CONFIDENCE INTERVALS BASED ON RANK TESTS

BY

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Chapter 1. Nonparametric Confidence Regions: Construction and Efficiency

1.1 Introduction and Outline

The theory of inference occupies a central place in mathematical statistics, and is generally subdivided into the three areas of point estimation, interval estimation, and hypothesis testing. While the problems of point estimation and testing of hypotheses have been investigated very extensively and fruitfully, the area of interval estimation has tended to receive somewhat less attention; often, this has been due more to operational difficulties than theoretical ones. Moreover, it should be made clear that, far from being separate and distinct areas, these three subjects are intimately related with each other. In a certain sense, however, the subject of interval estimation is of the broadest interest of the three in that once we obtain a procedure which gives us a confidence region for a given unknown parameter we are then in a position to perform hypothesis tests concerning that parameter, as well as to obtain reasonable point estimates of the parameter.

Most of the work in statistical inference has been in the area of parametric techniques; that is, methods which are based on the assumption that the data come from one or more underlying distributions which belong to a restricted parametric family. In particular, the normal family of distributions has received the greatest amount of study. In recent years, however there has developed an increasing interest in nonparametric methods, particularly those based on rank statistics. These methods are valid under a much broader class of underlying distributions, usually the class of all continuous distributions. This
interest has derived from a number of causes. First, there has been a concern for questions of robustness of procedures, that is, the degree of their validity under departures from the assumed model. In practice, one is frequently confronted with situations where there is no compelling reason for assuming that any particular parametric model is true. Second, there are certain situations which arise, particularly in the behavioral sciences, where the data is of such a nature that it cannot really be handled by parametric methods, as in the case of ordinal data. Finally and perhaps most importantly, there is the question of efficiency robustness. A number of authors have shown that many procedures based on rank tests have good large sample properties relative to their parametric counterparts, even when the parametric procedure being compared is the best for the given situation. In particular, it was shown that the one- and two-sample normal scores tests have an asymptotic relative efficiency of at least one relative to the corresponding t-tests.

Thus the properties of guaranteed significance levels, wider applicability, and good efficiency robustness all serve to recommend the consideration of nonparametric procedures. Nevertheless, there still exists considerable reluctance on the part of many investigators to use nonparametric techniques. This is particularly true in respect to interval estimation, fostered principally by the belief that confidence regions based on rank statistics are difficult to construct in practice. We will show that, in many cases, this is not a fair assessment.
In this thesis we will deal primarily with the two-sample location problem. This problem is introduced in the next section where we describe the construction of confidence intervals for the location shift parameter based on linear rank tests. In section 1.3 we briefly examine the construction of nonparametric confidence regions for other classes of problems. Returning to the two-sample location problem, section 1.4 discusses how to reasonably define the efficiency of a confidence procedure. In this connection is cited a result of Sen [15] which relates the length of a confidence interval to the efficacy of the rank test on which it is based. In section 1.5 we introduce a flexible, or adaptive, confidence procedure which is based on using the shortest of several confidence intervals. It is demonstrated that the flexible procedure is asymptotically distribution-free and efficient relative to each of the original procedures.

Chapter 2 concerns itself with the lengths of nonparametric confidence intervals for the location shift parameter, and contains the principal theoretical results of this thesis. In Theorems 1 and 2, we give conditions under which the length, suitably normalized, is asymptotically normally distributed. Theorem 3 extends the result to the joint distribution of the lengths of several confidence intervals.

In Chapter 3, we consider a few specific examples of the results of Chapter 3 and apply these results to approximate the finite sample properties of the flexible procedure described in 1.5. Monte Carlo simulations are performed to evaluate the accuracy of the asymptotic approximations and to further study the properties of the flexible procedure for a few special cases.
1.2 **The Two Sample Location Problem.**

The two-sample location problem is one of the most frequently encountered in statistical applications. Most experiments involving a treatment group and a control group fall into this category. Formally, we have two random samples, \( X = (X_1, X_2, \ldots, X_m) \) and \( Y = (Y_1, \ldots, Y_n) \) where the \( X_i \) are drawn from a population with c.d.f. \( F(\cdot; \varphi) \) and the \( Y_i \) are drawn from \( F(\cdot; \psi) \). The location parameters \( \varphi \) and \( \psi \) are unknown, while \( F \) may or may not be known. We are interested in making inferences about \( \theta = \psi - \varphi \), the location shift of the population of \( Y \)'s relative to the population of \( X \)'s. We may wish to test that \( \theta \) has some particular value, often 0, or obtain a confidence interval for \( \theta \).

Consider the null hypothesis \( H_{\theta_0} : \theta = \theta_0 \). Under \( H_{\theta_0} \), \( X_1, \ldots, X_m, Y_1 - \theta_0, \ldots, Y_n - \theta_0 \) are independent and identically distributed. Define the \( N = m+n \) dimensional vector \( W(X, Y) = (w_1(X, Y), \ldots, w_N(X, Y)) \) as follows:

\[
    w_i(X, Y) = \begin{cases} 
    1 & \text{if the } i\text{-th smallest of } X_1, \ldots, X_m, Y_1, \ldots, Y_n \text{ is an } X, \\
    0 & \text{otherwise}.
    \end{cases}
\]

(1.1)

Let \( \mathbf{1}_n \) denote the \( n \)-dimensional vector consisting of all ones. Then assuming \( F \) is continuous, with probability one there will be no ties among \( (X, Y - \theta_0 \mathbf{1}_n) \) for a given \( \theta_0 \); hence \( W(X, Y - \theta_0 \mathbf{1}_n) \) will be uniquely defined and will consist of \( m \) ones and \( n \) zeroes. Under \( H_{\theta_0} \), \( W(X, Y - \theta_0 \mathbf{1}_n) \) takes on each of its \( \binom{N}{m} \) possible values with probability \( \binom{N}{m}^{-1} \). A \((1-\alpha)\)-level rank test of \( H_{\theta_0} \) rejects \( H_{\theta_0} \) if
is a member of a specified set of $\alpha(m) W$-vectors. Clearly, only certain values of $\alpha$ will make $\alpha(m)$ an integer, namely, all multiples of $(N^{-1})$.

A special class of rank tests is the class of tests based on linear rank statistics. These are statistics of the form

\begin{equation}
T_N(X, \bar{Y}) = \frac{1}{m} \sum_{i=1}^{N} a_{i,N} W_i(X, \bar{Y})
\end{equation}

where $(a_{i,N})_{i=1}^{N}$ are fixed constants referred to as scores. Under $H_0$, $T_N(X, \bar{Y})$ has a discrete distribution which is independent of $\theta_0$ and which is, in particular, the distribution of $T_0(X, \bar{Y})$ when the $X_i$ and $Y_j$ are i.i.d. We will call this distribution the null distribution of $T_N$. For fixed $\alpha$, define the lower and upper $\frac{\alpha}{2}$-critical points of the null distribution as follows:

\begin{equation}
T_{N, \alpha/2} = \sup \{ x : \Pr_0(T_N(X, \bar{Y}) \leq x) \leq \frac{\alpha}{2} \}
\end{equation}

\begin{equation}
T_{N, 1 - \alpha/2} = \inf \{ x : \Pr_0(T_N(X, \bar{Y}) \geq x) \leq \frac{\alpha}{2} \}.
\end{equation}

Let

\begin{equation}
\alpha_N = \Pr_0(T_N(X, \bar{Y}) < T_{N, \alpha/2}) + \Pr_0(T_N(X, \bar{Y}) > T_{N, 1 - \alpha/2}).
\end{equation}
Then it is easily seen that \( \alpha_N \leq \alpha \). Thus an exact \((1-\alpha_N)\)-level test of 
\( H_{\theta} \) 
has as its critical region the set of all \( W \)-vectors for which 
\[
T_{N, \frac{\alpha}{2}} \leq T_N(\bar{X}_n, Y-n - \theta \bar{1}_n) \leq T_{N, \frac{1-\alpha}{2}}.
\]

We can define a \((1-\alpha_N)\)-level confidence region for \( \theta \) based on \( T_N \) 
to be

\[
\left\{ \theta; T_{N, \frac{\alpha}{2}} \leq T_N(\bar{X}_n, Y-n - \theta \bar{1}_n) \leq T_{N, \frac{1-\alpha}{2}} \right\}
\]

For the location shift problem, a reasonable class of linear rank statistics 
to consider is the class of statistics having monotone and nonconstant 
sequences of scores \( \{a_{i, N}\}_{i=1}^{N} \). In this case the confidence region 
in (1.6) will always be an interval, which we will denote by \((\theta^L, \theta^U)\).

1.2.1 Constructing Confidence Intervals for the Location Shift.

For a given sample \((\bar{X}_n, \bar{Y}_n)\), let us consider the rank statistic 
\( T_N(\bar{X}_n, Y-n - \theta \bar{1}_n) \) as a function of \( \theta \). As \( \theta \) varies, the value of \( T_N \) can 
change only when the vector \( W(\bar{X}_n, Y-n - \theta \bar{1}_n) \) changes. For this to occur, 
there must be a switch in the rankings of an \( X_i \) and a \( Y_j - \theta \). Thus 
the vector \( W(\bar{X}_n, Y-n - \theta \bar{1}_n) \) changes only at values of \( \theta \) where \( X_i = Y_j - \theta \), 
for some \( i \) and \( j \).

Let us now order each of the samples \( X(1) < X(2) < \cdots < X(m) \) and 
\( Y(1) < Y(2) < \cdots < Y(n) \). Let \( d_{ij} = Y(j) - X(i) \), \( i = 1, \ldots, m, \ j = 1, \ldots, n \). 
From the argument above we know that for a given \((\bar{X}_n, \bar{Y}_n)\), 
\( T_N(\bar{X}_n, Y-n - \theta \bar{1}_n) \) 
considered as a function of \( \theta \) is a step function with possible jumps.
only at values of $\theta$ equal to the \(mn\) values of \(d_{ij}\). If we assume that
the scores \(\{a_{i,j}\}_{i=1}^{N}\) are nondecreasing (we may do this without loss of
generality, since multiplication of the scores by -1 does not alter the
test), then \(T_N(X, Y ; \theta_{R})\) is an increasing step function of \(\theta\), with
jumps at \(\theta = d_{ij}, i=1, \ldots, m, j=1, \ldots, n\). Note that \(T_N(X, Y ; \theta_{R})\) is
undefined at \(\theta = d_{ij}\) since at these points there are ties. For con-
venience, and without introducing difficulties, we can arbitrarily define
\(T_N(X, Y ; \theta_{R})\) at these points so as to make \(T_N\) right continuous. Now
the effect of going from \(\theta = d_{ij}^-\) to \(\theta = d_{ij}^+\) is to change the
\((i+j-1)^{th}\) and \((i+j)^{th}\) components of \(W(X, Y ; \theta_{R})\) from 1,0 to 0,1
and thus to change the value of \(T_N(X, Y ; \theta_{R})\) by the amount
\(\frac{1}{m}(a_{i+j,N}-a_{i+j-1,N})\). This suggests the following simple procedure for
obtaining the confidence interval \((\theta^L, \theta^U)\):

1. Order the individual samples \(X(1) < \cdots < X(m)\) and \(Y(1) < \cdots < Y(n)\).
2. Calculate \(d_{ij} = Y(j)-X(i)\) \(i=1, \ldots, m, j=1, \ldots, n\).
3. Order the \(d_{ij}\), \(d(1) < d(2) < \cdots < d(mn)\), keeping track of the
original subscripts of \(d(k), i(k)\) and \(j(k), k=1, \ldots, mn\). Let
\(d(0) = -\infty, d(mn+1) = +\infty\).
4. Letting \(T_N(\theta_k)\) denote the constant value of \(T_N(X, Y ; \theta_{R})\) for
\(\theta \in (d(k), d(k+1))\), calculate \(T_N(\theta_k)\) iteratively using the
recursive formula

\[
(1.7) \quad T_N(\theta_k) = T_N(\theta_{k-1}) + (a_{i(k),j(k)}-a_{i(k),j(k)-1,N})/m
\]
starting with \( T_{N_1}(0) = \frac{1}{m} \sum_{i=1}^{m} a_{i,N} \). Stop when \( k = k^* \) where \( k^* \) satisfies \( T_{N_1}(k^* - 1) < T_{N_1}^{\alpha_{2}} \leq T_{N_1}(k^*) \). Then \( \theta^L = d(k^*) \).

Similarly, beginning with \( T_{N_1}(mn) = \frac{1}{m} \sum_{i=m+1}^{N} a_{i,N} \) calculate \( T_{N_1}(k) \) from \( T_{N_1}(k+1) \) using (1.7) . Stop when \( k = k^{**} \), where \( T_{N_1}(k^{**}) \leq T_{N_1}^{1-\alpha_{2}} < T_{N_1}(k^{**}+1) \). Then \( \theta^U = d(k^{**}+1) \).

Alternatively, in step 4, instead of beginning the iterations at \( k = 0 \) and \( k = mn \) one could start at some other values \( k = k_1 \) and \( k = k_2 \), based perhaps on some initial guess of \( \theta^L \) and \( \theta^U \). For example, \( d(k_1) \) and \( d(k_2) \) could be chosen to be the endpoints of the confidence interval based on the Wilcoxon statistic, which are particularly easy to obtain, as seen below. \( T_{N_1}(k_1) \) and \( T_{N_1}(k_2) \) would be evaluated directly from the definition of \( T_{N_1}(X_{\alpha_2} - \theta_{\alpha_2}) \) in (1.2) , using any value of \( \theta \) in \((d(k_1), d(k_1+1))\) and \((d(k_2), d(k_2+1))\) respectively. One would then proceed iteratively as in step 4, increasing or decreasing \( k \) according as \( T_{N_1}(k_1) \) is less than or greater than the corresponding critical value.

Confidence intervals for \( \theta \) based on the two-sample Wilcoxon test were considered by Lehmann [11] and a graphical construction procedure described by Moses [12]. The intervals in this case are particularly easy to obtain, since the scores for the Wilcoxon statistic are \( a_{i,N} = 1 \) and so the jumps in \( T_N \) are of constant size \( \frac{1}{m} \). Since

\[
T_{N_1}(0) = \frac{1}{m} \sum_{i=1}^{m} i = \frac{m+1}{2},
\]

it follows that \( T_{N_1}(k) = \frac{k}{m} + \frac{m+1}{2} \) and hence that

\[
k^* = mT_{N_1}^{\alpha_{2}} - \frac{m(m+1)}{2}.
\]

Similarly, \( k^{**} = \frac{m(N+n+1)}{2} - mT_{N_1}^{\alpha_{2}} \), using the
symmetry of the null distribution of $t_N$. Thus $\theta^L$ and $\theta^U$ are simply fixed order statistics of the $d_{ij}$. 
1.3 Confidence Regions For Other Types of Problems.

In the two-sample location problem we have seen that the boundary points of confidence regions for the location shift parameter based on any rank test must be among the points \( d_{ij} = Y(j)-X(i) \), \( i=1, \ldots, m \) \( j=1, \ldots, n \). In a similar way, the boundaries of confidence regions based on rank tests for parameters of other types of problems can be characterized. Bauer [2] characterizes these confidence regions for the two-sample location, two-sample scale with known location, and one-sample center of symmetry problems, and describes procedures for constructing the regions. Thus, some of the discussion in this and the previous section duplicates Bauer's results. However, our methods of construction in these cases differ somewhat from those of Bauer and are, we believe, more straightforward to apply. In addition, we consider the two-sample scale with unknown common location problem, the general location and scale problem, and the linear regression problem.

1.3.1 Two-Sample Scale Problem, Known Common Location.

We assume that \( X_1-\mu, X_2-\mu, \ldots, X_m-\mu, (Y_1-\mu)/\sigma, \ldots, (Y_n-\mu)/\sigma \) are i.i.d. according to some unknown absolutely continuous c.d.f. \( F \), where \( \mu \) is known and we are interested in the value of \( \sigma > 0 \). Under \( H_0: \sigma = \sigma_0 \), the \( \binom{m+n}{m} \) possible values of \( W(X-\mu_{1m}, \frac{1}{\sigma_0} (X-\mu_{1n})) \) are equally likely, where \( W(\cdot, \cdot) \) is as defined in (1.1). Hence, rank tests for the two sample scale problem are defined in the same way as for the two-sample location problem. If we consider linear rank statistics \( T_N \) as defined in (1.2), then \( T_N(X-\mu_{1m}, \frac{1}{\sigma} (X-\mu_{1n})) \) considered as a function of \( \sigma \)
is a step function with possible jumps at values of \( \sigma \) satisfying
\[ X_i - \mu = \frac{1}{\sigma} (Y_j - \mu), \quad i=1, \ldots, m, \quad j=1, \ldots, n \]
and \( \sigma = \frac{Y_j - \mu}{X_i - \mu} \). Since \( \sigma \) is assumed positive, we need only consider those pairs \((i,j)\) for which
\[ \text{sign} (Y_j - \mu) = \text{sign} (X_i - \mu). \]
Thus any confidence region for \( \sigma \) based on a rank test can have as its boundary points only points from among the finite set
\[ \{ r_{ij} = \frac{Y_j - \mu}{X_i - \mu}, \quad \text{sign}(Y_j - \mu) = \text{sign}(X_i - \mu), \quad i=1, \ldots, m, \quad j=1, \ldots, n \} \cup \{0, \infty\}. \]

Letting \( p \) be the number of \( r_{ij} \) in the set above, and letting
\[ r^{(1)} < r^{(2)} < \cdots < r^{(p)} \]
be the ordered values of the \( r_{ij} \), with
\[ r^{(0)} = 0, \quad r^{(p+1)} = \infty, \]
and defining \( i(k), j(k) \) and \( T_{N_i}(k) \) in the same way as in Section 1.2 (substituting \( r_i \)'s for \( d_i \)'s), it follows that
\[ T_{N_i}(k) = T_{N_i}(k-1) + \frac{1}{m} (a_i(k)+j(k), N-a_i(k)+j(k)-1, N) \text{sign}(X_i(k)-\mu). \]

Thus we can construct the confidence region for \( \sigma \) by evaluating \( T_{N_i}(k) \) iteratively using (1.8). However, it is clear from (1.8) that the jumps in \( T_N \) may not all be of the same sign and hence, for a given pair of samples \( X_i, X_j \), \( T_N \) may not be a monotonic step function of \( \sigma \).

Consequently the confidence region may not be an interval, and thus, unlike the two-sample location problem, we must evaluate all of the \( T_{N_i}(k), \quad k=1, \ldots, p \). To illustrate, consider the following hypothetical example:
\[ \bar{X} = (-2, 2, 3) \]
\[ \bar{X} = (-5, 4, 7, 8, 9) \]
\[ \mu \text{ is assumed to be } 0. \]
The Ansari-Bradley [1] statistic $S$ is a well known linear rank statistic for the two-sample scale problem, with scores $a_{i,N} = \left| i - \frac{N+1}{2} \right|$. Suppose we want to obtain a confidence region for $\sigma$ based on the one-sided test with critical region $38 \leq 3 \frac{1}{2}$. This test has significance level $\frac{50}{56}$. (There are $\binom{8}{3} = 56$ possible $W$-vectors and $38 \leq 3 \frac{1}{2}$ for 6 of them). Calculating the $r_{ij}$, ordering them, and calculating the values of $3S_{Ni}(k)$, we obtain:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(k)$</td>
<td>0</td>
<td>1.33</td>
<td>2</td>
<td>2.33</td>
<td>2.5</td>
<td>2.67</td>
<td>3</td>
<td>3.5</td>
<td>4</td>
<td>4.5</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$3S_{Ni}(k)$</td>
<td>$\frac{4}{2}$</td>
<td>$\frac{4}{2}$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{4}{2}$</td>
<td>$\frac{5}{2}$</td>
<td>$\frac{6}{2}$</td>
<td>$\frac{7}{2}$</td>
<td>$\frac{7}{2}$</td>
<td>$\frac{8}{2}$</td>
<td>$\frac{9}{2}$</td>
<td></td>
</tr>
</tbody>
</table>

Thus we see that $38 \leq 3 \frac{1}{2}$ only for $2 < \theta < 2.33$ and the confidence region for $\sigma$ is $(0,2) \cup (2.33, \infty)$.

Now the Ansari-Bradley statistic has scores which decrease as $i$ goes from 1 to $\left\lfloor \frac{N+1}{2} \right\rfloor$, and increase thereafter. Hence it is clear that $S$ ought to have reasonably good power when the location parameter $\mu$ is close to the median of $X$. On the other hand, if $P(X \leq \mu) = F(0)$ is near 1 or 0, then $S$ will have relatively low power. For example, if $F$ is negative exponential, then all observations lie to the right of the location parameter. In this case we would want to use a linear rank test which has monotone scores. The reason we do not get an interval in the example above is essentially that the assumed common location 0 is not close to the median of the particular observed combined sample.
One way to guarantee that our confidence region for $\sigma$ is always an interval is by choosing the linear rank statistic $T_N^*$ conditional on $N^*$, the number of observations in the combined sample less than $\mu$, in such a way that its scores $a_{i,N}^*$ are increasing for $i$ from 1 to $N^*$, and decreasing for the remaining $i$. Now under the null hypothesis the $W$-vector is independent of $N^*$ (in fact it is independent of the order statistics of the combined sample). Thus if $T_{N,N^*}^{N,N^*}$ is the rank statistic used when $N^*$ is the number of observations in the joint sample less than $\mu$, the conditional null distribution of $T_{N,N^*}^{N,N^*}$ given $N^*$ is the same as its unconditional null distribution. Of course such a conditional procedure involves obtaining the critical points of $T_{N,N^*}^{N,N^*}$ for each value of $N^*$. From (1.8), it follows that $T_{N,N^*}^{N,N^*}(k)$ will then be decreasing in $k$, and thus we will always get an interval as our confidence region for $\sigma$. One possible choice of scores for $T_{N,N^*}^{N,N^*}$ is $a_{i,N}^* = |i-(N^* + \frac{1}{2})|$. This is the "obvious" extension of the Ansari-Bradley statistic to the non-symmetric situation. When $N$ is even and $N^* = \frac{N}{2}$, $T_{N,N^*}^{N,N^*}$ reduces to $S$.

Suppose we apply this procedure to the data above. Since $N^* = 2$, our scores $a_{i,N}^*$ are in order of increasing $i$: $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}$. Starting in the lower tail, the null distribution of

\[ 3T_{8,2} = \sum_{i=1}^{8} a_{i,8} w_i \] is

\[
\begin{align*}
5 & \quad 2 \frac{1}{2} & 3 \frac{1}{2} & 4 \frac{1}{2} & \ldots \\
\frac{2}{56} & \quad \frac{3}{56} & \quad \frac{5}{56} & \ldots
\end{align*}
\]

$P(3T_{8,2} \leq t)$
Hence, in this case, we cannot achieve exact significance level \( \frac{6}{56} \). The closest achievable significance level is \( \frac{5}{56} \), obtained by rejecting if \( T_{8,2} \geq \frac{1}{2} \). Let us now obtain the corresponding confidence region for \( \sigma \). Letting \( T_{8,2}(k) \) be the value of \( T_{8,2} \) for \( \theta=\{r(k),r(k+1)\} \), we get

\[
\begin{array}{ccccccccccccc}
  k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  r(k) & 0 & 1.33 & 2 & 2.33 & 2.5 & 2.67 & 3 & 3.5 & 4 & 4.5 & \infty \\
  T_{8,2}(k) & \frac{a_1}{2} & \frac{a_1}{2} & \frac{a_1}{2} & \frac{a_1}{2} & \frac{a_1}{2} & \frac{a_1}{2} & \frac{a_1}{2} & \frac{a_1}{2} & \frac{a_1}{2} & \frac{a_1}{2} & \frac{a_1}{2} \\
\end{array}
\]

Hence our confidence region for \( \sigma \) is the interval \( (2,\infty) \).

1.3.2 Two-Sample Scale Problem, Unknown Common Location.

As in 1.3.1, we assume that \( X_1-\mu, \ldots, X_m-\mu, \frac{Y_1-\mu}{\sigma}, \ldots, \frac{Y_n-\mu}{\sigma} \) are i.i.d. according to some unknown absolutely continuous c.d.f. \( F \). However, now both \( \mu \) and \( \sigma \) are unknown. Using two-sample rank tests, we can obtain joint confidence regions for \( \mu \) and \( \sigma \) in the \((\mu, \sigma)\)-plane. Considering \( W(\frac{X_i-\mu}{\sigma}, \frac{Y_j-\mu}{\sigma}) \) as a function of \( \mu \) and \( \sigma \), we know that \( W \) changes only at points where ties occur, namely when for some \( i \) and \( j \), \( X_i-\mu = \frac{1}{\sigma}(Y_j-\mu) \) or

\[(\sigma-1)(\mu-\mu_X) = \mu_X-\mu_Y.\]

This determines a rectangular hyperbola in the \((\mu, \sigma)\)-plane. Considering only the half-plane \( \sigma > 0 \), the \( mn \) hyperbolas \((\sigma-1)(\mu-\mu_X) = \mu_X-\mu_Y, i=1, \ldots, m, j=1, \ldots, n \) partition this half-plane into connected regions over each of which the \( W \)-vector is constant, and hence any joint confidence region for \((\mu, \sigma)\) based on a rank test is a union of some of these regions. Now in such the same
way as before, we can easily express the relationship between the values of \( W(\bar{x}_i - \mu_{i,m}, \frac{1}{\sigma}(X - \mu_{i,n}) ) \) in two adjacent regions by noting the subscripts of the \( X \) and \( Y \) which determine the common boundary segment. Of course, now there is no natural ordering of the regions and so we cannot specify a general algorithm for evaluating the \( W \)-vectors in all the regions. If we consider a confidence region based on a linear rank test, then as in the previous model the choice of scores, or at least their shape, will be dictated by our feelings about the value of 

\[
P(X \leq \mu) = P(Y \leq \mu) = F(\sigma).
\]

It is worth observing in this model that since all the bounding hyperbolas \( (\sigma-1)(\mu-X_i) = X_i - Y_j \) have \( \sigma = 1 \) as an asymptote, the entire line \( \sigma = 1 \) lies in a single region and hence any joint confidence region will either include or exclude \( \sigma = 1 \) for all values of \( \mu \).

1.3.3 General Two-Sample Location and Scale Problem.

In this model we assume that \( X_1, \ldots, X_m, (\bar{Y}_1 - \mu)/\sigma, \ldots, (\bar{Y}_n - \mu)/\sigma \) are i.i.d. according to some unknown absolutely continuous c.d.f. \( F \), where both \( \mu \) and \( \sigma \) are unknown. As in the model of 1.3.2, we can obtain joint confidence regions for \( (\mu, \sigma) \) based on two-sample rank tests. In this case, the regions in the \( (\mu, \sigma) \)-plane of constant \( W \) will be formed by the lines

\[
X_i = \frac{Y_j - \mu}{\sigma}, \quad \text{or} \quad \sigma = -\frac{1}{X_i} \mu + \frac{Y_j}{X_i}, \quad i=1, \ldots, m, \quad j=1, \ldots, n.
\]

These are straight lines and thus any confidence region will be a union of polygonal regions in the \( \sigma > 0 \) half-plane. As before, we can easily obtain relations between the values of \( W(\bar{x}, \frac{1}{\sigma}(X - \mu_{i,n}) ) \) for \( (\mu, \sigma) \) in contiguous regions. For this problem,
however, the choice of the underlying rank test is unclear; presumably it will depend somewhat on which of the parameters \( \mu \) and \( \sigma \) we are more interested in.

1.3.4 One-Sample Center of Symmetry Problem.

In this model, also commonly known as the one-sample location problem, we assume that \( X_1, \ldots, X_N \) are i.i.d. according to an unknown absolutely continuous c.d.f. \( F \) with unknown center of symmetry \( \mu \), and we want to make inferences about \( \mu \). Instead of the \( W \)-vector, we now consider a \( Z \)-vector defined as follows:

\[
Z(\bar{X}) = (z_1(\bar{X}), \ldots, z_N(\bar{X})) \quad \text{where}
\]

\[
z_i(\bar{X}) = \begin{cases} 
1 & \text{if } X_j > 0, \text{ where } |X_j| \text{ is } i\text{-th smallest of } |X_1|, \ldots, |X_N| \\
0 & \text{if } X_j < 0, 
\end{cases}
\]

(1.9)

Under \( H_0: \mu = \mu_0 \), \( Z(\bar{X}_n) \) takes on each of its \( 2^N \) possible values with equal probability. A rank test of \( H_0 \) of significance level \( 1-\alpha \) rejects if \( Z(\bar{X}_n) \) takes on any one of a particular set of \( \alpha \cdot 2^N \) \( Z \)-vectors, which represents the critical region. Now if we consider \( Z(\bar{X}_n) \) as a function of \( \mu \) for a given \( \bar{X} \), it is easily seen that \( Z \) changes only at values of \( \mu \) equal to \( \bar{X}_i + \bar{X}_j \) for some \( 1 \leq i \leq j \leq N \). These \( \frac{N(N+1)}{2} \) values are usually referred to as Walsh averages. Thus any confidence region for \( \mu \) based on a rank test consists of a disjoint union of intervals whose endpoints are Walsh averages.
The most common type of rank test is one which is based on a linear rank statistic of the form

\[ V_N(\cdot) = \frac{1}{N} \sum_{i=1}^{N} a_{i,N} z_i(\cdot) . \]

Defining \( V_{N; \alpha} \), \( V_{N; 1 - \alpha} \), and \( \alpha_N \) as in (1.3) and (1.4)

define the \( 1-\alpha_N \) confidence region for \( \mu \) to be

\[ S(\mu) = \left\{ \mu \left| \frac{V_{N; \alpha}}{N} \leq V_N(\bar{X} - \mu_{\alpha_N}) \leq V_{N; 1 - \alpha} \right. \right\} . \]

Let \( X_{(1)} < X_{(2)} < \cdots < X_{(N)} \) be the order statistics of the sample.

Let \( m_{ij} = \frac{X_{(i)} + X_{(j)}}{2} \), \( 1 \leq i \leq j \leq N \). Let us consider what happens to

\( Z(\bar{X} - \mu_{\alpha_N}) \) as \( \theta \) goes from \( m_{ij}^- \) to \( m_{ij}^+ \). If \( i = j \), then the first component of \( Z \) changes from 1 to 0 and all other components are unchanged. If \( i < j \), the \( (j-i) \)-th and \( (j-i+1) \)-th components change from 0,1 to 1,0. Thus, if we define \( a_{ij, N} = 0 \), the corresponding change in \( V_N(\bar{X} - \mu_{\alpha_N}) \) is \( \frac{1}{N}(s_{j-i,N} - s_{j-i+1,N}) \). It follows that if the scores \( a_{i,N} \) are monotone in \( i \) then \( V_N(\bar{X} - \mu_{\alpha_N}) \) is a monotone step function in \( \mu \) and hence the confidence region in (1.11) is always an interval \((\mu^L, \mu^U)\). To obtain \( \mu^L \) and \( \mu^U \), we can use the following simple algorithm:

1. Order the sample \( X_{(1)} < \cdots < X_{(N)} \).

2. Calculate \( m_{ij} = \frac{X_{(i)} + X_{(j)}}{2} \), \( 1 \leq i \leq j \leq N \).
3. Order the \( m_{ij}, m(1) < m(2) < \cdots < m_{(N(N+1)/2)} \), keeping track of the original subscripts of \( m(k), i(k) \) and \( j(k) \), \( k=1, \ldots, N(N+1)/2 \). Let \( m(o) = -\infty \), \( m_{(N(N+1)/2 + 1)} = +\infty \).

4. Letting \( V_{N; 1}(k) \) denote the constant value of \( V_{N}(X - \mu_{N}) \) for \( \mu \in (d(k), d(k+1)) \), calculate \( V_{N; 1}(k) \) recursively using the formula

\[
V_{N; 1}(k) = V_{N; 1}(k-1) + \frac{1}{N} (a_{j(k)-1(k), N} - a_{j(k)-1(k)+1, N})
\]

where \( a_{o, N} = 0 \), starting with \( V_{N; 1}(o) = \frac{1}{N} \sum_{i=1}^{N} a_{i, N} \). Stop when \( k = k^* \) where \( k^* \) satisfies \( V_{N; 1}(k^*-1) < \frac{V_{N; 1}}{N; 2} \leq V_{N; 1}(k^*) \).

Then \( \mu = m(k^*) \).

Similarly, beginning with \( V_{N; 1}(N(N+1)/2) = 0 \), calculate \( V_{N; 1}(k) \) from \( V_{N; 1}(k+1) \) using (1.12). Stop when \( k = k^{**} \), where

\[
V_{N; 1}(k^{**}) \leq \frac{V_{N; 1}}{N; 1 - \alpha/2} < V_{N; 1}(k^{**}+1). \text{ Then } \mu = m(k^{**}+1) \).

1.3.5 Linear Regression Problem.

Suppose we observe independent random variables \( Y_1, \ldots, Y_N \) where \( Y_i \) is assumed to be of the form \( Y_i = \alpha + \beta x_i + e_i, \ i=1, \ldots, N \), where \( x_1, \ldots, x_N \) are known constants, \( e_1, \ldots, e_N \) are i.i.d. with unknown absolutely continuous distribution symmetric about 0, and we are interested in the unknown regression parameters \( \alpha \) and \( \beta \). Then we can
test $H_0: \alpha = \alpha_0, \beta = \beta_0$ by performing a one-sample rank test on $Z(\alpha, \beta) = Y - \alpha_0 X_1 - \beta_0 X_2$. Consider the joint confidence region for $(\alpha, \beta)$ based on such a rank test. We wish to determine at which values of $\alpha, \beta$ the vector $Z(\alpha, \beta)$ changes. Now as the components of the argument of $Z$ vary continuously, $Z$ changes only when one component changes sign, or when two components of opposite sign switch ranks in absolute value. In other words, $Z(\alpha, \beta)$ changes only at those $(\alpha, \beta)$ where either $e_i(\alpha, \beta) = 0$ for some $i$ or $e_i(\alpha, \beta) = -e_j(\alpha, \beta)$, for some $1 \leq i < j \leq N$. Now $e_i(\alpha, \beta) = 0 \Rightarrow Y_{i1} - \alpha x_i = 0$ or $\alpha = -x_i \beta + Y_{i1}$, and $e_j(\alpha, \beta) = -e_j(\alpha, \beta) \Rightarrow Y_{j1} - \beta x_j = -Y_{j1} + \alpha + \beta x_j$ or $\alpha = -\frac{x_i + x_j}{2} \beta + \frac{Y_{i1} + Y_{j1}}{2}$. Hence if we consider the $\frac{N(N+1)}{2}$ straight lines in the $(\alpha, \beta)$-plane given by $\alpha = -\frac{x_i + x_j}{2} \beta + \frac{Y_{i1} + Y_{j1}}{2}$, $1 \leq i \leq j \leq N$, it follows from the remarks above that these lines partition the $(\alpha, \beta)$-plane into polygonal regions over each of which the vector $Z(\alpha, \beta)$ is constant. Any confidence region based on $Z$ will therefore be a union of some of these regions. Again, we can obtain easy relations between the values of $Z(\alpha, \beta)$ in two contiguous regions by noting which $i$ and $j$ subscripts determine the common bounding line segment. Particular types of rank tests for the regression problem have been considered by a number of authors, including Theil [16], Hemelrijk [7] and Daniels [4].
1.4 Large Sample Properties in the Two-Sample Location Problem.

In order to speak meaningfully of large sample properties, we must consider sequences of rank tests indexed by sample size \( N \) which show stable behaviour. The class of linear rank statistics \( T_N = \frac{1}{m_N} \sum_{i=1}^{N} n_i w_i \) has proven particularly amenable to study. Under various conditions on the sequence \( \{\lambda_N\} \), \( \lambda_N = \frac{m_N}{N} \), and the scores \( \{a_{i,N}\} \), the asymptotic normality of the distribution of \( T_N \) under the null hypothesis as well as various sequences of alternative hypotheses has been shown by many investigators, including Wald and Wolfowitz [17], Hoeffding [8], Noether [13], Chernoff and Savage [5], and Hajek [6].

Suppose we want to compare tests of significance level \( 1-\alpha \) based on two different sequences of statistics \( \{T_N\} \) and \( \{T_N^*\} \). Consider testing the null hypothesis \( H_0: \theta = 0 \) against a sequence of alternatives \( H_N: \theta = \theta_N = cN^{-1/2} \), where \( c \) is an arbitrary constant. If we let \( N_1(N) \) and \( N_2(N) \) be the sample sizes at which the tests based on \( T \) and \( T^* \) respectively achieve a fixed power \( \gamma \) under \( H_N \), and if

\[
\lim_{N \to \infty} \frac{N_2(N)}{N_1(N)}
\]

exists and is independent of \( \alpha, \gamma, \) and \( c \) then this limit is defined to be the \textit{Pitman efficiency} or \textit{asymptotic relative efficiency (ARE)} of \( \{T_N\} \) with respect to \( \{T_N^*\} \), which we will denote by \( e_T|_{T^*} \). This concept of efficiency was introduced by Pitman [14], who showed that if we have

\[
\frac{T_N - \mu_N(\theta)}{\sigma_N(\theta)} \quad \text{and} \quad \frac{T_N^* - \mu_N^*(\theta)}{\sigma_N^*(\theta)}
\]

each converging in distribution to a \( N(0,1) \) distribution uniformly in \( \theta \) for \( \theta \) in some interval \([0, \theta_{\circ}]\), where \( \mu_N'(0) \neq 0 \) and \( \mu_N^*(0) \neq 0 \), and if

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\[ \lim_{N \to \infty} \frac{\sigma_N(\theta_N)}{\bar{\sigma}(0)}, \quad \lim_{N \to \infty} \frac{\sigma_N^*(\theta_N)}{\bar{\sigma}(0)}, \quad \lim_{N \to \infty} \frac{\mu'_N(\theta_N)}{\mu^{*'}_N(0)}, \text{ and } \lim_{N \to \infty} \frac{\mu^{*'}_N(\theta_N)}{\mu^{*'}_N(0)} \] are each 1, then

\[ e_{T^*} = \lim_{N \to \infty} \left[ \frac{\mu'_N(0)}{\sigma_N(0)} \right]^2 / \left[ \frac{\mu^{*'}_N(0)}{\sigma_N(0)} \right]^2 \]

(1.13)

The quantity \( \left[ \frac{\mu'_N(0)}{\sigma_N(0)} \right]^2 \) is called the efficacy of \( T_N \) for testing \( H_0: \theta = 0 \). Generally it will be \( O(N) \).

Now, taking the approach of Chernoff and Savage [3], let us assume that the scores \( \{a_{i,N} \} \) are generated by a score function \( J_N(u), 0 \leq u \leq 1 \), in the following manner:

(1.14) \[ a_{i,N} = J_N \left( \frac{i}{N} \right), i=1,\ldots,N \]

where the sequence of score functions \( \{J_n(u)\} \) converges to some function \( J(u) \), called the limiting score function, for \( 0 < u < 1 \). If we assume further that \( \lambda_N \) is bounded away from 0 and 1 and that the conditions of Chernoff-Savage Theorem 1 (proving the asymptotic normality of \( T_N \)) hold for both \( \{T_N\} \) and \( \{T^*_N\} \), then it follows that

(1.15) \[ \left[ \frac{\mu'_N(0)}{\sigma_N(0)} \right]^2 / N \lambda_N(1-\lambda_N) \frac{B^2(F)}{A^2} \rightarrow 1 \text{ as } N \rightarrow \infty \]

where
\[ B(F) = \int J'(F(x))f(x)df(x) \]

and

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\[ A^2 = \int J^2(u) du - (\int J(u) du)^2. \]

Hence in this case we have

\[
(1.16) \quad e_{T|T^*} = \frac{B^2(F)}{A^2} \left/ \frac{B^{*2}(F)}{A^{*2}} \right..
\]

If we now consider two sequences of confidence procedures \( \{P_N\} \) and \( \{P_N^*\} \) for obtaining confidence regions of significance level \( 1-\alpha \) for \( \theta \), we would like to define a reasonable measure of their relative efficiency. One approach is to consider a sequence of false values of \( \theta \), \( \theta_N = \theta_0 + c N^{-1/2} \) (assuming \( \theta = \theta_0 \) is the true value), where \( c \) is an arbitrary constant, and look at the probability of covering the false value. If \( S_N \) denotes the confidence region given by procedure \( P_N \), then if \( P_N \) is consistent, \( \Pr(S_N \text{ covers } \theta) \to 0 \quad \forall \theta \neq \theta_0 \). For a given \( \beta \) satisfying \( 0 < \beta < 1-\alpha \), let \( N_1(N) \) be the smallest sample size for which \( \Pr(S_{N_1} \text{ covers } \theta_N) \leq \beta \), and let \( N_2(N) \) be the smallest sample size for which \( \Pr(S_{N_2} \text{ covers } \theta_N) \leq \beta \).

Then if \( \lim_{N \to \infty} \frac{N_2(N)}{N_1(N)} \) exists and is independent of \( \alpha, \beta, \) and \( c \) we could define the asymptotic relative efficiency of \( \{P_N\} \) to \( \{P_N^*\} \) to be equal to this limit, and denote it by \( e_{P|P^*} \).

If the procedures \( P_N \) and \( P_N^* \) are based on linear rank tests \( T_N \) and \( T_N^* \) respectively, and if \( \{T_N\} \) and \( \{T_N^*\} \) each satisfy the conditions which were sufficient for (1.15), it follows that \( e_{P|P^*} = e_{T|T^*} \).

In the case where the confidence regions are intervals, we could consider the interval length as an inverse measure of efficiency; that is, the shorter the confidence intervals on the average, the more efficient
the procedure. More precisely, letting $\mathcal{L}_N$ and $\mathcal{L}^*_N$ be the lengths of the intervals $S_N$ and $S^*_N$, if $\{N^*_N\}$ is a sequence of sample size satisfying $E(\mathcal{L}_N)/E(\mathcal{L}^*_N) \to 1$ as $N \to \infty$, then we could define $e_{\mathcal{L}}|_{\mathcal{L}^*}$ to be equal to $\lim_{N \to \infty} \frac{N^*_N}{N}$ if it exists.

Lehmann [11] showed that for confidence intervals based on the Wilcoxon test, when $\lambda_N \to \lambda$ for $0 < \lambda < 1$,

$$\frac{1}{N^2} \mathcal{L}_N \xrightarrow{D} \frac{\frac{z_{\alpha/2}}{\sqrt{3}}}{[\lambda(1-\lambda)]^{1/2}} \int f^2(x)dx \quad \text{as } N \to \infty \quad (1.17)$$

where $z_{\alpha/2}$ is the upper $\frac{\alpha}{2}$ critical point of the standard normal distribution. The limiting value in (1.17) is equal to $\frac{2z_{\alpha/2}}{[\lambda(1-\lambda)]^{1/2}} \frac{A}{B(\mathcal{F})}$.

This result was generalized by Sen [15] to a large class of confidence intervals based on linear rank tests. Assuming the scores $\{a_{i,N}\}_{i=1}^N$ are generated by monotone score functions $J_N(u)$ as in (1.14) , and that $\{J_N(u)\}$ converges to some $J(u)$ on $0 < u < 1$, Sen proved that under certain regularity conditions on $\{J_N\}$, $J$, and $\mathcal{F}$,

$$\frac{1}{N^2} \mathcal{L}_N \xrightarrow{D} \frac{2z_{\alpha/2}}{[\lambda(1-\lambda)]^{1/2}} \frac{A}{B(\mathcal{F})} \quad \text{as } N \to \infty \quad (1.18)$$

where $A$ and $B(\mathcal{F})$ are defined in (1.15). Hence if $\{T_N\}$ and $\{T^*_N\}$ satisfy the regularity conditions, we have

$$\frac{\mathcal{L}_N}{\mathcal{L}^*_N} \xrightarrow{D} \frac{B(\mathcal{F})/A}{B^*(\mathcal{F})/A^*} = e_T|_{T^*} \quad \text{as } N \to \infty \quad (1.19)$$

Also, from (1.19),
\[ \frac{\mathcal{L}^*_N}{\mathcal{L}^*_{N}} \rightarrow 1 \text{ implies } \left( \frac{N^*}{N} \right)^{1/2} \frac{B(F)/A}{B^*(F)/A^*} \rightarrow 1, \]
or

\[ \frac{N^*}{N} \rightarrow \left( \frac{B(F)/A}{B^*(F)/A^*} \right)^2 = e_{T|T^*}. \]

Thus we see that our alternative formulation of \( e_{P|P^*} \), namely in terms of the lengths of the confidence intervals, again reduces to \( e_{T|T^*} \).

Furthermore, we note that the squared ratio of lengths \( \left( \frac{\mathcal{L}^*_N}{\mathcal{L}^*_N} \right)^2 \) is a consistent estimate of \( e_{P|P^*} = e_{T|T^*} \).
1.5 A Robust Flexible Procedure.

If we wish to obtain a nonparametric confidence interval for the
two-sample location shift parameter $\theta$, a natural family to consider
is the family of confidence intervals based on linear rank tests $T_N$
with monotone scores. The efficiency of our procedure will depend on
our choice of $T_N$ and on the underlying family $F$, which is unknown.
Since we know from the previous section that the length of the confi-
dence interval gives us useful information about the efficiency, we
could consider employing the following flexible procedure:

(i) Choose in advance $k > 1$ procedures $\{F_N^{(1)}, F_N^{(2)}, \ldots, F_N^{(k)}\}$ of
significance level $1-\alpha$, based on $k$ linear rank tests
$\{T_N^{(1)}, \ldots, T_N^{(k)}\}$ respectively, where $T_N^{(i)}$ have monotone scores.
Ideally, these should be chosen to "cover" with high efficiency a
large class of $F$'s.

(ii) For the given sample $(x, y)$, obtain the confidence interval for
each of the $k$ procedures, and calculate their lengths $L_1, \ldots, L_k$.

(iii) Use as final confidence interval the one with shortest length.

Now in general, such a flexible procedure will no longer be distri-
bution-free, as its confidence level $1-\alpha'$ will be less than $1-\alpha$ and
will depend on $F$. However, we will now show that in a certain sense the
flexible procedure is asymptotically distribution-free, with asymptotic
level $1-\alpha$. 

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Let us assume that \( \{T^{(i)}_N\} \) has scores generated by score functions 
\( \{J^{(i)}_N(u)\} \), with limiting score function \( J^*_i(u) \), for \( i=1,\ldots,k \). Let 
the values of \( A, B(F) \) corresponding to \( \{T^{(i)}_N\} \) be denoted by \( A_i, B_i(F) \),
and let 
\[ E_i(F) = \frac{B_i(F)}{A_i}, \quad i=1,\ldots,k. \]

**Theorem.** Suppose \( \{J^{(i)}_N(u)\}_{N=1}^{\infty}, J^*_i(u), i=1,\ldots,k \) satisfy the conditions
of Sen [15], Theorem 1. Then for all \( F \) satisfying the conditions of Sen's
Theorem 1 and such that the set \( \{E_1(F),\ldots,E_k(F)\} \) has a unique maximum,
we have \( 1-\alpha' \to 1-\alpha \) as \( N \to \infty \).

**Proof.** Let \( I_i \) denote the confidence interval given by \( p^{(i)}_i, i=1,\ldots,k \).
Let \( j \) be the subscript satisfying \( E_j(F) > E_i(F), i=1,\ldots,k, i \neq j \).

\[
\Pr(\mathcal{L}_j < \mathcal{L}_i, i=1,\ldots,k, i \neq j) = 1-\Pr(\mathcal{L}_j \geq \mathcal{L}_i \text{ for some } i \neq j)
\]
\[
= 1-\Pr(\bigcup_{i=1 \atop i \neq j}^{k} \mathcal{L}_j \geq \mathcal{L}_i)
\]
\[
\geq 1 - \sum_{i=1 \atop i \neq j}^{k} \Pr(\mathcal{L}_j \geq \mathcal{L}_i)
\]
(1.21)

Now by Sen's Theorem 1, \( N^{1/2} L_i \xrightarrow{p} \frac{2z_{\alpha/2}}{[\lambda(1-\lambda)]^{1/2}} E^{-1}_i(F) \) as \( N \to \infty \).

Hence \( N^{1/2}(\mathcal{L}_j - \mathcal{L}_i) \xrightarrow{p} \frac{2z_{\alpha/2}}{[\lambda(1-\lambda)]^{1/2}} [E^{-1}_j(F)-E^{-1}_i(F)] < 0 \) by assumption.

Thus \( \Pr(\mathcal{L}_j \geq \mathcal{L}_i) = \Pr(N^{1/2}(\mathcal{L}_j - \mathcal{L}_i) \geq 0) \to 0 \) as \( N \to \infty \). Substituting in
(1.21), we get...
\begin{align*}
\text{(1.22)} \quad & \Pr(\mathcal{L}_j < \mathcal{L}_1, \ i=1, \ldots, \ k, \ i \neq j) \to 1 \quad \text{as} \quad N \to \infty \\
\text{i.e.} \quad & \Pr(\mathcal{I}_j \text{ is shortest}) \to 1 \quad \text{as} \quad N \to \infty.
\end{align*}

Now if \( \theta_o \) is the true value of \( \theta \), we have

\[
1-\alpha = \Pr(\mathcal{I}_j \text{ covers } \theta_o)
\]

\begin{align*}
\text{(1.23)} \quad & = \Pr(\mathcal{I}_j \text{ covers } \theta_o | \mathcal{I}_j \text{ shortest}) \Pr(\mathcal{I}_j \text{ shortest}) \\
& \quad + \Pr(\mathcal{I}_j \text{ covers } \theta_o | \mathcal{I}_j \text{ not shortest}) (1-\Pr(\mathcal{I}_j \text{ shortest})).
\end{align*}

Taking \( \lim_{N \to \infty} \) on both sides of (1.23), and using (1.22), we get

\begin{align*}
\text{(1.24)} \quad & \Pr(\mathcal{I}_j \text{ covers } \theta_o | \mathcal{I}_j \text{ shortest}) \to 1-\alpha.
\end{align*}

Now

\begin{align*}
\text{(1.25)} \quad & 1-\alpha' = \sum_{i=1}^{k} \Pr(\mathcal{I}_i \text{ covers } \theta_o | \mathcal{I}_i \text{ shortest}) \Pr(\mathcal{I}_i \text{ shortest}).
\end{align*}

But from (1.22),

\[
\Pr(\mathcal{I}_i \text{ shortest}) \to \begin{cases} 
1 & \text{if } i=j \\
0 & \text{if } i \neq j.
\end{cases}
\]

Hence from (1.24) and (1.25), we get \( 1-\alpha' \to 1-\alpha \) as \( N \to \infty \). Q.E.D.

Thus if the individual procedures \( \{ P_N^{(i)} \} \) satisfy Sen's theorem then the flexible procedure is asymptotically distribution-free over all \( F's \)
which satisfy Sen's theorem and, more importantly, its asymptotic relative efficiency with respect to each of the individual procedures is no less than 1 over this class of $F$'s.

In order to study the finite sample properties of the flexible procedure, we need to know something about the joint distribution of $\mathcal{L}_1, \ldots, \mathcal{L}_k$. In the next chapter we consider the distribution of the lengths of confidence intervals based on rank tests. In particular, we show that under certain conditions, the joint distribution of $\mathcal{L}_1, \ldots, \mathcal{L}_k$, suitably normalized, converges to a k-variate normal distribution. In Chapter 3, we examine the moderate sample size behaviour of the flexible procedure for some particular cases.
Chapter 2. Lengths of Nonparametric Confidence Intervals for the Two-Sample Location Shift.

2.1 Preliminaries.

In section 1.2 we described a confidence procedure for the location shift parameter based on a linear rank statistic \( T_N = \frac{1}{m} \sum_{i=1}^{N} a_{i,N} \). When the sequence of scores \( \{a_{i,N}, i=1, \ldots, N\} \) is monotone, the resulting confidence region is always an interval. Because of the discreteness of \( T_N \), only certain coverage probabilities \( 1-\alpha_N \) can be achieved exactly. However, as we consider \( N \to \infty \), we can choose the coverage probabilities \( 1-\alpha_N \) so that they converge to a particular value \( 1-\alpha \).

Let us make the assumption that the scores \( \{a_{i,N}, i=1, \ldots, N\} \) are generated by score functions \( J_N(u), 0 \leq u \leq 1 \) according to (1.14), where \( J_N(u) \) converges to \( J(u) \) on \( (0,1) \). Let us further assume that the hypotheses of Chernoff-Savage Theorem 1 are satisfied. Letting

\[
\lambda_N = \frac{m}{N}, \\
N_0 = N \lambda_N (1-\lambda_N) \\
\mu = \int_0^1 J(u) du \\
A^2 = \int_0^1 J^2(u) du - \mu^2
\]

(2.1)

it follows that the null distribution of \( \frac{1}{N_0} (T_N - \mu) / A(1-\lambda_N) \) converges to that of a standard normal random variable. Hence,

\[
S_N = \left\{ \theta \mid \mu - \frac{A(1-\lambda_N)}{N_0^{1/2}} z_{\alpha/2} \leq T_N (\chi, \lambda, \theta) \leq \mu + \frac{A(1-\lambda_N)}{N_0^{1/2}} z_{\alpha/2} \right\}
\]

(2.2)
is a confidence region for $\theta$ with coverage probability converging to $1-\alpha$ as $N \to \infty$. Let us assume now that $T_N$ has nondecreasing scores $a_{i,N}$. Then $S_N$ is an interval $[\theta_{N}^{L}, \theta_{N}^{U}]$, where

$$\theta_{N}^{L} = \inf \left\{ \theta | T_N(x, x_0 - \theta a_n) \geq \mu - \frac{A(1-\lambda_N)}{N^{1/2}} z_{\alpha/2} \right\}$$

(2.3)

$$\theta_{N}^{U} = \sup \left\{ \theta | T_N(x, x_0 - \theta a_n) \leq \mu + \frac{A(1-\lambda_N)}{N^{1/2}} z_{\alpha/2} \right\}.$$

Let $L_N = \theta_{N}^{U} - \theta_{N}^{L}$ be the length of this confidence interval.

In the next section we consider the asymptotic distribution of $L_N$. In Theorem 1, we give conditions under which $L_N$, suitably normalized, converges in distribution to a normal distribution. The result is seen to apply to confidence intervals based on a large class of linear rank tests, and generalizes a result of Jurečková [9], where the author considers lengths of confidence intervals for a regression parameter based on Wilcoxon-type linear rank tests.
2.2 Asymptotic Distribution of $\mathcal{L}_N$.

We have already stated that under essentially the conditions of the Chernoff-Savage theorem, Sen [15] showed that

$$\sqrt{N}\mathcal{L}_N \overset{D}{\to} 2z_{\alpha/2} \frac{A}{B(F)} \quad \text{as} \quad N \to \infty,$$

where

$$A = \left[ \int J^2(u)du - (\int J(u)du)^2 \right]^{1/2}$$

and $B(F) = \int J'(F)f \, dF$, where $f$ is the density of $F$.

Before proceeding to the main result of this section concerning the deviation of $\sqrt{N}\mathcal{L}_N$ from its probability limit, let us introduce some notation. Let $F_m(x)$ be the empirical c.d.f. of $X$, i.e.,

$$(2.4) \quad F_m(x) = \frac{1}{m} \sum_{i=1}^{m} I(X_i \leq x), \quad \text{where} \quad I_A = \begin{cases} 1 & \text{when } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function of the event $A$.

Let $G_n(x)$ be the empirical c.d.f. of $X$.

Let $H_N^2(x) = \frac{\lambda_n}{N} F_m(x) + (1-\lambda_n) G_n(x+N_0^{-1/2}a)$. Thus $H_N^2(x)$ represents the empirical c.d.f. of the combined sample $(X, X^{-N_0^{-1/2}a})$.

Let $H^2(x) = \lambda_n F(x) + (1-\lambda_n) F(x+N_0^{-1/2}a)$. Under $H_0: \theta=0$, $E(H_N^2(x)) = H^2(x)$ \( \forall x \).
We now state the main result of this section.

**Theorem 1.**

If
1. \(0 < \lambda_0 \leq \frac{1}{2}\) and \(\lambda_0 \leq \lambda_N \leq 1 - \lambda_0\) for all sufficiently large \(N\),
2. \(J(u) = \lim_{N \to \infty} J_N(u), 0 \leq u \leq 1\) is continuous, monotone, and non-constant with \(J(u) + J(1-u) = \text{constant}\),
3. \(F\) has density \(f\) which is symmetric about some point,
4. \(f\) is twice differentiable on its support, and \(u > 0, K > 0, \exists\ |f^{(i)}(x+u)| \leq K[F(x)(1-F(x))^{2/3}, \forall x\) and for \(|u| \leq U, i = 0,1,2,\)
5. \(x^6 F(x)(1-F(x)) \to 0 \text{ as } |x| \to \infty,\)
6. \(K > 0, \delta > 0\) and
   \[|J^{(i)}(u)| \leq K[u(1-u)]^{-1+\delta}, 0 < u < 1, i = 1,2,3,\]
7. For any \(a,b\)
   \[
   \frac{1}{m} \sum_{i=1}^{N} (J_{N}^{(i)}(u))^2 \leq \left( \frac{1}{N^2} \sum_{i=1}^{N} a_i \right)^2 - \left( \frac{1}{N^2} \sum_{i=1}^{N} b_i \right)^2 + \frac{1}{N} \frac{A}{E(F)} \to N(0,1) \text{ as } N \to \infty
   \]

then

\[
\frac{1}{N^2 \sigma^2} \left( \frac{A}{E(F)} \right) \to N(0,1) \text{ as } N \to \infty
\]

where
\[ \sigma^2_N = 4z^2_{\alpha/2} \frac{A^2}{B_0(F)} \left\{ \lambda_N (1-\lambda_N) \left[ \int (Q(x)-2J^*(F(x)))f(x)^2 \, dF(x) \right] \right. \\
\left. - \left( \int (Q(x)-2J^*(F(x)))f(x) \, dF(x) \right)^2 \right\} \right. \\
\left. + \left(1-4\lambda_N (1-\lambda_N) \right) \left[ \int (J^*(F))^2 \, dF - (\int J^*(F) \, f \, dF)^2 \right] \right\} \right] \]

and \( Q(x) = \int_0^x J''(F(y))f(y) \, dF(y) \), provided \( \lambda_n \neq 0 \).

Remarks:

(a) Condition (1), that the ratio of the two sample sizes remain bounded away from 0 and \( \infty \), is the same as in the Chernoff-Savage theorem.

(b) Condition (2) essentially states that \( J'(u) \) is skew symmetric around the point \( u = \frac{1}{2} \), and implies that the distribution of \( T_N \) is asymptotically symmetric.

(c) The symmetry of \( f \) seems to be needed in general, but we will see in the proof that in the special case of the Wilcoxon test \( J(u)=u \), symmetry is unnecessary.

(d) Condition (4) is a smoothness condition on the density \( f \).

(e) Condition (5) holds for any distribution possessing finite sixth moment.

(f) Condition (6) is a strengthened version of condition (4) of the Chernoff-Savage theorem.

The boundedness of the function \( J(u) \) is implied by the condition

\[ |J'(u)| \leq K[u(1-u)]^{-1+\varepsilon}, \text{ since } J(1) - J(0) = \int_0^1 J'(u) \, du < \infty. \]

(g) Condition (7) is an analogue of Chernoff-Savage condition (2).
The method of proof of Theorem 1 is essentially an extension to higher order terms of the method of Sen's result, involving an expansion of \( T_N(x; X - N_0^{-1/2} a_{1n}) \) along the lines of Chernoff-Savage and manipulations in order to express \( N_0^{1/2}(N_0^{1/2} \cdot \mathcal{L}_N - \frac{2A}{B(F)} \cdot z_{\alpha/2}) \) in terms of a manageable first order term plus other terms which are shown to be of higher order.

Proof of Theorem 1.

Let \( T_N^a = T_N(x; X - N_0^{-1/2} a_{1n}) \).

Then

\[
T_N^a = \frac{1}{m} \sum_{i=1}^{N} J_N^a(x_i, x - N_0^{-1/2} a_{1n})
\]

\[
= \int J_N^a(H_N^a(x))dF_m(x)
\]

(2.6)

\[
= \int J(H_N^a)dF_m + \int [J_N^a(H_N^a) - J(H_N^a)]dF_m
\]

\[
= \int_{H_N^a < 1} J(H_N^a)dF_m + \int_{H_N^a = 1} J(H_N^a)dF_m + \int [J_N^a(H_N^a) - J(H_N^a)]dF_m
\]

(Note: For simplicity we will suppress the argument of a function if it is \( x \), but will include any arguments which are different from \( x \).)

Let

\[
C_1^a = \int_{H_N^a = 1} J(H_N^a)dF_m
\]

and

\[
C_2^a = \int [J_N^a(H_N^a) - J(H_N^a)]dF_m
\]

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Consider the first integral on the RHS of (2.6)

\[
\int_{H_N^a < l} J(H_N^a) dF_m = \int_{H_N^a < l} [J(H_N^a) + (H_N^a - H^a) J'(H_N^a)] dF_m
\]

(2.7)

\[
+ \int_{H_N^a < l} \frac{1}{2} (H_N^a - H^a)^2 J''(\alpha H_N^a + (1-\alpha)H^a) dF_m
\]

where \(0 < \alpha < 1\).

Let

\[
C^a_3 = \int_{H_N^a < l} \frac{1}{2} (H_N^a - H^a)^2 J''(\alpha H_N^a + (1-\alpha)H^a) dF_m .
\]

The first integral on the RHS of (2.7) is:

\[
\int_{H_N^a < l} [J(H_N^a) + (H_N^a - H^a) J'(H_N^a)] dF_m
\]

(2.8)

\[
= \int_{H_N^a = 1} [J(H_N^a) + (H_N^a - H^a) J'(H_N^a)] dF_m + \int_{H_N^a = 1} [-J(H_N^a) - (H_N^a - H^a) J'(H_N^a)] dF_m .
\]

Let

\[
C^a_4 = \int_{H_N^a = 1} [-J(H_N^a) - (H_N^a - H^a) J'(H_N^a)] dF_m .
\]

Expanding the first integral on the RHS of (2.8)
\[ \int [J(H^a) + (H_N^a - H^a)J'(H^a)] \, dF_m = \int J(H^a) \, dF + \int J(H^a) \, d(F_m - F) \]

\[ + \int (H_N^a - H^a)J'(H^a) \, dF + \int (H_N^a - H^a)J'(H^a) \, d(F_m - F) \]

Let

\[ C_5^a = \int (H_N^a - H^a)J'(H^a) \, d(F_m - F), \quad A^a = \int J(H^a) \, dF, \quad B_1^a = \int J(H^a) \, d(F_m - F), \]

\[ B_2^a = \int (H_N^a - H^a)J'(H^a) \, dF. \]

Then combining equations (2.6)-(2.9), we have

\[ T_N^a = A^a + B_1^a + B_2^a + \sum_{i=1}^{5} C_i^a. \]

Let us now expand \( A^a \). First, expanding \( H^a \) in a Taylor series, we have

\[ H^a = \lambda_n F^+(1 - \lambda_n) N_0^2 a \]

\[ = F^+(1 - \lambda_n) \left[ N_0^2 a f(x) + \frac{N_0^2 a^2}{2} f'(x) + \frac{N_0^2 a^3}{6} f''(x + \theta N_0^2 a) \right] \]

where \( 0 < \theta < 1 \).

Thus, expanding \( J(H^a) \), we get

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\[ J(\mathbf{H}^a) = J(F) + (1-\lambda_N) \left[ N_o^{-1/2} a f + \frac{N^{-1/2}}{2} f' + \frac{N^{-1/2}}{6} f''(x+\theta N_o a) \right] J'(F) \]
\[ + \frac{(1-\lambda_N)^2}{2} \left[ N_o^{-1/2} a f + \frac{N^{-1/2}}{2} f' + \frac{N^{-1/2}}{6} f''(x+\theta N_o a) \right] J''(F) \]
\[ + \frac{(1-\lambda_N)^3}{6} \left[ N_o^{-1/2} a f + \frac{N^{-1/2}}{2} f' + \frac{N^{-1/2}}{6} f''(x+\theta N_o a) \right] J'''(F) \]

where \( 0 < \gamma < 1 \)

\[ (2.12) \]
\[ = J(F) + N_o^{-1/2} a (1-\lambda_N) \int f J'(F) + N_o^{-1/2} \frac{(1-\lambda_N)}{2} \left[ f' J'(F) + (1-\lambda_N) f^2 J''(F) \right] \]
\[ + N_o^{-1/2} a^3 (1-\lambda_N) \left[ \frac{1}{2} f''(x+\theta N_o a) J'(F) + (1-\lambda_N) f \left( \frac{1}{2} f' + \frac{N_o^{-1/2}}{6} f''(x+\theta N_o a) \right) J'(F) \right] \]
\[ + N_o^{-1/2} \frac{(1-\lambda_N)}{2} \left[ \frac{1}{2} f' + \frac{N_o^{-1/2}}{6} f''(x+\theta N_o a) \right] J''(F) \]
\[ + \frac{(1-\lambda_N)^2}{6} \left( f' + \frac{N_o^{-1/2}}{2} f' + \frac{N_o^{-1/2}}{6} f''(x+\theta N_o a) \right) J'''(F) \]

Let \( \xi^a(x) \) denote the last term in square brackets. Substituting in \( A^a \) gives

\[ A^a = \int J(F) dF + N_o^{-1/2} a (1-\lambda_N) \int f J'(F) dF \]
\[ + N_o^{-1/2} \frac{(1-\lambda_N)}{2} \left[ f' J'(F) + (1-\lambda_N) f^2 J''(F) \right] dF \]
\[ + N_o^{-1/2} a^3 (1-\lambda_N) \int \xi^a(x) dF \]

\[ (2.13) \]
\[ = \mu + N_o^{-1/2} a (1-\lambda_N) B(F) + N_o^{-1/2} \frac{(1-\lambda_N)}{2} \left[ f' J'(F) + (1-\lambda_N) f^2 J''(F) \right] dF \]
\[ + N_o^{-1/2} a^3 (1-\lambda_N) \int \xi^a(x) dF . \]
Now \( \int [f'' J'(F) + (1-\lambda_N) f^2 J''(F)] dF \) is finite because of conditions (4) and (6). For \( \int f^2 J''(F) dF = [f^2 J'(F)]_{F=0}^{F=1} - 2 \int J'(F(x)) f(x) f''(x) dx \).

But \( f^2 J'(F) \leq K [F(1-F)]^{4/3} - 1 + \delta \), by conditions (4) and (6).

Note: \( K \) and \( \delta \) will always be used as generic positive constants which do not depend on \( N \).

Hence

\[
[f^2 J'(F)]_{F=0}^{F=1} = 0 , \quad \text{and} \quad \int f^2 J''(F) dF = -2 \int J'(F) f' f'' dx = -2 \int J'(F) f' dF .
\]

Thus

\[
\int [f' J'(F) + (1 - \lambda_N) f^2 J''(F)] dF = [1 - 2(1-\lambda_N)] \int J'(F) f' dF = (2\lambda_N - 1) \int J'(F) f' dF .
\]

(2.14)

We now show that in fact \( \int J'(F) f' dF = 0 \). Condition (2) that \( J(u) + J(1-u) \) is constant for all \( 0 < u < 1 \) implies that \( J'(u) = J'(1-u) \).

By condition (3), \( f \) is symmetric about some point, \( t \) say, so \( f(t+x) = f(t-x) \) \( \forall x \) and \( f'(t+x) = -f'(t-x) \). Now \( F(t+x) = 1 - F(t-x) \), so

\[
J'(F(t+x)) f'(t+x) f(t+x) = J'(1-F(t-x)) (-f'(t-x)) f(t-x) = -J'(F(t-x)) f'(t-x) f(t-x) .
\]

Hence \( J'(F(t+x)) f'(t+x) f(t+x) \) is an odd function of \( x \), so

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\[ 0 = \int_{-\infty}^{\infty} J'(F(t+x))f'(t+x)f(t+x)dx = \int_{-\infty}^{\infty} J'(F(t+x))f'(t+x)f(t+x)dF(t+x) \]

\[ = \int_{-\infty}^{\infty} J'(F(x))f'(x)dF(x) \]

Substituting in (9), we obtain

\[ (2.15) \quad A^a = \mu + \frac{1}{2} \int_{-\infty}^{\infty} J'(F(x))f'(x)dF(x) \]

[Remark: For \( J(u) = u \) (i.e., the Wilcoxon statistic), \( \int_{-\infty}^{\infty} J'(F(x))f'(x)dF(x) \)

\[ = \int_{-\infty}^{\infty} f'(x)f(x)dx = \int_{F^{-1}(1)}^{F^{-1}(0)} f'(x)f(x)dx = \frac{1}{2} \int_{F^{-1}(1)}^{F^{-1}(0)} f^2(x)dx \]

\[ = \frac{1}{2} [f^2(F^{-1}(1)) - f^2(F^{-1}(0))] . \]

But the condition \( f(x) \leq K[F(x)(1-F(x))]^{2} \) implies that \( f(F^{-1}(1)) = 0 = f(F^{-1}(0)) \). Hence in this case, \( \int_{-\infty}^{\infty} J'(F)F'F^2 = 0 \)

even without the condition on the symmetry of \( f \). \( \int_{-\infty}^{\infty} J'(F(x))f'(x)dF(x) \)

will be shown to be finite in the discussion of the remainder terms.

Now consider the term \( B_{1}^{a} \) from the line following (2.9):

\[ B_{1}^{a} = \int J(h^a)d(F_m-F) = -\int (F_m-F)J'(h^a)d\mu^a \]

\[ \text{(2.16)} \]

\[ = -\lambda \int (F_m-F)J'(h^a)dF -(1-\lambda_N)\int (F_m-F)J'(h^a)dF(x+F_{0}^{1/2}a) \]
\[ B_2^a = \int (H_N^a - H^a) J' (H^a) dF \]

(2.17) \[ = \lambda_a \int (F_m - F) J' (H^a) dF + (1 - \lambda_N) \int (C_n (x + N a) - F(x + N a)) J' (H^a) dF \cdot \]

Hence

\[ B_1^a + B_2^a = \]

(2.18) \[ (1 - \lambda_N) \left[ \int (C_n (x + N a) - F(x + N a)) J' (H^a) dF - \int (F_m - F) J' (H^a) dF (x + N a) \right] \]

Let \( B_n^a, B_m^a \) represent the two terms inside the square brackets respectively. Substituting

\[ T_N^a = \mu + N a (1 - \lambda_N) B(F) \]

(2.19) \[ + (1 - \lambda_N) (B_n^a - B_m^a) + N a (1 - \lambda_N) \int \xi^a(x) dF(x) + \sum_{i=1}^{5} c_i^a, \]

or after some manipulation,

\[ \frac{1}{N o} a - \frac{1}{(1 - \lambda_N) B(F)} (T_N^a - \mu) = \frac{1}{B(F)} (B_m^a - B_n^a) \]

(2.20) \[ - \frac{N o a}{B(F)} \int \xi^a dF - \frac{1}{(1 - \lambda_N) B(F)} \sum_{i=1}^{5} c_i^a. \]

Equation (2.20) is an identity in \( a \). Thus letting \( b \) be another arbitrary constant with \( b < a \), we obtain by subtraction
\[
\frac{1}{N_o^2}(a-b) - \frac{1}{(1-\lambda_N)B(F)} (T^a_N - T^b_N) = \\
(2.21) \frac{1}{B(F)} \left[ (B^a_{m_m} - B^b_{m_n}) - \frac{1}{B(F)} \left( a^3 \int c^a dF - b^3 \int c^b dF \right) \right] \\
- \frac{1}{(1-\lambda_N)B(F)} \sum_{i=1}^{\lambda} (c^a_i - c^b_i).
\]

We now expand \( B^a_n \) further.

\[
B^a_n = \int (G_n(x + N_o^2 a) - F(x + N_o^2 a)) J'(H^a) dF \\
= \frac{1}{n} \sum_{i=1}^{\lambda} \left\{ \int_{-\infty}^{\infty} (y + \frac{1}{2} N_o^2 a) J'(H^a) dF + \int_{y_1 - \frac{1}{2} N_o^2 a}^{\infty} (1 - F(x + N_o^2 a)) J'(H^a) dF \right\} \\
= \lambda \frac{N_o}{n} \sum_{i=1}^{\lambda} \left\{ \int_{-\infty}^{\infty} (y + \frac{1}{2} N_o^2 a) J'(H^a) dF + \int_{y_1 - \frac{1}{2} N_o^2 a}^{\infty} (1 - F(x + N_o^2 a)) J'(H^a(y_1)) f(y_1) \right\} \\
- \frac{N_o}{2} \left( \hat{f}(y_1 + (1-\frac{1}{2} N_o^2 a)) J'(H^a(y_1 + \frac{1}{2} N_o^2 a)) f(y_1 - \frac{1}{2} N_o^2 a) \right) \\
+ \hat{f}(y_1 + (1-\frac{1}{2} N_o^2 a)) J'(H^a(y_1 + \frac{1}{2} N_o^2 a)) f(y_1 - \frac{1}{2} N_o^2 a) \\
+ \hat{f}(y_1 + (1-\frac{1}{2} N_o^2 a)) J'(H^a(y_1 + \frac{1}{2} N_o^2 a)) f(y_1 - \frac{1}{2} N_o^2 a) \right\} \\
+ \frac{\lambda \frac{N_o}{n}}{\sum_{i=1}^{\lambda} \int_{y_1}^{\infty} \left( 1 - F(x + N_o^2 a) \right) J'(H^a) dF + \frac{1}{2} f(y_1 - \frac{1}{2} N_o^2 a) J'(H^a(y_1)) f(y_1) \right\} \\
- \frac{N_o}{2} \left( \hat{f}(y_1 + (1-\eta) N_o^2 a)) J'(H^a(y_1 + \frac{1}{2} N_o^2 a)) f(y_1 - \frac{1}{2} N_o^2 a) \right) +
\]
\[
\left. \begin{align*}
+ \left(1 - F(Y_1 + (1-\eta)N_0 \sigma a)\right)J''(H^a(Y_1 - \eta N_0 \sigma a))h^a(Y_1 - \eta N_0 \sigma a)f(Y_1 - \eta N_0 \sigma a) \\
+ \left(1 - F(Y_1 + (1-\eta)N_0 \sigma a)\right)J'(H^a(Y_1 - \eta N_0 \sigma a))f'(Y_1 - \eta N_0 \sigma a)
\end{align*} \right) \right] \}
\]

where \(0 < \delta, \eta < 1\).

(2.22)

Let \(D^a_{N,1}(Y_1), D^a_{N,2}(Y_1)\) denote the coefficients of \(-\frac{N_0^{-1}a^2}{2}\) in the first and second summations in (2.22).

Then we have

\[
\mathcal{B}^a_n = \frac{\lambda}{N_0} \sum_{i=1}^{\eta} \left[ \int_{-\infty}^{\infty} \left( G_{i,1}(x) - F(x + N_0^{-1/2}a) \right) J'(H^a) dF \\
+ N_0^{-1/2} a \int_{-\infty}^{\infty} \left( G_{i,1}(x) - F(x + N_0^{-1/2}a) \right) J'(H^a) dF \\
- \frac{N_0^{-1}a^2}{2} \left( D^a_{N,1}(Y_1) + D^a_{N,2}(Y_1) \right) \right].
\]

(2.23)

Taking two- and three-term Taylor expansions of \(F(x + N_0^{-1/2}a)\), we have

(2.24)

\[
F(x + N_0^{-1/2}a) = F(x) + \frac{1}{2} N_0^{-1/2} a f(x + N_0^{-1/2}a), \quad 0 < \mu < 1
\]

and

(2.25)

\[
F(x + N_0^{-1/2}a) = F(x) + \frac{1}{2} N_0^{-1/2} a f(x) + \frac{N_0^{-1}a^2}{2} f'(x + N_0^{-1/2}a), \quad 0 < \beta < 1.
\]

Using each of these in \(H^a(x)\) we have
\[ (2.26) \quad H^a(x) = F(x) + (1-\lambda_N)N_O \frac{1}{2} a f(x+\mu N_O^{-}\frac{1}{2} a) \]

and

\[ (2.27) \quad H^a(x) = F(x) + (1-\frac{1}{2} N_O^{-}\frac{1}{2} a f(x)) + N_O^{-}\frac{1}{2} \frac{1}{2} \frac{1}{2} a^2 f'(x+\beta N_O^{-}\frac{1}{2} a)] . \]

Hence, expanding \( J'(H^a) \) around \( J'(F) \), we get

\[ (2.28) \quad J'(H^a) = J'(F) + (1-\lambda_N)N_O^{-}\frac{1}{2} a f(x+\mu N_O^{-}\frac{1}{2} a)J''(H^{**}) \]

where \( H^{**} \) lies between \( F \) and \( H^a \),

as well as

\[ J'(H^a) = J'(F) + (1-\lambda_N)N_O^{-}\frac{1}{2} a f + N_O^{-}\frac{1}{2} \frac{1}{2} a f'(x+\beta N_O^{-}\frac{1}{2} a)]J''(F) \]

\[ + \frac{(1-\lambda_N)^2}{2} \frac{1}{2} N_O^{-}\frac{1}{2} a f + N_O^{-}\frac{1}{2} \frac{1}{2} a f'(x+\beta N_O^{-}\frac{1}{2} a)]^2 J'''(H^*) \]

where \( H^* \) lies between \( F \) and \( H^a \)

\[ (2.29) \quad = J'(F) + N_O^{-}\frac{1}{2} a(1-\lambda_N)J''(F) + N_O^{-}\frac{1}{2} a \frac{(1-\lambda_N)}{2} R_N^a \]

where \( R_N^a = f'(x+\beta N_O^{-}\frac{1}{2} a)J''(F)+(1-\lambda_N)(f + \frac{1}{2} a f'(x+\beta N_O^{-}\frac{1}{2} a)]^2 J'''(H^*) \).

Substituting from (2.24), (2.28), and (2.29) into (2.23) we have
\[ P_n^a = \frac{\lambda_n}{N_o} \sum_{i=1}^{n} \left\{ \frac{1}{2} \left( G_{i,1} - F \right) \right\} + \frac{1}{2} \left( N_o - 1 \right) a f(x + \beta N_o a) J''(H^a) dF \]

\[ + \frac{1}{2} N_o a \left[ J'(F(Y_1)) + N_o a(1 - \lambda) f(Y_1 + \mu N_o a) J''(H^{**}(Y_1)) \right] f(Y_1) \]

\[ - \frac{1}{2} N_o a \left\{ \sum_{i=1}^{n} \left\{ (G_{i,1} - F) J'(F) + N_o a(1 - \lambda) \right\} \right\} f(Y_1) \]

\[ - \frac{1}{2} N_o a \left[ J'(F) + N_o a(1 - \lambda) f(x + \mu N_o a) J''(H^{**}) \right] dF \]

\[ - \frac{1}{2} N_o a \left\{ \sum_{i=1}^{n} \left\{ (G_{i,1} - F) J'(F) + N_o a(1 - \lambda) \right\} \right\} dF \]

\[ + \frac{1}{2} N_o a \left[ J'(F(Y_1)) f(Y_1) + N_o a(1 - \lambda) f(Y_1 + \mu N_o a) f(Y_1) J''(H^{**}(Y_1)) \right] \]

\[ - \frac{1}{2} D_{N,1}(Y_1) - \frac{1}{2} D_{N,2}(Y_1) \]
Let the coefficient of $N_0^{-1}a_i^2$ in (2.30) be denoted by $\chi^a_N(Y_i)$. Replacing $a$ with $b$ in (2.30) and subtracting equations, we obtain cancellation of the leading term, namely

$$B_n^a - B_n^b = \frac{\chi^a_N}{N_0} \left\{ \frac{-1}{N_0(a-b)} \sum_{i=1}^{n} \left[ (1-\lambda_N) \int (G_{1,i} - F) J''(F) dF \right. \right.$$

$\left. \left. + J'(F(Y_i))f(Y_i) - \int J'(F) dF \right] + N_0^{-1} \sum_{i=1}^{n} (a^2 \chi_N^a(Y_i) - b^2 \chi_N^b(Y_i)) \right\}$

(2.31)

Now let

$$T(Y_i) = (1-\lambda_N) \int (G_{1,i} - F) J''(F) dF + J'(F(Y_i))f(Y_i) - \int J'(F) dF$$

(2.32)

Then

$$N_0(B_n^a - B_n^b) = \frac{1}{\lambda_N^2(a-b)n^{-1/2}} \sum_{i=1}^{n} T(Y_i) + \frac{1}{\lambda_N} \sum_{i=1}^{n} (a^2 \chi_N^a(Y_i) - b^2 \chi_N^b(Y_i))$$

(2.33)

We will show later in the discussion of the remainders that the second term on the RHS is $o_P(1)$. Now by the Central Limit Theorem,

$$\lambda_N^{1/2}(a-b)n^{-1/2} \sum_{i=1}^{n} T(Y_i)$$

is asymptotically normally distributed. We will now compute the mean and variance.

$$E(T(Y)) = E[(1-\lambda_N) \int (G_{1} - F) J''(F) dF + J'(F(Y))f(Y) - \int J'(F) dF]$$

$$= (1-\lambda_N) \int E(G_{1} - F) J''(F) dF + \int J'(F) dF - \int J'(F) dF$$

$$= 0$$
\[ \text{Var}(T(Y)) = E(T^2(Y)) \]
\[ = (1-\lambda_n)^2 E(\int J''(F)(G_1 - F) \, df) \]
\[ + 2(1-\lambda_n) E(J'(F(Y))f(Y) \int J''(F)(G_1 - F) \, df) \]
\[ + \int (J'(F)f)^2 \, df - (\int J'(F) \, df)^2 \]

(2.34)

Now

\[ E(\int J''(F)(G_1 - F) \, df)^2 = \]
\[ = \int \int (G_1(x) - F(x))(G_1(y) - F(y))J''(F(x))J''(F(y))f(x)f(y) \, df(x) \, df(y) \]
\[ = 2 \int \int \left( E(G_1(x) - F(x))(G_1(y) - F(y))J''(F(x))J''(F(y))f(x)f(y) \, df(x) \, df(y) \right) \]
\[ = 2 \int \int F(x)(1-F(y))J''(F(x))J''(F(y))f(x)f(y) \, df(x) \, df(y) \]
\[ = 2 \int \int \int \int J''(F(x))J''(F(y))f(x)f(y) \, df(x) \, df(y) \, df(u) \, df(v) \]
\[ = \int \int J''(F(x))J''(F(y))f(x)f(y) \, df(x) \, df(y) \, df(v) \, df(u) \]
\[ = \int (Q(v) - Q(u))^2 \, df(u) \, df(v), \quad \text{where } Q(t) = \int_0^t J''(F(z))f(z) \, df(z) \]
\[ = \frac{1}{2} \int (Q(v) - Q(u))^2 \, df(u) \, df(v) \]
\[ = \frac{1}{2} \left[ 2 \int Q^2(t) \, df(t) - 2(\int Q(t) \, df(t))^2 \right] \]

(2.35) \[ = \int Q^2 \, df - (\int Q \, df)^2 \]
\[
E(J'(F(Y))f(Y) \int J''(F)(G_1 - F)f\,dF)
\]

\[
= E \int J'(F(Y))f(Y)(G_1(x) - F(x))J''(F(x))f(x)\,dF(x)
\]

\[
= \int E[J'(F(Y))f(Y)(G_1(x) - F(x))J''(F(x))f(x)\,dF(x)
\]

\[
= \int \left\{ \int_{-\infty}^{x} J'(F(y))f(y)(1 - F(x))\,dF(y) + \right. \\
+ \left. \int_{x}^{\infty} J'(F(y))f(y)(-F(x))\,dF(y) \right\} J''(F(x))f(x)\,dF(x)
\]

\[
= \int_{y < x} J'(F(y))f(y)J''(F(x))f(x)(1 - F(x))\,dF(y)\,dF(x)
\]

\[
- \int_{x < y} J'(F(y))f(y)J''(F(x))f(x)F(x)\,dF(y)\,dF(x)
\]

\[
= \int_{y < x < v} J'(F(y))f(y)J''(F(x))f(x)\,dF(v)\,dF(y)\,dF(x)
\]

\[
- \int_{v < x < y} J'(F(y))f(y)J''(F(x))f(x)\,dF(v)\,dF(y)\,dF(x)
\]

\[
= \int_{y < v} [Q(v) - Q(y)]J'(F(y))f(y)\,dF(y)\,dF(v)
\]

\[
- \int_{v < y} [Q(y) - Q(v)]J'(F(y))f(y)\,dF(y)\,dF(v)
\]

\[
= \int [Q(v) - Q(y)]J'(F(y))f(y)\,dF(y)\,dF(v)
\]

\[
= \int_{Q} dF \int J'(F)\,dF - \int Q J'(F)f\,dF
\]

\[
= -[\int Q J'(F)f\,dF - \int Q dF \int J'(F)\,dF]
\]

(2.36)
Substituting in (2.34)

\[ \text{Var } T(Y) = (1-\lambda_N)^2 \left[ \int Q^2 dF - (\int Q dF)^2 \right] - 2(1-\lambda_N) \left[ \int Q J'(F) f dF - \int Q dF \int J'(F) f dF \right] + \int (J'(F) f)^2 dF - (\int J'(F) f dF)^2 \]

(2.37)

\[ = \int [(1-\lambda_N) Q - J'(F) f]^2 dF - (\int [(1-\lambda_N) Q - J'(F) f] dF)^2 \]

where \( Q(t) = \int_0^t J''(F) f dF \).

Now consider \( B_m^a \), defined following (2.18)

\[ B_m^a = \int (F_m - F) J'(H^a(x - \frac{1}{2} a)) dF(x + \frac{1}{2} a) = \int (\frac{1}{2} - \frac{1}{2} F_m(x - \frac{1}{2} a) - F(x - \frac{1}{2} a)) J'(H^a(x - \frac{1}{2} a)) dF(x). \]

Now \( H^a(x - \frac{1}{2} a) = \frac{1}{2} F_m(x - \frac{1}{2} a) + (1-\lambda_N) F(x) \).

Thus \( H^a(x - \frac{1}{2} a) \) is like \( H^{-a}(x) \), with the roles of \( \lambda_N \) and \( 1-\lambda_N \) reversed. But

\[ B_m^a = \int (G_n(x - \frac{1}{2} a) - F(x + \frac{1}{2} a)) J'(H^a) dF. \]

Thus \( B_m^a \) is of the same form as \( B_n^{-a} \), with the identifications \( n \leftrightarrow m \), \( \lambda_N \leftrightarrow 1-\lambda_N \), and \( a \leftrightarrow -a \). Hence we can get an expansion of
\[ B_{m}^{a} - B_{m}^{b} \text{ similar to the expansion of } B_{n}^{a} - B_{n}^{b} \text{ in (2.31)} \]

\[
B_{m}^{a} - B_{m}^{b} = \frac{1-\lambda_{N}}{N_{o}} N_{o}^{-\frac{1}{2}} (a-b) \sum_{i=1}^{m} \left\{ \lambda_{N} \int_{-\infty}^{\infty} J''(F) (F_{1,i} - F) f dF \right. \\
+ \left. J'(F(X_{i})) f(X_{i}) - \int_{-\infty}^{\infty} J'(F) f dF \right\} \\
+ \frac{1-\lambda_{N}}{N_{o}} N_{o}^{-1} \sum_{i=1}^{m} \left( a_{N}^{a}(X_{i}) - b_{N}^{b}(X_{i}) \right) \\
\]

where \( F_{1,i} \) is the empirical c.d.f. of the single observation \( X_{i} \).

\[
\therefore N_{o}(B_{m}^{a} - B_{m}^{b}) = -(1-\lambda_{N})^{\frac{1}{2}} (a-b) m^{-\frac{1}{2}} \sum_{i=1}^{m} S(X_{i}) \\
+ (1-\lambda_{N}) N_{o}^{-1} \sum_{i=1}^{m} \left( a_{N}^{a}(X_{i}) - b_{N}^{b}(X_{i}) \right) \\
\]

where

\[
S(X_{i}) = \lambda_{N} \int J''(F) (F_{1,i} - F) f dF + J'(F(X_{i})) f(X_{i}) - \int J'(F) f dF \\
\]

\( \xi_{N}^{a} \) is of the same form as \( \chi_{N}^{a} \) and so the proof that the last term in (2.33) is \( o_{p}(1) \) will carry over to the last term in (2.39)

As before, the leading term on the RHS of (2.39) is asymptotically normally distributed. In this case, we have \( E(S(X)) = 0 \) and

\[
\Var S(X) = \int \left[ \lambda_{N} - J'((F)) f \right]^{2} dF - \left( \int \left[ \lambda_{N} - J'((F)) f \right] dF \right)^{2} \\
\]

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Thus

\[ N \left[ (\frac{a^a}{m} - \frac{b^b}{n}) \right] \]

\[ = -(a-b) \left[ (1-\lambda_N)^\frac{1}{2} m - \frac{1}{2} \sum_{i=1}^{m} S(x_i) + \lambda_N \frac{1}{2} n - \frac{1}{2} \sum_{i=1}^{n} T(y_i) \right] \]

(2.42)

\[ + N^{-\frac{1}{2}} \left[ (1-\lambda_N)^\frac{1}{2} m \sum_{i=1}^{m} (\frac{2}{n} a^a_{N}(x_i) - b^b_{N}(x_i)) \right] \]

\[ + \lambda_N \sum_{i=1}^{n} (\frac{2}{m} a^a_{N}(y_i) - b^b_{N}(y_i)) \].

Let

(2.43)

\[ B(x, \bar{y}) = (1-\lambda_N)^\frac{1}{2} m - \frac{1}{2} \sum_{i=1}^{m} S(x_i) + \lambda_N \frac{1}{2} n - \frac{1}{2} \sum_{i=1}^{n} T(y_i) \]

Then \( B(x, \bar{y})/\sqrt{\text{Var} B(x, \bar{y})} \) converges in distribution to a standard normal distribution, provided \( \text{Var} B(x, \bar{y}) \neq 0 \).

\[ \text{Var} B(x, \bar{y}) = (1-\lambda_N) \text{Var} S(x) + \lambda_N \text{Var} T(y) \]

\[ = (1-\lambda_N) \int [\lambda_N Q-J'(F)\bar{f}]^2 dF + \lambda_N \int [(1-\lambda_N)Q-J'(F)\bar{f}]^2 dF \]

\[ - (1-\lambda_N)(\int [\lambda_N Q-J'(F)\bar{f}] dF)^2 - \lambda_N (\int [(1-\lambda_N)Q-J'(F)\bar{f}] dF)^2 \]

\[ = [(1-\lambda_N)\lambda_N^2 + \lambda_N(1-\lambda_N)^2] \int Q^2 dF - (\int Q dF)^2 \]

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\[-4\lambda_N (1-\lambda_N) \int Q\, J'(F) f dF - \int Q dF \int J'(F) f dF \]
\[+ (1-\lambda_N + \lambda_N) \left( \int (J'(F) f)^2 dF - (\int J'(F) f dF)^2 \right) \]

\[= \lambda_N (1-\lambda_N) \left[ \int Q^2 dF - (\int Q dF)^2 \right] - 4\lambda_N (1-\lambda_N) \int Q J'(F) f dF - \int Q dF \int J'(F) f dF \]
\[+ (4\lambda_N (1-\lambda_N) + 1-4\lambda_N (1-\lambda_N)) \int (J'(F) f)^2 dF - (\int J'(F) f dF)^2 \]

\[= \lambda_N (1-\lambda_N) \left[ \int (Q - 2J'(F) f)^2 dF - (\int (Q - 2J'(F) f) dF)^2 \right] \]
\[+ (1-4\lambda_N (1-\lambda_N)) \int (J'(F) f)^2 dF - (\int J'(F) f dF)^2 \]

\[(2.44) \]

Let us briefly look at when \( \text{Var} \, B(X,Y) \) is zero. This will occur only when \( \text{Var} \, S(X) \) and \( \text{Var} \, T(X) \) are both zero. Now

\[\text{Var} \, S(X) = \text{Var} (\lambda_N Q(X) - J'(F(X)) f(X)) = 0 \quad \text{only if} \]

\[\lambda_N Q(X) - J'(F(X)) f(X) = \text{constant \ a.s. } (X) \]

\[\Rightarrow \lambda_N Q'(X) - J''(F(X)) f^2(X) - J'(F(X)) f'(X) = 0 \quad \text{a.s. } (X) \]

But \( Q'(X) = J''(F(X)) f^2(X) \).

\[\Rightarrow J''(F(X)) f^2(X) = -\frac{1}{1-\lambda_N} J'(F(X)) f'(X) \quad \text{a.s. } (X) \]

\[\Rightarrow \text{either } \quad \frac{J''(F(X))}{J'(F(X))} f(X) = -\frac{1}{1-\lambda_N} \frac{f'(X)}{f(X)} \quad \text{a.s. } (X) \]

or \( J'(F(X)) = 0 \)
\[ \frac{\partial}{\partial x} (\log J'(F(X)) + \frac{1}{1-\lambda_N} \log f(X)) = 0 \quad \text{on} \quad J'(F(X)) \neq 0 \quad \text{a.s.} \]

\[ \frac{1}{1-\lambda_N} = \text{constant on} \quad J'(F(X)) \neq 0 \quad \text{a.s. (X)} \]

But \( \text{Var} \ T(X) = \text{Var} \left( (1-\lambda_N)Q(X) - J'(F(X))f(X) \right) \). Hence \( \text{Var} \ T(X) = 0 \)

\[ \Rightarrow \quad J'(F(X))[f(X)]^{1/\lambda_N} = \text{constant on} \quad J'(F(X)) \neq 0 \quad \text{a.s. (X)} \]

Thus for both \( \text{Var} \ S(X) \) and \( \text{Var} \ T(X) \) to be \( 0 \), we must have both \( J'(F(X))[f(X)]^{1/\lambda_N} \) and \( J'(F(X))[f(X)]^{1/\lambda_N} \) constant a.s. on \( J'(F(X)) \neq 0 \). For \( \lambda_N \neq \frac{1}{2} \), this implies by dividing one equation by the other, that \( f(x) \) is constant a.e. for \( x \) in the region where \( J'(F(x)) \neq 0 \), and hence that \( J'(F(x)) \) is constant a.e. whenever it is not \( 0 \).

For \( \lambda_N = \frac{1}{2} \), both equations reduce to

\[ J'(F(X))f^2(X) = \text{constant} \quad \text{a.s. (X)} \quad \text{on} \quad J'(F(x)) \neq 0 . \]

Let us now return to equation (2.21) multiplying both sides by \( N_0 \) and substituting from (2.42) and (2.43):

\[ \frac{1}{N_0} (a-b) - \frac{N_0}{(1-\lambda_N)B(F)} \left( T_N^a - T_N^b \right) = \frac{a-b}{B(F)} B(x, x) \]

\[ (2.45) \quad + \frac{1}{N_0 B(F)} \left[ (1-\lambda_N) \left( \sum_{i=1}^{m} (a^N_{x_i} - b^N_{x_i}) \right)^2 \right] + \lambda_N \left( \sum_{i=1}^{n} (a^N_{y_i} - b^N_{y_i}) \right) \]

\[ - \frac{1}{N_0} \int a^3 dF - \frac{b^3 \int b^3 dF}{(1-\lambda_N)B(F)} - \frac{N_0}{(1-\lambda_N)B(F)} \sum_{i=1}^{5} (c^a_{i} - c^b_{i}) . \]
We will show in the section on remainders that the last three terms on the RHS of (2.45) are \( o_p(1) \) uniformly in \( a \) and \( b \) for \( a \) and \( b \) both lying in any bounded interval. Hence,

\[
(2.46) \quad \frac{1}{N_o} (a-b) - \frac{N_o}{(1-\lambda_f)B(F)} (T_N^a - T_N^b) = - \frac{a-b}{B(F)} B(\bar{\chi}, \bar{\chi}) + o_p(1).
\]

Now (2.46) is an identity in \( a \) and \( b \) and so we can plug in \( a = N_0^{1/2} \theta_N^U \), \( b = N_0^{1/2} \theta_N^L \). Since, by Sen's theorem, \( N_0^{1/2} \theta_N^U \) and \( N_0^{1/2} \theta_N^L \) are bounded in probability (in fact they are asymptotically normally distributed), and the remainder term in (2.46) is \( o_p(1) \) uniformly in \( a \) and \( b \) for \( a \) and \( b \) in any bounded interval, it follows that the remainder term is again \( o_p(1) \) after the substitution. Hence we have

\[
N_0 (\theta_N^U - \theta_N^L) - \frac{N_o}{(1-\lambda_f)B(F)} (T_N^a - T_N^b) = - \frac{1}{B(F)} N_0 (\theta_N^U - \theta_N^L) B(\bar{\chi}, \bar{\chi}) + o_p(1)
\]

or

\[
(2.47) \quad N_0 \mathcal{L}_N - \frac{N_o}{(1-\lambda_f)B(F)} (T_N^a - T_N^b) = - \frac{1}{B(F)} N_0 \mathcal{L}_N B(\bar{\chi}, \bar{\chi}) + o_p(1)
\]

Now we showed that \( B(\bar{\chi}, \bar{\chi})/\sqrt{\text{Var} B(\bar{\chi}, \bar{\chi})} \) converges in distribution to a standard normal distribution. Also, by Sen's result, \( N_0^{1/2} \mathcal{L}_N \overset{p}{\to} 2z_{\alpha/2} \frac{A}{B(F)} \). Hence by a well known result
(2.48) \[ \frac{1}{N} \frac{1}{\sigma_N^2} \mathcal{L}_N \sim B(\xi, \chi) \xrightarrow{d} N(0,1) \text{ as } N \to \infty \]

where

(2.49) \[ \sigma_N^2 = 4z_{\alpha/2}^2 \frac{A^2}{B^2(F)} \frac{1}{B^2(F)} \text{ Var } B(\xi, \chi) \]

and \( \text{Var } B(\xi, \chi) \) is given by (2.44).

Next, we will show that

\[ \left( \frac{1}{N} \theta^U_N - \frac{1}{N} \theta^L_N \right) = 2 \frac{A(1-\lambda_N)}{\sqrt{N}} z_{\alpha/2} + o(N^{-1}) \]

Now

\[ \frac{1}{N} \theta^L_N = \frac{1}{N} \theta^L_N \quad \text{and} \quad \frac{1}{N} \theta^U_N = \frac{1}{N} \theta^U_N \]

But

\[ \theta^L_N = \inf \{ \theta \mid T_N(\xi, \chi - \theta \xi_n) \geq \mu - \frac{A(1-\lambda_N)}{\sqrt{N}} z_{\alpha/2} \} \]

and

\[ \theta^U_N = \sup \{ \theta \mid T_N(\xi, \chi - \theta \xi_n) \leq \mu + \frac{A(1-\lambda_N)}{\sqrt{N}} z_{\alpha/2} \} \]
Thus we have \( T_N(\bar{X}, \bar{X} - (\bar{\theta}_N^L)_{1n}) < \mu - \frac{A(1-\lambda_N)}{N_0^{1/2}} z_{\alpha/2} \leq T_N(\bar{X}, \bar{X} - \bar{\theta}_N^L) \)

and \( T_N(\bar{X}, \bar{X} - (\bar{\theta}_N^U)_{1n}) < \mu + \frac{A(1-\lambda_N)}{N_0^{1/2}} z_{\alpha/2} \leq T_N(\bar{X}, \bar{X} - \bar{\theta}_N^U) \)

(See figure 2.1). Hence...
\[
|T_N(X, x - \theta^U_{N, i}) - T_N(x, x - \theta^L_{N, i})| - \frac{2A(1 - \lambda_N)}{N^{1/2}} |z_{\alpha/2}|
\]

\[
= |T_N(x, x - \theta^U_{N, i}) - (\mu - \frac{A(1 - \lambda_N)}{N^{1/2}} z_{\alpha/2})| - |T_N(x, x - \theta^L_{N, i}) - (\mu + \frac{A(1 - \lambda_N)}{N^{1/2}} z_{\alpha/2})|
\]

\[
\leq |T_N(x, x - \theta^U_{N, i}) - (\mu - \frac{A(1 - \lambda_N)}{N^{1/2}} z_{\alpha/2})| + |T_N(x, x - \theta^L_{N, i}) - (\mu + \frac{A(1 - \lambda_N)}{N^{1/2}} z_{\alpha/2})|
\]

\[
\leq |T_N(x, x - \theta^U_{N, i}) - T_N(x, x - \theta^L_{N, i})| + |T_N(x, x - \theta^L_{N, i}) - T_N(x, x - \theta^L_{N, i})|
\]

Now we know from Chapter 1 that \( \theta^U_N \) and \( \theta^L_N \) belong to the set \( \{d_{ij}, i=1, \ldots, m, j=1, \ldots, n\} \), where \( d_{ij} = x_j - x_i \cdot |T_N(x, x - d_{ij}) - T_N(x, x - d_{ij})| \)
represents the jump in the rank statistic due to a switching of ranks between \( x_i \) and \( x_j \). As we saw, this jump is of size

\[
\frac{1}{m} (J_N(i, i+1) - J_N(i, i)) \text{, for some } i \in \{1, 2, \ldots, N-1\}.
\]

A well-known result, sometimes known as Polya's lemma, states that

if a sequence of monotone functions converges pointwise to a bounded monotone function, then the convergence is uniform. Hence \( J_N(u) \) converges uniformly to \( J(u) \) and so for any sequence \( \{i_N, N=1, 2, \ldots\} \) where \( i_N \in \{1, 2, \ldots, N\}, \quad |J_N(i_N) - J_N(i_N)| = o(1). \)

Furthermore, since \( J(u) \) is continuous on the closed interval \([0,1]\), it is uniformly continuous, and so

\[
|J_N(i_N) - J_N(i_N)| = o(1). \quad \text{But}
\]

\[
\frac{1}{m} |J_N(i_N) - J_N(i_N)|
\]

\[
\leq \frac{1}{m} |J_N(i_N) - J_N(i_N)| + \frac{1}{m} |J_N(i_N) - J_N(i_N)| + \frac{1}{m} |J_N(i_N) - J_N(i_N)|
\]

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(2.51) \quad = o(N^{-1}), \text{ by the observations above.}

Thus

(2.52) \quad |T_N(x, x - d_{i,j}) - T_N(x, x - d_{i,j})| = o(N^{-1}).

Thus it follows from (2.50) that

(2.53) \quad N_o \frac{2A(1-\lambda_n)}{N^{1/2} \sigma_{x/2}} = o(1).

Substituting in (2.47) gives

(2.54) \quad N_o \frac{2A}{B(F)} \frac{z_{x/2}}{\sigma_{x/2}} = - \frac{1}{N^{1/2} \sigma_{x/2}} B(x, x) + o_p(1).

Hence, from (2.48) we obtain that

(2.55) \quad \frac{1}{N^{2}} \frac{1}{N} \frac{2A}{B(F)} \frac{z_{x/2}}{\sigma_{x/2}} \to N(0,1) \text{ as } N \to \infty.

where

\[ \sigma_N = 2z_{x/2} \frac{A}{B^2(F)} \left\{ \lambda_n \left[ \int (Q - 2J'(F)F)^2 dF - \left( \int (Q - 2J'(F)F) dF \right)^2 \right] \right. \]

(2.56) \quad + \left. (1 - 4n \lambda_n (1 - \lambda_n)) \left[ \int (J'(F)F)^2 dF - \left( \int J'(F)F dF \right)^2 \right] \right\}^{1/2} \]
Hence, subject to the proof of statements concerning the remainder terms, we are done. In the next section we consider the remainder terms in detail.
2.3 Remainder Terms - Proof of Their Asymptotic Negligibility.

The remainder term we are interested in is, from (2.45)

\[
\frac{1}{N_0 B(F)} \left[ (1-\lambda_N) \sum_{i=1}^{m} \left(a^{2 \frac{\partial}{\partial N}}(X_i) - b^{2 \frac{\partial}{\partial N}}(X_i) \right) + \lambda_N \sum_{i=1}^{n} \left(a^{2 \frac{\partial}{\partial N}}(Y_i) - b^{2 \frac{\partial}{\partial N}}(Y_i) \right) \right]
\]

\[
- \frac{1}{N_0^{1/2} B(F)} \left[ a \int \frac{a}{dF} - b \int \frac{b}{dF} \right] - \frac{N_0}{(1-\lambda_N) B(F)} \sum_{i=1}^{5} (c_i - c_i).
\]

We want to show that this term is \(o_p(1)\) uniformly in \(a\) and \(b\) for \(a\) and \(b\) bounded. The uniformity of the convergence follows from the fact that \(a\) and \(b\) only enter into our expressions through \(N_0^{-1/2} a\) and \(N_0^{-1/2} b\). Since we are considering \(N_0 \to \infty\), and hence \(N_0^{-1/2} a \to 0\), any statement which is true for \(N_0^{-1/2} T < \delta\) and hence \(N_0 > \frac{T^2}{\delta^2}\) will also be true for any \(|a| < T\), since \(N_0^{-1/2} T < \delta \Rightarrow N_0^{-1/2} |a| < \delta\) for \(|a| < T\).

Before proceeding with the proof, let us establish a number of useful lemmas. As before, we assume throughout that \(\theta = 0\).

**Lemma 1.** For any \(\epsilon > 0\), \(\exists \eta(\epsilon) > 0\) \(\exists\)

\[
P(X_i \in I_N(\eta), i=1,\ldots,m, Y_j \in I_N(\eta), j=1,\ldots,n) > 1-\epsilon
\]

for all \(N\) sufficiently large, where \(I_N(\eta) = \{x | F(x)(1-F(x)) > \frac{\eta}{N}\}\).
Proof. $I_N(\eta) = (a_N, b_N)$, where $a_N = F^{-1}(\frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4\eta}{N}})$ and $b_N = F^{-1}(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4\eta}{N}})$

$$P(X_i \in I_N(\eta), i = 1, \ldots, m, Y_j \in I_N(\eta), j = 1, \ldots, n) = [F(b_N) - F(a_N)]^N$$

$$= (\sqrt{1 - \frac{4\eta}{N}})^N > \left(1 - \frac{4\eta}{N}\right)^N \geq 1 - 4\eta \quad \text{for} \quad N > 4\eta.$$ 

Hence choosing $\eta(\epsilon) = \frac{\epsilon}{4}$ gives us the result.

Corollary. Let $(G_n)$ be any sequence of c.d.f.'s and let $H_N(x) = \lambda_N^2 F(x) + (1 - \lambda_N^2) G_N(x)$, where $\lambda_N$ satisfies condition (1) of Theorem 1. Then for all sufficiently large $N$, where $I_N(\eta) = \{x \mid H_N(x)(1 - H_N(x)) > \frac{\eta}{N}\}$.

Proof. From condition (1) we have that $H_N(1 - H_N) \geq \lambda_N^2 F(1 - F)$. Thus

$$F(1 - F) > \frac{\eta}{N} \implies H_N(1 - H_N) > \frac{\lambda_N^2 \eta}{N}.$$ 

Hence $I_N(\eta') \subseteq I_N(\lambda_N^2 \eta')$ and therefore from Lemma 1, choosing $\eta = \lambda_N^2 \frac{\epsilon}{4}$ gives the result.

Lemma 2. Under conditions (1) and (4) of Theorem 1, $\forall \epsilon > 0$, $\exists \eta(\epsilon) > 0$ where

$$P(Y_j - N \overset{1}{\sim} \frac{1}{2} a \in I_N(\eta), j = 1, \ldots, n) > 1 - \epsilon$$

for all $N$ sufficiently large.
Proof. From Lemma 1, $I_n(\eta) = (F^{-1}(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\eta}{N}}), F^{-1}(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4\eta}{N}}))$

Now, $\sqrt{1 - \frac{4\eta}{N}} > 1 - \frac{4\eta}{N}$

$\therefore I_n(\eta) \supset (F^{-1}(\frac{2\eta}{N}), F^{-1}(1 - \frac{2\eta}{N}))$

$P(Y - N a \in I_n(\eta)) \geq P(F^{-1}(\frac{2\eta}{N}) + N_{\alpha} \leq Y < F^{-1}(1 - \frac{2\eta}{N}) + N_{\alpha})$

$= F(F^{-1}(1 - \frac{2\eta}{N}) + N_{\alpha}) - F(F^{-1}(\frac{2\eta}{N}) + N_{\alpha})$

$= 1 - \frac{4\eta}{N} + N_{\alpha} [F(F^{-1}(1 - \frac{2\eta}{N}) + N_{\alpha}) - F(F^{-1}(\frac{2\eta}{N}) + N_{\alpha})]$

where $0 < \alpha < 1$

$\geq 1 - \frac{4\eta}{N} - \frac{|a|}{N^{1/2}} \frac{2}{K} [F(F^{-1}(1 - \frac{2\eta}{N}))(1 - F(F^{-1}(1 - \frac{2\eta}{N})))^3$

$\geq 1 - \frac{4\eta}{N} - \frac{|a|}{N^{1/2}} \frac{2}{K} [F(F^{-1}(\frac{2\eta}{N}))(1 - F(F^{-1}(\frac{2\eta}{N})))^3$

for $N$ sufficiently large, by condition (4)

$= 1 - \frac{4\eta}{N} - \frac{|a|}{N^{1/2}} \frac{2}{K} \left[\frac{2\eta}{N} \left(1 - \frac{2\eta}{N}\right)^3\right]$

$\geq 1 - \frac{4\eta}{N} - \frac{|a|}{N^{1/2}} \left(\frac{2}{N}\right)^3$ , for $N > 4\eta$

$= 1 - \frac{4\eta}{N} - \frac{2/3}{N^{1/6}} \frac{|a|}{N^{1/6}}$

$\geq 1 - \frac{4\eta \eta^{2/3}}{N}$ for $\frac{1}{N^6} > 2|a|K$
Choose $\eta(\varepsilon)$ so that $4\eta + \frac{2}{\eta} = \varepsilon$.

Then we have, for $N$ sufficiently large,

$$P(Y_j - N_0 \leq a \in I_N(\eta), j=1,\ldots,n) \geq (1 - \frac{c}{N})^n \geq (1 - \frac{c}{N})^N \geq 1 - \varepsilon.$$ Q.E.D.

Lemma 3. Let $g(x)$ be any nonnegative function. Then the following inequality holds:

$$E \int |F_m - F| g \, dF_m \leq \frac{1}{m} \int g \, dF + \frac{1}{m^{\frac{1}{2}}} \int g[F(1-F)]^\frac{1}{2} \, dF.$$ 

Proof.

$$E \int |F_m - F| g \, dF_m = E[\frac{1}{m} \sum_{i=1}^m |F_m(X_{(i)}) - F(X_{(i)})|g(X_{(i)})],$$ where $X_{(i)}$ = $i$-th order statistic of $X$.

$$= \frac{1}{m} \sum_{i=1}^m E(\frac{1}{m} - F(X_{(i)})|g(X_{(i)}))$$

$$= \frac{1}{m} \sum_{i=1}^m \int |\frac{1}{m} - F| g \frac{m!}{(i-1)! (m-i)!} F^{i-1}(1-F)^{m-i} \, dF$$

$$= \frac{1}{m} \int g \left\{ \sum_{i=1}^m |\frac{1}{m} - F| \frac{F^i}{i!} (1-F)^{m-i} \right\} \, dF$$

$$\leq \frac{1}{m} \int g \left\{ \sum_{i=1}^m |\frac{1}{m} - F| (|1-mF| + mF) \frac{F^i}{i!} (1-F)^{m-i} \right\} \, dF$$

$$= \frac{1}{m} \int g \left\{ \sum_{i=1}^m (\frac{1}{m} - F) \frac{(m)!}{i!} F^i (1-F)^{m-i} + mF \sum_{i=1}^m |\frac{1}{m} - F| \frac{(m)!}{i!} F^i (1-F)^{m-i} \right\} \, dF$$
\[ \leq \int \frac{g}{F} \left\{ \sum_{i=0}^{m} \left( \frac{i}{m} - F \right)^{2} \left( \frac{m}{i} \right)^{i} \left( 1 - F \right)^{m-i} \right\} dF \]

\[ \leq \int \frac{g}{F} \left\{ \frac{F(1-F)}{m} + \frac{F(1-F)}{m} \right\} dF \]

\[ \leq \frac{1}{m} \int g \, dF + \frac{1}{m^{1/2}} \int g[F(1-F)]^{1/2} dF \]

Q.E.D.

**Lemma 4.** Let \( g(x) \) be any nonnegative function. Then the following identity holds:

\[ E \int \left( \frac{F}{m} - F \right)^{2} g \, dF \left[ m \int \frac{(1-F)(1-2F)}{m^{2}} + \frac{F(1-F)}{m} \right] g \, dF \]

**Proof.**

\[ E \int \left( \frac{F}{m} - F \right)^{2} g \, dF \left[ m \sum_{i=1}^{m} \left( \frac{F}{m} - F \right)^{2} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} \right] g \, dF \]

\[ = \frac{1}{m} \sum_{i=1}^{m} E \left\{ \left( \frac{i}{m} - F \right)^{2} g(X_{i}) \right\} \]

\[ = \frac{1}{m} \sum_{i=1}^{m} \int \left( \frac{i}{m} - F \right)^{2} g \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} dF \]

\[ = \frac{1}{m} \int \frac{g}{F} \left\{ \sum_{i=1}^{m} \left( \frac{i}{m} - F \right)^{2} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} \right\} dF \]

\[ = \frac{1}{m} \int \frac{g}{F} \left\{ \sum_{i=1}^{m} \left( \frac{i}{m} - F \right)^{3} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} + \sum_{i=1}^{m} \left( \frac{i}{m} - F \right)^{2} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} \right\} dF \]

\[ = \int \frac{g}{F} \left\{ \sum_{i=0}^{m} \left( \frac{i}{m} - F \right)^{3} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} - \left( \frac{i}{m} - F \right)^{2} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} \right\} dF \]

\[ = \int \frac{g}{F} \left\{ \sum_{i=0}^{m} \left( \frac{i}{m} - F \right)^{3} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} - \left( \frac{i}{m} - F \right)^{2} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} \right\} dF \]

\[ = \int \frac{g}{F} \left\{ \sum_{i=0}^{m} \left( \frac{i}{m} - F \right)^{3} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} - \left( \frac{i}{m} - F \right)^{2} \left( \frac{m}{i} \right)^{i-1} \left( 1 - F \right)^{m-i} \right\} dF \]
\[
\begin{align*}
&= \int g \left\{ \frac{F(1-F)(1-2F)}{m^2} + \frac{F^3(1-F)^m}{m} + F\left(\frac{F(1-F)}{m} - \frac{F^2(1-F)^m}{m}\right) \right\} \, dF \\
&= \int g \left\{ \frac{(1-F)(1-2F)}{m^2} + \frac{F(1-F)}{m} \right\} \, dF \quad \text{Q.E.D.}
\end{align*}
\]

Lemma 5. Let \( g(x) \) be any nonnegative function. Then we have the following inequality:

\[
E \int |F_m - F|^3 g \, dF_m \leq \int \left\{ \frac{\sqrt{3} \left[ F(1-F) \right]^2}{m^{3/2}} + \frac{2F(1-F)}{m^2} + \frac{1}{m^2} \right\} g \, dF
\]

Proof.

\[
E \int |F_m - F|^3 g \, dF_m = \frac{1}{m} \sum_{i=1}^m E\{ |F_m(X(i)) - F(X(i))|^3 g(X(i)) \}
\]

\[
= \frac{1}{m} \sum_{i=1}^m \int |\frac{1}{m} - F|^3 g(\frac{1}{m} F(1-F)^{m-i}) \, dF
\]

\[
\leq \frac{1}{m} \int \frac{g}{F} \sum_{i=1}^m \left\{ \frac{1}{m} - F \right\}^3 \left( |i - mF| + mF \right)^i (1-F)^{m-i} \, dF
\]

\[
= \frac{1}{m} \int \frac{g}{F} \left\{ \sum_{i=1}^m \left( \frac{1}{m} - F \right)^i (1-F)^{m-i} + mF \sum_{i=1}^m |\frac{1}{m} - F|^3 (1-F)^{m-i} \right\} \, dF
\]

\[
\leq \frac{1}{m} \int \frac{g}{F} \left\{ \sum_{i=0}^m \left( \frac{1}{m} - F \right)^i (1-F)^{m-i} + mF \sum_{i=0}^m |\frac{1}{m} - F|^3 (1-F)^{m-i} \right\} \, dF
\]

\[
\leq \int \frac{g}{F} \left\{ \frac{3[F(1-F)]^2}{m^2} + F(1-F)(1-6F(1-F)) + F[\mu \mu_j] \right\} \, dF
\]

where \( \mu_j \) is the \( j \)-th moment about the mean of a binomial proportion with parameters \( m \) and \( F \).
\[
= \int \frac{g}{F} \left\{ \frac{3[F(1-F)]^2}{m^2} + \frac{F(1-F)(1-6F(1-F))}{m^3} + F[\frac{F(1-F)}{m}]^2 \right\} \ dF \\
\leq \int \frac{g}{F} \left\{ \frac{3[F(1-F)]^2}{m^2} + \frac{F(1-F)}{m^3} \right\} + \frac{g}{F} \left\{ \frac{3[F(1-F)]^3}{m^3} + \frac{[F(1-F)]^2}{m^4} \right\} \frac{1}{2} \ dF \\
\leq \int \frac{g}{F} \left\{ \frac{F(1-F)}{m^2} + \frac{1}{m^3} + \sqrt{\frac{3}{2}} \frac{[F(1-F)]^2}{m^{3/2}} + \frac{F(1-F)}{m^2} \right\} \ dF \\
= \int \frac{g}{F} \left\{ \frac{\sqrt{3}}{m^{3/2}} \frac{[F(1-F)]^2}{m^{3/2}} + \frac{2F(1-F)}{m^2} + \frac{1}{m^3} \right\} \ dF \\
\text{Q.E.D.}
\]

**Lemma 6.** Let \( r > -1, \ s \leq -1, \ r+s > -2 \). Then we have

\[
\int \int_{0 < u < v < 1} u^r(1-u)^s v^s(1-v)^r \ du \ dv < \infty.
\]

**Proof.** Partition the region of integration into 3 regions, A, B, C, as illustrated in the following figure.

Now \( \int \int_{B} \) is finite since the integrand is bounded on B. On A, \( u < v < \epsilon \implies (1-u)^s < (1-v)^s \), since s is negative. Hence
\[ \iiint_A u^{r-(1-u)^s} v^{s-(1-v)^r} \,du \,dv \leq \iiint_A u^r v^s (1-v)^{r+s} \,du \,dv \]

\[ = \int_0^e v^s (1-v)^{r+s} \int_0^v u^r \,du \,dv \]

\[ = \frac{1}{r+1} \int_0^e v^{r+s+1} (1-v)^{r+s} \,dv . \]

Now if \( r+s \geq 0, \ (1-v)^{r+s} \leq 1, \) and if \( r+s < 0, \ (1-v)^{r+s} \leq (1-e)^{r+s} \).

Thus \( (1-v)^{r+s} \) is bounded, and so the last integral \( \leq K \int_0^e v^{r+s+1} \,dv < \infty \)
since \( r+s+1 > -1 \). \( \iiint_B \) is bounded in the same way. Q.E.D.

Lemma 7. Let \( g(x,y) \) be any function. Then we have

\[ \mathbb{E} \iiint_{x<y} g(x,y) \,d(F_m(x)-F(x))d(F_m(y)-F(y)) = -\frac{1}{m} \iiint_{x<y} g(x,y) \,dF(x) \,dF(y), \]

assuming the RHS is finite.

Proof.

\[ \iiint_{x<y} g(x,y) \,d(F_m(x)-F(x))d(F_m(y)-F(y)) \]

\[ = \iiint_{x<y} g(x,y) \,dF_m(x) \,dF_m(y) - \iiint_{x<y} g(x,y) \,dF_m(x) \,dF_m(y) \]

\[ - \iiint_{x<y} g(x,y) \,dF_m(x) \,dF_m(y) + \iiint_{x<y} g(x,y) \,dF_m(x) \,dF_m(y) \]

\[ = I_1 - I_2 - I_3 + I_4 . \]
Now by Fubini's theorem, it follows that $E(I_2) = E(I_3) = I_4$. Now

$$I_1 = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} g(x_i, x_j),$$

where $x_1, \ldots, x_m$ are i.i.d. with c.d.f. $F$. Now the summation consists of $\binom{m}{2}$ identically distributed terms, each of which is distributed as $g(x(1), x(2))$, where $x(1), x(2)$ are the order statistics of a sample of size 2 from $F$.

$$E(I_1) = \frac{m(m-1)}{2m^2} \iint_{x < y} 2g(x, y)dF(x)dF(y) = \frac{m-1}{m} \iint_{x < y} g(x, y)dF(x)dF(y) = \frac{m-1}{m} I_4$$

Hence $E(I_1 - I_2 - I_3 + I_4) = \frac{m-1}{m} I_4 - I_4 = -\frac{1}{m} I_4$, and we are done.

**Lemma 8.** Let $\{X_{i,n}, i=1, \ldots, n, n=1,2,\ldots\}$ be a triangular array of random variables which are i.i.d. within each row. Assume $E(X_{i,n}) = 0 \forall n$, and that $\{X_{i,n}, n=1,2,\ldots\}$ are uniformly integrable. Then we have

$$\frac{1}{n} \sum_{i=1}^{n} X_{i,n} \to 0 \text{ as } n \to \infty.$$

**Proof.** For any random variable $Y$, let $Y^{(s)}$ represent the truncated random variable defined by

$$Y^{(s)} = \begin{cases} Y & \text{if } |Y| \leq s \\ 0 & \text{if } |Y| > s \end{cases}$$
Then by the weak law of large numbers (c.f. Feller, Vol. II [5], theorem 1, p. 316), a sufficient condition for \( \frac{1}{n} \sum_{i=1}^{n} X_{1,n} \to 0 \) is that \( \forall \eta > 0 \) and \( \forall \) truncation level \( s > 0 \),

(i) \( n \mathbb{P}\left( \frac{|X_{1,n}|}{n} > \eta \right) \to 0 \) as \( n \to \infty \)

(ii) \( n \text{Var}\left( \frac{X_{1,n}}{n} \right) \to 0 \)

Now \( n \mathbb{P}(|X_{1,n}| > \eta n) \leq \int_{|X_{1,n}| > \eta n} |X_{1,n}| \, d\mathbb{P} < \varepsilon \) for \( n \) sufficiently large, by uniform integrability. Hence (i) is satisfied. Let \( F_n(x) \) be the c.d.f. of \( X_{1,n} \), and let \( \{a_n\} \) be a sequence with \( a_n \uparrow \infty \), \( a_n = o(n) \).

\[
\begin{aligned}
\text{Var}\left( \frac{X_{1,n}}{n} \right) &\leq \frac{1}{n} \int_{|x| \leq \eta n} x^2 \, dF_n(x) \\
&= \frac{1}{n} \int_{|x| \leq \eta n} x^2 \, dF_n(x) + \frac{1}{n} \int_{a_n \leq |x| < \eta n} x^2 \, dF_n(x) \\
&\leq \frac{a_n}{n} \int_{|x| \leq \eta n} x \, dF_n(x) + \frac{\eta n}{n} \int_{a_n \leq |x| < \eta n} x \, dF_n(x) \\
&\leq \frac{a_n}{n} \mathbb{E}|X_{1,n}| + s \int_{|x| > a_n} x \, dF_n(x).
\end{aligned}
\]
By uniform integrability, $E|X_{1,n}|$ is uniformly bounded in $n$, and

$$\int |x| dF_n(x) \to 0 \text{ as } n \to \infty.$$ Since $\frac{a_n}{n} \to 0$, we have

$$n \text{ Var}(\frac{X_{1,n}(s)}{n}) \to 0,$$ and we are done.

We are now ready to show the asymptotic negligibility of the remainder terms, term by term. First, consider the terms involving $c_i^a - c_i^b$. We will show that $c_i^a - c_i^b$ is $o_p(N^{-1})$, $i=1,\ldots,5$.

$$c_i^a - c_i^b = o_p(N^{-1}).$$

From the line following (2.6), $c_i^a = \int J(H_N^a) dF_m$. In defining $c_i^b$, we could define it either as $\int J(H_N^b) dF_m$ or $\int J(H_N^a) dF_m$. Identity (2.21) will hold in either case, so long as we use the same convention for all the $c_i^b$ terms involving $H_N$ in the range of integration. Our results will be true in either case, but we will arbitrarily choose to work with $H_N^a$ in the range of integration.

Thus

$$(2.57) \quad c_i^a - c_i^b = \int (J(H_N^a) - J(H_N^b)) dF_m.$$ 

Now

$$(2.58) \quad \int J(H_N^a) dF_m = \begin{cases} \frac{1}{m} J(1) & \text{if } X_{max} > Y_{max}^N \to \frac{1}{2} a_n \\ \end{cases}$$

$$0 \text{ otherwise}.$$
and

\[
(2.59) \quad \int J(H_N^b) \, dP = \begin{cases} 
\frac{1}{m} J(1) & \text{if } X_{\text{max}} > Y_{\text{max}} - N_0^{-\frac{1}{2}}b \\
0 & \text{otherwise}
\end{cases}
\]

Hence, since \( a > b \),

\[
(2.60) \quad c_{1-1}^{a-b} = \begin{cases} 
\frac{1}{m} J_N(1) & \text{if } Y_{\text{max}} - N_0^{-\frac{1}{2}}a < X_{\text{max}} < Y_{\text{max}} - N_0^{-\frac{1}{2}}b \\
0 & \text{otherwise}
\end{cases}
\]

\[
\therefore \quad P_N(\left|c_{1-1}^{a-b}\right| > 0) = P_N(Y_{\text{max}} - N_0^{-\frac{1}{2}}a < X_{\text{max}} < Y_{\text{max}} - N_0^{-\frac{1}{2}}b)
\]

\[
(2.61) \quad = \frac{1}{m} n \int \left[ F_m(y-N_0^{-\frac{1}{2}}b) - F_m(y-N_0^{-\frac{1}{2}}a) \right] f^{n-1} \, dF
\]

\[
= m \frac{1}{n} N_0^{-\frac{1}{2}}(a-b) \int F_{m-1}(y-N_0^{-\frac{1}{2}}c) f(y-N_0^{-\frac{1}{2}}c) f^{n-1} \, dF
\]

where \( c \in (b, a) \).

For \( N \) sufficiently large, \( |N_0^{-\frac{1}{2}}c| < u \), in which case, by condition (4),
\[ P_N(\mid c_1^a - c_1^b \mid > 0) \leq K \frac{3}{N_0} \int [F(1-F)]^{\frac{2}{3}} F^{-1} dF \]

\[ \leq K \frac{3}{N_0} 0(\frac{1}{N_0^{\frac{1}{3}}} ) \text{ by lemma 7} \]

\[ = 0(\frac{1}{N_0^{\frac{1}{6}}} ) \]

Thus \( c_1^a - c_1^b \) is \( o_p(N^{-r}) \) for any \( r \), and in particular it is \( o_p(N^{-1}) \).

\( c_2^a - c_2^b = o_p(N^{-1}) \).

This is true by assumption (7) of the theorem.

\( c_4^a - c_4^b = o_p(N^{-1}) \).

From the line following (2.8),

\[ (2.63) \quad c_4^a = - \int_{H_N^a = 1} (J(H^a) + (H_N^a - H^a)J'(H^a)) dF_m. \]

Thus

\[ c_4^a - c_4^b = - \int_{H_N^a = 1} [J(H^a) - J(H^b) + (H_N^a - H^a)J'(H^a) - (H_N^b - H^b)J'(H^b)] dF_m \]

\[ = - \int_{H_N^a = 1} (J(H^a) - J(H^b)) dF_m - \int_{H_N^a = 1} (H_N^a - H_N^b + H^b)J'(H^a) dF_m \]

\[ - \int_{H_N^a = 1} (H_N^b - H^b)[J'(H^a) - H'(H^b)] dF_m \]
\[ \triangleq -J_1 - J_2 - J_3 \]

\[ J_4 = \begin{cases} 
\frac{1}{m} \left[ J(H^a(x_{\text{max}})) - J(H^b(x_{\text{max}})) \right] & \text{if } x_{\text{max}} > y_{\text{max}} - \frac{1}{2} a \\
0 & \text{otherwise}
\end{cases} \]

\[ \therefore E |J_4| \leq \frac{1}{m} \int |J(H^a) - J(H^b)| m F^{-1} dF \]

\[ = \int (H^a - H^b) |J'(H^*)| F^{-1} dF \quad \text{where} \quad H^b < H^* < H^a \]

\[ = (1-\lambda_N) \int (F(x + \frac{1}{2} a) - F(x + \frac{1}{2} b)) |J'(H^*)| F^{-1} dF \]

\[ = N_o^{-\frac{1}{2}} (a-b)(1-\lambda_N) \int f(x + \frac{1}{2} c) |J'(H^*)| F^{-1} dF \quad \text{where} \quad c \in (b, a) \]

\[ \leq \frac{K}{N_o^{1/2}} \int [F(1-F)]^{2/3} [H^*(1-H^*)]^{-1+\delta} F^{-1} dF \quad \text{for} \quad N_o \ \text{sufficiently}
\]

\[ \text{large by condition (4)} \quad \text{and (6)}. \]

Now since \( H^b < H^* < H^a \),

\[ H^*(1-H^*) \geq \min[H^a(1-H^a), H^b(1-H^b)] > \lambda_o^2 F(1-F). \]

Hence

\[ [H^*(1-H^*)]^{-1+\delta} < K[F(1-F)]^{-1+\delta}. \]
Thus

\[ E|A_1| \leq \frac{K}{N_o^{1/2}} \int [F(1-F)]^{-\frac{1}{2} + \delta} \, F^{m-1} \, df = \frac{K}{N_o^{1/2}} \frac{\Gamma(\frac{2}{3} + \delta) \Gamma(m - \frac{1}{3} + \delta)}{\Gamma(m + \frac{1}{3} + \delta)} \]

(2.65)

= \frac{K}{N_o^{1/2}} \frac{0(\frac{1}{m^{2/3} + 6})}{0(\frac{1}{m^{2/3} + 6})} \text{ by Stirling's formula}

= o(\frac{1}{N_o^{7/6} + 6}) = o(N^{-1}).

Hence, by Markov's inequality, \( A_1 = o_p(N^{-1}). \)

\[ A_2 = \int_{H_N^a=1} (H_N^a - H_N^b + H^a) J'(H^a) \, df_m \]

(2.66)

= \int_{H_N^a=1} (1 - H_N^b) J'(H^a) \, df_m - \int_{H_N^a=1} (H_N^a - H_N^b) J'(H^a) \, df_m

\[ A = A_{21} - A_{22} \]

\[ |A_{21}| > 0 \text{ iff } Y_{max}^{\frac{1}{2} - N_o a} < X_{max} < Y_{max}^{\frac{1}{2} - N_o b}. \]

In (2.61) and (2.62) we showed that the probability of this event approaches 0 as \( N \to \infty. \) Hence \( A_{21} = o_p(N^{-\nu}) \forall \nu. \)
$\mathcal{G}_{22} = o_p(N^{-1})$ follows in exactly the same way as $\mathcal{G}_1$ above.

Hence $\mathcal{G}_2 = o_p(N^{-1})$.

$$\mathcal{G}_3 = \int_{H^a_N = 1}^{H^b_N} (H^b_N - H^a_N)[J'(H^a_N) - J'(H^b_N)]dF_m$$

$$= \int_{H^a_N = 1}^{H^b_N} (H^b_N - H^a_N)J''(H^*)dF_m, \text{ where } H^b_N < H^* < H^a_N.$$  

$$= \frac{1}{m} (H^b_N(X_{\text{max}}) - H^b_N(X_{\text{max}}))(H^a_N(X_{\text{max}}) - H^b_N(X_{\text{max}}))J''(H^*(X_{\text{max}}))I\{X_{\text{max}} \geq N_{o}^{-\frac{1}{2}} a < X_{\text{max}}\}$$  

(2.67)

Now by Lemma 1, there exists $\eta > 0$ such that $P_N(X_{\text{max}} \geq \eta I_N(\eta)) > 1 - \epsilon$.

Letting $\mathcal{G}^*_3 = \mathcal{G}_3 \cdot I\{X_{\text{max}} \geq N_{o}^{-\frac{1}{2}} a \}$, we have that $P_N(\mathcal{G}^*_3 = \mathcal{G}_3) > 1 - \epsilon$ for $N$ sufficiently large. Hence it suffices to show that $\mathcal{G}^*_3 = o_p(N^{-1})$.

Now

$$H^b_N(X_{\text{max}}) - H^b_N(X_{\text{max}}) = \lambda_N(1-F(X_{\text{max}})) + (1-\lambda_N)(G_n(X_{\text{max}} + N_{o}^{-\frac{1}{2}} a) - F(X_{\text{max}} + N_{o}^{-\frac{1}{2}} b))$$

and

$$H^a(x) - H^b(x) = (1-\lambda_N)N_{o}^{-\frac{1}{2}} (a-b) F(x + N_{o}^{-\frac{1}{2}} c), \text{ where } c \in (b,a).$$

Therefore

$$E(\mathcal{G}_3^2 | X_{\text{max}}) \leq \frac{K}{N_{o}^{\frac{1}{2}}}
\left\{(1-F(X_{\text{max}}))^{\frac{1}{2}} f^2(X_{\text{max}} + N_{o}^{-\frac{1}{2}} c) \right\}^{\frac{1}{2}} \left\{J''(H^*(X_{\text{max}}))\right\}^2$$

$$+ (1-F(X_{\text{max}}))E\left\{|G_n(X_{\text{max}} + \frac{1}{2} b) - F(X_{\text{max}} + \frac{1}{2} b)| | X_{\text{max}}\right\}$$

$$\cdot f^2(X_{\text{max}} + N_{o}^{-\frac{1}{2}} c) [J''(H^*(X_{\text{max}}))]^2$$
\[ + E\left( (G_n(X_{\text{max}} + \frac{1}{2}b) - F(X_{\text{max}} + N_0 c))^2 \right| X_{\text{max}} \right) \]

\[ \cdot \left( \frac{1}{2} \right) \left( X_{\text{max}} - \frac{1}{2} c \right) [J''(H^*(X_{\text{max}}))]^2 \right) . \]

(2.68)

Now

\[ E\left( |G_n(X_{\text{max}} + N_0 c) - F(X_{\text{max}} + N_0 c)| \right| X_{\text{max}} \right) \leq \left[ \text{Var} G_n(X_{\text{max}} + N_0 c) \right| X_{\text{max}} \right] \]

\[ = \left[ \left( \frac{1}{2} \right) \left( X_{\text{max}} - \frac{1}{2} c \right) [J''(H^*(X_{\text{max}}))]^2 \right)^{\frac{1}{2}} \]

and

\[ E\left( (G_n(X_{\text{max}} + N_0 b) - F(X_{\text{max}} + N_0 b))^2 \right| X_{\text{max}} \right) = \frac{1}{n} \left( \frac{1}{2} \right) \left( X_{\text{max}} - \frac{1}{2} b \right) [J''(H^*(X_{\text{max}}))]^2 \]

Hence

\[ E(\mathcal{G}_{31}^2) = E(E(\mathcal{G}_{31}^2 \right| X_{\text{max}})) \]

\[ \leq \frac{K}{N_0} \left\{ \int_{\mathcal{X}} (1-F)^2 f^2(x+N_0 c) [J''(H^*)]^2 mF \right\} \sum_{m=1}^{m-1} dF \]

\[ + \int_{\mathcal{X}} (1-F) \left[ \frac{F(x+N_0 b) - F(x+N_0 c)}{n} \right] \frac{1}{2} \left( \frac{1}{2} \right) \left( x + \frac{1}{2} c \right) [J''(H^*)]^2 mF \right\} \]

\[ + \int_{\mathcal{X}} (1-F) \left[ \frac{F(x+N_0 b) - F(x-N_0 b)}{n} \right] \frac{1}{2} \left( \frac{1}{2} \right) \left( x - \frac{1}{2} c \right) [J''(H^*)]^2 mF \right\} \]

(2.69) \[ \Delta \frac{K}{N_0} \left\{ \mathcal{G}_{31} + \mathcal{G}_{32} + \mathcal{G}_{33} \right\} . \]

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Now
\[ H^b < H^* < H^a \implies H^*(1-H^*) > \min\{H^a(1-H^b), H^b(1-H^a)\} > \lambda_0^2 F(1-F). \]

Hence by condition (6),
\[ J''(H^*) \leq K[F(1-F)]^{-2+\delta}. \]

Also, for \( N_0 \) sufficiently large, by condition (4),
\[ f^2(x+N_0) \leq K[F(x)(1-F(x))]^{-\frac{4}{3}}, \]
\[ 1-F(x+N_0) = 1-F(x) + \frac{1}{2} f(x+N_0) \leq 1-F(x) + \frac{1}{2} f(x) + \frac{1}{2} f(x+N_0) \]
for \( 0 < \alpha < 1 \)
\[ \leq 1-F(x) + \frac{K}{N_0^{1/2}} [F(x)(1-F(x))]^{2/3} \]
for \( N \) sufficiently large.

Hence
\[ J_{\beta_1} \leq K N_0 \int_{\mathbb{R}} (1-F)^2 [F(1-F)]^{-\frac{4}{3}} \left[ F^{m-1} dF \right] \leq K N_0 \int_{\mathbb{R}} F^{m-4} (1-F)^{-\frac{2}{3}} + \delta dF \]
\[ \leq K N_0 \frac{K}{N_0^{1/3+\delta}} = K N_0^{\frac{2}{3} - \delta} \]
(2.70)

Hence
\[ (2.71) \]
\[ \frac{K}{N_0^3} J_{\beta_1} = o(N^{-2}) \]
\[ G_{32} \leq \frac{1}{N^0} \int_{I_N(\eta)} (1-F)[F(1-F)]^{-\frac{4}{3} + \frac{1}{3} + 8} \left[ 1 - F + \frac{K}{N^{1/2}} (F(1-F))^3 \right]^{\frac{2}{3}} F^{m-1} dF \]

\[ \leq \frac{1}{N^0} \int_{I_N(\eta)} [F(1-F)]^{-\frac{8}{3} + 8} (1-F)^{m-1} \left[ (1-F)^{\frac{1}{2}} + \frac{K}{N^{1/4}} (F(1-F))^3 \right]^{\frac{1}{3}} dF \]

\[ \leq \frac{1}{N^0} \int_{I_N(\eta)} F^{m-\frac{5}{2}} [F(1-F)]^{-\frac{7}{6} + 8} + \frac{1}{N^0} \int_{I_N(\eta)} F^{m-2} [F(1-F)]^{-\frac{4}{3} + 8} dF \]

\[ \leq \frac{1}{N^0} \frac{K}{N^{1/6 + 8}} + \frac{1}{N^0} \frac{K}{N^{-1/3 + 8}} \]

\[ \leq \frac{2}{N^0} - \delta \]

(2.72)

Hence

(2.73)

\[ \frac{K}{N^0} G_{32} = o(N^{-2}) \]

\[ G_{33} \leq K \int_{I_N(\eta)} [F(1-F)]^{-\frac{4}{3} + \frac{1}{3} + 8} F^{m-1} \left[ 1 - F + \frac{K}{N^{1/2}} (F(1-F))^3 \right] dF \]

(2.74)

\[ \leq K \int_{I_N(\eta)} F^{m-2} [F(1-F)]^{-\frac{5}{3} + 8} dF + \frac{K}{N^{1/2}} \int_{I_N(\eta)} F^{m-1} [F(1-F)]^{-2 + 8} dF \]

\[ \leq \frac{K}{N^{-2/3 + 8}} + \frac{K}{N^{1/2 - 1 + 8}} \leq \frac{2}{N^0} - \delta \]

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Hence

\[ (2.75) \quad \frac{K}{N^3} J_{33} = o(N^{-2}) . \]

Combining (2.69), (2.71), (2.73), and (2.75) gives \( E(\mathcal{J}_3^*) = o(N^{-2}) \). By Chebychev's inequality it follows that \( \mathcal{J}_3^* = o_p(N^{-1}) \). Hence, going back to (2.64), we have that \( c_4^{a} - c_4^{b} = o_p(N^{-1}) \).

\[ c_5^{a} - c_5^{b} = o_p(N^{-1}) . \]

From the line following (2.9) \( c_5^{a} = \int (H_{N}^{a} - H_{N}^{b}) J'(H^a) d(F_m - F) \),

\[ c_5^{a} - c_5^{b} = \int [(H_{N}^{a} - H_{N}^{b}) J'(H^a) - (H_{N}^{b} - H_{N}^{b}) J'(H^b)] d(F_m - F) \]

\[ = \int (H_{N}^{a} - H_{N}^{b} - H_{N}^{a} + H_{N}^{b}) J'(H^a) d(F_m - F) + (H_{N}^{b} - H_{N}^{b}) (J'(H^a) - J'(H^b)) d(F_m - F) \]

\[ = (1 - \lambda_N) \int \left( G_n(x + N_a) - G_n(x + N_b) - F(x + N_a) + F(x + N_b) \right) J'(H^a) d(F_m - F) \]

\[ + \lambda_N \int (F_m - F) (J'(H^a) - J'(H^b)) d(F_m - F) \]

\[ + (1 - \lambda_N) \int \left( G_n(x + N_a) - F(x + N_b) \right) J'(H^a) J'(H^b) d(F_m - F) \]

\[ \triangleq (1 - \lambda_N) J_1 + \lambda_N J_2 + (1 - \lambda_N) J_3 . \]

(2.76)

Let us split \( J_1 \) into two integrals, \( \mathcal{J}_1 = \int_{I_N(\eta)} \) and \( \mathcal{J}_1 = \int_{I_N(\eta)} \).
\[ \mathcal{G}_1 = \int_{I_N(\eta)} \left( G_n(x+N_0^{-\frac{1}{2}}a)-G_n(x+N_0^{-\frac{1}{2}}b)-F(x+N_0^{-\frac{1}{2}}a)+F(x+N_0^{-\frac{1}{2}}b) \right) J'(H^a) \, d(F_m-F) \]

\[ E(\mathcal{G}_1) = E \quad \int \int_{x,y \in I_N(\eta)} \left( G_n(x+N_0^{-\frac{1}{2}}a)-G_n(x+N_0^{-\frac{1}{2}}b)-F(x+N_0^{-\frac{1}{2}}a)+F(x+N_0^{-\frac{1}{2}}b) \right) \]

\[ \cdot \left( G_n(y+N_0^{-\frac{1}{2}}a)-G_n(y+N_0^{-\frac{1}{2}}b)-F(y+N_0^{-\frac{1}{2}}a)+F(y+N_0^{-\frac{1}{2}}b) \right) \]

\[ \cdot J'(H^a(x))J'(H^a(y)) \, d(F_m(x)-F(x)) \, d(F_m(y)-F(y)) \]

\[ = E \left[ E \int \int_{x,y \in I_N(\eta)} \text{same thing} \right] \]

\[ = E \int \int_{x,y \in I_N(\eta)} E \left( G_n(x+N_0^{-\frac{1}{2}}a)-G_n(x+N_0^{-\frac{1}{2}}b)-F(x+N_0^{-\frac{1}{2}}a)+F(x+N_0^{-\frac{1}{2}}b) \right) \]

\[ \cdot \left( G_n(y+N_0^{-\frac{1}{2}}a)-G_n(y+N_0^{-\frac{1}{2}}b)-F(y+N_0^{-\frac{1}{2}}a)+F(y+N_0^{-\frac{1}{2}}b) \right) \]

\[ J'(H^a(x))J'(H^a(y)) \, d(F_m(x)-F(x)) \, d(F_m(y)-F(y)) \].

Let

\[ K_n(x) = G_n(x+N_0^{-\frac{1}{2}}a)-G_n(x+N_0^{-\frac{1}{2}}b)-F(x+N_0^{-\frac{1}{2}}a)+F(x+N_0^{-\frac{1}{2}}b) \]
\[ E(A_1^2) = 2E \int_{x<y} E[K_n(x)K_n(y)] J'(N^a(x)) J'(N^a(y)) d(F_m(x) - F(x)) d(F_m(y) - F(y)) \]

\[ x, y \in I_N(\eta) \]

\[ + \frac{1}{m} E \int_{I_N(\eta)} E(K_n^2(x)) [J'(N^a(x))]^2 \, dF_m(x) \]

(2.77)

Consider \( E[K_n(x)K_n(y)] \) for \( x < y \).

Case (i): \( x + N_o^{-1/2}a < y + N_o^{-1/2}b \)

Then

\[ \begin{pmatrix}
\frac{1}{2} \, G_n(x+N_o^{-1/2}a) - G_n(x+N_o^{-1/2}b) \\
\frac{1}{2} \, G_n(y+N_o^{-1/2}a) - G_n(y+N_o^{-1/2}b)
\end{pmatrix} \]

are jointly distributed as a trinomial vector \((U, V)\) with parameters \( n \) and

\[ p_1 = F(x+N_o^{-1/2}a) - F(x+N_o^{-1/2}b), \quad p_2 = F(y+N_o^{-1/2}a) - F(y+N_o^{-1/2}b) . \]

Then

\[ E[K_n(x)K_n(y)] = \frac{\text{Cov}(U, V)}{n^2} = - \frac{p_1 p_2}{n} = \]

\[ \frac{1}{n} (F(x+N_o^{-1/2}a) - F(x+N_o^{-1/2}b))(F(y+N_o^{-1/2}a) - F(y+N_o^{-1/2}b)) \]

(2.78)

Case (ii): \( x + N_o^{-1/2}a > y + N_o^{-1/2}b \)
Then
\[ n \left( \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2}
\end{array} \right) \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2}
\end{array} \\
\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2}
\end{array} \left( \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2}
\end{array} \right) \text{ is distributed as } \begin{array}{c}
U+V \\
V+W
\end{array} \]

where \((U,V,W)^T\) is a quadrinomial vector with parameters \(n\) and

\[ p_1 = F(y+N_0 b) - F(x+N_0 b), \quad p_2 = F(x+N_0 a) - F(y+N_0 b), \]

\[ p_3 = F(y+N_0 a) - F(x+N_0 a). \]

Hence

\[ E[K_n(x)K_n(y)] = \frac{1}{n^2} \{ \text{Cov}(U,V+W)+\text{Var}(V)+\text{Cov}(V,W) \} \]

\[ = \frac{1}{n} \left( -p_1(p_2+p_3) + p_2(1-p_2) - p_2p_3 \right) = \frac{1}{n} \left( -(p_1+p_2)(p_2+p_3)+p_2 \right) \]

\[ = \frac{1}{n} \left( -(F(x+N_0 a)-F(x+N_0 b))(F(y+N_0 a)-F(y+N_0 b)) \right. \]

\[ + \left. (F(x+N_0 a)-F(y+N_0 b)) \right) \]

(2.79)

Combining (2.78) and (2.79) gives
\[
E[K_n(x)K_n(y)] = - \frac{\frac{1}{2} F(x+N_{\frac{1}{2}}a) - F(x+N_{-\frac{1}{2}}b) \cdot F(y+N_{\frac{1}{2}}a) - F(y+N_{-\frac{1}{2}}b)}{n} + \frac{F(x+N_{-\frac{1}{2}}a) - F(y+N_{-\frac{1}{2}}b)}{n} \cdot I\{y < x + N_{\frac{1}{2}}(a-b)\}.
\]

(2.80)

Also,

\[
E(K_n^2(x)) = \frac{\frac{1}{2} F(x+N_{\frac{1}{2}}a) - F(x+N_{-\frac{1}{2}}b) \cdot (1 - F(x+N_{\frac{1}{2}}a) + F(x+N_{-\frac{1}{2}}b))}{n}
\]

(2.81)

Substituting (2.80) and (2.81) into (2.77),

\[
E(\mathcal{J}^2_1) = - \frac{2}{n} E \iint_{x < y \in I_N(\eta)} (F(x+N_{\frac{1}{2}}a) - F(x+N_{-\frac{1}{2}}b)) J'(H^a(x)) \, d(F_m(x) - F(x)) \, d(F_m(y) - F(y))
\]

\[
+ \frac{2}{n} E \iint_{x < y < x + N_{-\frac{1}{2}}(a-b) \in I_N(\eta)} (F(x+N_{\frac{1}{2}}a) - F(y+N_{-\frac{1}{2}}b)) J'(H^a(x)) J'(H^a(y)) \, d(F_m(x) - F(x)) \, d(F_m(y) - F(y))
\]
\[ + \frac{1}{m} \mathbb{E} \int_{I_n(\eta)} \left( F(x+N_0^{\frac{1}{2}}a) - F(x+N_0^{\frac{1}{2}}b) \right) \left( 1 - F(x+N_0^{\frac{1}{2}}a) + F(x+N_0^{\frac{1}{2}}b) \right) [J'(H^a)]^2 \, dF_m \]

\[ = - \frac{2}{n} \mathbb{E}(\mathcal{G}_{11}) + \frac{2}{n} \mathbb{E}(\mathcal{G}_{12}) + \frac{1}{mn} \mathbb{E}(\mathcal{G}_{13}). \]

(2.82)

By Lemma 7

\[ \mathbb{E}(\mathcal{G}_{11}) = - \frac{1}{m} \iint_{x<y} (F(x+N_0^{\frac{1}{2}}a) - F(x+N_0^{\frac{1}{2}}b)) J'(H^a(x)) \]

\[ x, y \in I_N(\eta) \]

\[ (F(y+N_0^{\frac{1}{2}}a) - F(y+N_0^{\frac{1}{2}}b)) J'(H^a(y)) dF(x) dF(y) \]

\[ = - \frac{1}{2m} \int_{I_N(\eta)} (F(x+N_0^{\frac{1}{2}}a) - F(x+N_0^{\frac{1}{2}}b)) J'(H^a) dF)^2 \]

(2.83)

Similarly,

\[ \mathbb{E}(\mathcal{G}_{12}) = - \frac{1}{m} \iint_{x<y<x+N_0^{\frac{1}{2}}(a-b)} (F(x+N_0^{\frac{1}{2}}a) - F(y+N_0^{\frac{1}{2}}b)) \]

\[ x, y \in I_N(\eta) \]

\[ J'(H^a(x)) J'(H^a(y)) dF(x) dF(y) \]

(2.84)
\[ E(\mathcal{G}_{13}) = \int_{I_N(\eta)} \frac{1}{2} \left[ F(x + N_o^a) - F(x + N_o^b) \right] \left[ 1 - F(x + N_o^a) + F(x + N_o^b) \right] [J'(H^a)]^2 dF \]

(2.85)

From (2.83)

\[ \left| -\frac{2}{n} E(\mathcal{G}_{11}) \right| \leq \frac{K}{N_o^2} \left[ \int_{I_N(\eta)} \frac{1}{2} (a-b) f(x + N_o^c) K[H^a(1-H^a)]^{-1+\delta} dF \right]^2 \]

where \( c(c(b,a)) \)

\[ \leq \frac{K}{N_o^2} \left[ \int [F(1-F)]^{\frac{2}{3}} (1+\delta) dF \right]^2 \text{ for } N_o \text{ sufficiently large} \]

(2.86)

\[ \left( \frac{2}{n} E(\mathcal{G}_{12}) \right) \leq \frac{K}{N_o^2} = o(N_o^{-2}). \]

From (2.84)

\[ \left| \frac{2}{n} E(\mathcal{G}_{12}) \right| \leq \frac{K}{N_o^2} \iint_{x < y < x + N_o^a (a-b),} F(x + N_o^a) - F(y + N_o^b) \left| J'(H^a(x)) \right| dF(x) dF(y) \]

Now on \( x < y < x + N_o^a (a-b), \)

\[ F(x + N_o^a) - F(y + N_o^b) < \frac{1}{2} \]

and
\[ F(x+N_o^{1/2}a) - F(y+N_o^{1/2}b) < F(x+N_o^{1/2}a) - F(x+N_o^{1/2}b) \]

Hence

\[ F(x+N_o^{1/2}a) - F(y+N_o^{1/2}b) < \left[ (F(x+N_o^{1/2}a) - F(x+N_o^{1/2}b))(F(y+N_o^{1/2}a) - F(y+N_o^{1/2}b)) \right]^{1/2} \]

Thus we have

\[ \frac{2}{n} E(\mathcal{I}_{L_2}) \leq \frac{K}{N_o^{1/2}} \int \int_{x < y, x, y \in \Gamma_N(\eta)} \frac{1}{2} (F(x+N_o^{1/2}a) - F(x+N_o^{1/2}b))^2 J'(H^a(x)) \]

\[ \leq \frac{K}{N_o^{1/2}} \int_{\Gamma_N(\eta)} (F(x+N_o^{1/2}a) - F(x+N_o^{1/2}b))^2 J'(H^a) dF(x) dF(y) \]

\[ \leq \frac{K}{N_o^{1/2}} \left[ \int_{\Gamma_N(\eta)} |f(x+N_o^{1/2}a)| \frac{1}{2} |H^a(1-H^a)|^{-1+\delta} dF \right]^2 \]

\[ \leq \frac{K}{N_o^{5/2}} \left[ \int_{F(1-F)} \frac{1}{3} - 1+\delta dF \right]^2 \text{ for } N_o \text{ sufficiently large} \]

(2.87) \[ \leq \frac{K}{N_o^{5/2}} = o(N_o^{-1/2}) \]

From (2.85)

\[ \frac{1}{n} E(\mathcal{I}_{L_2}) \leq \frac{K}{N_o^{1/2}} \int_{\Gamma_N(\eta)} \frac{1}{2} (F(x+N_o^{1/2}a) - F(x+N_o^{1/2}b))[H^a(1-H^a)]^{-1+\delta} dF \]
\[
\begin{align*}
\mathcal{J} \leq \frac{K}{N_o^{5/2}} \int_{I_N(\eta)} \left[ F\left(1 - F\right) \right]^{\frac{2}{3} - 2 + \delta} \, dF \\
\mathcal{J} \leq \frac{K}{N_o^{5/2}} \left( \frac{K}{N_o} \right)^{\frac{1}{3} + \delta} = \frac{K}{N_o^{13/6 + \delta}} = o(N_o^{-2}) .
\end{align*}
\]

(2.88)

Substituting (2.86), (2.87), (2.88) into (2.82), we get

\[E(\mathcal{J}^2_1) = o(N_o^{-2})\]

and hence by Chebyshev's inequality,

(2.89) \hspace{2cm} \mathcal{J}_1 = o_p\left( N_o^{-1} \right) .

Let us now consider \( \mathcal{\bar{J}}_1 \).

\[\mathcal{\bar{J}}_1 = \int_{I_N(\eta)} \left( c_n\left(x + \frac{1}{2}a\right) - c_n\left(x + \frac{1}{2}b\right) - F\left(x + \frac{1}{2}a\right) + F\left(x + \frac{1}{2}b\right) \right) dF \, dF .\]

From Lemmas 1 and 2, it follows that we can choose an \( \eta > 0 \) such that for \( N \) sufficiently large \( P(X_i \in I_N(\eta)), i = 1, \ldots, m, Y_i - N_o^{-1/2}a \in I_N(\eta), Y_j - N_o^{-1/2}b \in I_N(\eta), j = 1, \ldots, n \) \( > 1 - \epsilon \). Hence on this set,
\[ |\overline{g}_1| = \int_{I_N^1(\eta)^N} \frac{1}{2} \left( F(x + N_0^{-1-a}) - F(x + N_0^{-1-b}) \right) J'(\mathcal{H}^a) d\mathcal{F} \]

\[ \leq \frac{K}{N_0^{1/2}} \int_{I_N^1(\eta)} F(1-F)^{2/3 - 1+\delta} dF \text{ for } N_0 \text{ sufficiently large} \]

\[ \leq \frac{K}{N_0^{1/2}} \left\{ \int_0^{2n/N} [u(1-u)]^{-1/3 + \delta} du + \int_{1-2n/N}^{1} [u(1-u)]^{-1/3 + \delta} du \right\} \]

\[ (2.90) \quad \leq \frac{K}{N_0^{1/2}} \left( \frac{K}{N_0} \right)^{2/3 + \delta} = \frac{K}{N_0^{7/6 + \delta}} = o(N_0^{-1}) . \]

Thus

\[ (2.91) \quad \overline{g}_1 = o_p(N_0^{-1}) . \]

Consider

\[ \mathcal{J}_2 = \int (F_m - F)(J'(\mathcal{H}^a) - J'(\mathcal{H}^b)) d(F_m - F) . \]

In an analogous way to the proof of identity (7.4) in Chernoff and Savage [3], pp. 986-7, it can be shown that

\[ \mathcal{J}_2 = \frac{1}{2} \int (J'(\mathcal{H}^a) - J'(\mathcal{H}^b)) d(F_m - F)^2 + \frac{1}{m} \int (J'(\mathcal{H}^a) - J'(\mathcal{H}^b)) dF_m \]

\[ \triangle \quad \frac{1}{2} [\mathcal{J}_{21} + \mathcal{J}_{22}] \]

\[ (2.92) \]
\[ J_{21} = \int (J'(H^a) - J'(H^b)) d(F_m - F)^2 \]

\[ = \int (H^a - H^b) J''(H^*) d(F_m - F)^2 \quad \text{where} \quad H^b < H^* < H^a \]

\[ = -\left( F_m - F \right)^2 (H^a - H^b) J'''(H^*) dH^* - \int (F_m - F)^2 J''(H^*)(H^a - H^b) dx \]

where \( h^a, h^b \) are the densities corresponding to \( H^a, H^b \) respectively.

\[ \triangle = -J_{211} - J_{212} \]

(2.93)

Partition \( J_{211} \) into integrals over \( I_N^*(\eta) \) and \( \overline{I}_N^*(\eta) \), where \( I_N^*(\eta) = \{ x | H^*(1-H^*) > \frac{\eta}{N} \} \), and denote these integrals by \( J_{211} \) and \( \overline{J}_{211} \) respectively.

\[ J_{211} = \int_{I_N^*(\eta)} (F_m - F)^2 (H^a - H^b) J'''(H^*) dH^* \]

\[ E|J_{211}| \leq E \int_{I_N^*(\eta)} (F_m - F)^2 (H^a - H^b) |J'''(H^*)| dH^* \]

\[ = \int_{I_N^*(\eta)} \frac{F(1-F)}{m} (H^a - H^b) |J'''(H^*)| dH^* \]

\[ \leq \frac{K}{N^{3/2}} \int_{I_N^*(\eta)} \left[ F(1-F) \right]^{1+\frac{2}{3}} \left[ H^*(1-H^*) \right]^{-3+\delta} dH^* \quad \text{for} \quad N_0 \text{ sufficiently large} \]
\[
\frac{K}{N^{3/2}} \int_{I_N^*(\eta)}^{\frac{4}{3} + \delta} [H^*(1-H^*)]^{-\frac{1}{2}} + \delta \ dH^*
\]

(2.94) \[
\frac{K}{N^{3/2}} \left( \frac{K}{N^{\delta}} \right) = \frac{K}{N^{7/6 + \delta}} = o(N^{-1}) .
\]

Thus

(2.95) \[
\mathcal{J}_{211} = o_p(N^{-1}) .
\]

Consider

\[
\mathcal{J}_{211} = \int_{I_N^*(\eta)} (F_m^2-F^2) (H^a-H^b)J''(H^*) dH^* .
\]

By the Corollary to Lemma I, \( \eta \) can be chosen so that for \( N \) sufficiently large,

\[
P\left\{ \text{no } X_i \in I_N^*(\eta) \right\} > 1-\epsilon .
\]

Let

\[
(a_N, b_N) = I_N^*(\eta) .
\]

Now on the set \( A_N = \{ \text{no } X_i \in I_N^*(\eta) \} \),

\[
(F_m^2-F^2) = \begin{cases} 
F^2 & \text{for } x < a_N \\
(1-F)^2 & \text{for } x > b_N 
\end{cases}
\]

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Since \( F(a_n) \leq \frac{1}{\lambda_0} H^*(a_n) \leq \frac{2n}{\lambda_0 n} \) and \( 1 - F(b_n) \leq \frac{1}{\lambda_0} (1 - H^*(b_n)) \leq \frac{2n}{\lambda_0 n} \), \( K > 0 \Rightarrow \)

\[ F^2 < K[F(1-F)]^2 \quad \text{for } x < a_n \quad \text{and} \quad (1-F)^2 < K[F(1-F)]^2 \quad \text{for } x > b_n. \]

Hence on \( A_n \),

\[ (F_m - F)^2 \leq K[F(1-F)]^2 \]

and thus

\[ |\bar{J}_{211}| \leq K \int_{\bar{I}_N^*(\eta)} [F(1-F)]^2 (H^a - H^b) J''(H^*) dH^* \]

\[ \leq \frac{K}{N_0^{1/2}} \int_{\bar{I}_N^*(\eta)} [F(1-F)]^2 \frac{2}{3} (H^*(1-H^*))^{-3+\delta} dH^* \]

\[ \leq \frac{K}{N_0^{1/2}} \int_{\bar{I}_N^*(\eta)} [H^*(1-H^*)]^{-\frac{1}{3}+\delta} dH^* \]

\[ \leq \frac{K}{N_0^{1/2}} \left[ 2 \int_0^{2n/N} \right] [u(1-u)]^{-\frac{1}{3}+\delta} du ] \]

\[ (2.96) \quad \leq \frac{K}{N_0^{1/2}} \cdot \frac{K}{N_0^{2/3+\delta}} = \frac{K}{N_0^{7/6+\delta}} = o(N_0^{-1}). \]

Thus

\[ (2.97) \quad \bar{J}_{211} = o_p(N^{-1}). \]

Consider
\[ J_{212} = \int (F_m - F)^2 J''(H^*)(h^a - h^b) \, dx. \]

Partition \[ J_{212} \] into integrals over \[ I_N(\eta) \] and \[ \overline{I}_N(\eta) \], denoted by \[ J_{212}^I \] and \[ J_{212}^O \].

\[ J_{212}^I = \int \frac{(F_m - F)^2 J''(H^*)(h^a - h^b)}{I_N(\eta)} \, dx \]

\[ = \int \frac{F(1-F)}{m} \cdot \frac{1}{|J''(H^*)|} \cdot \frac{1}{|h^a - h^b|} \, dx \]

\[ \leq \frac{K}{N^2} \int \frac{F(1-F)(H^*(1-H^*))^{-2+8} |f(x+N^{1/2}a) - f(x+N^{1/2}b)|}{I_N(\eta)} \, dx \]

Now \( H^b < H^* < H^a \Rightarrow H^*(1-H^*) \geq \min(H^a(1-H^a),H^b(1-H^b)) \geq \chi_0^2 F(1-F) \)

\[ \leq \frac{K}{N^3/2} \int \frac{F(1-F)}{I_N(\eta)} \left[ F(1-F) \right]^{1-2+\frac{2+8}{3}} \, dx \text{ for } N \text{ sufficiently large} \]

\[ = \frac{K}{N^{3/2}} \int \frac{F(1-F)}{I_N(\eta)} \left[ F(1-F) \right]^{-\frac{1}{3}+8} \, dx. \]

Since \( F(1-F) > \frac{1}{N} \) on \( I_N(\eta) \) and we can assume that \( -\frac{1}{3}+8 < 0 \), we have, letting \( I_N(\eta) = (a_N, b_N) \)

\[ (2.98) \quad E|J_{212}| \leq \frac{K}{N^{3/2-1/3+8}} \cdot \frac{1}{N^{7/6+8}} \cdot (b_N - a_N). \]
Now
\[ a_N > F^{-1}\left(\frac{n}{N}\right) \quad \text{and} \quad b_N < F^{-1}\left(1 - \frac{n}{N}\right) . \]

Hence
\[ b_N - a_N \leq F^{-1}\left(1 - \frac{n}{N}\right) - F^{-1}\left(\frac{n}{N}\right) . \]

If \( F \) has bounded support, then \( b_N - a_N \) is bounded and \( E(D_{212}) = o(\frac{1}{N}) \).

If \( F \) has unbounded support, then \( F^{-1}\left(\frac{n}{N}\right) \to \infty \), \( F^{-1}\left(1 - \frac{n}{N}\right) \to \infty \) as \( N \to \infty \).

Let \( d_N = F^{-1}\left(\frac{n}{N}\right) \). \( F(d_N) = \frac{n}{N} \).

By condition (5),
\[ d_N F(d_N) \to 0 \quad \text{as} \quad N \to \infty \]

\[ \implies \frac{n}{N} \left[F^{-1}\left(\frac{n}{N}\right)\right]^6 \to 0 \quad \text{as} \quad N \to \infty \]

\[ \implies \frac{K}{\sqrt[1/6]{N}} F^{-1}\left(\frac{n}{N}\right) = o(1) . \]

Hence

\[ 2.99 \quad \frac{K}{\sqrt[7/6+5]{N}} (-a_N) = o(N^{-1}) . \]

By symmetry of \( F \), \( \frac{K}{\sqrt[1/6]{N}} F^{-1}\left(1 - \frac{n}{N}\right) = o(1) \) also. Hence

\[ 2.100 \quad \frac{K}{\sqrt[7/6+5]{N}} b_N = o(N^{-1}) . \]
From (2.98) it follows that $E|\mathcal{J}_{212}| = o(N^{-1})$ and thus

\[(2.101) \quad \mathcal{J}_{212} = o_p(N^{-1}).\]

Now consider

\[
\bar{\mathcal{J}}_{212} = \int_\overline{I}_N(\eta) (F^{-1}_n - F)^2 J''(h^*) (h^a - h^b) dx.
\]

As before, $\eta$ can be chosen so that for $N$ sufficiently large, the event $A_N$ that there are no $X_1$ in $\overline{I}_N(\eta)$ has probability greater than $1-\epsilon$, and on $A_N$ we have

\[
|\bar{\mathcal{J}}_{212}| \leq K \int_\overline{I}_N(\eta) [F(1-F)]^{2\delta} [H^*(1-H^*)]^{-2+\delta} |h^a - h^b| dx
\]

\[
\leq K \int_\overline{I}_N(\eta) [F(1-F)]^{-2+\delta} (h^a + h^b) dx \quad \text{since} \quad H^*(1-H^*) \geq \lambda_0^2 p(1-F)
\]

\[(2.102) \quad \leq K \left[ \int_\overline{I}_N(\eta) [F(1-F)]^\delta dH^a + \int_\overline{I}_N(\eta) [F(1-F)]^\delta dH^b \right]
\]

Since $F(1-F) < \frac{h}{N}$ on $\overline{I}_N(\eta)$, $[F(1-F)]^\delta < \frac{K}{N^2}$.

\[\therefore \quad |\bar{\mathcal{J}}_{212}| \leq K \frac{N}{N^2} \left[ \int_\overline{I}_N(\eta) dH^a + \int_\overline{I}_N(\eta) dH^b \right]
\]

\[
\leq K \frac{N}{N^2} \left[ H^a(F^{-1}(\frac{2\eta}{N})) + 1 - H^a(F^{-1}(1 - \frac{2\eta}{N})) \right]
\]

\[(2.103) \quad \leq H^b(F^{-1}(\frac{2\eta}{N})) + 1 - H^b(F^{-1}(1 - \frac{2\eta}{N})) \right].
\]

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Now
\[ H^a(x) = F(x) + \left(1 - \lambda_n \right) N_0^{\frac{1}{2}} a f(x + \alpha N_0^{\frac{1}{2}} a) \]
\[ \leq F(x) + N_0^{\frac{1}{2}} [F(x)(1-F(x))]^{\frac{2}{3}} \quad \text{for some } K > 0 \text{ and } N \text{ sufficiently large.} \] (2.104)

Also,
\[ 1 - H^a(x) \leq 1 - F(x) + N_0^{\frac{1}{2}} [F(x)(1-F(x))]^{\frac{2}{3}} \] (2.105)

Hence
\[ H^a(F^{-1}(\frac{2n}{N})) \leq \frac{2n}{N} + N_0^{\frac{1}{2}} \left( \frac{2n}{N} \right)^{\frac{2}{3}} \leq \frac{K}{N_0} \quad \text{for some } K > 0 \]

and
\[ 1 - H^a(F^{-1}(1 - \frac{2n}{N})) \leq \frac{2n}{N} + N_0^{\frac{1}{2}} \left( \frac{2n}{N} \right)^{\frac{2}{3}} + 8 \leq \frac{K}{N_0} \quad \text{for some } K > 0. \]

Similar statements hold for \( H^b(F^{-1}(\frac{2n}{N})) \) and \( 1 - H^b(F^{-1}(1 - \frac{2n}{N})) \).

Thus from (2.103) we have
\[ |\bar{g}_{212}| \leq \frac{K}{N_0^{1+8}} = o(N_0^{-1}) \quad \text{on the set } A_N. \]

Hence
\[ \bar{g}_{212} = o_p(N^{-1}). \] (2.106)
Combining (2.93), (2.95), (2.97), (2.101), and (2.106) we have

\[(2.107) \quad \mathcal{J}_{21} = o_p(N^{-1}).\]

Next from (2.92) consider

\[\mathcal{J}_{22} = \frac{1}{m} \int (J'(H^a) - J'(H^b)) dF_m.\]

By Lemma 1, \(\exists \eta > 0 \quad \exists \quad \text{for } N \text{ sufficiently large}\)

\[\mathbb{E}\left[ \frac{1}{m} \int_{I_N(\eta)} (J'(H^a) - J'(H^b)) dF_m \right] \leq \frac{1}{m} E \int_{I_N(\eta)} (H^a - H^b) \left| J''(H^*) \right| dF_m = \frac{1}{m} \int_{I_N(\eta)} (H^a - H^b) \left| J''(H^*) \right| dF \quad \text{where } H^b < H^* < H^a\]

\[\leq \frac{K}{N_0^{3/2}} \int_{I_N(\eta)} [(F(I-F)]^{2/3} [F^*(1-F)]^{-2+\delta} dF \quad \text{for } N \text{ sufficiently large}\]

\[\leq \frac{K}{N_0^{3/2}} \int_{I_N(\eta)} (F(I-F)]^{\frac{4}{3} + \delta} dF\]

\[\leq \frac{K}{N_0^{1/3 + \delta}} = \frac{K}{N_0^{7/6 + \delta}} = o(N_0^{-1}).\]

\[(2.108)\]
Hence

\begin{equation}
J_{22} = o_p(N^{-1})
\end{equation}

(2.107) and (2.109) substituted in (2.92) gives

\begin{equation}
\frac{4}{\sigma_2} = o_p(N^{-1}).
\end{equation}

Consider now from (2.76)

\[ J_3 = \int (G_n(x+N_0^{-1}b)-F(x+N_0^{-1}b))(J'(H^a)-J'(H^b))d(F_m-F). \]

Partition this into \( J_3 \) and \( \bar{J}_3 \), representing integrals over \( I_N(\eta) \) and \( \bar{I}_N(\eta) \) respectively.

\[ E(J_3^2 | X) = E \iint_{x,y \in I_N(\eta)} (G_n(x+N_0^{-1}b)-F(x+N_0^{-1}b))(G_n(y+N_0^{-1}b)-F(y+N_0^{-1}b)) \]
\[ \cdot (J'(H^a(x))-J'(H^b(x)))(J'(H^a(y))-J'(H^b(y)))d(F_m(x)-F(x)) 
\]
\[ \cdot d(F_m(y)-F(y)) \]
\[ = 2 \iint_{x \leq y \in I_N(\eta)} E[(G_n(x+N_0^{-1}b)-F(x+N_0^{-1}b))(G_n(y+N_0^{-1}b)-F(y-N_0^{-1}b))] \]
\[ \cdot (J'(H^a(x))-J'(H^b(x)))(J'(H^a(y))-J'(H^b(y)))d(F_m(x)-F(x)) 
\]
\[ + \frac{1}{m} \int_{I_N(\eta)} E(G_n(x+N_0^{-1}b)-F(x+N_0^{-1}b))^2(J'(H^a)-J'(H^b))^2 dF_m. \]
\[
\begin{align*}
&= 2n \iint_{x < y, x, y \in I_N(\eta)} \frac{1}{2} F(x + N_0 b)(1 - F(x + N_0 b))(J'(H^a(x)) - J'(H^b(x))) \\
&\quad + \frac{1}{mn} \int_{I_N(\eta)} F(x + N_0 b)(1 - F(x + N_0 b))(J'(H^a) - J'(H^b))^2 dF_m \\
&= \frac{2n}{n} J_{31} + \frac{1}{mn} J_{32} \\
\end{align*}
\]

(2.111)

By Lemma 7,

\[
E(J_{31}) = -\frac{1}{m} \iint_{x < y, x, y \in I_N(\eta)} \frac{1}{2} F(x + N_0 b)(1 - F(x + N_0 b))(J'(H^a(x)) - J'(H^b(x))) \\
- \frac{1}{2} (J'(H^a(y)) - J'(H^b(y))) dF(x) dF(y).
\]

Now

\[
|J'(H^a) - J'(H^b)| = (H^a - H^b)|J''(H^*)| \quad \text{where} \quad H^b < H^* < H^a
\]

\[
\leq \frac{K}{N_0} \left[ F(1-F) \right]^\frac{2}{3} [H^*(1-H^*)]^{-2+\delta}
\]

for \( N_0 \) sufficiently large

\[
\leq \frac{K}{N_0} \left[ F(1-F) \right]^\frac{4}{3} + \delta.
\]

Also

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\[
F(x+N_0^{-\frac{1}{2}})F(y+N_0^{-\frac{1}{2}}) = F(x)(1-F(y)) + N_0^{-\frac{1}{2}}F(x+N_0^{-\frac{1}{2}})(1-F(y))
\]

\[
- N_0^{-\frac{1}{2}}F(x+N_0^{-\frac{1}{2}})F(x) - N_0^{-\frac{1}{2}}F(x+N_0^{-\frac{1}{2}})F(y)
\]

\[
\leq F(x)(1-F(y)) + \frac{K}{N_0^{1/2}} \left[ F(x)(1-F(y)) \right]^\frac{2}{3}(1-F(y)) + \frac{K}{N_0^{1/2}} \left[ F(y)(1-F(y)) \right]^\frac{2}{3}F(x)
\]

\[
\quad + \frac{K}{N_0} \left[ F(x)(1-F(x))F(y)(1-F(y)) \right]^\frac{2}{3}.
\]

Hence

\[
E(\mathcal{S}_{\delta_1}) \leq \frac{K}{N_0^{1/2}} \int_{x,y \in \mathbb{I}_N(\eta)} \left\{ F(x)(1-F(y)) + \frac{K}{N_0^{1/2}} \left[ F(x)(1-F(x)) \right]^\frac{2}{3}(1-F(y))
\]

\[
\quad + \frac{K}{N_0^{1/2}} \left[ F(y)(1-F(y)) \right]^\frac{2}{3}F(x) + \frac{K}{N_0} \left[ F(x)(1-F(x))F(y)(1-F(y)) \right]^\frac{2}{3}
\]

\[
\quad \left[ F(x)(1-F(x))F(y)(1-F(y)) \right]^\frac{1}{3} \delta F(x)\delta F(y)
\]

\[
\leq \frac{K}{N_0^{1/2}} \int_{0 < u < v < 1} \frac{1}{3} + \delta \left( \frac{1}{3} + \delta \right) \frac{4}{3} + \delta \left( 1-v \right) \frac{1}{3} + \delta \frac{1}{3} + \delta \left( 1-v \right) \frac{1}{3} + \delta 
\]

\[
+ \frac{K}{N_0^{5/2}} \int_{0 < u < v < 1} u \left( \frac{2}{3} + \delta \right) \frac{4}{3} + \delta \left( 1-v \right) \frac{1}{3} + \delta \frac{1}{3} + \delta 
\]

\[
+ \frac{K}{N_0^{5/2}} \int_{0 < u < v < 1} \left[ u \left( \frac{2}{3} + \delta \right) \frac{4}{3} + \delta \left( 1-v \right) \frac{1}{3} + \delta \right] \frac{1}{3} + \delta 
\]

\[
+ \frac{K}{N_0^{3}} \int_{0 < u < v < 1} \left[ u \left( \frac{2}{3} + \delta \right) \right] \frac{2}{3} + \delta 
\]

(2.112)
By Lemma 6, each of the four integrals is finite. Hence

\[ (2.113) \quad E(\mathcal{J}_{31}) \leq \frac{K}{N_0^2}. \]

Consider now

\[ \mathcal{J}_{32} = \int \frac{1}{I_N(\eta)} \left[ F(x + N_0^{-1}b)(1-F(x + N_0^{-1}b)) \right] \left( J'(H^a) - J'(H^b) \right)^2 dF_m. \]

\[ E(\mathcal{J}_{32}) = \int \frac{1}{I_N(\eta)} \left[ F(x + N_0^{-1}b)(1-F(x + N_0^{-1}b)) \right] \left( J'(H^a) - J'(H^b) \right)^2 dF \]

\[ \leq \int \frac{1}{I_N(\eta)} \left[ (F(1-F) + (1-2F)N_0^{-1}b - 2F(x + \theta N_0^{-1}b)(H^a - H^b)^2 \{ J''(H^x) \}^2 \right] dF \]

where \( 0 < \theta < 1, \ H^b < H^x < H^a. \)

\[ \leq \frac{K}{N_0} \int \frac{1}{I_N(\eta)} \left[ F(1-F) + \frac{K}{N_0^{1/2}} \left[ F(1-F) \right]^{2/3} + \frac{K}{N_0} \left[ F(1-F) \right]^{4/3} \right] \left[ F(1-F) \right]^{4/3} - 4 + \delta \] \[ \leq \frac{K}{N_0} \int \frac{1}{I_N(\eta)} \left[ F(1-F) \right]^{2/3} \left[ F(1-F) \right]^{2/3} \left[ F(1-F) \right]^{2/3} \left[ F(1-F) \right]^{2/3} \] \[ \left( \text{since} \ \frac{1}{N_0} \left[ F(1-F) \right]^{2/3} \text{ is dominated by} \ \frac{1}{N_0^{1/2}} \left[ F(1-F) \right]^{2/3} \right) \]

\[ \leq \frac{K}{N_0} \int \frac{1}{I_N(\eta)} \left[ F(1-F) \right]^{-\frac{5}{3} + \delta} dF + \frac{K}{N_0^{3/2}} \int \frac{1}{I_N(\eta)} \left[ F(1-F) \right]^{2/3} - 2 + \delta dF \]

\[ \leq \frac{K}{N_0^{1/2} + 2/3 + \delta} \quad \leq \frac{K}{N_0^{1/3 + \delta}} \]

\[ (2.114) \]
From (2.111), (2.113), and (2.114) we have

\[ E(\mathcal{J}_{2}^{\circ}) = E(E(\mathcal{J}_{2}^{\circ}|X) = \frac{2}{n} E(\mathcal{J}_{21}) + \frac{1}{mn} E(\mathcal{J}_{22}) \]

\[ \leq \frac{K}{N_{o}^{3}} + \frac{K}{N_{o}^{7/3+\delta}} \]

(2.115)

\[ = o(N_{o}^{-2}). \]

Hence

(2.116)

\[ \mathcal{J}_{2} = o_p(N_{o}^{-1}). \]

Consider now

\[ \mathcal{J}_{3}^{\tau} = \int \frac{(G_{n}(x+N_{o}^{-2}b)-F(x+N_{o}^{-2}b))(J'(H^{a})-J'(H^{b}))d\mathcal{F}}{\mathcal{T}_{N}(\eta)} \]

By Lemma 1, \( \exists \eta > 0 \) such that for \( N \) sufficiently large, \( P_{N}(A_{N}) > 1-\epsilon \), where

\[ A_{N} = \{ \text{no } X_{i} \in \mathcal{T}_{N}(\eta) \} . \]

Hence on \( A_{N} \), we have

\[ \mathcal{J}_{3}^{\tau} = -\int \frac{1}{\mathcal{T}_{N}(\eta)} (G_{n}(x+N_{o}^{-2}b)-F(x+N_{o}^{-2}b))(J'(H^{a})-J'(H^{b}))d\mathcal{F} \]

\[ \leq \int \frac{1}{\mathcal{T}_{N}(\eta)} |G_{n}(x+N_{o}^{-2}b)-F(x+N_{o}^{-2}b)|J''(H^{a})|d\mathcal{F} \]

\[ \leq \frac{1}{\mathcal{T}_{N}(\eta)} \left[ \frac{F(x+N_{o}^{-2}b)(1-F(x+N_{o}^{-2}b))}{J''(H^{a})} \right]^{\frac{1}{2}} \]

\[ \leq \frac{K}{N_{o}^{1/2}} \left[ F(1-F) \right]^{\frac{2}{3}} \]

\[ = o(N_{o}^{-2}). \]

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\[
\begin{align*}
\leq & \frac{K}{N_0} \int_{I_N(\eta)} [F(1-F)]^{-\frac{5}{6}+\delta} dF + \frac{K}{N_0^{5/4}} \int_{I_N(\eta)} [F(1-F)]^{-1+\delta} dF \\
\leq & \frac{K}{N_0} \left( \frac{K}{N_0} \right)^{\frac{1}{6}+\delta} + \frac{K}{N_0^{5/4}} \\
\leq & \frac{K}{N_0^{7/6+\delta}} = o(N_0^{-1}).
\end{align*}
\]

(2.117)

Hence

(2.118) \quad \bar{J}_3 = o_p(N_0^{-1}).

(2.116) and (2.118) imply that \( J_3 = o_p(N_0^{-1}) \). This combined with (2.89), (2.91) and (2.110), substituted in (2.76) gives us that \( c_5^a - c_5^b = o_p(N_0^{-1}) \).

\[c_5^a - c_5^b = o_p(N_0^{-1}).\]

From the line following (2.7)

\[c_5^a = \frac{1}{2} \int_{H_N^a < 1} (H_N^a - H^a)^2 J''(\alpha H_N^a + (1-\alpha)H^a) dF_m.\]

Hence

\[c_5^a - c_5^b = \frac{1}{2} \int_{H_N^a < 1} \left( (H_N^a - H^a)^2 J''(\alpha H_N^a + (1-\alpha)H^a) - (H_N^b - H^b)^2 J''(\beta H_N^b + (1-\beta)H^b) \right) dF_m\]

where \( 0 < \alpha, \beta < 1 \).

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\[
\frac{1}{2} \int \left( (H_N^a - H_N^b)^2 - (H_N^a - H_N^b)^2 \right) j''(\alpha H_N^a + (1 - \alpha) H_N^b) dF_m \left( H_N^a < 1 \right) + \frac{1}{2} \int \left( (H_N^a - H_N^b)^2 \right) j''(\alpha H_N^a + (1 - \alpha) H_N^a) - j''(\beta H_N^b + (1 - \beta) H_N^b) dF_m \left( H_N^a < 1 \right)
\]

\[\Delta = \frac{1}{2} f_1 + \frac{1}{2} f_2\]

(2.119)

Now expanding \( f_1 \), and letting \( H_N^* = \alpha H_N^a + (1 - \alpha) H_N^a \), we get

\[
f_1 = 2\lambda_N (1 - \lambda_N) \int \left( F_n - F_n \right) \left( G_n(\frac{-1}{2}a) - G_n(\frac{-1}{2}b) \right) + \left( F_n - F_n \right) \left( \frac{-1}{2}a \right) dF_m \left( H_N^a < 1 \right)
\]

\[+ (1 - \lambda_N)^2 \int \left( (G_n(\frac{-1}{2}a) - F_n(\frac{-1}{2}a))^2 - (G_n(\frac{-1}{2}b) - F_n(\frac{-1}{2}b))^2 \right) j''(H_N^*) dF_m \left( H_N^a < 1 \right)
\]

\[\Delta = 2\lambda_N (1 - \lambda_N) f_{11} + (1 - \lambda_N)^2 f_{12}\]

(2.120)

Consider \( f_{11} \). From equation (7.9) of Chernoff and Savage [3], p. 990, we obtain that

\[P_{\nu}(H_N^*(1 - H_N^*) > \nu H_N^*(1 - H_N^*) \text{ on } 0 < H_N^a < 1) > 1 - \epsilon\]

On this set,

\[|j''(H_N^*)| \leq K[H_N^*(1 - H_N^*)]^{-2+\delta} \leq K[H_N^*(1 - H_N^*)]^{-2+\delta} \leq K[F(1 - F)]^{-2+\delta}.
\]

Then
\[ |f_{11}| \leq K \int_{H^a_N < 1} |F_m - F| |G_n(x + \frac{1}{2}a) - G_n(x + \frac{1}{2}b) - F(x + a) + F(x + b)| [F(1-F)]^{-2+\delta} dF_m \]

Furthermore, \( \exists \eta > 0 \) such that for \( N \) sufficiently large, \( P(\text{all } X_i \in I_N(\eta)) > 1 - \epsilon \).

Hence \( \exists \eta > 0, K > 0 \) with probability \( > 1 - \epsilon \), for \( N \) sufficiently large,

\[ |f_{11}| \leq K \int_{H^a_N < 1} |F_m - F| |G_n(x + \frac{1}{2}a) - G_n(x + \frac{1}{2}b) - F(x + a) + F(x + b)| [F(1-F)]^{-2+\delta} dF_m \]

\( (2.121) \)

\[ \Delta = f_{11}' \]

\[ E(f_{11}') = E(E(f_{11}' | \mathcal{G})) \]

\[ = K E \int_{H^a_N < 1} |F_m - F| |G_n(x + \frac{1}{2}a) - G_n(x + \frac{1}{2}b) - F(x + a) + F(x + b)| [F(1-F)]^{-2+\delta} dF_m \]

\[ \leq K E \int_{H^a_N < 1} |F_m - F| \left[ \left( F(x + \frac{1}{2}a) - F(x + \frac{1}{2}b) \right) \left( 1 - F(x + \frac{1}{2}a) + F(x + \frac{1}{2}b) \right) \right]^{1/2} [F(1-F)]^{-2+\delta} dF_m \]

\[ \leq \frac{K}{N^{1/2}} E \int_{I_N(\eta)} |F_m - F| \left[ \frac{2}{N^{1/2}} \left[ F(1-F) \right]^{2+\delta} \right] dF_m \]

for \( N \) sufficiently large
\[
\leq \frac{K}{N_0^{3/4}} \int_{I_N(\eta)} |F_m - F| \left( \int [F(1-F)]^{-\delta/2 + \delta} \, dF \right) \leq \frac{K}{N_0^{7/4}} \int_{I_N(\eta)} \left( \int [F(1-F)]^{-\delta/2 + \delta} \, dF \right) + \frac{K}{N_0^{5/4}} \int_{I_N(\eta)} \left( \int [F(1-F)]^{-\delta/2 + \delta} \, dF \right), \text{ by Lemma 3}
\]

\[
\leq \frac{K}{N_0^{7/4 - 2/3 + \delta}} + \frac{K}{N_0^{5/4 - 1/6 + \delta}}
\]

\[
\leq \frac{K}{N_0^{13/12 + \delta}} = o(N_0^{-1})
\]

(2.122)

Hence

(2.123) \quad J_{11} = o_N(N^{-1}).

Consider

\[
J_{12} = \int_{H_N^a < 1} (G_n(x + N_0^{-2/3}a) - F(x + N_0^{-2/3}a))^2 - (G_n(x + N_0^{-1/3}b) - F(x + N_0^{-1/3}b))^2 \, dF_m
\]

As above, \( \exists \quad \eta > 0, K > 0 \) for \( N \) sufficiently large, \( P_N(J_{12} = J'_{12}) > 1 - \epsilon \), where

\[
J'_{12} = \int_{H_N^a < 1} (G_n(x + N_0^{-2/3}a) - F(x + N_0^{-2/3}a))^2 - (G_n(x + N_0^{-1/3}b) - F(x + N_0^{-1/3}b))^2 \, dF_m\]
\[ E|\mathbf{g}'_{12}| = E(E|\mathbf{g}'_{12}||\mathbf{x}) \leq KE \int_{H_N < 1} E(|G_n(x+N_0^{-1/2}a) - G_n(x+N_0^{-1/2}b) - F(x+N_0^{-1/2}a) + F(x+N_0^{-1/2}b)|). \]

\( I_N(\eta) \]

\[ |G_n(x+N_0^{-1/2}a) + G_n(x+N_0^{-1/2}b) - F(x+N_0^{-1/2}a) - F(x+N_0^{-1/2}b)| |F(1-F)|^{-2} + d^2 \]

Now let

\[ \left( \begin{array}{c}
G_n(x+N_0^{-1/2}b) \\
\frac{-1}{2} \\
G_n(x+N_0^{-1/2}a) - G_n(x+N_0^{-1/2}b) 
\end{array} \right) = \begin{bmatrix} U \\ V \end{bmatrix}. \]

Then \( \begin{bmatrix} U \\ V \end{bmatrix} \) is distributed as a trinomial vector with parameters \( n, \)

\[ p_1 = F(x+N_0^{-1/2}b), \quad p_2 = F(x+N_0^{-1/2}a) - F(x+N_0^{-1/2}b). \quad \] Hence

\[ |G_n(x+N_0^{-1/2}a) - G_n(x+N_0^{-1/2}b) - F(x+N_0^{-1/2}a) + F(x+N_0^{-1/2}b)| |G_n(x+N_0^{-1/2}a) + G_n(x+N_0^{-1/2}b) - F(x+N_0^{-1/2}a) - F(x+N_0^{-1/2}b)| \]

\[ = \frac{1}{n^2} |v - E(v)||2u + v - E(2u + v)| \]

\[ \leq \frac{2}{n^2} |v - E(v)||u - E(u)| + \frac{1}{n^2} (v - E(v))^2. \]

\( (2.125) \)

Taking expectations on both sides, we get
\[ E(\text{LHS}) \leq \frac{2}{n^2} \text{E}[V-E(V)\mid U-E(U)] + \frac{1}{n^2} \text{Var} V \]

\[ \leq \frac{2}{n^2} \text{Var} V \text{Var} U \frac{1}{2} + \frac{1}{n^2} \text{Var} V \]

\[ = \frac{2}{n} \left[ p_1 p_2 (1-p_1)(1-p_2) \right]^{\frac{1}{2}} + \frac{1}{n} p_2 (1-p_2) \]

\[ \leq \frac{2}{n} \left[ F(x+N_o^{-2} b)(1-F(x+N_o^{-2} b)) \right] \frac{1}{2} \]

\[ + \frac{1}{n} \left( F(x+N_o^{-2} a) - F(x+N_o^{-2} b) \right) \]

\[ \leq \frac{K}{N_o} \left[ \left\{ F(1-F) + \frac{K}{N_o^{1/2}} \left[ F(1-F) \right]\right\} + \frac{K}{N_o^{1/2}} \left[ F(1-F) \right]\right] + \frac{K}{N_o^{3/2}} \left[ F(1-F) \right]\]

\[ \text{for } N_o \text{ sufficiently large} \]

\[ \leq \frac{K}{N_o^{5/4}} \left[ F(1-F) \right] + \frac{K}{N_o^{3/2}} \left[ F(1-F) \right]\]

(2.126)

Substituting (2.126) into (2.124), we get

\[ E|d_{12}^t| \leq \frac{K}{N_o^{5/4}} \int_{N(\eta)} \left[ F(1-F) \right]^{\frac{5}{6} - 2 + \delta} dF + \frac{K}{N_o^{3/2}} \int_{N(\eta)} \left[ F(1-F) \right]^{\frac{2}{3} - 2 + \delta} dF \]

(2.127)

\[ \leq \frac{K}{N_o^{5/4 - 1/6 + \delta}} + \frac{K}{N_o^{3/2 - 1/3 + \delta}} \]

\[ \leq \frac{K}{N_o^{13/12 + \delta}} = o(N_o^{-1}) . \]
Hence

\[(2.128) \quad \mathcal{J}_{12} = o_p(N^{-1}).\]

Substituting (2.123) and (2.128) into (2.120) gives

\[(2.129) \quad \mathcal{J}_1 = o_p(N^{-1}).\]

Consider now

\[\mathcal{J}_2 = \int_{H_N^{a} < 1} (H_N^{b} - H_N^{a})^2 (j''(\alpha H_N^{a} + (1-\alpha)H_N^{a}) - j''(\beta H_N^{b} + (1-\beta)H_N^{b}))dF_m\]

Now \(\exists \eta > 0 \in \mathbb{R}\) for \(N\) sufficiently large, \(P_N(J_2 = J_2') > 1-\epsilon\), where

\[\mathcal{J}_2' = \int_{H_N^{a} < 1, I_N(\eta)} (H_N^{b} - H_N^{a})^2 (j''(\alpha H_N^{a} + (1-\alpha)H_N^{a}) - j''(\beta H_N^{b} + (1-\beta)H_N^{b}))dF_m\]

\[= \int_{H_N^{a} < 1, I_N(\eta)} (H_N^{b} - H_N^{a})^2 [\alpha(H_N^{a} - H_N^{a}) - \beta(H_N^{b} - H_N^{b}) + (H^a - H^b)]j''(H_N^{a})dF_m\]

where \(H^a_N\) lies between \(\alpha H_N^{a} + (1-\alpha)H^{a}\) and \(\beta H_N^{b} + (1-\beta)H_N^{b}\).

As before, \(\exists \nu > 0 \in \mathbb{R}\) \(P_N(j''(\alpha H_N^{a} + (1-\alpha)H_N^{a}) - j''(\beta H_N^{b} + (1-\beta)H_N^{b})) > \nu F(1-F)\) and \((\beta H_N^{b} + (1-\beta)H_N^{b})(1-\beta H_N^{b} - (1-\beta)H_N^{b}) > \nu F(1-F)\) on \(0 < H_N^{a}, H_N^{b} < 1\) \(> 1-\epsilon\). Since
lies between $\alpha H_N^a + (1-\alpha)H_N^b$ and $\beta H_N^b + (1-\beta)H_N^b$, it follows that

$$P_N\{H_N^*(1-H_N^*) > v F(1-F) \text{ on } 0 < H_N^a, H_N^b < 1\} > 1-\epsilon .$$

On this set, then, we have

$$|\mu_2'| \leq K \int_{H_N^a < 1} \int_{I_N(\eta)} (H_N^b - H_N^a)^2 |\alpha(H_N^a - H_N^a) - \beta(H_N^b - H_N^b) + (H_N^a - H_N^b)| [F(1-F)]^{3+\delta} dF_m$$

$$\leq K \int_{H_N^a < 1} \int_{I_N(\eta)} (H_N^b - H_N^a)^2 |H_N^a - H_N^b - H_a + H_b| [F(1-F)]^{3+\delta} dF_m$$

$$+ K \int_{H_N^a < 1} \int_{I_N(\eta)} |H_N^b - H_N^b|^3 [F(1-F)]^{-3+\delta} dF_m$$

$$+ K \int_{H_N^a < 1} \int_{I_N(\eta)} (H_N^b - H_N^b)^2 (H_N^a - H_N^b) [F(1-F)]^{3+\delta} dF_m$$

$$\Delta \mu_1 + \mu_2 + \mu_3 .$$

(2.130)
\[ I_k = K \int_{I_N(\eta)} \left( \frac{\lambda^2}{(F_m - F)^2 + 2\lambda_N(1 - \lambda_N)(F_m - F)(F_n(x + N_o b) - F(x + N_o b))} + (1 - \lambda_N)^2 \right) \\
\times \frac{1}{2} \left( G_n(x + N_o b) - G_n(x + N_o b) \right) (F(x + N_o b)) \right)^2 \|
\]

\[ \leq K \int_{I_N(\eta)} (F_m - F)^2 \left| G_n(x + N_o a) - G_n(x + N_o b) - F(x + N_o a) + F(x + N_o b) \right| [F(1 - F)]^{-3+6} dF_m \]

\[ + K \int_{I_N(\eta)} \left| G_n(x + N_o b) - F(x + N_o b) \right|^2 \left| G_n(x + N_o a) - G_n(x + N_o b) - F(x + N_o a) + F(x + N_o b) \right| [F(1 - F)]^{-3+6} dF_m \]

\[ + K \int_{I_N(\eta)} \left| G_n(x + N_o b) - F(x + N_o b) \right| \left| G_n(x + N_o a) - G_n(x + N_o b) - F(x + N_o a) + F(x + N_o b) \right| [F(1 - F)]^{-3+6} dF_m \]

\[ = I_{11} + I_{12} + I_{13} \]

(2.131)

\[ E(I_{11} | X) = K \int_{I_N(\eta)} (F_m - F)^2 \left| G_n(x + N_o a) - G_n(x + N_o b) - F(x + N_o a) + F(x + N_o b) \right| [F(1 - F)]^{-3+6} dF_m \]

\[ \leq K \int_{I_N(\eta)} (F_m - F)^2 \left[ \frac{\left( F(x + N_o a) - F(x + N_o b) \right) \left( 1 - F(x + N_o a) + F(x + N_o b) \right)}{n} \right]^\frac{1}{2} \]

\[ [F(1 - F)]^{-3+6} dF_m \]
\[
\frac{K}{N_0^{1/2}} \int I_N(\eta) (F_m - F)^2 \left[ \frac{K}{N_0} [F(1-F)]^{1/2} \right]^2 \frac{1}{[F(1-F)]^{-3+8}} \, dF_m \\
\leq \frac{K}{N_0^{3/4}} \int I_N(\eta) (F_m - F)^2 [F(1-F)]^{-\frac{8+8}{3}} \, dF_m
\]

(2.132)

\[
E(\mathcal{J}_{11}) = E(E(\mathcal{J}_{11} | \mathcal{X})) \leq \frac{K}{N_0^{3/4}} E \int I_N(\eta) (F_m - F)^2 [F(1-F)]^{-\frac{8+8}{3}} \, dF_m \\
\leq \frac{K}{N_0^{11/4}} \int I_N(\eta) \left\{ \frac{K}{N_0} (1-F) + \frac{K}{N_0} F(1-F) [F(1-F)] \right\} \frac{8+8}{3} \, dF, \text{ by Lemma 4} \\
\leq \frac{K}{N_0^{11/4}} \int I_N(\eta) \left[ F(1-F) \right]^{-\frac{8+8}{3}} \, dF + \frac{K}{N_0^{7/4}} \int I_N(\eta) \left[ F(1-F) \right]^{-\frac{8+8}{3}} \, dF \\
\leq \frac{K}{N_0^{11/4-5/3+8}} + \frac{K}{N_0^{7/4-2/3+8}} \\
= \frac{K}{N_0^{13/12+8}} = o(N_0^{-1}).
\]

(2.133)

(2.134) \quad \therefore \quad \mathcal{J}_{11} = o_p(N^{-1}).

Now
$$E(\mathcal{I}_{12} | \mathcal{X}) = K \int \frac{1}{I_N(\eta)} \left\{ \sum \left[ \frac{1}{4} E\left( G_n\left( x+N_0 / 2 b \right) - F\left( x+N_0 / 2 a \right) \right) \right] G_n\left( x+N_0 / 2 a \right) - G_n\left( x+N_0 / 2 b \right) \right\}^{1/2} \left[ F\left( 1-F \right) \right]^{-3/6} dF_m$$

$$\leq K \int \frac{1}{I_N(\eta)} \left\{ \frac{1}{2} E\left( G_n\left( x+N_0 / 2 b \right) - F\left( x+N_0 / 2 a \right) \right) \right\} \left[ F\left( 1-F \right) \right]^{-3/6} dF_m$$

$$\leq K \int \frac{1}{I_N(\eta)} \left\{ \frac{3}{n^2} \left[ F\left( x+N_0 / 2 b \right) \right] \left[ 1-F\left( x+N_0 / 2 a \right) \right] \right\} \left[ F\left( 1-F \right) \right]^{-3/6} dF_m$$

$$\leq K \int \frac{1}{I_N(\eta)} \left\{ \frac{1}{n} \left[ F\left( x+N_0 / 2 a \right) - F\left( x+N_0 / 2 b \right) \right] \left[ 1-F\left( x+N_0 / 2 b \right) \right] \right\} \left[ F\left( 1-F \right) \right]^{-3/6} dF_m$$

$$\leq K \int \frac{1}{I_N(\eta)} \left\{ \frac{K_{N_0 / 2}}{N_0} \left[ F\left( x+N_0 / 2 b \right) \right] \left[ 1-F\left( x+N_0 / 2 b \right) \right] \right\} \left[ F\left( 1-F \right) \right]^{-3/6} dF_m$$

$$\leq \frac{K_{N_0 / 2}}{N_0^{1/2}} \int \left\{ F\left( 1-F \right) \right\} \left[ F\left( 1-F \right) \right]^{-3/6} dF_m$$

$$\leq \frac{K_{N_0 / 2}}{N_0^{1/2}} \int \left\{ F\left( 1-F \right) \right\} \left[ F\left( 1-F \right) \right]^{-3/6} dF_m$$
\[+ \frac{K}{N_0^{9/4}} \int_{I_N(\eta)} \left\{ \text{F}(1-F)^{1/2} + \frac{K}{N_0^{1/4}} \left[ \text{F}(1-F) \right]^{1/2} \right\} \left[ \text{F}(1-F) \right]^{1/2-3+8} \text{dF} \]

\[(2.135)\]

for \( N_0 \) sufficiently large.

Now

\[E(J_{12}) = E(E(J_{12} | X)) \]

\[\leq \frac{K}{N_0^{7/4}} \int_{I_N(\eta)} \left[ \text{F}(1-F) \right]^{-\frac{5+8}{3}} \text{dF} + \frac{K}{N_0^{9/4}} \int_{I_N(\eta)} \left[ \text{F}(1-F) \right]^{-2+8} \text{dF} \]

\[+ \frac{K}{N_0^{9/4}} \int_{I_N(\eta)} \left[ \text{F}(1-F) \right]^{-\frac{13}{6}+8} \text{dF} + \frac{K}{N_0^{10/4}} \int_{I_N(\eta)} \left[ \text{F}(1-F) \right]^{-\frac{7}{3}+8} \text{dF} \]

\[\leq \frac{K}{N_0^{7/4-2/3+8}} + \frac{K}{N_0^{9/4-1+8}} + \frac{K}{N_0^{9/4-7/6+8}} + \frac{K}{N_0^{10/4-4/3+8}} \]

\[\leq \frac{K}{N_0^{13/12+8}} = o(N_0^{-1}) \]

\[(2.136)\]

Hence

\[(2.137)\]

\[J_{12} = o_p(N_0^{-1}). \]
Consider now $\mathcal{J}_{13}$.

$$E(\mathcal{J}_{13} | X) = K \int_{I_N(\eta)} |F_m - F| \left\{ E\left\{ G_n(x+N_0^{-\frac{1}{2}}b) - F(x+N_0^{-\frac{1}{2}}b) \right\} \right\} [F(1-F)]^{-3+\delta} \, dF_m$$

$$\leq K \int_{I_N(\eta)} |F_m - F| \left\{ E\left( G_n(x+N_0^{\frac{1}{2}}b) - F(x+N_0^{\frac{1}{2}}b) \right)^2 \right\}$$

$$\cdot E\left( G_n(x+N_0^{\frac{1}{2}}a) - G_n(x+N_0^{\frac{1}{2}}b) - F(x+N_0^{\frac{1}{2}}a) + F(x+N_0^{\frac{1}{2}}b) \right)^2 \frac{1}{2} [F(1-F)]^{-3+\delta} \, dF_m$$

$$\leq K \int_{I_N(\eta)} |F_m - F| \left\{ \frac{F(x+N_0^{-\frac{1}{2}}b)(1-F(x+N_0^{-\frac{1}{2}}b))}{n} \right\} \frac{1}{2} \left\{ \frac{F(x+N_0^{\frac{1}{2}}a) - F(x+N_0^{\frac{1}{2}}b)}{n} \right\} \frac{1}{2}$$

$$\cdot [F(1-F)]^{-3+\delta} \, dF_m$$

$$\leq K \int_{I_N(\eta)} \left\{ \left[ F(1-F) \right]^2 + \frac{K_{1/4}}{N_o} \left[ F(1-F) \right]^{\frac{5}{2}} \right\} \left\{ \frac{K_{1/4}}{N_o} \left[ F(1-F) \right]^2 \right\} \frac{1}{2}$$

$$\cdot [F(1-F)]^{-3+\delta} \, dF_m$$

$$\leq \frac{K}{N_o^{5/4}} \int_{I_N(\eta)} |F_m - F| \left\{ \left[ F(1-F) \right]^2 \frac{8}{3} \right\}^{\frac{1}{2}} + \frac{K_{1/4}}{N_o} \left[ F(1-F) \right]^{\frac{1}{2}} - \frac{8}{3} + 8 \} \, dF_m$$

(2.138)

Now
\( E(\mathcal{I}_{13}) = E(E(\mathcal{I}_{13} | \mathcal{F})) \)

\[
\leq \frac{K}{N_0^{1/4}} \left\{ \frac{K}{N_0} \int_{I_N(\eta)} \left\{ \left[ F(1-F) \right]^{-\frac{13}{6}+6} \right. \right. \\
+ \left. \left. \frac{K}{N_0^{1/4}} \left[ F(1-F) \right]^{-\frac{7}{3}+6} \right\} dF \right\}

\leq \frac{K}{N_0^{9/4-7/6+6}} + \frac{K}{N_0^{5/2-4/3+6}} + \frac{K}{N_0^{7/4-2/3+6}} + \frac{K}{N_0^{2-5/6+6}}

\leq \frac{K}{N_0^{13/12+6}} = o(N^{-1})

(2.139)

Hence

(2.140) \quad \mathcal{I}_{13} = o_p(N^{-1}).

Substituting (2.140), (2.137), and (2.134) into (2.131) gives us

(2.141) \quad \mathcal{I}_1 = o_p(N^{-1}).

Consider now from (2.130)

\[
\mathcal{I}_2 = K \int_{H_N^b < 1} \int_{I_N(\eta)} |H_N^b - h^b| \left[ F(1-F) \right]^{-3+6} dF_m.
\]
Expanding $|h_N^b - H^b|^3$, we get

$$
\mathcal{J}_2 \leq K \int_{I_N^0(\eta)} |F_m - F|^3 [F(1-F)]^{-3+8} \, dF_m
$$

$$
+ K \int_{I_N^0(\eta)} (F_m - F)^2 |G_n(x + N_0^2 b) - F(x + N_0^2 b)| [F(1-F)]^{-3+8} \, dF_m
$$

$$
+ K \int_{I_N^0(\eta)} |F_m - F|^3 |G_n(x + N_0^2 b) - F(x + N_0^2 b)|^2 [F(1-F)]^{-3+8} \, dF_m
$$

$$
+ K \int_{I_N^0(\eta)} |G_n(x + N_0^2 b) - F(x + N_0^2 b)|^3 [F(1-F)]^{-3+8} \, dF_m
$$

$$
= \mathcal{J}_{21} + \mathcal{J}_{22} + \mathcal{J}_{23} + \mathcal{J}_{24}
$$

(2.142)

By Lemma 5,

$$
E(\mathcal{J}_{21}) \leq \frac{K}{N_0^{3/2}} \int_{I_N^0(\eta)} [F(1-F)]^{3/2} - 3 + 8 \, dF + \frac{K}{N_0^2} \int_{I_N^0(\eta)} [F(1-F)]^{1-3+8} \, dF
$$

$$
+ \frac{K}{N_0^3} \int_{I_N^0(\eta)} [F(1-F)]^{-3+8} \, dF
$$

$$
\leq \frac{K}{N_0^{3/2-1/2+6}} + \frac{K}{N_0^{2-1+6}} + \frac{K}{N_0^{3-2+6}} \leq \frac{K}{N_0^{1+6}} = o(N_0^{-1})
$$

(2.143)
(2.144) 

\[ \mathcal{J}_{2l} = o_p(N^{-1}) \]

\[
E(\mathcal{J}_{22} | \mathcal{X}) = K \int_{I_N(\eta)} (F_m - \bar{F})^2 \left\{ (n - N_0) \left[ \frac{1}{2} \right] + \frac{1}{2} \right\} [F(1-F)]^{-3+\varepsilon} dF_m
\]

\[
\leq K \int_{I_N(\eta)} (F_m - \bar{F})^2 \left\{ \frac{1}{2} \left\{ \frac{1}{2} \right\} + \frac{K}{N_0^{1/4}} \right\} [F(1-F)]^{-3+\varepsilon} dF_m
\]

\[
E(\mathcal{J}_{22}) = E(E(\mathcal{J}_{22} | \mathcal{X}))
\]

\[
\leq \frac{K}{N_0^{1/2}} \int_{I_N(\eta)} \left\{ \frac{K}{N_0^{1/4}} \right\} [F(1-F)]^{-\frac{5}{2}+\varepsilon} + \frac{K}{N_0^{1/4}} [F(1-F)]^{-\frac{8}{3}+\varepsilon} dF
\]

by Lemma 4

\[
\leq \frac{K}{N_0^{5/2}} \left\{ \frac{K}{N_0^{-3/2+\varepsilon}} + \frac{K}{N_0^{1/4-5/3+\varepsilon}} \right\} + \frac{K}{N_0^{5/2}} \left\{ \frac{K}{N_0^{-1/2+\varepsilon}} + \frac{K}{N_0^{1/4-2/3+\varepsilon}} \right\}
\]

\[
\leq \frac{K}{N_0^{1+\varepsilon}} = o(N_0^{-1})
\]

(2.145)
Thus

\[(2.146) \quad \theta_{22} = o_p(N^{-1})\]

\[
E(\theta_{23}|\chi) = K \int_{I_N(\eta)} |F_m - F| E(G_n(x+N_0^{-\frac{1}{2}}b)-F(x+N_0^{-\frac{1}{2}}b))^2[F(1-F)]^{-3+\delta} dF_m
\]

\[
= K \int_{I_N(\eta)} |F_m - F| \left[ \frac{1}{n} \frac{F(x+N_0^{-\frac{1}{2}}b)(1-F(x+N_0^{-\frac{1}{2}}b))}{n} \right] [F(1-F)]^{-3+\delta} dF_m
\]

\[
\leq \frac{K}{N_0} \int_{I_N(\eta)} |F_m - F| \left\{ F(1-F) + \frac{K}{N_0^{1/2}} [F(1-F)]^2 \right\} [F(1-F)]^{-3+\delta} dF_m
\]

\[
E(\theta_{23}) = E(E(\theta_{23}|\chi))
\]

\[
\leq \frac{K}{N_0} \int_{I_N(\eta)} \left\{ \frac{K}{N_0^{1/2}} + \frac{K}{N_0^{1/2}} [F(1-F)]^{\frac{1}{2}} \right\} \left\{ [F(1-F)]^{-2+\delta} + \frac{K}{N_0^{1/2}} [F(1-F)]^{-\frac{7}{3}+\delta} \right\} dF
\]

by Lemma 3

\[
\leq \frac{K}{N_0^{1+\delta}} \left\{ \frac{K}{N_0^{1+\delta}} + \frac{K}{N_0^{1/2-4/3+\delta}} \right\} + \frac{K}{N_0^{3/2}} \left\{ \frac{K}{N_0^{1/2+\delta}} + \frac{K}{N_0^{1/2-5/6+\delta}} \right\}
\]

\[
\leq \frac{K}{N_0^{1+\delta}} = o(N_0^{-1})
\]

\[(2.147)\]

Hence

\[(2.148) \quad \theta_{23} = o_p(N^{-1})\]
\[ E(\mathcal{G}_{24}) = E(E(\mathcal{G}_{24} | \mathcal{X})) \]

\[ = KE \int_{\mathcal{N}(\eta)} E|G_n(x+N_{o-b})-F(x+N_{o-b})|^3 [F(1-F)]^{-3+8} dF_m \]

\[ = K \int_{\mathcal{N}(\eta)} E|G_n(x+N_{o-b})-F(x+N_{o-b})|^3 [F(1-F)]^{-3+8} dF \]

Now for any random variable \( U \)

\[ E|U-E(U)|^3 \leq [E(U-E(U))^2 E(U-E(U))^4]^{\frac{1}{2}} \]

Hence

\[ E|G_n(x+N_{o-b})-F(x+N_{o-b})|^3 \]

\[ \leq \left( \frac{F(x+N_{o-b})(1-F(x+N_{o-b}))}{n} \right)^{\frac{1}{2}} \left( \frac{3[F(x+N_{o-b})(1-F(x+N_{o-b}))]^2}{n^3} \right) \]

\[ + \frac{F(x+N_{o-b})(1-F(x+N_{o-b}))}{n^3} \left( 1-6F(x+N_{o-b})(1-F(x+N_{o-b})) \right) \]

\[ \leq \frac{K}{N_o^{1/2}} \left\{ F(1-F)^{\frac{1}{2}} + \frac{K}{N_o^{1/4}} [F(1-F)]^{\frac{1}{3}} \right\} \left\{ \frac{K}{N_o} (F(1-F) + \frac{K}{N_o^{1/2}} [F(1-F)]^2) \right\} \]

\[ + \frac{K}{N_o^{3/2}} \left\{ [F(1-F)]^{\frac{1}{2}} + \frac{K}{N_o^{1/4}} [F(1-F)]^{\frac{1}{3}} \right\} \]

\[ \leq \frac{K}{N_o^{3/2}} \left\{ K[F(1-F)]^{\frac{2}{2}} + \frac{K}{N_o^{1/4}} [F(1-F)]^{\frac{4}{3}} + \frac{K}{N_o^{1/2}} [F(1-F)]^{\frac{7}{6}} + \frac{K}{N_o^{5/4}} F(1-F) \right\} \]

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\[
\frac{K}{N_0^{3/2}} \left\{ K F(1-F) + \frac{K}{N_0^{1/4}} \left[ F(1-F) \right]^{5/6} + \frac{K}{N_0^{1/2}} \left[ F(1-F) \right]^{2/3} \right\} \\
\leq \frac{K}{N_0^{3/2}} \left[ F(1-F) \right]^{3/2} + \frac{K}{N_0^{7/4}} \left[ F(1-F) \right]^{4/3} + \frac{K}{N_0^{3/2}} F(1-F) + \frac{K}{N_0^{9/4}} \left[ F(1-F) \right]^{5/6} \\
+ \frac{K}{N_0^{5/2}} \left[ F(1-F) \right]^{2/3}
\]

(2.150)

Substituting (2.150) into (2.149) we get

\[
E(\mathcal{G}_{24}) \leq \frac{K}{N_0^{3/2}} \int_{\eta} \left[ F(1-F) \right]^{-3/2+6} dF + \frac{K}{N_0^{7/4}} \int_{\eta} \left[ F(1-F) \right]^{-5/2+6} dF \\
+ \frac{K}{N_0^{3/2}} \int_{\eta} \left[ F(1-F) \right]^{-2+6} dF + \frac{K}{N_0^{9/4}} \int_{\eta} \left[ F(1-F) \right]^{-13/6+6} dF \\
+ \frac{K}{N_0^{5/2}} \int_{\eta} \left[ F(1-F) \right]^{-7/3+6} dF \\
\leq \frac{K}{N_0^{3/2-1/2+6}} + \frac{K}{N_0^{7/4-2/3+6}} + \frac{K}{N_0^{2-1+6}} + \frac{K}{N_0^{9/4-7/6+6}} + \frac{K}{N_0^{5/2-4/3+6}}
\]

\[
\leq \frac{K}{N_0^{1+6}} = o(N_0^{-1})
\]

(2.151)
Hence

\[(2.152) \quad \mathcal{J}_{24} = o_p(N^{-1}).\]

Combining (2.152), (2.148), (2.146), and (2.144) in (2.142) gives

\[(2.153) \quad \mathcal{J}_2 = o_p(N^{-1}).\]

Finally, let us consider from (2.130)

\[
\mathcal{J}_3 = K \int_{H_N^a<H_N^b \leq 1 \atop I_N(\eta)} \left( F_{m-F}^a F_{m-F}^b \right)^2 \left( H_N^a - H_N^b \right)^2 \left[ F(1-F) \right]^{-3+\delta} dF_m
\]

\[
\leq K \lambda_N^2 \int_{I_N(\eta)} \left( F_{m-F}^a F_{m-F}^b \right)^2 \left( H_N^a - H_N^b \right)^2 \left[ F(1-F) \right]^{-3+\delta} dF_m
\]

\[
+ K 2 \lambda_N (1-\lambda_N) \int_{I_N(\eta)} \left( F_{m-F}^a F_{m-F}^b \right)^2 \left( H_N^a - H_N^b \right)^2 \left[ F(1-F) \right]^{-3+\delta} dF_m
\]

\[
+ K (1-\lambda_N)^2 \int_{I_N(\eta)} \left( F_{m-F}^a F_{m-F}^b \right)^2 \left( H_N^a - H_N^b \right)^2 \left[ F(1-F) \right]^{-3+\delta} dF_m
\]

\[
= K \lambda_N^2 J_{31} + 2K \lambda_N (1-\lambda_N) J_{32} + K (1-\lambda_N)^2 J_{33}
\]

\[(2.154)\]

Now

\[
H_N^a - H_N^b = (1-\lambda_N) \left( F(x+N_{\infty}^a a) - F(x+N_{\infty}^a b) \right) \leq \frac{K}{N_{\infty}^{1/2}} \left[ F(1-F) \right]^\frac{2}{3} \text{ for } N_{\infty} \text{ sufficiently large.}
\]
\[ E(\mathcal{G}_{31}) \leq \frac{K}{N_0^{1/2}} \int_{I_N(\eta)} \left( F - F \right)^2 \left[ F(1 - F) \right] \frac{dF}{N_0^{1/2}} \leq \frac{K}{N_0^{1/2}} \int_{I_N(\eta)} \left[ \frac{K}{N_0^3} \right]^2 \left[ F(1 - F) \right] \left[ F(1 - F) \right] \frac{dF}{N_0^{1/2}} \text{ by Lemma 4} \]

\[ \leq \frac{K}{N_0^{5/2 - 4/3 + 3}} + \frac{K}{N_0^{3/2 - 1/3 + 3}} \leq \frac{K}{N_0^{7/6 + 3}} = o(N_0^{-1}) \]

(2.155)

\[ E(\mathcal{G}_{32}) = E(E(\mathcal{G}_{32} | \mathcal{X})) \]

Now

\[ E(\mathcal{G}_{32} | \mathcal{X}) = \int_{I_N(\eta)} (F - F) E(G_n(x + N_0^{-1/2}b) - F(x + N_0^{-1/2}b))(H - H)(F(1 - F))^{-3 + 8} dF \]

\[ = 0 \]

(2.156) \quad \therefore E(\mathcal{G}_{32}) = 0

\[ E(\mathcal{G}_{33}) = E(E(\mathcal{G}_{33} | \mathcal{X})) \]

\[ = E \int_{I_N(\eta)} E(G_n(x + N_0^{-1/2}b) - F(x + N_0^{-1/2}b))^2(H - H)(F(1 - F))^{-3 + 8} dF \]

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\[ \begin{align*}
&\leq E \int_{I_N} \frac{1}{n} \left( \frac{F(x+N_0b)(1-F(x+N_0b))}{F(1-F)} \right)^{\frac{2}{3}-\frac{3}{8}} \frac{K}{N_0^{1/2}} \left[ F(1-F) \right]^{\frac{2}{3}} \frac{dF}{N_0^{1/2}} \\
&\leq \frac{K}{N_0^{3/2}} \int_{I_N(\eta)} \left\{ F(1-F) + \frac{K}{N_0^{1/2}} \left[ F(1-F) \right]^{\frac{2}{3}} \right\} \left[ F(1-F) \right]^{-\frac{7}{8}} dF \\
&\leq \frac{K}{N_0^{3/2-1/3+\delta}} + \frac{K}{N_0^{2-2/3+\delta}} \\
&\leq \frac{K}{N_0^{7/6+\delta}} = o(N_0^{-1}).
\end{align*} \]

(2.157)

Substituting (2.157), (2.156) and (2.155) in (2.154) gives

(2.158) \hspace{1cm} E(\theta_3) = o(N_0^{-1}).

Since \( \theta_3 \geq 0 \), it follows that

(2.159) \hspace{1cm} \theta_3 = o_p(N^{-1}).

Combining (2.159), (2.153) and (2.141) in (2.130) gives \( J_2 = o_p(N^{-1}) \) and hence

(2.160) \hspace{1cm} J_2 = o_p(N^{-1}).

This combined with (2.129) and (2.119) gives

(2.161) \hspace{1cm} c_3^{a} - c_3 = o_p(N^{-1})
\[-\frac{1}{N Q^{1/2} B(F)} \int_{\mathcal{O}} a^2 \int_{\mathcal{O}} b^{1/2} \int_{\mathcal{O}} c \cdot dF = o_p(1)\]

It suffices to show that \( \int \zeta dF \) and \( \int \xi dF \) are both uniformly bounded in \( N \). Now from (2.12) we have

\[
\zeta^a(x) = \frac{1}{6} \left( \frac{1}{2} f''(x + \theta N^{-2}_o a) + \frac{1}{6} \frac{1}{2} f''(x + \theta N^{-2}_o a) \right) J''(F) + \frac{1}{2} \left( \frac{1}{2} f''(x + \theta N^{-2}_o a) \right) J''(F)
\]

(2.162) \[
+ \frac{\lambda}{2} (1 - 1)_N^2 \left( \frac{1}{2} f' \right) + \frac{1}{6} \frac{1}{2} f''(x + \theta N^{-2}_o a) \right) J''(F)
\]

\[+ \left( 1 - 1_N \right) \left( f + \frac{1}{2} \frac{1}{2} f' + \frac{1}{6} f''(x + \theta N^{-2}_o a) \right) J''(F) (\gamma F + (1 - \gamma) H^a)\]

Using condition (4) of the theorem, we have that for \( N \) sufficiently large

\[|f''(x + \theta N^{-2}_o a)| \leq K[F(1 - F)]^{2/3}\]

\[\left| \frac{1}{2} f' + \frac{1}{6} f''(x + \theta N^{-2}_o a) \right| \leq K[F(1 - F)]^{2/3}\]

\[\left( \frac{1}{2} f' + \frac{1}{6} f''(x + \theta N^{-2}_o a) \right)^2 \leq K[F(1 - F)]^{4/3}\]

\[|f + \frac{1}{2} f' + \frac{1}{6} f''(x + \theta N^{-2}_o a)|^3 \leq K[F(1 - F)]^2\]

Also, \( \gamma F + (1 - \gamma) H^a \geq \lambda_o F \) and \( 1 - \gamma F + (1 - \gamma) H^a \geq \lambda_o (1 - F) \). Hence by condition (6),

\[J''(\gamma F + (1 - \gamma) H^a) \leq K[(\gamma F + (1 - \gamma) H^a)(1 - \gamma F + (1 - \gamma) H^a)]^{3+6}\]

\[\leq K[F(1 - F)]^{-3+8}\]

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Combining these results, we get

\[ \left| \int_{\zeta}^{a} \zeta \, d\zeta \right| \leq \int_{\zeta}^{a} |\zeta| \, d\zeta \]

\[ \leq K \int \left[ (1-F) \right]^{\frac{2}{3} - 1 + \delta} \, dF + K \int \left[ (1-F) \right]^{\frac{2}{3} + \frac{2}{3} - 2 + \delta} \, dF \]

(2.163)

\[ + \frac{K}{N^1/2} \int \left[ (1-F) \right]^{\frac{4}{3} - 2 + \delta} \, dF + K \int \left[ (1-F) \right]^{2 - 3 + \delta} \, dF \]

\[ \leq K \cdot \]

Since \( \zeta^b \) is of the same form, \( \int |\zeta^b| \, d\zeta \) is similarly uniformly bounded.
\[
\frac{1}{N O_B(F)} \left[ (1-\lambda_N) \sum_{i=1}^{m} (a^{\xi_N(X_i)} - b^{\xi_N(X_i)}) + \lambda_N \sum_{i=1}^{n} (a^{\chi_N(Y_i)} - b^{\chi_N(Y_i)}) \right] = o_p(1) \]

It suffices to show that \( \frac{1}{m} \sum_{i=1}^{m} \xi_N^{a}(X_i) \), \( \frac{1}{m} \sum_{i=1}^{m} \xi_N^{b}(X_i) \), \( \frac{1}{n} \sum_{i=1}^{n} \chi_N^{a}(Y_i) \), and \( \frac{1}{n} \sum_{i=1}^{n} \chi_N^{b}(Y_i) \) are each \( o_p(1) \). Since all of these terms are essentially of the same form, we will only prove the result for one of them, and the rest will follow in exactly the same way.

\[
\frac{1}{n} \sum_{i=1}^{n} \chi_N^{a}(Y_i) = o_p(1) \]

From equation (2.30) we get

\[
E_n^a = \int (G_n - F)'(F)dF + N_a^2 (1-\lambda_N) \int (G_n - F)^{''}(F)dF + \frac{1}{n} \sum_{i=1}^{n} (F(Y_i))^F(0) \]

(2.164)

\[
= \int J'(F)'dF \]

\[
+ N_a^2 \frac{1}{n} \sum_{i=1}^{n} \chi_N^{a}(Y_i) \, .
\]

Now

\[
E[\int (G_n - F)'(F)dF] = 0 \, , \, \, E[\int (G_n - F)^{''}(F)dF] = 0 \, ,
\]

and

\[
E[\frac{1}{n} \sum_{i=1}^{n} (F(Y_i))^F(0) - \int J'(F)'dF] = 0 \, .
\]

Also, from (2.22) we have
Thus, taking expectations in (2.164) gives us $E(\sum_{i=1}^{n} \chi_{N}^{a}(Y_{i})) = 0$ and hence

$$E(\chi_{N}^{a}(Y_{i})) = 0, \ i = 1, \ldots, n.$$  

By Lemma 8, a sufficient condition for $\frac{1}{n} \sum_{1}^{n} \chi_{N}^{a}(Y_{i}) = o_{p}(1)$ is that $\chi_{N}^{a}(Y_{i})$ be uniformly integrable in $N$.

In fact, we will show that $\{\chi_{N}^{a}(Y), N = 1, 2, \ldots\}$ are uniformly bounded by an integrable random variable.

From (2.30),

$$\chi_{N}^{a}(Y) = \frac{1-\lambda_{N}}{2} \int (G_{1} - F) R_{N}^{a} dF - (1-\lambda_{N}) \int J''(H^{**}) f(x + \mu N_{o}^{a}) f dF - \frac{1}{2} \int J'(H^{a}) f'(x + \beta N_{o}^{a}) a dF$$

$$+ (1-\lambda_{N}) J''(H^{**}(Y)) f(Y + \mu N_{o}^{a}) f(Y) - \frac{1}{2} D_{N,1}^{a}(Y) - \frac{1}{2} D_{N,2}^{a}(Y)$$  

(2.166)

where $G_{1}$ is the empirical c.d.f. of the single observation $Y$, $H^{*}, H^{**}$ both lie between $F$ and $H^{a}, R_{N}^{a}$ is as defined in (2.29) and the $D$'s are as defined following (2.22).
Consider the first term on the RHS in (2.166), \( \int (G_1 - F) R_N^a dF \). From (2.29),

\[
R_N^a = J''(F) f'(x + \beta N_o^{-a}) + (1 - \lambda N_o) J''(H^*)(f + \frac{N_o^{-a}}{2}) f'(x + \beta N_o^{-a})^2
\]

\[
\therefore |R_N^a| \leq K[F(1-F)]^{-2 + \frac{2}{3} + \delta} + K[F(1-F)]^{-3 + \frac{4}{3} + \delta}, \text{ for } N_o \text{ sufficiently large}
\]

\[
\leq K[F(1-F)]^{-\frac{5}{3} + \delta}
\]

(2.167)

\[
\therefore |\int (G_1 - F) R_N^a dF| \leq |G_1 - F||R_N^a| dF
\]

\[
\leq \int |G_1 - F|[F(1-F)]^{-\frac{5}{3} + \delta} \text{ dF, which is a r.v. independent of } N.
\]

\[
E \int |G_1 - F|[F(1-F)]^{-\frac{5}{3} + \delta} \text{ dF} = 2 \int [F(1-F)][F(1-F)]^{-\frac{5}{3} + \delta} \text{ dF} \leq K
\]

(2.168)

Consider the second term in (2.166), \( \int J''(H^{**}) f(x + \mu N_o^{-a}) f dF \), which is non-stochastic. We want to bound it uniformly in \( N_o \).

\[
|\int J''(H^{**}) f(x + \mu N_o^{-a}) f dF| \leq \int |J''(H^{**})| f(x + \mu N_o^{-a}) f dF
\]

\[
\leq K \int [F(1-F)]^{-2 + \frac{1}{3} + \delta} \text{ dF for } N_o \text{ sufficiently large}
\]

\[
\leq K.
\]

(2.169)
The third term in (2.166), \[ \int J'(H^a) f'(x+\frac{1}{2} a) dF, \] is also non-stochastic.

\[ \left| \int J'(H^a) f'(x+\frac{1}{2} N_o a) dF \right| \leq \int |J'(H^a)| \left| f'(x+\frac{1}{2} N_o a) \right| dF \]

\[ \leq K \int [F(1-F)]^{-1+\frac{2}{3}+\delta} dF, \text{ for } N_o \text{ sufficiently large} \]

(2.170) \[ \leq K. \]

Consider the fourth term, \[ J''(H^{**}(y)) f(y+\frac{1}{2} N_o a) f(y). \]

\[ \left| J''(H^{**}(y)) f(y+\frac{1}{2} N_o a) f(y) \right| \]

\[ \leq K[F(1-F)]^{-2+\frac{4}{3}+\delta} \text{ for } N_o \text{ sufficiently large}, \]

which is independent of N.

(2.171) \[ E K[F(1-F)]^{\frac{2}{3}+\delta} = K \int [F(1-F)]^{\frac{2}{3}+\delta} dF \leq K. \]

From (2.22),

\[ D_{N_o}^a(Y) = f(y+(1-\delta) N_o a) J'(H^a(y-\frac{1}{2} N_o a)) f(y-\frac{1}{2} N_o a) \]

(2.172) \[ + F(y_1+(1-\delta) N_o a) J''(H^a(y_1-\frac{1}{2} N_o a)) f(y_1-\frac{1}{2} N_o a) \]

\[ + F(y_1+(1-\delta) N_o a) J'(H^a(y_1-\frac{1}{2} N_o a)) f'(y_1-\frac{1}{2} N_o a) \]

Now \( H^a(y-\frac{1}{2} N_o a) \) lies between \( H^a(y) \) and \( H^a(y-N_o \frac{1}{2} a) = (1-\lambda_N) F(Y) + \lambda_N F(Y-N_o \frac{1}{2} a). \)
But 
\[ H^a(Y-N^{-1/2}_0a) \geq (1-\lambda^a_Y)F(Y) \geq \lambda^a_Y F(Y) \] 
and 
\[ 1-H^a(Y-N^{-1/2}_0a) \geq \lambda^a_Y (1-F(Y)). \]

Hence 
\[ H^a(Y-N^{-1/2}_0a)(1-H^a(Y-N^{-1/2}_0a)) \geq \lambda^a_Y F(Y)(1-F(Y)). \]

Also 
\[ H^a(Y)(1-H^a(Y)) \geq \lambda^a_Y F(Y)(1-F(Y)). \]

Therefore

\[ (2.173) \quad H^a(Y-N^{-1/2}_0a)(1-H^a(Y-N^{-1/2}_0a)) \geq \min\{H^a(Y)(1-H^a(Y)), H^a(Y-N^{-1/2}_0a)(1-H^a(Y-N^{-1/2}_0a))\} \]

\[ \geq \lambda^a_Y F(Y)(1-F(Y)). \]

Hence for \( N_0 \) sufficiently large, we have,

\[ |D^a_{N,1}(Y)| \leq K[F(Y)(1-F(Y))]^{-1+\frac{4}{3}+\delta} + K[F(Y)(1-F(Y))]^{-2+\frac{4}{3}+\delta} + K[F(Y)(1-F(Y))]^{-1+\frac{2}{3}+\delta} \]

\[ \leq K[F(Y)(1-F(Y))]^{-\frac{2}{3}+\delta}, \] which is independent of \( N \).

\[ (2.174) \quad E K[F(Y)(1-F(Y))]^{-\frac{2}{3}+\delta} = K \int [F(1-F)]^{-\frac{2}{3}+\delta} \, dF \leq K \]

In a similar manner we can show that \( \{|D^a_{N,2}(Y)|\} \) are uniformly bounded by an integrable random variable.

Hence, combining these results in (2.166), we have that \( \{|X^a_N(Y)|\} \) are uniformly bounded by an integrable random variable, and hence uniformly integrable. This completes the discussion of the remainder terms.
2.4 Two Special Cases of Theorem 1.

The hypothesis of Theorem 1 involves a condition on the sequence of score functions $\{J_N(u)\}$ which converges to $J(u)$, namely condition (7). We will now show that this condition is indeed satisfied for two common forms of $\{J_N\}$.

Theorem 2. For the cases

(i) \[ J_N\left(\frac{1}{N}\right) = J\left(\frac{1}{N+1}\right) \]

and

(ii) \[ J_N\left(\frac{1}{N}\right) = J\left(\frac{1-\frac{1}{2}}{N}\right) , \]

under conditions (1) to (6) of Theorem 1, we have condition (7) of Theorem 1 satisfied, and thus the conclusion of Theorem 1 holds.

Proof of Theorem 2. We want to show that

\[ \frac{1}{m} \sum_{i=1}^{N} \left( J_N\left(\frac{1}{N}\right) - J\left(\frac{1}{N}\right) \right) \left( w_i(X_i - X_N) - \frac{1}{2} \right) = o_P(N^{-1}) \]

Let us denote this quantity by $\mathcal{J}$. Then

(2.175) \[ \mathcal{J} = \int [J_N^{(a)}(H^a_N) - J^{(a)}(H^a_N) - J_N^{(b)}(H^b_N) + J^{(b)}(H_N)] \, dF_m . \]

Case (i): \[ J_N\left(\frac{1}{N}\right) = J\left(\frac{1}{N+1}\right) . \]

\[ \mathcal{J} = \int [J\left(\frac{N}{N+1}\right)^{(a)}(H^a_N) - J^{(a)}(H^a_N) - J\left(\frac{N}{N+1}\right)^{(b)}(H^b_N) + J^{(b)}(H_N)] \, dF_m . \]
Let us partition $\mathcal{I}$ into $\mathcal{I}_1 = \int_{H_N^a < 1} I_N^a \mathrm{d}F$ and $\mathcal{I}_2 = \int_{H_N^a = 1} I_N^a \mathrm{d}F$. We will show that $\mathcal{I}_1$ and $\mathcal{I}_2$ are each $o_p(N^{-1})$.

Now by Lemma 1, \exists \eta > 0 \exists \mathcal{I}$ for $N$ sufficiently large, $P(X_{1:i} \in I_N^a(\eta), i = 1, \ldots, m) > 1 - \frac{\varepsilon}{2}$, where $I_N^a(\eta) = \{x \mid F(1-F) > \frac{\eta}{N}\}$. Also, by a result mentioned previously, \exists \eta^* > 0 \exists \mathcal{I}$ such that $P(H_N^a(l-h_N^a) > \eta H_N^a(l-h_N^a)$ on $0 < H_N^a < l) > 1 - \frac{\varepsilon}{2}$. Let this event be denoted by $A_N(\eta^*)$. Then

$$\mathcal{I}_1 = \int_{H_N^a < 1} [J(\frac{N}{N+1} H_N^a) - J(H_N^a) - J(\frac{N}{N+1} H_N^b) + J(H_N^b)] \mathrm{d}F$$

with probability $> 1 - \varepsilon$.

Let the integral above be denoted by $\mathcal{I}^*$. Then to show that $\mathcal{I}_1 = o_p(N^{-1})$, it suffices to show $\mathcal{I}^*_1 = o_p(N^{-1})$. Now

$$|\mathcal{I}^*_1| = \int_{H_N^a < 1} \left[ - \frac{H_N^a}{N+1} J'(\frac{N+\theta}{N+1} H_N^a) + \frac{H_N^b}{N+1} J'(\frac{N+\theta}{N+1} H_N^b) \right] \mathrm{d}F,$$

where $0 < \theta, \alpha < 1$.

(2.176)

$$\leq \int_{H_N^a < 1} \frac{H_N^a - H_N^b}{N+1} \left| J'(\frac{N+\theta}{N+1} H_N^a) \right| \mathrm{d}F + \int_{H_N^a < 1} \frac{H_N^b}{N+1} \left| J'(\frac{N+\theta}{N+1} H_N^a) - J'(\frac{N+\alpha}{N+1} H_N^a) \right| \mathrm{d}F.$$

$$\leq I_N^a(\eta) A_N(\eta^*) \mathcal{I}_1 + \mathcal{I}_2.$$
\[
\frac{N+\theta}{N+1} H_N^a(1 - \frac{N+\theta}{N+1} H_N^a) \geq \frac{1}{2} H_N^a (1 - H_N^a)
\]

\[
(2.177) \quad \geq \frac{\eta^*}{2} H_N^a (1 - H_N^a), \quad \text{on } A_N(\eta^*)
\]

\[
\geq \frac{\eta^* \lambda^2}{2} F(1-F)
\]

Hence

\[
(2.178) \quad J_1 \leq \frac{K}{N_0} \int_{I_N(\eta)} (G_n(x+N_0 2a) - G_n(x+N_0 2b)) [F(1-F)]^{-1+\delta} \, dF_m
\]

\[
E(J_1) = E(E(J_1 | X)) \leq \frac{K}{N_0} E \int_{I_N(\eta)} E(G_n(x+N_0 2a) - G_n(x+N_0 2b)) [F(1-F)]^{-1+\delta} \, dF_m
\]

\[
= \frac{K}{N_0} \int_{I_N(\eta)} (F(x+N_0 2a) - F(x+N_0 2b)) [F(1-F)]^{-1+\delta} \, dF
\]

\[
(2.179) \quad \leq \frac{K}{N_0^{3/2}} \int [F(1-F)]^{-1+\delta} \, dF \quad \text{for } N_0 \text{ sufficiently large}
\]

\[
\leq \frac{K}{N_0^{3/2}}.
\]

Consider \( J_2 \).
\[ J_2 = \int_{H_N^a < 1} \left| J'\left(\frac{N+\theta}{N+1} H_N^a\right) - J'\left(\frac{N+\alpha}{N+1} H_N^b\right) \right| dF_m \]

\[ \leq \int_{H_N^a < 1} \frac{H_N^b}{N+1} \left| \frac{N+\theta}{N+1} H_N^a - \frac{N+\alpha}{N+1} H_N^b \right| |J''(\tilde{H}_N)| dF_m, \text{ where } \tilde{H}_N \text{ lies between } \frac{N+\theta}{N+1} H_N^a \text{ and } \frac{N+\alpha}{N+1} H_N^b. \]

(2.180)

Now

\[ \left| \frac{N+\theta}{N+1} H_N^a - \frac{N+\alpha}{N+1} H_N^b \right| \leq \frac{N+\theta}{N+1} (H_N^a - H_N^b) + \frac{\theta - \alpha}{N+1} H_N^b \leq (H_N^a - H_N^b) + \frac{H_N^b}{N}. \]

and

\[ \frac{N}{N+1} H_N^b \leq \tilde{H}_N \leq H_N^a. \]

Hence

\[ \tilde{H}_N(1-\tilde{H}_N) \geq \min\left\{ \frac{N}{N+1} H_N^b(1 - \frac{N}{N+1} H_N^b), H_N^a(1-H_N^a) \right\} \]

\[ \geq \frac{1}{2} \min\{H_N^b(1-H_N^b), H_N^a(1-H_N^a)\} \]

\[ \geq \frac{1}{2} \eta^* F(1-F) \text{ on } 0 < H_N^b, H_N^a < 1, \text{ on } A_N(\eta^*). \]

Hence, substituting in (2.180), we get
\[ J_2 \leq \frac{K}{N_0} \int \frac{H_N^{b}(H_N^{a} - H_N^{b})}{I_N(\eta)} [F(1-F)]^{-2+8} dF_m + \frac{K}{N_0^2} \int \frac{(H_N^{b})^2}{I_N(\eta)} [F(1-F)]^{-2+8} dF_m \]

\[
(2.181) \leq \frac{K}{N_0} \int \frac{1}{I_N(\eta)} [G_n(x+N_0 a) - G_n(x+N_0 b)][F(1-F)]^{-2+8} dF_m + \frac{K}{N_0^2} \int \frac{1}{I_N(\eta)} [F(1-F)]^{-2+8} dF_m
\]

\[ E(J_2) \leq \frac{K}{N_0} \int \frac{1}{I_N(\eta)} [F(x+N_0 a) - F(x+N_0 b)][F(1-F)]^{-2+8} dF + \frac{K}{N_0^2} \int \frac{1}{I_N(\eta)} [F(1-F)]^{-2+8} dF \]

\[
\leq \frac{K}{N_0^{3/2}} \int \frac{2^{-2+8}}{I_N(\eta)} dF + \frac{K}{N_0^{2-1+8}} \text{ for } N_0 \text{ sufficiently large.}
\]

\[
(2.182) \leq \frac{K}{N_0^{3/2-1/3+8}} + \frac{K}{N_0^{1+8}}
\]

\[
\leq \frac{K}{N_0^{1+8}}
\]

From (2.176), (2.179), and (2.182) we conclude that

\[
(2.183) \quad E|J_{1}^*| \leq \frac{K}{N_0^{1+8}}
\]

Then \( J_{1}^* = o_p(N^{-1}) \) and hence by the earlier remarks we have

\[
(2.184) \quad J_{1} = o_p(N^{-1}).
\]

Now consider
\[ J_2 = \int_{H_N^a=1}^{H_N^b} \left[ J\left( \frac{N}{N+1} H_N^a \right) - J(H_N^a) - J\left( \frac{N}{N+1} H_N^b \right) + J(H_N^b) \right] dF_m \]

(2.185)

\[ = \int_{H_N^a=1}^{H_N^b} \left[ J\left( \frac{N}{N+1} \right) - J(1) - J\left( \frac{N}{N+1} H_N^b \right) + J(H_N^b) \right] dF_m. \]

Now \( F_m \) has no jumps in the region \( H_N^a=1 \) unless \( X_{\text{max}} > Y_{\text{max}} - \frac{1}{2} a \).

Then

\[ J_2 = \begin{cases} J\left( \frac{N}{N+1} \right) - J(1) - J\left( \frac{N}{N+1} H_N^b(X_{\text{max}}) \right) + J(H_N^b(X_{\text{max}})) & \text{if } X_{\text{max}} \geq Y_{\text{max}} - \frac{1}{2} a \\ 0 & \text{otherwise} \end{cases} \]

(2.186)

But \( H_N^b(X_{\text{max}}) = 1 \) unless \( X_{\text{max}} < Y_{\text{max}} - \frac{1}{2} b \).

Thus \( |J_2| > 0 \) only if \( Y_{\text{max}} - \frac{1}{2} b < X_{\text{max}} < Y_{\text{max}} - \frac{1}{2} a \). But we saw already in Section 2.3 in considering \( c_1^a - c_1^b \) (equations (2.61) and (2.62)) that the probability of this event converges to 0. Hence \( J_2 = o_p(N^{-1}) \) \( \forall r \), in particular

(2.187)

\[ J_2 = o_p(N^{-1}) \]

Combining (2.184) and (2.187) gives the result

(2.188)

\[ J = o_p(N^{-1}) \]
Case (ii). \( J_N(\frac{1}{N}) = \int \left[ \frac{1}{2N} \right] \).

\[
\mathcal{J} = \int [J(H^a_N - \frac{1}{2N}) - J(H^b_N - \frac{1}{2N}) - J(H^a_N - \frac{1}{2N}) - J(H^b_N - \frac{1}{2N})]dF_m
\]

(2.189) \[= \mathcal{J}_1 + \mathcal{J}_2, \text{ where } \mathcal{J}_1 = \int_{H^a_N < 1}, \mathcal{J}_2 = \int_{H^a_N = 1} \]

Now \( \mathcal{J}_2 = o_p(N^{-1}) \) follows exactly as in Case (i). To show \( \mathcal{J}_1 = o_p(N^{-1}) \), it suffices to show that \( \mathcal{J}^* = o_p(N^{-1}) \), where

(2.190) \[\mathcal{J}^*_1 = \int_{H^a_N < 1} \frac{1}{2N} \int_{I_N(\eta)} [J(H^a_N - \frac{1}{2N}) - J(H^b_N - \frac{1}{2N}) - J(H^b_N - \frac{1}{2N})]dF_m \]

where \( I_N(\eta) \) and \( A_N(\eta^*) \) are as in Case (i).

Letting \( \mathcal{R}_N \) be the region of integration in (2.190), we have

\[\mathcal{J}^*_1 = \int_{\mathcal{R}_N} \left\{ -\frac{1}{2N} J'(H^a_N - \frac{\alpha}{2N}) + \frac{1}{2N} J'(H^b_N - \frac{\beta}{2N}) \right\}dF_m \text{ where } \theta < \alpha, \beta < 1.\]

\[-\frac{1}{2N} \int_{\mathcal{R}_N} \frac{1}{2N} (H^a_N - \frac{\alpha}{2N})J''(H^*_N)dF_m \text{ where } H^*_N \text{ lies between} \]

(2.191) \[H^a_N - \frac{\alpha}{2N} \text{ and } H^b_N - \frac{\beta}{2N} \]

\[-\frac{1}{2N} \int_{\mathcal{R}_N} \left\{ (1 - \lambda_N)(g_n(x + N \frac{\ldots}{2} a) - g_n(x + N \frac{\ldots}{2} b)) - \frac{\alpha - \beta}{2N} J''(H^*_N) \right\}dF_m \]
Now \( H_N^* \) lies between \( H_N^a - \frac{1}{2N} \) and \( H_N^b - \frac{1}{2N} \).

\[ (2.192) \quad \Rightarrow \quad H_N^b - \frac{1}{2N} < H_N^* < H_N^a. \]

Now on \( 0 < H_N^b, H_N^b - \frac{1}{2N} > 0 \) \( H_N^b \). Also \( 1 - H_N^b + \frac{1}{2N} > 1 - H_N^b \). Hence, on \( 0 < H_N^b < 1 \)

\[ (H_N^b - \frac{1}{2N})(1 - H_N^b + \frac{1}{2N}) \geq \frac{1}{2} H_N^b(1 - H_N^b) \geq \frac{\eta_*^2}{2} H_N^b(1 - H_N^b), \text{ on } A_N(\eta^*) \]

\[ (2.193) \]

\[ \frac{\eta_*^2}{2} \leq \frac{1}{2} F(1 - F). \]

Also, on \( 0 < H_N^a < 1 \),

\[ H_N^a(1 - H_N^a) \geq \eta_*^2 H_N^a(1 - H_N^a), \text{ on } A_N(\eta^*) \]

\[ (2.194) \]

\[ \geq \frac{\eta_*^2}{2} F(1 - F). \]

From (2.192)-(2.194) we conclude that on \( 0 < H_N^b, H_N^a < 1 \), we have

\[ (2.195) \quad H_N^*(1 - H_N^*) \geq \frac{\eta_*^2}{2} F(1 - F), \text{ on } A_N(\eta^*). \]

Returning to (2.191),

\[ E|g^*| \leq K_N \cdot E \sum_{N} \left( |G_n(x + N \cdot a) - G_n(x + N \cdot b)| \right) \left| J''(H_N^*) \right| dF_m \]

\[ + \frac{K_N}{2} \cdot E \sum_{N} \left| J''(H_N^*) \right| dF_m \]

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\[ \leq \frac{K}{N_0} \int \frac{1}{I_N(\eta)} \left( F(x+N_0^{-1/2}a) - F(x+N_0^{-1/2}b) \right) [F(1-F)]^{-2+\delta} \, dF + \frac{K}{N_0^2} \int \frac{1}{I_N(\eta)} [F(1-F)]^{-2+\delta} \, dF \]

\[ \leq \frac{K}{N_0^{3/2}} \int \frac{1}{I_N(\eta)} [F(1-F)]^{-2+\delta} \, dF + \frac{K}{N_0^{2-1+\delta}} \quad \text{for } N_0 \text{ sufficiently large} \]

\[ \leq \frac{K}{N_0^{3/2-1/3+\delta}} + \frac{K}{N_0^{1+\delta}} \]

\[ \leq \frac{K}{N_0^{1+\delta}} \]

(2.196)

\[ \therefore \mathcal{J}_1^* = o_p(N^{-1}) \quad \text{and thus} \]

(2.197) \[ \mathcal{J}_1 = o_p(N^{-1}). \]

This combined with \( \mathcal{J}_2 = o_p(N^{-1}) \) gives us \( \mathcal{J} = o_p(N^{-1}). \) \( \text{Q.E.D.} \)
2.5 Asymptotic Joint Distribution of Lengths of Several Confidence Intervals.

In order to study the properties of a flexible procedure as outlined in Section 1.5, we need to know something about the joint distribution of the lengths of several confidence intervals, each based on a different rank test. Theorems 1 and 2 provide us a means of approximating the length distribution marginally for each of a class of confidence intervals, but we need an approximation for the joint distribution of lengths of several intervals. This is provided by the next result.

Let $T_{N,1}$ and $T_{N,2}$ be two linear rank statistics, defined as in the preliminaries to Theorem 1. Let the corresponding confidence intervals of asymptotic level $1-\alpha$ have lengths $\mathcal{L}_{N,1}$ and $\mathcal{L}_{N,2}$.

**Theorem 3.** If the conditions of Theorem 1 hold for both $T_{N,1}$ and $T_{N,2}$ and $\lambda_N \to \lambda$, $0 < \lambda < 1$, as $N \to \infty$, then we have

$$
N\lambda(1-\lambda) \left( \begin{array}{c} \mathcal{L}_{N,1} - \frac{2z_{\alpha/2} A_1}{N_{1/2} B_1(F)} \\ \frac{2z_{\alpha/2} A_1}{B_1^2(F)} \end{array} \right), \quad \left( \begin{array}{c} \mathcal{L}_{N,2} - \frac{2z_{\alpha/2} A_2}{N_{1/2} B_2(F)} \\ \frac{2z_{\alpha/2} A_2}{B_2^2(F)} \end{array} \right) \mathcal{N}_2 \left( \begin{array}{cc} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{array} \right)
$$

(2.198)

as $N \to \infty$,

where
\[
\sigma_i^2 = \lambda(1-\lambda)\left\{ \int (q_1-2j_1(F)f)^2 dF - \left(\int (q_1-2j_1(F)f)\right)^2 \right\}, \quad i=1,2
\]

\[
+ (1-4\lambda(1-\lambda))\left\{ \int (j_1(F)f)^2 dF - \left(\int j_1(F)f dF\right)^2 \right\}, \quad i=1,2
\]

\[
\sigma_{12} = \lambda(1-\lambda)\left\{ \int (q_1-2j_1(F)f)(q_2-2j_2(F)f)dF - \int (q_1-2j_1(F)f)dF \int (q_2-2j_2(F)f)dF \right\}
\]

\[
+ (1-4\lambda(1-\lambda))\left\{ \int j_1(F)f^2 dF - \int j_1(F)f dF \int j_2(F)f dF \right\}
\]

and

\[
Q_i(x) = \int_0^x J_1^*(F) J_2(F) f(y) dF(y), \quad i=1,2.
\]

\[(2.199)\]

**Proof.** In order to prove the joint asymptotic normality of

\[
N\left( \mathcal{L}_{N,1}^2 - \frac{2z\alpha/2}{N_0^{1/2}B_1(F)} \right) \quad \text{and} \quad N\left( \mathcal{L}_{N,2}^2 - \frac{2z\alpha/2}{N_0^{1/2}B_2(F)} \right),
\]

it suffices by the Cramer-Wold theorem to prove the asymptotic normality of arbitrary linear combinations of these two random variables. Now from equation (2.54) in the proof of Theorem 1, we have

\[(2.200)\]

\[
N_0\left( \mathcal{L}_{N,1}^2 - \frac{2z\alpha/2}{N_0^{1/2}B_1(F)} \right) = -\frac{N_0^{1/2}}{B_1(F)} \mathcal{L}_{N,1} B_1(\chi, \chi) + o_p(1), \quad i=1,2.
\]

Dividing both sides by \( \lambda_N(1-\lambda_N) \), we get

\[(2.201)\]

\[
N\left( \mathcal{L}_{N,1} - \frac{2z\alpha/2}{N_0^{1/2}B_1(F)} \right) = -\frac{N_0^{1/2}}{\lambda(1-\lambda)B_1(F)} \frac{\lambda_N(1-\lambda)}{\lambda_N(1-\lambda_N)} \mathcal{L}_{N,1} B_1(\chi, \chi) + o_p(1), \quad i=1,2.
\]

Now from (2.43) we have
(2.202) \[ B_1(Y_i, Y_j) = (1-\lambda_N) \left[ \frac{1}{2} m \sum_{j=1}^{m} S_1(Y_j) + \frac{1}{2} n \sum_{j=1}^{n} T_1(Y_j) \right], \quad i=1,2 \]

where

\[ S_1(X) = \lambda_N \int j_1''(F) (F_1 - F) f dF + J_1'(F(X)) f(X) - \int J_1'(F) f dF, \quad i=1,2 \]

and

\[ T_1(X) = (1-\lambda_N) \int j_1''(F) (F_1 - F) f dF + J_1'(F(X)) f(X) - \int J_1'(F) f dF, \]

\( F_1 \) being the empirical c.d.f. of the observation \( X \).

Thus, letting \( u \) and \( v \) be arbitrary nonzero constants, we have

\[
\begin{align*}
&N \left[ u \left( \frac{2z_{\alpha/2} A_1}{N_1^{1/2} B_1(F)} \right) + v \left( \frac{2z_{\alpha/2} A_2}{N_2^{1/2} B_2(F)} \right) \right] \\
&= -\frac{\lambda (1-\lambda_N)}{\lambda_N} \left\{ \frac{N_1^{1/2}}{\lambda (1-\lambda) B_1(F)} \left[ (1-\lambda_N) \frac{1}{2} m \sum_{j=1}^{m} u S_1(Y_j) + \frac{1}{2} \sum_{j=1}^{n} u T_1(Y_j) \right] \\
&\quad + \frac{N_2^{1/2}}{\lambda (1-\lambda_N) B_2(F)} \left[ (1-\lambda_N) \frac{1}{2} m \sum_{j=1}^{m} v S_2(Y_j) + \frac{1}{2} \sum_{j=1}^{n} v T_2(Y_j) \right] \right\} \\
&\quad + o_p(1).
\end{align*}
\]

\[
\begin{align*}
&= -\frac{\lambda (1-\lambda_N)}{\lambda_N} \left\{ \frac{2z_{\alpha/2} A_1}{\lambda (1-\lambda) B_1(F)} \left[ (1-\lambda_N) \frac{1}{2} m \sum_{j=1}^{m} u S_1(Y_j) + \frac{1}{2} \sum_{j=1}^{n} u T_1(Y_j) \right] \\
&\quad + \frac{2z_{\alpha/2} A_2}{\lambda (1-\lambda) B_2(F)} \left[ (1-\lambda_N) \frac{1}{2} m \sum_{j=1}^{m} v S_2(Y_j) + \frac{1}{2} \sum_{j=1}^{n} v T_2(Y_j) \right] \right\} \\
&\quad + o_p(1).
\end{align*}
\]
\[
- \frac{\lambda(1-\lambda)}{\lambda_n(1-\lambda_n)} \left( \frac{N_0^{1/2}}{\lambda(1-\lambda)B_1(F)} \frac{A_1}{\lambda(1-\lambda)B_2(F)} \right) \left( (1-\lambda_n)^{1/2} \frac{1}{m} \sum_{j=1}^{m} u S_1(X_j) \right) + \frac{1}{n} \frac{1}{2} \sum_{j=1}^{n} u T_1(Y_j) \\
+ \frac{1}{n} \frac{1}{2} \sum_{j=1}^{n} u T_2(Y_j) \\
- \frac{\lambda(1-\lambda)}{\lambda_n(1-\lambda_n)} \left( \frac{N_0^{1/2}}{\lambda(1-\lambda)B_1(F)} \frac{A_2}{\lambda(1-\lambda)B_2(F)} \right) \left( (1-\lambda_n)^{1/2} \frac{1}{m} \sum_{j=1}^{m} v S_2(X_j) \right) + \frac{1}{n} \frac{1}{2} \sum_{j=1}^{n} v T_2(Y_j) \\
+ o_p(1)
\]

Now by Sen's theorem,
\[
\frac{N_0^{1/2}}{\lambda(1-\lambda)B_1(F)} \frac{A_1}{\lambda(1-\lambda)B_2(F)} = o_p(1), \quad i=1,2.
\]

Also, by the Central Limit Theorem,
\[
(1-\lambda_n)^{1/2} \frac{1}{m} \sum_{j=1}^{m} u S_1(X_j) + \frac{1}{n} \frac{1}{2} \sum_{j=1}^{n} v T_1(Y_j)
\]
and
\[
(1-\lambda_n)^{1/2} \frac{1}{m} \sum_{j=1}^{m} v S_2(X_j) + \frac{1}{n} \frac{1}{2} \sum_{j=1}^{n} v T_2(Y_j)
\] are each \( o_p(1) \).

Hence, we have
\begin{align*}
N \left[ \frac{\mathcal{L}_{N,1} - \frac{2\alpha/2}{N^{1/2}} A_1}{B_1(F)} \right] + \frac{\mathcal{L}_{N,2} - \frac{2\alpha/2}{N^{1/2}} A_2}{B_2(F)} \right] \\
= - \frac{\lambda(1-\lambda)}{\lambda_N(1-\lambda_N)} \left\{ \frac{2\alpha/2}{\lambda(1-\lambda)E_1^2(F)} \left( (1-\lambda_N)^{1/2} \sum_{j=1}^{m} u S_1(X_j) + \lambda^2 \sum_{j=1}^{n} u T_1(Y_j) \right) \\
+ \frac{2\alpha/2}{\lambda(1-\lambda)E_2^2(F)} \left( (1-\lambda_N)^{1/2} \sum_{j=1}^{m} v S_2(X_j) + \lambda^2 \sum_{j=1}^{n} v T_2(Y_j) \right) \right\} \\
+ o_p(1)
\end{align*}

(2.204)

\begin{align*}
= - \frac{\lambda(1-\lambda) 2\alpha/2}{\lambda_N(1-\lambda_N) \lambda(1-\lambda)} \left\{ (1-\lambda_N)^{1/2} \sum_{j=1}^{m} \left( u \frac{A_1}{E_1^2(F)} S_1(X_j) + v \frac{A_2}{E_2^2(F)} S_2(X_j) \right) \\
+ \lambda^2 \sum_{j=1}^{n} \left( u \frac{A_1}{E_1^2(F)} T_1(Y_j) + v \frac{A_2}{E_2^2(F)} T_2(Y_j) \right) \right\} + o_p(1).
\end{align*}

Let

(2.205) \quad \sigma^2 = \text{Var} \left( u \frac{A_1}{E_1^2(F)} S_1(X) + v \frac{A_2}{E_2^2(F)} S_2(X) \right)

and

(2.206) \quad \tau^2 = \text{Var} \left( u \frac{A_1}{E_1^2(F)} T_1(Y) + v \frac{A_2}{E_2^2(F)} T_2(Y) \right).

Then by the Central Limit Theorem, and using the fact that \( \frac{\lambda(1-\lambda)}{\lambda_N(1-\lambda_N)} \rightarrow 1 \),

we have
\[ N\lambda(1-\lambda)\left[ u\left( L_{N,1} - \frac{2z_0/2}{\sqrt{N - A_1}} B_1(F) \right) + v\left( L_{N,2} - \frac{2z_0/2}{\sqrt{N - A_2}} B_2(F) \right) \right] / \left[ (1-\lambda_N)^2 + \lambda_N^2 \right]^{1/2} \]

(2.207)

\[ \mathcal{L}_{N,0} \sim N(0, 4z_0^2) \text{ as } N \to \infty. \]

All that remains is to evaluate the asymptotic covariance matrix. The asymptotic variances follow from Theorem 1, noting that \( \lambda_N \to \lambda \). In order to obtain the covariance of the asymptotic distribution, we need to first evaluate \( \text{Cov}(T_1(Y), T_2(Y)) \) and \( \text{Cov}(S_1(X), S_2(X)) \). Now

\[ T_1(Y) = (1-\lambda_N) \int (G_1-F)j''_1(F)df + J'_1(F(Y))f(Y) - \int J'_1(F)df \]

(2.208) where \( G_1 \) is the empirical c.d.f. of the observation \( Y \)

\[ \text{E}(T_1(Y)) = 0 \]

\[ \therefore \quad \text{Cov}(T_1(Y), T_2(Y)) = \]

\[ \text{E}(T_1(Y)T_2(Y)) = \]

\[ = (1-\lambda_N)^2 \int \int (G_1(x)-F(x))(G_1(y)-F(y))j''_1(F(x))j''_2(F(y))f(x)f(y)dF(x)dF(y) \]

\[ + (1-\lambda_N)\text{E}[J'_1(F(Y))f(Y)] \int (G_1-F)j''_2(F)df \]

(2.209)

\[ + (1-\lambda_N)\text{E}[J'_2(F(Y))f(Y)] \int (G_1-F)j''_1(F)df \]

\[ + \text{E}[J'_1(F(Y))f(Y)J'_2(F(Y))f(Y)] - \int J'_1(F)df \cdot \int J'_2(F)df \]

\[ = (1-\lambda_N)^2 E_1 + (1-\lambda_N)E_2 + (1-\lambda_N)E_3 + \int J'_1(F)j'_2(F)df - \int J'_1(F)df \cdot \int J'_2(F)df \]

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Now, for $x < y$

$$\mathbb{E}|G_1(x)-F(x)||G_1(y)-F(y)| = F(x)F(y)(1-F(y)) + F(x)(1-F(y))(F(y)-F(x)) + (1-F(x))(1-F(y))F(x)$$

$$(2.210)$$

$$= F(x)(1-F(y))(1+2(F(y)-F(x)))$$

$$\leq 3 F(x)(1-F(y)).$$

Hence

$$\iint \mathbb{E}|G_1(x)-F(x)||G_1(y)-F(y)||J_1''(F(x))|J_2''(F(y))|f(x)f(y)dF(x)dF(y)$$

$$\leq K \iint_{x < y} F(x)(1-F(y))[F(x)(1-F(x))]^{-2} + \frac{2}{3} + \frac{5}{8}[F(y)(1-F(y))]^{-2} + \frac{2}{3} + \frac{5}{8}dF(x)dF(y)$$

$$< \infty, \text{ by Lemma 6.}$$

Hence, by Fubini's theorem,

$$E_1 = \iint \mathbb{E}[(G_1(x)-F(x))(G_1(y)-F(y))]|J_1''(F(x))|J_2''(F(y))|f(x)f(y)dF(x)dF(y)$$

$$= \iint [F(\min(x,y))-F(x)F(y)]J_1''(F(x))|J_2''(F(y))|f(x)f(y)dF(x)dF(y)$$

$$= \iint_{x < y} F(x)(1-F(y))[J_1''(F(x))|J_2''(F(y))|f(x)f(y)dF(x)dF(y)$$

$$+ \iint_{y < x} F(y)(1-F(x))[J_1''(F(x))|J_2''(F(y))|f(x)f(y)dF(x)dF(y)$$

$$= \iiint_{u < x < y < v} J_1''(F(x))|J_2''(F(y))|f(x)f(y)dF(u)dF(v)dF(x)dF(y)$$

$$+ \iiint_{u < y < x < v} J_1''(F(x))|J_2''(F(y))|f(x)f(y)dF(u)dF(v)dF(x)dF(y)$$

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\[
= \mathcal{J}_1^\prime(F(x))f(x)J_2^\prime(F(x))f(y)df(x)df(y)df(u)df(v)
\]
\[
= \mathcal{J}_1^\prime(F(x))f(x)df(x)\mathcal{J}_2^\prime(F(y))f(y)df(y)df(u)df(v)
\]
\[
= \left[ \mathcal{J}_1^\prime(F(x))f(x)df(x) \right] \left[ \mathcal{J}_2^\prime(F(y))f(y)df(y) \right] df(u)df(v)
\]
\[
= \mathcal{J}_1^\prime(F(x))f(x)df(x)\mathcal{J}_2^\prime(F(y))f(y)df(y)df(u)df(v)
\]
\[
= \frac{1}{2} \iint (Q_1(v) - Q_1(u))(Q_2(v) - Q_2(u))df(u)df(v)
\]
\[
= \frac{1}{2} \left[ \iint Q_1(v)Q_2(v)df(u)df(v) + \iint Q_1(u)Q_2(u)df(u)df(v) - \iint Q_1(u)Q_2(v)df(u)df(v) - \iint Q_1(v)Q_2(u)df(u)df(v) \right]
\]
\[
= \frac{1}{2} \left[ 2 \int Q_1 Q_2 df - 2 \int Q_1 df \cdot \int Q_2 df \right]
\]
\[
(2.211)
= \int Q_1 Q_2 df - \int Q_1 df \int Q_2 df
\]

\[
E_2 = E[J_1^\prime(F(y))f(y)J_2^\prime(F)df]
\]
\[
= E \int J_1^\prime(F(y))f(y)(G_1(x) - F(x))J_2^\prime(F(x))f(x)df(x)
\]
\[
(2.212)
\]

Now
\[
E(J_1^\prime(F(y))f(y)(G_1(x) - F(x))| f(x))
\]
\[
\leq \frac{1}{2} (J_1^\prime(F(y))f(y))^2 E\left( (G_1(x) - F(x))^2 \right)
\]
\[
\leq \frac{1}{2} (K[F(y)(1-F(y))]^{2(-1 + \frac{2}{3} + 8)}[F(x)(1-F(x))]^{\frac{1}{2}})
\]
\[
\leq K[F(x)(1-F(x))]^{\frac{1}{2}}
\]

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Hence

\[
\int E|J_1'(F(Y))f(Y)(G_1(x)-F(x))||J_2''(F(x))|f(x)dF(x) \\
\leq K \int [F(1-F)]^\frac{1}{2} + \frac{2}{3} + \delta dF \\
< \infty .
\]

(2.213)

Thus Fubini's theorem applies again, and we have from (2.212)

\[
E_2 = \int E[J_1'(F(Y))f(Y)(G_1(x)-F(x))]J_2''(F(x))f(x)dF(x) \\
= \int E[J_1'(F(Y))f(Y)(I(Y \leq x)-F(x))]J_2''(F(x))f(x)dF(x) \\
= \int \int x J_1'(F(Y))f(y)(l-F(x))dF(y) \\
- \int x J_1'(F(Y))f(y)F(x)dF(y)J_2''(F(x))f(x)dF(x) \\
= \int \int \int J_1'(F(Y))f(y)J_2''(F(x))f(x)dF(v)dF(x)dF(y) \\
- \int \int \int (Q_2(v)-Q_2(y))J_1'(F(Y))f(y)dF(v)dF(y) \\
= \int \int (Q_2(v)-Q_2(y))J_1'(F(Y))f(y)dF(v)dF(y) \\
- \int \int (Q_2(y)-Q_2(v))J_1'(F(Y))f(y)dF(y)dF(v) \\
= \int \int (Q_2(v)-Q_2(y))J_1'(F(Y))f(y)dF(y)dF(v) \\
- \int Q_2(v)dF(v) \int J_1'(F(Y))f(y)dF(y) - \int Q_2(y)J_1'(F(Y))f(y)dF(y) \\
= -\left[\int Q_2J_1'(F)dF - \int Q_2dF \int J_1'(F)dF\right] .
\]

Similarly,

\[
E_3 = -\left[\int Q_1J_2'(F)dF - \int Q_1dF \int J_2'(F)dF\right] .
\]

(2.215)

Substituting in (2.209) we have
\[ \text{Cov}(T_1(Y), T_2(Y)) = (1-\lambda_N)^2 \left\{ \int Q_1 Q_2 \, dF - \int Q_1 \, dF \int Q_2 \, dF \right\} \]

\[ - (1-\lambda_N) \left\{ \int Q_1 \hat{J}_2(F) \, dF + \int Q_2 \hat{J}_1(F) \, dF \right\} \]

\[ - \int Q_1 \, dF \int \hat{J}_2(F) \, dF - \int Q_2 \, dF \int \hat{J}_1(F) \, dF \right\} \]

\[ + \int \hat{J}_1(F) \hat{J}_2(F) \, dF - \int \hat{J}_1(F) \, dF \int \hat{J}_2(F) \, dF \]

(2.216)

\[ = \int \left[ (1-\lambda_N)Q_1 - \hat{J}_1(F) \right] \left[ (1-\lambda_N)Q_2 - \hat{J}_2(F) \right] \, dF \]

\[ - \int \left[ (1-\lambda_N)Q_1 - \hat{J}_1(F) \right] \, dF \int \left[ (1-\lambda_N)Q_2 - \hat{J}_2(F) \right] \, dF . \]

Similarly,

\[ \text{Cov}(S_1(X), S_2(X)) = \int \left[ \lambda N Q_1 - \hat{J}_1(F) \right] \left[ \lambda N Q_2 - \hat{J}_2(F) \right] \, dF \]

(2.217)

\[ - \int \left[ \lambda N Q_1 - \hat{J}_1(F) \right] \, dF \int \left[ \lambda N Q_2 - \hat{J}_2(F) \right] \, dF . \]

Now from (2.204) it follows that

\[ N \lambda (1-\lambda) \left( \frac{\mathcal{L}}{N} - \frac{2 \alpha / 2}{N^{1/2} B_1(F)} \right) \]

\[ = (1-\lambda_N) \frac{1}{2} \frac{1}{2} \sum_{j=1}^m S_i(X_j) + \frac{1}{2} \frac{1}{2} \sum_{j=1}^n T_i(Y_j) + o_p(1) \]

(2.218)

Hence the asymptotic covariance of
\[
\begin{align*}
\frac{N\lambda(1-\lambda)}{2z_{\alpha/2}} & \leq \frac{A_1}{\sqrt{B_1(F)}} \\
\frac{2z_{\alpha/2} A_2}{\sqrt{B_2(F)}} & > N^{1/2} O(1)
\end{align*}
\]

and

\[
\begin{align*}
\frac{N\lambda(1-\lambda)}{2z_{\alpha/2}} & \leq \frac{A_2}{\sqrt{B_2(F)}} \\
\frac{2z_{\alpha/2} A_1}{\sqrt{B_1(F)}} & > N^{1/2} O(1)
\end{align*}
\]

is

\[
\text{Cov}((1-\lambda_N)^{1/2} \sum_{j=1}^{m} S_j(X_j) + \lambda_N \sum_{j=1}^{n} T_j(Y_j), (1-\lambda_N)^{1/2} \sum_{j=1}^{m} S_j(X_j) + \lambda_N \sum_{j=1}^{n} T_j(Y_j))
\]

\[
= (1-\lambda_N)\text{Cov}(S_1(X), S_2(X)) + \lambda_N \text{Cov}(T_1(Y), T_2(Y))
\]

\[
\begin{align*}
&= [(1-\lambda_N)^2 + \lambda_N (1-\lambda_N)] \left[ \int q_1 q_2 dF - \int q_1 dF \int q_2 dF \right] \\
&- [(1-\lambda_N)^2 + \lambda_N (1-\lambda_N)] \left[ \int q_1 J_2(F) f dF + \int q_2 J_1(F) f dF \right] \\
&- \left[ \int q_1 dF \cdot \int J_2(F) f dF - \int q_2 dF \cdot \int J_1(F) f dF \right] \\
&+ [(1-\lambda_N)^2 + \lambda_N] \left[ \int J_1(F) J_2(F) f^2 dF - \int J_1(F) f dF \cdot \int J_2(F) f dF \right]
\end{align*}
\]
\[ \begin{align*}
&= \lambda_n(1-\lambda_n)\left[ \int q_1 q_2 dF - \int q_1 dF \int q_2 dF \right] \\
&\quad - 2\lambda_n(1-\lambda_n)\left[ \int q_1 J_2'(F) r dF + \int q_2 J_1'(F) r dF - \int q_1 dF \int J_2'(F) r dF - \int q_2 dF \int J_1'(F) r dF \right] \\
&\quad + \left[ \int J_1'(F) J_2'(F) r^2 dF - \int J_1'(F) r dF \int J_2'(F) r dF \right] \\
&= \lambda_n(1-\lambda_n)\left[ (q_1 - 2J_1'(F) r dF)(q_2 - 2J_2'(F) r dF) \right] \\
\end{align*} \]

(2.219) \[ \begin{align*}
&\quad - \int (q_1 - 2J_1'(F) r dF) \cdot \int (q_2 - 2J_2'(F) r dF) \\
&\quad + (1-\lambda_n(1-\lambda_n))\left[ \int J_1'(F) J_2'(F) r^2 dF - \int J_1'(F) r dF \cdot \int J_2'(F) r dF \right] \\
\end{align*} \]

Noting that \( \lambda_n(1-\lambda_n) \rightarrow \lambda(1-\lambda) \neq 0 \), we get the desired asymptotic covariance.

Q.E.D.

We observe, of course, that the result of Theorem 3 can be extended in the obvious way to the k-procedure case.
Chapter 3. Special Cases and Applications

3.1 Some Special Cases of Theorems 1-3.

Let us specialize the general asymptotic results of Chapter 2 to a few specific cases. In particular, we examine the following three score functions:

\[ J_1(u) = u - \frac{1}{2}, \quad 0 \leq u \leq 1 \quad \text{(Wilcoxon)} \]

\[ J_2(u) = -\cos \pi u, \quad 0 \leq u \leq 1 \quad \text{(Cosine)} \]

\[ J_3(u) = (u - \frac{1}{2})^3, \quad 0 \leq u \leq 1 \quad \text{(Cubic)} \]

Each of these score functions is continuous and increasing, with \( J(u) = -J(1-u) \). Hence condition 2 of Theorem 1 is satisfied. Since they each have bounded first three derivatives on \([0,1]\), condition 6 of the Theorem is satisfied. Moreover, we will consider samples from the following three densities:

(i) \( f(x) = (2\pi)^{-\frac{1}{2}} e^{-x^2/2} \) - normal distribution

(ii) \( f(x) = \frac{1}{2} e^{-|x|} \) - Laplace or double exponential distribution

(iii) \( f(x) = \frac{e^{-x}}{(1+e^{-x})^2} \) - logistic distribution
Each of these distributions is symmetric and possesses moments of all orders, hence conditions 3 and 5 of Theorem 1 are satisfied. The only nontrivial condition left to verify is condition 4, namely that

$$\exists \ U > 0, \ K > 0 \ \exists$$

$$|f^{(1)}(x+u)| \leq K[F(x)(1-F(x))]^{2/3} \ \forall x \ \text{and for} \ |u| < U, \ i=0,1,2.$$

Since $f$, $f'$, and $f''$ are easily seen to be founded in each case, it suffices to show that for some $x_o > 0$ and $U > 0$, $\exists \ K > 0$ \ \exists

$$|f^{(1)}(x+u)| \leq K \ \text{for all} \ x \geq x_o \ \text{and} \ |u| \leq U, \ i=0,1,2.$$

Choosing any $U$, $x_o$ satisfying $0 < U < x_o$, we have in the Laplace case

$$\frac{f(x+u)}{[1-F(x)]^{2/3}} = \frac{1}{2} e^{-\frac{x+u}{2}} = 2^{-1/3} e^{-x/3} e^{-u} \leq 2^{-1/3} e^U.$$

Since $|f''(x)| = |f'(x)| = f(x) \ \forall x$, (3.1) is satisfied by choosing $K = 2^{-1/3} e^U$.

In the logistic case, we observe that $f = F'(1-F)$. Hence,

$$|f'| = |1-2F|f \leq f \ \text{and}$$

$$|f''| = \left|\frac{d}{dx} (1-2F)f\right| = |1-2F|^2 (1-2F)f' \leq 2F^2 + |f'|$$

$$\leq 2F(1-F) + f \leq f/2 + f = \frac{3}{2} f.$$
Thus it suffices to verify (3.1) for \( i = 0 \). Now \( F(x) = \frac{1}{1+e^{-x}} \), so

\[
(3.3) \quad \frac{f(x+u)}{[1-F(x)]^{2/3}} = \frac{e^{-x-u}}{(1+e^{-x-u})^2} \cdot \frac{(1+e^{-x})^{2/3}}{e^{-2x/3}} \leq e^{-u} e^{-\frac{x}{3}} (1+e^{-x})^{2/3} \leq 2^{2/3} e^U.
\]

Choosing \( K = 2^{2/3} e^U \) gives us (3.1).

In the normal case, we know from the expansion of Mills' ratio (e.g. [10], p. 137) that for \( x > 0 \),

\[
(3.4) \quad \frac{1-F(x)}{f(x)} = \frac{1}{x} - \frac{1}{x^2} + R(x), \quad \text{where} \quad |R(x)| \leq \frac{1}{x^3}.
\]

Hence for \( x > 0 \),

\[
(3.5) \quad \frac{1}{x} \left(1 - \frac{2}{x^2}\right) \leq \frac{1-F(x)}{f(x)} \leq \frac{1}{x}
\]

which implies

\[
(3.6) \quad x \leq \frac{f(x)}{1-F(x)} \leq \frac{x}{1-2/x^2} \quad \text{for} \quad x^2 > 2.
\]

But for \( x > 2 \), \( \frac{1}{1-2/x^2} < \frac{1}{1-\frac{1}{2}} = 2 \). Hence, for \( x > 2 \),

\[
(3.7) \quad x \leq \frac{f(x)}{1-F(x)} \leq 2x.
\]

Now
\[
\frac{f(x+u)}{(1-F(x))^{2/3}} = \frac{f(x)e^{-xu-u^2/2}}{(1-F(x))^{2/3}} = \left[\frac{f(x)}{1-F(x)}\right]^{2/3} \frac{1}{3} x^2 e^{-xu-u^2/2} \\
\leq (2x)^{2/3} \frac{e^{-x^2/6-xu}}{(2\pi)^{1/6}} \leq 2x^{2/3} e^{-xu-x^2/6}.
\]

But \(x^{2/3} e^{-xu-x^2/6} \to 0\) as \(x \to \infty\), hence \(\exists K \ni \frac{f(x+u)}{(1-F(x))^{2/3}} \leq K\)
for \(x > 2\) and \(|u| \leq U\).

Now, \(|f'(x)| = |x|f(x)|\) and \(|f''(x)| = |x^2-1|f(x)|\), hence we can bound
\[
\frac{|f'(x+u)|}{(1-F(x))^{2/3}} \quad \text{and} \quad \frac{|f''(x+u)|}{(1-F(x))^{2/3}} \quad \text{in essentially the same way.}
\]

Let us now evaluate \(Q(x)\) for each score function \(J_1(u)\) under each of the three distribution families above. Letting \(Q_1(x)\) correspond to \(J_1(u)\), we have by definition

\[
Q_1(x) = \int_0^x J_1'(F) f dF
\]

\(Q_1(x):\)

Since \(J_1''(u) = 0\), we have

\[
(3.9) \quad Q_1(x) = 0 \quad \text{for every } F.
\]

\(Q_2(x):\)

\(J''(u) = \pi^2 \cos \pi u\), hence

\[
(3.10) \quad Q_2(x) = \int_0^\pi \pi^2 \cos(\pi F) f dF.
\]

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In the normal case, this integral cannot be evaluated analytically. In the Laplace case we have, for \( x > 0 \), \( F(x) = 1 - \frac{1}{2} e^{-x} \) and hence

\[
Q_2(x) = \pi^2 \int_0^x \cos(\pi - \frac{\pi}{2} e^{-x}) \frac{1}{2} e^{-x} (e^{-x} - \frac{1}{2}) d(e^{-x})
\]

\[
= - \frac{\pi^2}{4} \int_{e^{-x}}^1 \cos(\frac{\pi}{2} y) y \, dy
\]

\[
= - \frac{\pi^2}{4} \left[ \frac{\pi}{2} y \sin(\frac{\pi}{2} y) \right]_{e^{-x}}^1 + \frac{\pi^2}{4} \int_{e^{-x}}^1 \frac{2}{\pi} \sin(\frac{\pi}{2} y) dy
\]

\[
= - \frac{\pi}{2} (1 - e^{-x} \sin(\frac{\pi}{2} e^{-x})) - [\cos(\frac{\pi}{2} y)]_{e^{-x}}^1
\]

\[
= - \frac{\pi}{2} + \frac{\pi}{2} e^{-x} \sin(\frac{\pi}{2} e^{-x}) + \cos(\frac{\pi}{2} e^{-x}).
\]

Since \( Q \) is an even function, it follows that for all \( x \),

\[
(3.11) \quad Q_2(x) = \frac{\pi}{2} e^{-|x|} \sin\left(\frac{\pi}{2} e^{-|x|}\right) + \cos\left(\frac{\pi}{2} e^{-|x|}\right) - \frac{\pi}{2}, \text{ for f Laplace}.
\]

In the logistic case, using the fact \( f = F(1-F) \), we have

\[
(3.12) \quad Q_2(x) = \pi^2 \int_0^x \cos(\pi F) F(1-F) dF
\]

\[
= \pi^2 \int_{1/2}^{F(x)} \cos(\pi u) (u-u^2) du
\]

\[
= \frac{\pi^2}{2} \left[ \frac{1}{\pi} \cos(\pi u) + \frac{u \sin(\pi u)}{\pi^2} \right]_{1/2}^{F(x)} - \pi^2 \left[ \frac{2 u \cos(\pi u)}{\pi^2} + \frac{u^2 - 2}{\pi^3} \sin(\pi u) \right]_{1/2}^{F(x)}
\]
\[ \begin{align*}
&= \cos(\pi F) + \pi F \sin(\pi F) - \frac{\pi}{2} F \cos(\pi F) - \pi F^2 \sin(\pi F) + \frac{2}{\pi} \sin(\pi F) \\
&\quad + \frac{\pi}{4} - \frac{2}{\pi} \\
&= (\pi F(1-F) + \frac{2}{\pi}) \sin(\pi F) + (1-2F) \cos(\pi F) - \frac{\pi}{2} \frac{2}{\pi} \\
&= \left( \frac{\pi e^{-x}}{(1+e^{-x})^2} + \frac{2}{\pi} \right) \sin(-\frac{\pi}{1+e^{-x}}) + \left( 1 - \frac{2e^{-x}}{(1+e^{-x})^2} \right) \cos(-\frac{\pi}{1+e^{-x}}) - \frac{\pi}{2} - \frac{2}{\pi} \\
&\text{for } F \text{ logistic.}
\end{align*} \]

\underline{Q_3(x):}

Since \( J''(u) = 6(u - \frac{1}{2}) \), we have

\[(3.13) \quad Q_3(x) = 6 \int_0^x (F - \frac{1}{2}) \, dF.\]

Again, in the normal case, we cannot evaluate \( Q_3 \) analytically. In the Laplace case, we have for \( x > 0 \)

\[
\begin{align*}
Q_3(x) &= 6 \int_0^x \frac{1}{2} (1-e^{-x}) \frac{1}{2} e^{-x} (-\frac{1}{2}) d(e^{-x}) \\
&= \frac{3}{4} \int_{e^{-x}}^1 (y-y^2) \, dy = \frac{3}{4} \left[ \frac{1}{2}(1-e^{-2x}) - \frac{1}{3}(1-e^{-3x}) \right] \\
&= \frac{1}{3}(1-3e^{-2x} + 2e^{-3x}) .
\end{align*}
\]

Since \( Q \) is even, we have for all \( x \)

\[(3.14) \quad Q_3(x) = \frac{1}{3}(1-3e^{-2|x|} + 2e^{-3|x|}), \quad F \text{ Laplace.}\]

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Under logistic \( F \),

\[
(3.15) \quad Q_3(x) = 6 \int_0^x (F - \frac{1}{2})F(1-F)dF = 6 \int_{\frac{1}{2}}^{F(x)} (-\frac{1}{2}u + \frac{3}{2}u^2 - u^3)du \\
= 6[-\frac{1}{4}(F^2 - \frac{1}{4}) + \frac{1}{2}(F^3 - \frac{1}{3}) - \frac{1}{4}(F^4 - \frac{1}{16})] \\
= 6[\frac{1}{24} - \frac{F^2}{2} + \frac{F^3}{2} - \frac{F^4}{4}] \\
= \frac{3}{32} - \frac{1}{4(1+e^{-x})^2} + \frac{1}{2(1+e^{-x})^3} - \frac{1}{4(1+e^{-x})^4}, \quad F \text{ logistic}.
\]

We are now in a position to evaluate the parameters of the asymptotic distribution of interval lengths, as given by (2.198) and (2.199).

First of all, the values of \( A^2 \) for the three score functions are

\[
A_{1}^2 = \frac{1}{12}, \quad A_{2}^2 = \frac{1}{2}, \quad A_{3}^2 = \frac{1}{448}.
\]

Defining

\[
\begin{align*}
&\quad a_{ij} = \int (Q_i - 2J_i(F)f)(Q_j - 2J_j(F)f)dF - \int (Q_i - 2J_i(F)f)dF \int (Q_j - 2J_j(F)f)dF \\
&\quad b_{ij} = \int J_i'(F)J_j'(F)f^2dF - \int J_i'(F)f^2dF \int J_j'(F)f^2dF
\end{align*}
(3.16)
\]

we obtain Table 3.1.
Table 3.1

<table>
<thead>
<tr>
<th>F</th>
<th>Laplace</th>
<th>Logistic</th>
<th>Normal *</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1(F)$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{2\sqrt{\pi}} = .282$</td>
</tr>
<tr>
<td>$b_2(F)$</td>
<td>$\frac{2}{\pi}$</td>
<td>$\frac{4}{\pi^2}$</td>
<td>$\frac{1}{40} = .025$</td>
</tr>
<tr>
<td>$b_3(F)$</td>
<td>$\frac{1}{32}$</td>
<td>$\frac{1}{40}$</td>
<td>$\frac{1}{2\sqrt{\pi}} = .282$</td>
</tr>
<tr>
<td>$a_{11}$</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{45}$</td>
<td>$\frac{1}{2\sqrt{3}-3}/3\pi = .0492$</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>$\frac{2(\pi-2)}{\pi^2}$</td>
<td>$\frac{8(12-\pi^2)/3\pi^4}{3\pi^4}$</td>
<td>$\frac{1}{120} = .0083$</td>
</tr>
<tr>
<td>$a_{13}$</td>
<td>$\frac{1}{150}$</td>
<td>$\frac{1}{120}$</td>
<td>$\frac{1}{40} = .0258$</td>
</tr>
<tr>
<td>$a_{22}$</td>
<td>$(\pi^2+6)/24$</td>
<td>$(\pi^2-9\pi+9)/24\pi^2$</td>
<td>$\frac{1}{2240} = .00134$</td>
</tr>
<tr>
<td>$a_{23}$</td>
<td>$\frac{3(48-12\pi-\pi^2)/\pi^4}{3\pi^4}$</td>
<td>$\frac{1}{2240}$</td>
<td>$\frac{1}{1400} = .00714$</td>
</tr>
<tr>
<td>$a_{33}$</td>
<td>$\frac{3}{2240}$</td>
<td>$\frac{1}{1400}$</td>
<td>$\frac{1}{2\sqrt{3}-3}/12\pi = .0123$</td>
</tr>
<tr>
<td>$b_{11}$</td>
<td>$\frac{1}{48}$</td>
<td>$\frac{1}{180}$</td>
<td>$\frac{1}{2\sqrt{3}-3}/12\pi = .0123$</td>
</tr>
<tr>
<td>$b_{12}$</td>
<td>$(3\pi-8)/2\pi^2$</td>
<td>$\frac{2(72-7\pi^2)/3\pi^4}{3\pi^4}$</td>
<td>$\frac{1}{180} = .00556$</td>
</tr>
<tr>
<td>$b_{13}$</td>
<td>$-\frac{1}{640}$</td>
<td>$-\frac{1}{1600}$</td>
<td>$\frac{1}{1600} = .000595$</td>
</tr>
<tr>
<td>$b_{22}$</td>
<td>$(\pi^4+6\pi^2-56)/24\pi^2$</td>
<td>$\frac{1}{256}$</td>
<td>$\frac{1}{256} = .0762$</td>
</tr>
<tr>
<td>$b_{23}$</td>
<td>$-\frac{1}{16\pi} + \frac{3(48-12\pi-\pi^2)/\pi^4}{3\pi^4}$</td>
<td>$-\frac{1}{1600}$</td>
<td>$\frac{1}{11200} = .000268$</td>
</tr>
<tr>
<td>$b_{33}$</td>
<td>$\frac{15}{35210}$</td>
<td>$\frac{3}{11200}$</td>
<td>$\frac{3}{11200} = .000268$</td>
</tr>
</tbody>
</table>

* Blank entries represent values that were unobtainable
Recalling the asymptotic joint length distribution from (2.198) we have that the centering term for \( N_0 \mathcal{L}_{N,i} \) is \( 2z_{\alpha/2} N_0^{1/2} \frac{A_1}{B_1(F)} \). Table 3.2 gives the values of \( \frac{A_1}{B_1(F)} \) for the cases under consideration.

### Table 3.2

<table>
<thead>
<tr>
<th>( F )</th>
<th>Laplace ( \frac{A_1}{B_1(F)} )</th>
<th>Logistic ( \frac{A_2}{B_2(F)} )</th>
<th>Normal ( \frac{A_3}{B_3(F)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2}{\sqrt{3}} )</td>
<td>1.155</td>
<td>( \frac{\sqrt{3}}{\sqrt{3}} )</td>
<td>1.732</td>
</tr>
<tr>
<td>( \frac{\sqrt{2}}{4} )</td>
<td>1.111</td>
<td>( \frac{\pi}{\sqrt{3}} )</td>
<td>1.745</td>
</tr>
<tr>
<td>( \frac{4}{\sqrt{7}} )</td>
<td>1.512</td>
<td>( \frac{5}{\sqrt{7}} )</td>
<td>1.890</td>
</tr>
</tbody>
</table>

We observe from Table 3.2 that in terms of asymptotic efficiency, the procedure based on the cubic score function is considerably less efficient than either the Wilcoxon or the cosine procedure under both Laplace and logistic samples, with the Wilcoxon slightly better than the cosine under the logistic (in fact, we know that in this case the Wilcoxon is the locally most powerful rank test), and vice versa under the Laplace.

The asymptotic variance of \( N_0 \mathcal{L}_{N,i} \) is

\[
\sigma_i^2 = 4z_{\alpha/2}^2 \frac{A_1^2}{B_1(F)} \left[ \lambda (1-\lambda) a_{i1} + (1-\lambda^2) b_{i1} \right]
\]
and the asymptotic covariance of $N_o L_{N,i}$ and $N_o L_{N,j}$ is

$$
\sigma_{ij} = 4z^2 \frac{A_i A_j}{B_i^2(F) B_j^2(F)} \left[ \lambda(1-\lambda)a_{ij} + (1-4\lambda(1-\lambda))b_{ij} \right].
$$

Denoting the covariance matrix by $\Sigma$, we have tabulated in Table 3.3 the values of $\frac{1}{2z^2 \alpha/2} \Sigma$ under the Laplace and logistic models for the special case of equal sample sizes ($\lambda = \frac{1}{2}$), as well as the value of $\frac{1}{2} \sigma_1^2$ under the normal model. It is worth pointing out here that the value of $\sigma_1^2$ (Wilcoxon procedure) does not depend on $\lambda$ regardless of the underlying $F$.

<table>
<thead>
<tr>
<th>$F$</th>
<th>Laplace</th>
<th>Logistic</th>
<th>Normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2} \xi$</td>
<td>${1.778, 1.864, 1.397}$</td>
<td>${2.400, 2.609, 1.870}$</td>
<td>$\frac{1}{2} \sigma_1^2 = .648$</td>
</tr>
<tr>
<td>$\frac{1}{2z^2 \alpha/2}$</td>
<td>${1.864, 2.013, 1.121}$</td>
<td>${2.609, 3.018, 1.344}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>${1.397, 1.121, 3.135}$</td>
<td>${1.870, 1.344, 4.082}$</td>
<td></td>
</tr>
</tbody>
</table>
3.2 Monte Carlo Results.

To get some idea of the accuracy of the asymptotic approximation to the length distribution, Monte Carlo simulations were performed on the computer. For a particular pair of score functions, repeated samples of size \( m = n = 40 \) were generated from one of the three distributions considered in 3.1, and the 2-sided 80% and 50% confidence intervals based on each of the two score functions were obtained. The sampling means, variances, and covariances of the interval lengths were calculated. The aggregated results for the three score functions considered in the previous section are given in Table 3.4. Each entry is followed by a number in parentheses representing the number of replications on which the entry is based. Below each Monte Carlo value is the corresponding value based on the approximations in 3.1.

Table 3.4

Comparison of Monte Carlo results with approximation based on asymptotic distribution. \( m = n = 40 \).

Upper entry = Monte Carlo value, lower entry = asymptotic value.

<table>
<thead>
<tr>
<th>( \alpha = .20 )</th>
<th>20E(( L_2 ))</th>
<th>Var-Cov(20 ( L_2 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.60 (700)</td>
<td>3.50 (700)</td>
<td>3.40 (200)</td>
</tr>
<tr>
<td>13.24</td>
<td>2.92</td>
<td>3.06</td>
</tr>
<tr>
<td>13.23 (200)</td>
<td></td>
<td>3.60 (200)</td>
</tr>
<tr>
<td>12.74</td>
<td></td>
<td>3.31</td>
</tr>
<tr>
<td>17.51 (200)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17.34</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Laplace $\alpha = .50$</td>
<td>7.12 (700)</td>
<td>1.15 (700)</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>6.97</td>
<td>.81</td>
<td>.85</td>
</tr>
<tr>
<td>6.89 (200)</td>
<td></td>
<td>1.16 (200)</td>
</tr>
<tr>
<td>6.70</td>
<td></td>
<td>.92</td>
</tr>
<tr>
<td>9.21 (200)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9.12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Logistic $\alpha = .20$</td>
<td>20.12 (400)</td>
<td>5.12 (400)</td>
</tr>
<tr>
<td>19.86</td>
<td>3.94</td>
<td>4.29</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>20.01</td>
<td></td>
<td>4.96</td>
</tr>
<tr>
<td>21.88 (200)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21.67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Logistic $\alpha = .50$</td>
<td>10.54 (400)</td>
<td>1.80 (400)</td>
</tr>
<tr>
<td>10.45</td>
<td>1.09</td>
<td>1.19</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>10.53</td>
<td></td>
<td>1.37</td>
</tr>
<tr>
<td>11.34 (200)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11.40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(20E(\mathcal{L}_1))</td>
<td>(\text{Var}(20\mathcal{L}_1))</td>
<td></td>
</tr>
<tr>
<td>Normal $\alpha = .20$</td>
<td>11.82 (300)</td>
<td>1.46 (300)</td>
</tr>
<tr>
<td>11.73</td>
<td>1.07</td>
<td></td>
</tr>
<tr>
<td>Normal $\alpha = .50$</td>
<td>6.24 (300)</td>
<td>.52 (300)</td>
</tr>
<tr>
<td>6.17</td>
<td>.29</td>
<td></td>
</tr>
</tbody>
</table>
Examination of the results in Table 3.4 shows that at the sample sizes \( m = n = 40 \) \( (N_0 = 20) \), the asymptotic mean lengths are very good approximations to the actual mean lengths, as estimated by the Monte Carlo simulations; the asymptotic variances and covariances, however, are rather poorer approximations to their actual values, being consistently smaller than the Monte Carlo values by amounts from 8% to 45%. This difference in accuracy is not surprising, since the terms contributing to the asymptotic variance are of a higher order than the asymptotic mean. Nevertheless, probability plots of some of the Monte Carlo results indicate that the shape of the length distribution at these sample sizes is quite close to normal. Calculating pairwise correlations of the lengths from Table 3.4, we obtain Table 3.5.

**Table 3.5**

Comparison of length correlations of Monte Carlo results with correlations based on asymptotic distribution. \( m = n = 40 \)

<table>
<thead>
<tr>
<th></th>
<th>Laplace</th>
<th>Logistic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Asymptotic</td>
<td>Monte Carlo ( \alpha=.20 )</td>
</tr>
<tr>
<td>( \rho_{12} )</td>
<td>.98</td>
<td>.96</td>
</tr>
<tr>
<td>( \rho_{13} )</td>
<td>.59</td>
<td>.59</td>
</tr>
</tbody>
</table>
Thus we see that while the asymptotic variances and covariances do not give very good approximations to their actual values at $N_0 = 20$, the correlations of the asymptotic length distribution are considerably more accurate.
3.3 Properties of the Flexible Confidence Procedure—Some Special Cases.

In Section 1.5, we considered a general type of flexible procedure which was shown to have a robustness property with respect to asymptotic efficiency and was asymptotically distribution-free. In this section we will examine a few specific examples of this flexible procedure and show how the results of Chapter 2 can be used to approximate their finite sample properties.

In particular, let us consider flexible procedures with $k=2$, i.e., we use the shorter of two $(1-\alpha)$-level confidence intervals based on rank tests with asymptotic score functions $J_1(u)$ and $J_2(u)$. Assuming the conditions of Theorem 3 are satisfied, we can use the asymptotic joint normality of the interval lengths to obtain an approximation to the deviation from $1-\alpha$ of the coverage probability of the flexible procedure. We showed in Section 1.5 that this deviation tends to 0 as the sample sizes approach infinity.

Letting $1-\alpha'$ be the true coverage probability of the flexible procedure (which depends on $F$, $m$, and $n$), we assume as before that the location parameter $\theta$ is 0. Denoting the $(1-\alpha)$-level confidence interval based on $J_1(u)$ by $J_1$ and its midpoint by $\mathcal{M}_1$, we have

\begin{equation}
\alpha' - \alpha = 1-\alpha - (1-\alpha') = (1-\alpha - P(J_1 \text{ covers } 0 | \mathcal{L}_1 < \mathcal{L}_2)) P(\mathcal{L}_1 < \mathcal{L}_2)
+ (1-\alpha - P(J_2 \text{ covers } 0 | \mathcal{L}_2 < \mathcal{L}_1)) P(\mathcal{L}_2 < \mathcal{L}_1).
\end{equation}
\[ = P(\mathcal{I}_1 \text{ covers } 0) P(\mathcal{L}_1 < \mathcal{L}_2) - P(\mathcal{I}_1 \text{ covers } 0, \mathcal{L}_1 < \mathcal{L}_2) \\
+ P(\mathcal{I}_2 \text{ covers } 0) P(\mathcal{L}_2 < \mathcal{L}_1) - P(\mathcal{I}_2 \text{ covers } 0, \mathcal{L}_2 < \mathcal{L}_1) \]

Suppose we now assume that the asymptotic joint distribution of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) as given by Theorem 3 is exact for finite sample sizes. From Sen's theorem we have that \( N_0^{1/2} \mathcal{M}_i \) is asymptotically distributed as \( N(0, A_i^2/B_i^2(F)) \); let us also assume that this distribution is exact.

Furthermore, from the symmetry of \( F \) it follows that \( \mathcal{L}_1 \) and \( \mathcal{M}_j \) are uncorrelated, for any \( i \) and \( j \), and hence under the normality assumption we can take \( (\mathcal{L}_1, \mathcal{L}_2) \) to be independent of \( (\mathcal{M}_1, \mathcal{M}_2) \). Under these assumptions, we can evaluate the right hand side of (3.18).

Thus

\[(3.18) \quad \alpha' - \alpha = 2P(0 < 2\mathcal{M}_1 < \mathcal{L}_1)P(\mathcal{L}_1 < \mathcal{L}_2) - 2P(0 < 2\mathcal{M}_1 < \mathcal{L}_1 < \mathcal{L}_2) \\
+ 2P(0 < 2\mathcal{M}_2 < \mathcal{L}_2)P(\mathcal{L}_2 < \mathcal{L}_1) - 2P(0 < 2\mathcal{M}_2 < \mathcal{L}_2 < \mathcal{L}_1) \]

\[= 2\left(P(0 < 2\mathcal{M}_1 < \mathcal{L}_1) - \frac{1}{2}P(\mathcal{L}_1 < \mathcal{L}_2) - 2P(0 < 2\mathcal{M}_1 < \mathcal{L}_1 < \mathcal{L}_2) - P(\mathcal{M}_1 < 0, \mathcal{L}_1 < \mathcal{L}_2)\right) \\
+ 2\left(P(0 < 2\mathcal{M}_2 < \mathcal{L}_2) - \frac{1}{2}P(\mathcal{L}_2 < \mathcal{L}_1) - 2P(0 < 2\mathcal{M}_2 < \mathcal{L}_2 < \mathcal{L}_1) - P(\mathcal{M}_2 < 0, \mathcal{L}_2 < \mathcal{L}_1)\right) \]

\[= 2P(0 < 2\mathcal{M}_1 < \mathcal{L}_1)P(\mathcal{L}_1 < \mathcal{L}_2) - P(\mathcal{L}_1 < \mathcal{L}_2) - 2P(0 < 2\mathcal{M}_1 < \mathcal{L}_1) - P(2\mathcal{M}_1 < \mathcal{L}_1, \mathcal{L}_2 < \mathcal{L}_1) \\
+ 2 \cdot \frac{1}{2} P(\mathcal{L}_1 < \mathcal{L}_2) + 2P(0 < 2\mathcal{M}_2 < \mathcal{L}_2)P(\mathcal{L}_2 < \mathcal{L}_1) - P(\mathcal{L}_2 < \mathcal{L}_1) \\
- 2P(0 < 2\mathcal{M}_2 < \mathcal{L}_2) - P(2\mathcal{M}_2 < \mathcal{L}_2, \mathcal{L}_1 < \mathcal{L}_2) + 2 \cdot \frac{1}{2} P(\mathcal{L}_2 < \mathcal{L}_1) \]

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\[
-2F(2n_1 < L_1(L_1 < L_2)) + 2F(2n_1 < L_1, L_2 < L_1)
-2F(2n_2 < L_2)(1 - F(L_2 < L_1)) + 2F(2n_2 < L_2, L_1 < L_2)
= 2[F(2n_1 - L_1 < 0, L_2 - L_1 < 0) - F(2n_1 - L_1 < 0)F(L_2 - L_1 < 0)] + F(2n_2 - L_2 < 0, L_1 - L_2 < 0) - F(2n_2 - L_2 < 0)F(L_1 - L_2 < 0)].
\]

Now we use the normality assumptions to evaluate the RHS. If we denote the mean vector and covariance matrix of \((L_1, L_2)\) by \(\mu = (\mu_1, \mu_2)\) and \(\Sigma = \begin{pmatrix} \tau_1^2 & \tau_{12} \\ \tau_{12} & \tau_2^2 \end{pmatrix}\), then we have

\[
(3.19) \quad P(2n_1 - L_1 < 0) = P \left( \frac{2n_1 - L_1 - \mu_1}{\sqrt{4A_{11}/N_0B_{11}(F) + \tau_1^2}} < \frac{-\mu_1}{\sqrt{4A_{11}/N_0B_{11}(F) + \tau_1^2}} \right)
\]

\[
= 1 - \Phi \left( \frac{\mu_1}{\sqrt{4A_{11}/N_0B_{11}(F) + \tau_1^2}} \right)
\]

and

\[
(3.20) \quad P(L_i - L_j < 0) = \Phi \left( \frac{\mu_i - \mu_j}{\sqrt{\tau_{12}^2 + \tau_{11}^2 + \tau_{12}^2}} \right)
\]

Now \(2n_1 - L_1\) and \(L_2 - L_1\) have correlation

\[
(3.21) \quad \rho = \frac{\tau_{12}^2}{\sqrt{(4A_{11}/N_0B_{11}(F) + \tau_1^2)(\tau_{12}^2 + \tau_{11}^2 + \tau_{12}^2)}}.
\]

Hence
\[(3.22) \quad P(2\mu_1 - L_1 < 0, \, L_2 - L_1 < 0) \]
\[= P\left( X < \frac{\mu_1}{\sqrt{4A_1^2/N_0 E_1^2(P) + \tau_1^2}}, \; Y < \frac{\mu_1 - \mu_2}{\sqrt{\tau_1^2 + \tau_2^2 - 2\tau_{12}}} \right) \]

where \((X,Y) \sim N_2((0,0),(1/\rho_{11}))\).

Thus \(P(2\mu_1 - L_1 < 0, \, L_2 - L_1 < 0)\) can be evaluated with the aid of tables of the bivariate normal distribution (e.g. [18]).

Using (3.17) to (3.22), values of \(a' - \alpha\) and \(P(L_1 < L_j)\) were calculated for each of the three possible flexible procedures based on a pair chosen from the score functions \(J_1, J_2, J_3\) defined in 3.1, under the Laplace and logistic models for \(m = n = 40\) and for \(\alpha = .20\) and \(\alpha = .50\). The results are given in Table 3.6.

Because of the limits of accuracy involved in the required interpolations, the values for \(a' - \alpha\) are believed to be accurate to within .002.

For the purpose of comparison, Monte Carlo simulations of the flexible procedure with \(k=2\) were performed for some of the same cases as above. In each case, counts were made of the number of replications for which each of the two procedures in consideration produced the shorter interval as well as the number of times each interval covered 0, conditioning on which was shorter.

Let \(R = \) number of Monte Carlo replications.

\(R_i = \) number of replications for which \(J_i\) is shorter, \(i=1,2\).

\(r_{ij} = \) number of replications for which \(J_i\) is shorter and \(J_j\) covers 0, \(i,j=1,2\).
Now from (3.17), we have

\[(3.23) \quad \alpha' - \alpha = [P(I_1 \text{ covers } 0) - P(I_1 \text{ covers } 0 | I_1 \text{ shorter})] P(I_1 \text{ shorter}) \]
\[+ [P(I_2 \text{ covers } 0) - P(I_2 \text{ covers } 0 | I_2 \text{ shorter})] P(I_2 \text{ shorter}). \]

This suggests the following estimate of \( \alpha' - \alpha \):

\[(3.24) \quad \hat{\alpha'} - \alpha = \left[ \frac{r_{11} + r_{21}}{R} - \frac{r_{11}}{R_1} \right] \frac{R_1}{R} + \left[ \frac{r_{12} + r_{22}}{R} - \frac{r_{22}}{R_2} \right] \frac{R_2}{R} \]
\[= \left( \frac{r_{11} + r_{21}}{R} - \frac{r_{11}}{R_1} \right) \frac{R_1}{R} + \left( \frac{r_{12} + r_{22}}{R} - \frac{r_{22}}{R_2} \right) \frac{R_2}{R} \]
\[= \frac{(r_{21} - r_{22}) R_1 + (r_{12} - r_{11}) R_2}{R^2} \].

The Monte Carlo estimates of \( \alpha' - \alpha \) and \( P(\mathcal{L}_1 < \mathcal{L}_j) \) are given in Table 3.6 immediately below the corresponding value as calculated under the normal approximation. For each entry, the number of replications, \( R \), was 200. Blank entries are for cases where no simulations were performed.

Examining Table 3.6, we observe that while the asymptotic length distribution gives us a fairly good ballpark estimate of \( P(\mathcal{L}_1 < \mathcal{L}_j) \), it does not yet give us a very accurate estimate of this quantity at \( m = n = 40 \), tending somewhat to overestimate the probability that the more efficient procedure will give the shorter interval. This can be attributed mainly to the already observed fact that the asymptotic
Table 3.6

Properties of some particular flexible (k=2) procedures for m=n=40

Upper entry = value based on asymptotic length distribution

Lower entry = Monte Carlo value (R=200)

<table>
<thead>
<tr>
<th>F</th>
<th>Flexible Procedure</th>
<th>( \alpha = .20 )</th>
<th>( \alpha' - \alpha )</th>
<th>( \alpha = .50 )</th>
<th>( \alpha' - \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( J_i ) vs. ( J_j )</td>
<td>( P(\mathcal{L}_i &lt; \mathcal{L}_j) )</td>
<td>( \alpha' - \alpha )</td>
<td>( P(\mathcal{L}_i &lt; \mathcal{L}_j) )</td>
<td>( \alpha' - \alpha )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Laplace</td>
<td>Wilcoxon vs.</td>
<td>.07</td>
<td>.001</td>
<td>.07</td>
<td>.002</td>
</tr>
<tr>
<td></td>
<td>Cosine</td>
<td>.16</td>
<td>.00</td>
<td>.20</td>
<td>.00</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Wilcoxon vs.</td>
<td>.99</td>
<td>.002</td>
<td>.99</td>
<td>.003</td>
</tr>
<tr>
<td></td>
<td>Cubic</td>
<td>.97</td>
<td>.01</td>
<td>.95</td>
<td>.01</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Cosine vs.</td>
<td>.98</td>
<td>.003</td>
<td>.98</td>
<td>.003</td>
</tr>
<tr>
<td></td>
<td>Cubic</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Logistic</td>
<td>Wilcoxon vs.</td>
<td>.59</td>
<td>.005</td>
<td>.59</td>
<td>.014</td>
</tr>
<tr>
<td></td>
<td>Cosine</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Wilcoxon vs.</td>
<td>.80</td>
<td>.014</td>
<td>.80</td>
<td>.012</td>
</tr>
<tr>
<td></td>
<td>Cubic</td>
<td>.69</td>
<td>.00</td>
<td>.67</td>
<td>.01</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Cosine vs.</td>
<td>.73</td>
<td>.020</td>
<td>.73</td>
<td>.019</td>
</tr>
<tr>
<td></td>
<td>Cubic</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

distribution tends to overestimate the variances and covariances of the interval lengths. For \( \alpha' - \alpha \), the simulations do not give us a very good estimate, since we are estimating very small probabilities here on the basis of only 200 replications. Nevertheless, it appears safe to
say that the estimate of $\alpha' - \alpha$ based on the asymptotics is of the right order of magnitude. Under the Laplace sample, the estimate is never greater than 1%, and never greater than 2% under the logistic sample. Also, we observe that there does not appear to be a great difference between the values of $\alpha' - \alpha$ for $\alpha = .20$ and $\alpha = .50$, hence we might postulate that the relative error in significance level $\frac{\alpha' - \alpha}{\alpha}$ incurred by using a flexible procedure is smaller for larger $\alpha$.

As a rule of thumb, it is reasonable to say that the closer $P(\mathcal{L}_1 < \mathcal{L}_j)$ is to 1 or 0, the closer $1 - \alpha'$ will be to $1 - \alpha$. Hence it is a generally favorable circumstance that, even when the means of $\mathcal{L}_1$ and $\mathcal{L}_j$ are quite close (and hence the two competing procedures are of almost the same efficiency for the given $F$), the fact that $\mathcal{L}_1$ and $\mathcal{L}_j$ are positively correlated (and often highly so, viz. Table 3.5) tends to push $P(\mathcal{L}_1 < \mathcal{L}_j)$ away from $\frac{1}{2}$. 

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3.4 Concluding Remarks.

There are a number of possible directions in which the results of this thesis might be extended. On the theoretical level, it would be of interest to try to relax the conditions of Theorem 1 of Chapter 2 so as to include confidence intervals based on tests with unbounded score functions, such as the normal scores test. It should also be possible to obtain asymptotic results for the case of non-symmetric distribution families. Application of some of the recent work in weak convergence might prove fruitful in this area.

On the practical side, a much more thorough investigation of the finite sample properties of flexible confidence procedures remains to be done. Among the questions of interest here are:

(i) How does one wisely choose the competing score functions as well as the number of them?

(ii) For a given flexible procedure, what is the worst we can do in terms of loss in actual coverage probability?

(iii) In comparing flexible with non-flexible procedures, how does one weigh the loss in coverage probability (which may depend on the underlying $F$) against the improvement in efficiency robustness?

In connection with the issue of choosing the competing score functions for the flexible procedure, let us mention a family of score functions which might be called Winsorized Wilcoxon score functions. These are of the form
(3.25) \[ J(u) = \begin{cases} 
  p - \frac{1}{2} & \text{for } 0 \leq u \leq p \\
  u - \frac{1}{2} & \text{for } p \leq u \leq 1-p \\
  \frac{1}{2} - p & \text{for } 1-p \leq u \leq 1 
\end{cases} \]

Figure 3.1

Letting \( p \) range from 0 to \( \frac{1}{2} \) gives us a family of score functions, with the extreme cases \( p = 0 \) and \( p = \frac{1}{2} \) reducing to the score functions of the Wilcoxon and median tests respectively. Roughly speaking, as \( p \) increases, \( J(u) \) is good (i.e. efficient) for increasingly heavier-tailed distributions. One of the advantages of this family of score functions is that the corresponding confidence intervals are particularly easy to obtain, as their endpoints are simply fixed order statistics of certain subsets of the differences \( X_i - Y_j \).
While most of this thesis has been concerned with the two-sample location problem, one should be able to obtain analogous results for the one-sample problem, just as most previous results for two-sample rank tests have been shown to carry over to the one-sample case.

The emphasis of this thesis has been on interval estimation. However, the basic idea behind the flexible confidence intervals introduced in section 1.5 can be easily exploited to obtain flexible (or adaptive) point estimates. Under the conditions of Sen's theorem, we know that the midpoint $\mathcal{M}$ of a two-sided nonparametric confidence interval is a consistent estimate of the location shift parameter $\theta$ and, as pointed out in 3.3, that $\frac{1}{\sqrt{n}}(\mathcal{M} - \theta)$ converges to a normal distribution with mean 0 and variance $\frac{A^2}{E^2(F)}$. Since the length $\mathcal{L}$ of the confidence interval is a consistent estimate of a positive multiple of $A/E(F)$, it follows that the midpoint of the shortest of several different nonparametric confidence intervals will be an estimate with efficiency robustness superior to the non-adaptive estimates. The results of Chapter 2 could be used to study the finite-sample efficiency of such an adaptive estimate.
References


Technometrics, 7, 257-60.

Math. Statist. 20, 455-58.


Math. Statist. 37, 1759-70.

