \frac{1}{n} X_0 \), where \( X_0 \sim \tilde{N}(0,1) \); that is, the discrete Brownian motion is the discrete Brownian bridge plus a suitable multiple of a standard normal variable independent of the Brownian bridge. A representation of \( W_i \), corresponding to (14) above, is

\[
W_i = \sqrt{2} \sum_{j=1}^{n-1} \frac{\sin \frac{j\pi}{n}}{2n \sin \frac{j\pi}{2n}} X_j + \frac{i}{n} X_0;
\]

this expression may be compared to (14) for \( u = i/n \).

5 Serial Correlation Coefficients

In time series analysis serial correlation coefficients are used to estimate serial dependence and to test the null hypothesis of independence against the alternative of dependence of lag one. The simplest such serial correlation coefficient is

\[
r = \frac{\sum_{t=2}^{n-1} y_t y_{t-1}}{\sum_{t=1}^{n-1} y_t^2},
\]

where \( (y_1, \ldots, y_{n-1})' = y \) constitute an observed time series. To represent \( r \) in canonical form we want the eigenvalues of the numerator quadratic form.
Let

\[
A = \frac{1}{2} \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}.
\]

Then

\[
r = \frac{y' Ay}{y'y},
\]

and

\[
A = I - \frac{n}{2} \tilde{K}^{-1}.
\]

Hence the eigenvalues of \(A\) are

\[
1 - 2 \sin^2 \frac{j\pi}{2n} = \cos \frac{j\pi}{n}, \quad j = 1, \ldots, n - 1.
\]

When there is no serial dependence (\(\xi y' = \sigma^2 I\)), \(r\) is distributed as

\[
\frac{\sum_{j=1}^{n-1} \cos(j\pi/n) X_j^2}{\sum_{j=1}^{n-1} X_j^2},
\]

where \(X_j\) are i.i.d. \(N(0, 1)\).

An alternative to the simple serial correlation coefficient is to modify \(A\) defined above by the addition of \(\frac{1}{2}\) in the upper left corner and lower right corner to give \(A^*\), say. Then

\[
\frac{2y'(I - A^*)y}{y'y} = \frac{\sum_{t=1}^{n-1} (y_t - y_{t-1})^2}{\sum_{t=1}^{n-1} y_t^2},
\]

which is the ratio of the mean square successive difference to the variance. This was studied by von Neumann (1941) and formed the basis for the Durbin-Watson statistic. See Anderson (1948) and Section 6.5 of Anderson (1971) for more detail. Here

\[
2(I - A^*) = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 \\
0 & 0 & 0 & \cdots & -1 & 1
\end{bmatrix}.
\]
The eigenvalues are
\[ \lambda_j = 2(1 - \cos \frac{\pi j}{n - 1}) = 2\sin^2 \frac{\pi j}{2(n - 1)}, \quad j = 1, \ldots, n - 2, \]
and the normalized eigenvectors are \( x_j \) with components
\[ x_{ji} = \sqrt{\frac{2}{n - 1}} \cos \frac{\pi j(2i - 1)}{2(n - 1)}, \quad j = 1, \ldots, n - 2, \quad i = 1, \ldots, n - 1. \]

There is also an eigenvalue \( \lambda_{n-1} = 0 \), with eigenvector \( 1 = (1, 1, \ldots, 1)' \). The eigenvalues of \( A^* \) are \( \cos \pi j/(n - 1), \quad j = 1, \ldots, n - 2, \) and 1.

The autoregressive model of order 1, \( y_t = \phi y_{t-1} + u_t \), defines a stationary process if \( |\phi| < 1 \). When \( y_1, \ldots, y_{n-1} \) are observed, \( y_0 = 0 \), and the \( u_t \) are i.i.d. \( N(0, \sigma^2) \), the maximum likelihood estimator of \( \phi \) is \( \hat{\phi} = \frac{\sum_{t=2}^{n-1} y_t y_{t-1}}{\sum_{t=2}^{n-1} y_{t-1}^2} \), which differs from \( r \) by the absence of \( y_{n-1}^2 \) in the denominator. When \( \phi = 1 \) and \( y_0 = 0 \), the autoregressive process for \( t = 0, 1, \ldots, n \) is that of \( \sigma V_0 = \sigma \sqrt{n} W(0), \ldots, \sigma V_n = \sigma \sqrt{n} W(1) \). Then
\[ \hat{\phi} - \phi = \frac{\sum_{t=2}^{n-1} y_{t-1} u_t}{\sum_{t=2}^{n-1} y_{t-1}^2} \]
which is distributed as
\[ \frac{\sigma^2 V_{n-1}^2 - \sum_{t=1}^{n-1} u_t^2}{2\sigma^2 \sum_{t=1}^{n-2} V_t^2}. \]
The knowledge of discrete Brownian motion can be used to characterize the distribution of \( \hat{\phi} - \phi \), and the limiting distribution of \( n(\hat{\phi} - \phi) \) is the distribution of \( (W^2(1) - 1)/(2 \int_0^1 W^2(r)dr) \).


When a Gaussian process \( G(u) \) has mean zero, values \( G(0) = G(1) = 0 \), and a covariance function \( \rho(u, v) \) of the form \( \min(u, v) - uv - Z(u, v) \), it will be called a modified Brownian bridge. Modified Brownian bridges arise in goodness-of-fit tests when parameters must be estimated; see Stephens (1976) for examples in tests for distributions. They occur also in tests for time

The tests for time series are based on a comparison of the sample spectral distribution function with the distribution given by the model. Suppose $F_T(\lambda)$ is the standardized sample spectral distribution function, based on a sample of size $T$, of a stationary stochastic process. By a suitable transformation $\Lambda(u)$, the process $F_T(\lambda)$ may be transformed to a process on $[0, 1]$ with an asymptotic covariance function of the form

$$\rho_S(u, v) = \min(u, v) - uv + q(u)q(v), \quad (37)$$

where $q(u)$ depends on the true spectral distribution of the stochastic process, and $q(0) = q(1) = 0$ (Anderson, 1993).

In Anderson and Stephens (1993) two methods were used to approximate the eigenvalues of (2), with $\rho_S(u, v)$ as kernel. One method was to transform the above covariance function to the Fourier coefficients

$$\int_0^1 \int_0^1 [h(u, v) + q(u)q(v)]f_i(u)f_j(v)dudv = \frac{\delta_{ij}}{\bar{\theta}_i} + \alpha_i\alpha_j,$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$, $i \neq j$, $\bar{\theta}_j = \pi^2 j^2$, $j = 1, 2, \cdots$, and

$$\alpha_i = \int_0^1 q(u)f_i(u)du. \quad (38)$$

The last integral is computed by numerical integration as

$$a_i = \frac{1}{n} \sum_{j=0}^n q\left(\frac{i}{n}\right)f_i\left(\frac{j}{n}\right)$$

$$= \frac{1}{n} \left[\frac{1}{2}q(0)f_i(0) + \sum_{j=1}^{n-1} q\left(\frac{j}{n}\right)f_i\left(\frac{j}{n}\right) + \frac{1}{2}q(1)f_i(1)\right],$$

which is the trapezoidal rule, since $q(0) = q(1) = 0$. Because $f_i\left(\frac{j}{n}\right) = \sqrt{2}\sin\frac{\pi ji}{n}$, the Fast Fourier Transform can be used. Let

$$\Theta_{n-1}^* = \text{diag}(\bar{\theta}_1, \cdots, \bar{\theta}_{n-1}), \quad \alpha_{n-1}^* = (\alpha_1, \cdots, \alpha_{n-1})'.$$

Then the characteristic roots of $\Theta_{n-1}^{-1} + \alpha_{n-1}^*\alpha_{n-1}^*$ are the zeros $\nu$ of

$$|\Theta_{n-1}^{-1} + \alpha_{n-1}^*\alpha_{n-1}^* - \nu I_{n-1}| = |\Theta_{n-1}^{-1} - \nu I_{n-1}| T, \quad (39)$$

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where
\[
T = \left\{ 1 + \alpha_{n-1}^{*} \left( \Theta_{n-1}^{*-1} - \nu I_{n-1} \right)^{-1} \alpha_{n-1}^{*} \right\} \\
= \left\{ 1 + \sum_{j=1}^{n-1} \frac{\alpha_{j}^{2}}{\delta_{j}^{*-1} - \nu} \right\}.
\]

The zeros \( \nu \) of (39) approximate the reciprocals of the first \( n - 1 \) eigenvalues of \( h(u, v) + q(u)q(v) \).

The other method of Anderson and Stephens was to discretize \( h(u, v) + q(u)q(v) \), that is, approximate \( h(u, v) + q(u)q(v) \) by \( \frac{1}{n} \left[ h \left( \frac{i}{n}, \frac{j}{n} \right) + q \left( \frac{i}{n} \right)q \left( \frac{j}{n} \right) \right] \), \( i, j = 0, 1, \cdots, n \), find the characteristic roots of this matrix, and use the reciprocals of the nonzero roots as approximations to the first \( n - 1 \) eigenvalues of \( h(u, v) + q(u)q(v) \).

It follows from the above algebra that these two methods are the same, for the characteristic roots of \( \Theta_{n-1}^{*-1} + \alpha_{n-1}^{*} \alpha_{n-1}^{*} \) are the characteristic roots of the matrix whose \( i, j \)-th element is
\[
2 \sum_{g,k=1}^{n-1} \left[ \delta_{gk} + \alpha_{g} \alpha_{k} \right] \sin \pi i g \sin \pi j k = \frac{1}{n} \left[ h \left( \frac{i}{n}, \frac{j}{n} \right) + q \left( \frac{i}{n} \right)q \left( \frac{j}{n} \right) \right].
\]

The above description is an idealization of what Anderson and Stephens did. Actually \( n \) occurs in three roles: the number of terms in the numerical integration, the number of zeros of (39), and the order of \( \frac{1}{n} \left[ h \left( \frac{i}{n}, \frac{j}{n} \right) + q \left( \frac{i}{n} \right)q \left( \frac{j}{n} \right) \right] \).

In the above, \( n \) has been taken to be the same each time, but in Anderson and Stephens the \( n \)'s in these roles were different.

**Identities**

Before turning to further applications, it is interesting to observe that a number of identities (perhaps more entertaining than useful) can be derived from the above analysis. For example, the right-hand side of (21) must be identically equal to \( \min (i, j) \), and that of (31) must equal \( n \min (i, j) - ij \). Further identities come from (39). Briefly, when \( \alpha_{j} \) are all non-zero, the solutions \( \nu \) are given by setting \( T \) in (39) equal to zero. When however we already know the solutions, we have an identity. We illustrate with the example of the connection between Brownian motion and the Brownian Bridge, when both are continuous processes. The covariance for the Brownian Bridge (BB) is \( \min (u, v) - uv \), and that for Brownian Motion (BM) is
min(u, v); thus we have \( \text{cov}(BM) = \text{cov}(BB) + uv \), which is of the form (37) with \( q(u) = u \). The analysis goes through as above, but \( n \) is now infinite. Thus \( T = 1 + \sum_{j=1}^{\infty} \frac{\alpha_j^2}{(\hat{\vartheta}_j^{-1} - \nu)} \), with \( \hat{\vartheta}_j = \pi^2 j^2 \) and \( f_j(u) = \sqrt{2} \sin \pi j u \), the characteristic roots and corresponding characteristic functions for BB, from Section 2. From (38) we have \( \alpha_j^2 = \frac{2}{(\pi^2 j^2)} \), for \( j = 1, 2, \ldots \). These values are inserted into \( T \), and \( T \) set equal to zero, to give a solution \( \nu \). But a solution \( \nu \), is a value \( (2r - 1)^2 \pi^2 / 4 \), for \( r = 1, 2, \ldots \) given in Section 2. When a typical \( \nu \) is used in \( T \) we obtain the identity

\[
\frac{1}{2} = \sum_{j=1}^{\infty} \frac{1}{4[j/(2r - 1)]^2 - 1}.
\]

If the covariances are written \( \text{cov}(BB) = \text{cov}(BM) - q(u)q(v) \), and a similar analysis is made, \( T \) becomes \( 1 - \sum_{j=1}^{\infty} \frac{\alpha_j^*}{(\hat{\vartheta}_j^{-1} - \theta)} \), where \( \alpha_j^* \) is given by (38), with \( q(u) = u \) and \( f_j(u) = \sqrt{2} \sin \{(2j - 1)\pi u / 2\} \), the characteristic function for BM from Section 2. The integral gives \( \alpha_j^* = 32/((2j - 1)\pi^4) \). These values and the known solutions \( \theta_r = \pi^2 r^2 \) are inserted into \( T \) and \( T \) again set equal to zero, to give the identity

\[
\frac{\pi^2}{32} = \sum_{j=1}^{\infty} \frac{1}{4(2j - 1)^2 - (2j - 1)^4 / r^2}
\]

for any positive integer \( r \). When \( r = 1 \), the series converges to \( \pi^2 / 32 = 0.3084 \) to 4 d.p. in five terms; for higher values of \( r \) it converges more slowly. Both these identities may be verified by using partial fractions on the summand, and the well-known result (in the case of (40)), \( \pi^2 / 6 = \sum_{j=1}^{\infty} j^{-2} \). Other identities can be found from the discrete processes, but the integral for \( \alpha_j \) becomes a sum, and the algebra is much more complicated.

7 Applications in Goodness-of-Fit

7.1 Test for the uniform distribution

Suppose \( z_1, z_2, \ldots, z_N \) constitute a random sample of values \( z \) between 0 and 1. It is desired to test \( H_0 \): the distribution of \( z \) is the uniform distribution between 0 and 1, written \( U(0, 1) \). A statistic for testing the fit is the Cramér-von Mises statistic

\[
W^2 = N \int_{0}^{1} (F_N(z) - z)^2 dz,
\]

(41)
where \( F_N(z) \) is the empirical distribution function of the \( z_i \), defined by

\[
F_N(z) = \frac{\# \text{ of } z_i \leq z}{N}, \quad 0 \leq z \leq 1.
\]

When \( H_0 \) is true the limiting behavior of \( \sqrt{N}(F_N(z) - z) \), indexed by \( z \), is that of the Brownian bridge. Thus, in particular, the limiting distribution of \( W^2 \) is given by (12), and the mean is 1/6, from (13). (See Anderson and Darling, 1952, 1954.) The Kolmogorov-Smirnov statistic is \( D_N = \sup |F_N(z) - z| \); then \( \sqrt{N}D_N \) has a limiting distribution which is that of the supremum of the absolute value of the Brownian bridge.

The test for uniformity has considerable importance because any test for a completely specified continuous distribution can be reduced to a test for U(0,1). Also, a test for the exponential distribution with unknown scale may be reduced to such a test. (See Stephens, 1986a, b.)

An interesting adaptation of \( W^2 \) is the Watson statistic \( U^2 \) defined by

\[
U^2 = N \int_{0}^{1} [(F_N(z) - z) - \int_{0}^{1} F_N(y) - y] dy]^2 dz;
\]

\( U^2 \) can be expressed as \( \int_{0}^{1} Q^2(t) dt \) where \( Q(t) \) is the process \( B(t) - \int_{0}^{1} B(u) du; \) the covariance of \( Q(t) \) is then \( \min(s, t) - st + 1/12 - \{(s - s^2) + (t - t^2)\}/2 \). The process \( Q(t) \) is a modified Brownian bridge.

The interest in \( U^2 \) arises because it can (and, in general, should) be used for observations on a circle since its computed value does not depend on the origin used to calculate \( F_N(x) \).

### 7.2 Test for the discrete uniform distribution

The analogues of the statistics above have been explored for testing for discrete distributions by Choulakian, Lockhart and Stephens (1994). When the test is for the discrete uniform distribution with \( n \) cells, the limiting distributions of test statistics \( W^2 \) and \( U^2 \) depend on the discrete Brownian bridge. Suppose of \( N \) observations that \( o_i \) is the observed number of observations falling into cell \( i = 1, \cdots, n \). Let \( \Pr \{\text{observation is in cell } i\} = p_i \); then the expected number of observations is \( e_i = Np_i \). Define \( T_0 = 0 \), and

\[
T_j = \frac{1}{\sqrt{N}} \left( \sum_{i=1}^{j} o_i - \sum_{i=1}^{j} e_i \right), \quad j = 1, \cdots, n.
\]
The null hypothesis is \( p_i = 1/n, \ i = 1, \ldots, n \). Then the limiting distribution (as \( N \to \infty \)) of \( T_0, T_1, \ldots, T_n \) is the distribution of \( B_0, B_1, \ldots, B_n \), which is normal with covariances given by (22). The test statistics are

\[
W^2 = \frac{1}{n} \sum_{j=1}^{n-1} T_j^2 = \frac{1}{n} \mathbf{T}'\mathbf{T}
\]

and

\[
U^2 = \frac{1}{n} \mathbf{T}'(\mathbf{I} - 11'/n)(\mathbf{I} - 11'/n) \mathbf{T},
\]

where \( \mathbf{T} = (T_1, \ldots, T_n)' \). It follows from above that the limiting distribution of \( W^2 \) is given by (32).

To find the limiting distribution of \( U^2 \) we want the eigenvalues of the covariance matrix of \( (\mathbf{I} - 11'/n)\mathbf{T} \). Note that

\[
\mathcal{E} \mathbf{T} \mathbf{T}' = \frac{1}{n^2} \begin{bmatrix} \mathbf{K} & 0 \\ 0' & 0 \end{bmatrix} = \frac{1}{n^2} \mathbf{K}^*,
\]

say, which has eigenvalues \( 1/[4n \sin^2(j\pi/n)] \) from (27) and eigenvectors with components (28). Let \( \mathbf{B} = \mathbf{I} - 11'/n \) and \( \mathbf{Y} = \mathbf{B} \mathbf{T} \). Then \( U^2 = \mathbf{T}' \mathbf{B} \mathbf{B} \mathbf{T} = \mathbf{Y}'\mathbf{Y} \). The eigenvalues of \( \mathcal{E} \mathbf{Y} \mathbf{Y}' = \Sigma_\mathcal{Y} \) are the eigenvalues of \( \mathbf{B} \mathbf{K}^* \mathbf{B} \). Since \( \mathbf{B} \) is idempotent, the eigenvalues of \( \Sigma_\mathcal{Y} \) are those of \( \mathbf{B} \mathbf{K}^* \). From the analysis for \( W^2 \), we have \( \mathbf{K}^* \mathbf{x}_i = \lambda_i \mathbf{x}_i \), so \( \mathbf{B} \mathbf{K}^* \mathbf{x}_i = \lambda_i \mathbf{B} \mathbf{x}_i = \lambda_i (\mathbf{x}_i - \overline{x}_i \mathbf{1}) \), where \( \overline{x}_i = \sum_{j=1}^{n} x_{ij}/n \). The \( i \)th eigenvalue of \( \mathbf{K}^* \) is therefore the same as that of \( \mathbf{K}^* \) provided \( \overline{x}_i = 0 \).

For \( n \) odd \( n - 1 \) eigenvalues of \( \Sigma_\mathcal{Y} \) have multiplicity 2; their values are

\[
\lambda_i = \frac{1}{2n\{1 - \cos(i\pi/n)\}}, \quad i = 2, 4, \ldots, n - 1.
\]

For each value there are two eigenvectors. One of these is the \( \mathbf{x}_i \) for \( W^2 \) given above, and the other is \( \mathbf{x}_i^* \) whose \( j \)th component is

\[
x_{ij}^* = (2/n)^{1/2} \cos\{\pi ij/n\}, \quad j = 1, 2, \ldots, n.
\]

For \( n \) even, the eigenvalues are again given by (42) each with multiplicity 2, but for \( i = 2, 4, \ldots, n - 2 \); the eigenvectors are again \( \mathbf{x}_i \) and \( \mathbf{x}_i^* \). There is a further eigenvalue \( \lambda_{n-1} = 1/(4n^2) \) with corresponding eigenvector \( \mathbf{x}_{n-1} \).
\[(1/n)^{1/2}(-1,1,-1,\ldots,1)'\]. Let \(\lambda_i^*\) denote the \(i\)-th eigenvalue when the complete set of \(n - 1\) eigenvalues has been arranged in descending order, and let \(u_i^*\) be the corresponding eigenvector; the limiting distribution of \(U^2\) is then of the form \(\sum_{i=1}^{n} \lambda_i^* X_i^2\).

A heuristic explanation of the occurrence of matrix \(I - A^*\) in section 5 and matrix \(K^*\) above, one related to the inverse of the other, is that one enters into a statistic based on taking first differences, and the other into a statistic based on summing observed values. Because the statistics are quadratic forms, the differencing or summing processes occur twice; thus the eigenvalues and eigenvectors required in the distribution theory are those of the second-difference matrix and its inverse.

### 7.3 Test for the uniform distribution with censored data

Pettitt and Stephens (1976) have studied the effect of censoring on EDF tests for the continuous uniform distribution. Suppose the sample is censored with a fraction \(q\) missing from the lower end and a fraction \(1 - p\) missing from the upper end. The statistics will be called \(W^2_{q,p}\) and \(U^2_{q,p}\). The definition of \(W^2_{q,p}\) is the same as in (41) but with the limits of the integral from \(q\) to \(p\) instead of from 0 to 1. For \(U^2\) the definition is more complicated, and only \(U^2_{0,p}\) is considered. The authors gave computing formulas for these statistics.

The limiting distributions are now functionals of a truncated Brownian bridge. The solutions of the integral equation (2) are \(\lambda_i = m_i^2\), where \(m_i\) is a solution of

\[
\tan mp = m(p - q - 1)/(1 + m^2pq - m^2q);
\]

the (non-normalized) eigenfunctions are \(f_i(t) = a \cos m_it + b \sin m_it\), with \(a/b = \{m_iq - \tan(m_iq)/\{m_iq\tan(m_iq) + 1\}\}.\) For the special case of right-censoring, with \(q = 0\), we have the equation

\[
\tan mp = -m(1 - p)
\]

to give \(m_i\), and \(f_i(t) = \sqrt{2}A \sin m_it\) where the normalizing constant \(A\) is \(1/\{p - \sin m_ip \cos m_ip/m_i\}^{1/2}\).
For $U^2$, again with $q = 0$, the corresponding eigenvalues $\lambda_i$ are given by $4k_i^2$ where $k_i$ are given in two sets, the solutions of

$$\sin kp = 0$$

and of

$$\tan kp = -k(1 - p).$$

The solutions to (43) are $k_i = i\pi/p$, for $i = 1, 2, \ldots$ with corresponding non-normalized eigenfunction $f_i(t) = \cos 2k_i t$. If $k_i^*$ is a solution of (44), the corresponding non-normalized eigenfunction is

$$f_i(t) = \sin 2k_i^* t + k_i^*(1 - p) \cos 2k_i^* t.$$ 

When $p = 1$ these results are the same as those given for the non-truncated case.

### 7.4 Locally invariant tests

Watson (1993) has examined locally invariant tests, using the $U^2$ statistic, for the discrete uniform case. The alternative distribution to $H_0 : p_i = 1/n$ for all cells is $H_A : q_i = 1/n + \epsilon(p_i - 1/n)$, $i = 1, \ldots, n$, where $p'$ with components $p_i$ is a vector of probabilities and where $\epsilon$ will be allowed to tend to zero. Define the matrix $C$ to be the $n \times n$ shift-one circulant

$$C = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}$$

Suppose $M_i = C^i p$, and define $C^* = \sum_{i=1}^n M_i M_i - 11' / n$; Watson’s statistic for a locally most powerful invariant test is then $T = (p - e)' C^* (p - e) / N$. For its limiting distribution one must know the eigenvalues and eigenvectors of $C^*$. For $n$ odd, these occur in pairs, given by
\[ \lambda_j = \left( \sum_{i=1}^{n} p_i \cos(2\pi ji/n) \right)^2 + \left( \sum_{i=1}^{n} p_i \sin(2\pi ji/n) \right)^2, \]  

(45)

for \( j = 2, 4, \ldots, n - 1 \). For \( n \) even, the \( \lambda_j \) are again given by (45) for \( j = 2, 4, \ldots, n - 2 \), with an extra eigenvalue \( \lambda_n = (p_1 - p_2 + p_3 \ldots - p_n) \). The eigenvectors \( x_j \) and \( x_j^* \) corresponding to the two values of \( \lambda_j \) have components \( x_{ji} = \cos(2\pi ji/n) \), \( i = 1, \ldots, n \) and \( x_{ji}^* = \sin(2\pi ji/n) \), \( i = 1, \ldots, n \). The eigenvector corresponding to \( \lambda_n \) when \( n \) is even is cosine vector. Let \( \lambda_j^* \), \( j = 1, \ldots, n-1 \) denote the eigenvalues when the two sets are put together. The limiting distribution of \( nT \) is that of

\[ \sum_{j=1}^{n-1} \lambda_j^* X_j^2, \]

where, as before, \( X_j \) are i.i.d. standard normal variables. When \( n \) is odd this expression can be put in terms of i.i.d. exponential random variables, since the two random variables \( X_j^2 \) belonging to identical values of \( \lambda_j^* \) may be added together.

8 Simulation of a Gaussian process

The representations given above, in various applications, might be used to simulate a Gaussian process. For example, consider the representation (11) for the Brownian bridge. The functional of \( B(u) \) given by \( \sup_u |B(u)| \) is the Kolmogorov-Smirnov statistic for testing uniformity. In this application the distribution can be found exactly (for example, see Stephens, 1986a), but when tests are made using the probability integral transformation with estimated parameters, the distribution is, in general, not known. Then the covariance structure of \( \sqrt{N}{F_N(z) - z} \), where \( F_N(z) \) is defined in Section 7.1, is that of a modified Brownian bridge. It has the form \( \rho(u,v) = h(u,v) - Z(u,v) \), where \( h(u,v) \) is the Brownian bridge covariance and \( Z(u,v) \) is a function dependent on the distribution tested and on the parameter(s) estimated. Many examples are given in Stephens (1986a).

When \( Z(u,v) \) can be factored into \( Z(u,v) = q(u)q(v) \), one can still find the solutions of (2) analytically, but in any case they could be found numerically as described in Section 4. Then the process again could be represented
by the first equality in (10) with the appropriate eigenvalues \( \theta_j \) and corresponding eigenfunctions \( f_j(u) \). Of course, in practice, the sum must be truncated after a finite number of terms.

The alternative method of simulating a Gaussian process, and probably the most used, is to discretize the covariance as described in Section 4. To fix ideas, suppose one wishes to simulate the Brownian bridge. Its discrete version has the covariance \( K^* \), say, equal to \( K/n^2 \), where \( K \) is defined in (23). Suppose \( K^* = TT' \), where \( T \) is lower triangular, and suppose \( X \) is a vector of i.i.d. \( N(0,1) \) variables. The transformation \( Y = TX \) will produce a set of normal variables \( Y \) with covariance \( K^* \) and these can be used at the values \( y_i = i/n \), as an approximation to the Gaussian process.

In one of these methods one truncates the sum of normal variables, and in the other one first discretizes the covariance. Chandra, Singpurwalla, and Stephens (1981, 1983) have explored the second method, applied to find the distribution of the Kolmogorov-Smirnov statistic used in testing for the Extreme-Value or Weibull distribution. There are obvious difficulties in trying to find the supremum of a continuous process when only the values at a set of discrete points are known. When it is attempted to overcome these by simulating at more and more points, the covariance matrix becomes bigger and bigger and new problems arise in accurately making the decomposition \( K^* = TT' \), followed by the transformation \( Y = TX \). (See Chandra, Singpurwalla, and Stephens, 1983.) It may be that the representation by a sum of normals will give better results. Further work is needed on comparing these two procedures.

References


