WEAK CONVERGENCES WITH RANDOM INDICES

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RICHARD T. DURRETT and SIDNEY I. RESNICK

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ABSTRACT

Suppose \(\{X_n, n \geq 0\}\) are random variables such that for normalizing constants \(a_n > 0, b_n, n \geq 0\) we have \(Y_n(\cdot) = (X_n(\cdot) - b_n) / a_n\) \(\Rightarrow Y(\cdot)\) in \(D(0, \omega)\). Then \(a_n\) and \(b_n\) must vary in specific ways and the process \(Y\) possesses a scaling property. If \(\{N_n\}\) are positive integer valued random variables we discuss when \(Y_{N_{n}} \Rightarrow Y\) and \(Y'_{n} = (X_{N_{n}}(\cdot) - b_{n}) / a_{n} \Rightarrow Y'\). Results given subsume random index limit theorems for convergence to Brownian Motion, stable processes and extremal processes.

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1. Introduction

The main question we consider is the following: If there is a functional limit theorem for a sequence of random variables \( \{X_n, n \geq 0\} \) then how is the convergence affected when the indices are rv's.

To state the question precisely we introduce the following

**BASIC ASSUMPTION:** For some \( a_n > 0, b_n, n \geq 1 \), \( Y_n = (X_{\lfloor n \rfloor} - b_n)/a_n \) converges weakly as a sequence of random elements of \( D = D(0, \infty) \) -- the space of right continuous functions with finite left limits at each \( t > 0 \)--to a process \( Y \in D \) which has a nondegenerate distribution at each \( t > 0 \).

For what follows, \( Y_n \Rightarrow Y \) denotes weak convergence in the metric space \( D(0, \infty) \) which means \( Y_n \Rightarrow Y \) in \( D[0,s] \) for each \( 0 < r < s < \infty \) which are not fixed discontinuities of the limit \( Y \).

(See [1], [7].) Observe that the convergence we have postulated is the usual one encountered in the convergence of random walks to stable processes (see [1], [23], [24]) and partial maxima to extremal processes (see [28], [29], or [31]).

In Section 2 we show that under assumption 1 the norming constants \( a_n \) and \( b_n \) must vary in specific ways and \( Y \) must be stochastically continuous. These results owe much to Lamperti (1962) and Weissman (1975). Our contribution is that stochastic continuity of the limit is now a conclusion rather than an assumption.
In Section 3 we discuss some conditions for the convergence of \( Y_n \) and \( Y_n' = \frac{X_{n_n'} - b_n}{a_n} \) when the sequences \( \{X_n\} \) and \( \{N_n\} \) are independent and dependent. Of interest here is that no special properties of \( Y \) are needed to prove the results beyond those obtained from the basic assumption so that many random index limit theorems can be obtained as special cases. Section 4 deals with the convergence of \( Y_n(1) \) and \( Y_n'(1) \). Such results have been of interest in estimation theory using random sample sizes. We also give a counterexample to a conjecture of Mogyorodi and Giniasu.

2. **Consequences of the Basic Assumption**

**Theorem 1:** If the basic assumption holds, then for each \( s > 0 \) the following two limits exist

\[
\lim_{n \to \infty} \frac{s_{[ns]}}{a_n} = \alpha(s) > 0 \quad (a)
\]

\[
\lim_{n \to \infty} \frac{b_{[ns]} - b_n}{a_n} = \beta(s) \quad (b)
\]

and satisfy

\[\{Y(st); t > 0\} \overset{d}{=} \{\alpha(s) Y(t) + \beta(s); t > 0\} \quad (c)\]

Furthermore, for some constant \( \hbar \) one of the following possibilities holds:

(i) \( \alpha(s) = s^o, \rho > 0 \) \hspace{1cm} \( \beta(s) = h(s^o - 1) \)

(ii) \( \alpha(s) = 1 \) \hspace{1cm} \( \beta(s) = h \log s \)

(iii) \( \alpha(s) = s^o, \rho < 0 \) \hspace{1cm} \( \beta(s) = h(1-s^o) \)
Proof: Let \( T = \{ t > 0 : P[Y(t) \neq Y(t^-)] = 0 \} \). From [1], p. 124, \((0, \infty)\)-\(T\) is countable. Pick \( t, s > 0 \) so that \( t, st \in T \). Using Theorem 5.5 of [1] gives that if \( s_n \to s \)

\[
\left( X_{[ns_n]} - b_{[ns_n]} \right)/a_{[ns_n]} \Rightarrow Y(t)
\]

and

\[
\left( X_{[ns_n]} - b_{n} \right)/a_n \Rightarrow Y(st).
\]

Now by the convergence to types theorem there exist \( \alpha(s) > 0 \) and \( \beta(s) \) which satisfy (a), (b), and

\[
Y(st) \overset{d}{=} \alpha(s) Y(t) + \beta(s)
\]

so that

\[
\alpha(st) Y(1) + \beta(st) \overset{d}{=} \alpha(s) \alpha(t) Y(1) + \alpha(s) \beta(t) + \beta(s)
\]

(where we have assumed for convenience \( 1 \in T \)). Therefore for \( t, st \in T \)

\[
\alpha(st) = \alpha(s) \alpha(t)
\]

\[
\beta(st) = \alpha(s) \beta(t) + \beta(s)
\]

for which (i), (ii), (iii) are the only measurable solutions with \( \alpha(1) = 1 \).

To show equality of the finite dimensional distributions in (c) it suffices to repeat the above procedure using a multivariate analogue of the convergence to types theorem. See for example [3], p. 1154.
DeHaan (1970 or 1974) describes the variation in (ii) by saying $b_{[\cdot]}$ is $\pi$-varying (or in the class $\pi$) with auxiliary function $a_{[\cdot]}$. The function $a_{[\cdot]}$ in this case is slowly varying. Cases (i) and (iii) correspond to the regular variation of $b_{[\cdot]}$ and $b_{\infty} = b_{[\cdot]}$ respectively. Note that since (a) and (b) hold for $s_n \to s$ the convergence in (a) and (b) is uniform on compact subsets of $(0, \infty)$.

Examples of the convergence described in (i) are weak convergence to stable laws of index $1/\rho$ ($0 < 1/\rho < 2$, $\rho \neq 1$) or to the extremal process generated by $\Phi_{1/\rho}(x) = \exp(-x^{-1/\rho})$, $x > 0$.

The variation in (iii) arises in weak convergence to the extremal process generated by $\Psi_{1/\rho}(x) = \exp(-|x|^{1/\rho})$, $x \leq 0$, $= 1$, $x > 0$.

Finally situation (ii) arises when the limit $Y$ is an extremal process generated by $\Lambda(x) = \exp(-e^{-x})$ and also (as pointed out to us by Laurens de Haan) in connection with weak convergence to a stable law of index 1 as follows: If $\xi_n$, $n \geq 1$ and iid rv's in the domain of attraction of a stable law of index 1, set $\bar{X}_n = \sum_{i=1}^{n} \xi_i/n$. Then there exist $a_n$, $b_n$ varying as described in (ii) such that $(\bar{X}_{[n\cdot]} - b_n)/a_n \Rightarrow Y$.

Notice that for alternatives (ii) and (iii) if $h > 0$ we have $Y(t) \overset{P}{\to} -\infty$ as $t \to 0$, a situation which was excluded by Lamperti [6] who assumed convergence of the finite dimensional distributions $(Y(t_1), Y(t_2), \ldots, Y(t_k))$, $0 \leq t_1 < t_2 < \cdots < t_k$, for any $k$. 

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THEOREM 2: If \( \{X_n, n \geq 0\} \) satisfies the basic assumption, then \( Y \) is stochastically continuous.

Proof. If \( u \in T \), then for each \( \varepsilon > 0 \)

\[
\lim_{s \to u} P[|Y(s) - Y(u)| > \varepsilon] = 0
\]

From Theorem 1, \( \{Y(s^\tau), \tau > 0\} \overset{d}{=} (\alpha(s) Y(\tau) + \beta(s), \tau > 0) \). If \( t > 0 \) is chosen arbitrarily and \( s = t/u \) then letting \( v' = v/s \) gives

\[
\lim_{v' \to u} P[|Y(v') - Y(t)| > \varepsilon] = \lim_{v' \to u} P[|Y(sv') - Y(su)| > \varepsilon]
\]

Since \( (Y(sv'), Y(su)) \overset{d}{=} (\alpha(s) Y(v') + \beta(s), \alpha(s) Y(u) + \beta(s)) \), the above equals

\[
\lim_{v' \to u} P[|Y(v') - Y(u)| > \varepsilon/\alpha(s)] = 0
\]

(recall \( u \in T \)). Therefore \( Y \) is left stochastically continuous at \( t \) and since right stochastic continuity follows from \( Y \) being a.s. right continuous the proof is complete.

3. Functional Limit Theorems with Random Indices

Throughout this section we assume the basic assumption holds. We will be concerned with conditions that guarantee
(1) \[ Y_{N_n} \Rightarrow Y \]

and

(2) \[ Y'_n = \frac{X_{[N_n]}}{a_n} - b_n \Rightarrow Y' \]

where \([N_n]\) are integer valued random variables. The easiest and most familiar conditions arise when the random indices and sequence of processes are assumed independent.

**Theorem 3:** If for each \(n\), \(N_n\) and \(Y_n\) are independent then 
\[ N_n \xrightarrow{d} \infty \] is sufficient for (1) and 
\[ N_n/n \xrightarrow{d} N \] with 
\[ P(0 < N < \infty) = 1 \] is sufficient for (2).

**Proof:** To prove the first statement let \(A\) be a Borel subset of 
\[ D = D(O, \infty) \] such that \(P(Y \in \partial A) = 0\). Then as \(n \to \infty\), 
\[ P(Y_n \in A) \to P(Y \in A) \] so if \(N_n \xrightarrow{d} \infty\):

\[ P(Y_{N_n} \in A) = \sum_{k=0}^{\infty} P(Y_k \in A) P(N_n = k) \to P(Y \in A). \]

To prove the other result observe that by Theorem 3.2 in [1] 
\( (Y_n', N_n/n) \Rightarrow (Y,N) \). For \(t > 0\) and \(f \in D\) let \(\psi(f,t) = f(t) \in D\). \(\psi\) is continuous on \(D \times R\) so by the continuous mapping theorem 5.1 in [1], 
\[ Y'_n = \psi(Y_n', N_n/n) \Rightarrow Y(N) = Y(N') = \alpha(N) Y(\cdot) + \beta(N) \]
where the last equality follows from independence and Theorem 1.
Observe that for the proof of the second result it was enough that \((Y_n, N/n) \Rightarrow (Y, N)\) with \(P(0 < N < \infty) = 1\). This hypothesis is not sufficient to prove (1), but it is to show \(Y_n \Rightarrow Y\).

**THEOREM 4**: If \((Y_n, N/n) \Rightarrow (Y, N)\) with \(P(0 < N < \infty) = 1\) then \(Y_n' \Rightarrow Y(N')\) and \(Y_n \Rightarrow [Y(N') - \beta(N)]/\alpha(N)\).

**Proof.** The first result was observed above. To prove the second note that as above, the mapping \(\psi'(f, t) = (f(t'), t)\) is continuous \(D \times R_+ \to D \times R_+\) so that

\[
(Y_n', N/n) = \psi'(Y_n', N/n) \Rightarrow \psi'(Y, N) = (Y(N'), N).
\]

Applying the theorem for sequences of continuous mappings (Theorem 5.5 in [1]) with

\[
\psi_n(f, x) = f(t) \frac{a_n}{a_n[xn]} + \left(\frac{b_n - b[xn]}{a_n}\right) \frac{a_n}{a_n[xn]}
\]

and the remark on uniform convergence following Theorem 1 gives

\[
Y_n = \frac{a_n}{a_n N_n} Y_n' + \frac{b_n - b N_n}{a_n N_n} \Rightarrow \frac{Y(N') - \beta(N)}{\alpha(N)}
\]

In Theorem 4 if \(Y\) and \(N\) are dependent, it is not necessarily true that \((Y(N') - \beta(N))/\alpha(N) \overset{d}{=} Y\) as the following examples show:
EXAMPLE 1. Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with mean zero and variance one, and let $S_k = X_1 + \cdots + X_k$. From Donsker's theorem (Theorem 10.1 in [1]), $W_n = S_{\lfloor n \rfloor}/n^{1/2}$ converges to a Brownian motion $W$ as $n \to \infty$. Let $\varrho : \mathbb{D} \to \mathbb{R}$ be the time of the last zero coming before 1, i.e.,

$$\varrho(f) = \inf\{t : f(s) \neq 0 \text{ for all } t < s < 1\} \wedge 1.$$

$\varrho$ is continuous at each continuous $f$ with $f(1) \neq 0$ which is not $= 0$ on any $[\varrho(f) - \epsilon, \varrho(f)]$ so using the continuous mapping theorem for $\psi(f) = (f, \varrho(f))$ gives $(W_n, \varrho(W_n)) \Rightarrow (W,T)$ with $P(0 < T < \infty) = 1$.

From Theorem 4, if $N_n = n\varrho(W_n)$ then $W_n \Rightarrow W(T^*)/\sqrt{T}$. Since $W(T) = 0$ (1) does not hold.

EXAMPLE 2. Let $\{Z_n, n \geq 0\}$ be i.i.d. r.v.'s such that $\exists a_n > 0, b_n$ and

$$\Pr[\bigvee_{i=0}^{n} Z_i \leq a_n x + b_n] \to \mathbb{\Lambda}(x) = \exp(-e^{-x})$$

Set $Y_n(t) = \lfloor nt \rfloor \bigvee_{i=0}^{j-1} Z_i / a_n$ so that $Y_n \Rightarrow Y$ where $Y$ is the extremal process generated by $\mathbb{\Lambda}(x)$ (see [26],[27], [31]).

Say $Z_1$ is a record value if $Z_j > \bigvee_{i=0}^{j-1} Z_i$ and let $L_0 = 0 < L_1 < L_2 < \cdots$ be the indices at which records occur. Let $\mu(n)$ be the number of records among $Z_0, \ldots, Z_n$.

If $f \in \mathbb{D}$ define $\psi f = \inf\{s > 1 : f(s) - f(s- \neq 0) \text{ so that } \psi f$ is the time of the first jump in $f$ subsequent to 1. By the continuous mapping theorem
\[(Y_n', \psi Y_n) \implies (Y, \psi Y)\]

Note that
\[
\psi Y_n = \frac{\inf\{k > n | X_k \text{ is a record}\}}{n} = \frac{L_{\mu(n)+1}}{n}
\]

\[\text{index of first record after } n\]

Setting \(\psi Y = N\) we have
\[(Y_n', \frac{L_{\mu(n)+1}}{n}) \implies (Y, N)\]

and by Theorem 4
\[Y_{L_{\mu(n)+1}} \implies Y(N \cdot) - \log N\]

since for \(Y, \alpha(t) = 1, \beta(t) = \log t\). It is not the case that
\[Y(N \cdot) - \log N \not\sim Y\]

since \(Y\) is stochastically continuous but \(Y(N \cdot) - \log N\), has a fixed discontinuity at \(t = 1\).

If \(Y\) and \(N\) are independent then \((Y(N \cdot) - \beta(N))/\alpha(N) \not\sim Y\)

as a consequence of Theorem 1. A trivial condition for this is that
\[N_n/n \xrightarrow{P} \lambda, \text{ a positive finite constant.} \]

A general sufficient condition for this can be formulated using a mixing concept. The mixing results we will use can be found in [21]. See also [1], [12], [13].
DEFINITION. Suppose \( \{V_n; n \geq 0\} \) are random elements of a metric space \( S \) defined on \((\Omega, \mathcal{F}, P)\). The sequence \( V_n \) is R-mixing if for some \( V \), \( P(V \in \cdot | E) \rightarrow P(V \in \cdot) \) for all \( E \in \mathcal{F} \) such that \( P(E) > 0 \).

The reason for using this property can be seen in the following characterization [21]:

**Lemma 1.** If \( V_n \implies V \) then \( V_n \) is R-mixing if and only if for any sequence of random elements \( Z_n \) of a metric space \( S' \) such that \( Z_n \overset{P}{\rightarrow} Z \) we have \( (V_n, Z_n) \implies (V, Z) \) where \( V \) and \( Z \) are independent.

From this lemma we can immediately conclude

**Theorem 5.** If \( Y_n \) is R-mixing and \( N_n / n \overset{P}{\rightarrow} N \) with \( P(0 < N < \infty) = 1 \) then \( Y_n \overset{P}{\rightarrow} Y \) and \( Y_n' \overset{P}{\rightarrow} \alpha(N)Y + \beta(N) \).

**Proof.** From Theorem 4 and Lemma 1, \( Y_n \overset{P}{\rightarrow} [Y(N') - \beta(N)]/\alpha(N) \) where \( Y \) and \( N \) are independent so by Theorem 1 the limit has the same law as \( Y \). The second statement follows from the calculations in Theorem 4.

To apply Theorem 5 to examples the following result is often useful ([2], p. 44)

**Lemma 2.** Suppose \( \{V_n\} \) are random elements on \((\Omega, \mathcal{F}, P)\) with \( F = \bigcap_{n \geq 1} \mathcal{G}(V_j; j \geq n) \). If \( F \) is trivial under \( P \) then \( \forall \in F \)

\[
\lim_{n \to \infty} \sup_{A} |P(\mathcal{A}E) - P(\mathcal{A})P(E)| = 0
\]
where the sup is over \( A \in \mathcal{A} (V_j, j \geq n) \).

Observing that for R-mixing it suffices that \( P[V_n \varepsilon | E] \to P[V \varepsilon] \) for \( E \) in a semi-algebra generating \( \mathcal{A} (V_n, n \geq 0) \) and using Lemma 2 we see that if \( \{X_n; n \geq 0\} \) has a trivial tail \( \sigma \)-field then \( Y_n \) is R-mixing. Using the Hewitt-Savage zero-one law now gives the conclusions of Theorem 5 for random walks and partial maxima of i.i.d.r.v.'s. For examples of dependent sequences for which the mixing property can be verified and Theorem 5 applied see McLeish (1975).

For the results in section 2 we have divided \( N_n \) by \( n \) and assumed \( P[0 < N < \infty] = 1 \). For some applications, particularly to supercritical branching processes, these assumptions are undesirable. The next two theorems give the analogues of Theorem 5 without these two conditions.

**THEOREM 6.** If \( Y_n \) is R-mixing and \( N_n/n \xrightarrow{P} N \) with \( P[N = 0] < 1 \), then under \( Q(\cdot) := P[\cdot | N > 0] \), \( Y_n \equiv Y_n \xrightarrow{\alpha} Y, Y_n \equiv \alpha(N)Y + \beta(N) \).

**Proof.** From the definition of mixing \( Q[Y_n \varepsilon] \equiv P[Y \varepsilon] \). If \( Q(E) > 0 \) then \( Q[Y_n \varepsilon | E] = P[Y_n \varepsilon | E, N > 0] \equiv P[Y \varepsilon] \) so \( Y_n \) is R-mixing with respect to \( Q \). Furthermore \( Q[(N_n/n - N) > \varepsilon] \to 0, n \to \infty \). We can obtain the desired conclusions now from Theorem 5.

**Remark:** Theorem 6 has been shown for random sums of independent and dependent r.v.'s by Jagers [20] and Dion [19] respectively.
THEOREM 7. If $Y_n$ is $R$-mixing and there exist integers $C(n)$ such that $N_n/C(n) \xrightarrow{p} N$, $P(0 < N < \infty) = 1$ then

$$
(X_{[N_n/m]} - b_{C(m)})/a_{C(m)} \xrightarrow{d} \alpha(N)Y + B(N)
$$

and

$$
Y_{N_n} \xrightarrow{d} Y.
$$

Proof. Mixing entails $(Y_{C(M)}, N_{m/m}/C(m)) \Rightarrow (Y, N)$. Applying the continuous map $\psi$ of Theorem 4 gives the first result and the second result is obtained as in Theorem 4.

REMARKS: Most of the theorems in this section have been shown previously, at least in special cases. The first result in Theorem 3 is due to Renyi [22] and the second was shown by Thomas [15] aided by de Haan [10], assuming that $N_{m/m} \Rightarrow N$ with $P(0 < N < \infty) = 1$ and "the normalizing constants are computed from a c.d.f. $F$ which is in the domain of attraction of one of the extreme value distributions" (i.e., the conclusions of Theorem 1 hold). A proof of Theorem 4 can be extracted from proofs given by Billingsley for two special cases (Theorems 17.2 and 20.3 in [1]). Theorem 5 has been shown for the special cases of random walks converging to Brownian motion and partial maxima by Sreehari (1968) and Sen (1972) respectively.
4. Convergence of \( Y_n(1) \) and \( Y_n'(1) \)

To complete our study, we will derive conditions for the convergence of \( Y_n(1) \) and \( Y_n'(1) \). In some cases this will require additional hypotheses since the convergence of processes discussed above occurs in \( D(0,\infty) \) and so we only have convergence of one-dimensional distributions for times at which the limit process \( Y(N') \) is continuous in probability.

**EXAMPLE 3.** Let \( X_1, X_2, \ldots \) be i.i.d.r.v.'s and \( S_n = X_1 + \cdots + X_n \). Suppose there is a sequence \( a_n \) so that \( V_n = \frac{S_n}{a_n} \rightarrow V \), a stable process of index \( \alpha \), \( 0 < \alpha < 2 \). Let \( \mathcal{G}(f) = \inf \{ t \mid |f(t) - f(t^-)| \geq \varepsilon \} \), \( N_n = \mathcal{G}(V_n') + (-1)^n \) and \( N = \mathcal{G}(V) \). As in Example 1 it follows from the continuous mapping theorem that \( (V_n, \frac{N_n}{n}) \rightarrow (V, N) \) and from Theorem 4, \( V_n \rightarrow \frac{V(N) - \beta(N)}{\alpha(N)} \) but

\[
\begin{align*}
\lim_{n \to \infty} V_{2n-1} &\rightarrow (V(N) - \beta(N))/\alpha(N), \\
\lim_{n \to \infty} V_{2n} &\rightarrow (V(N) - \beta(N))/\alpha(N)
\end{align*}
\]

and since \( |V(N) - V(N^-)| \geq \varepsilon \), \( V_n(1) \) does not converge.

The reason that convergence fails in Example 3 is that \( P(V(N) \neq V(N^-)) > 0 \). If \( V \) and \( N \) are independent then since \( V \) is stochastically continuous \( P(V(N) \neq V(N^-)) = 0 \), and the convergence of \( V_n(1) \) will follow from that of \( (V_n', \frac{N_n}{n}) \) by the continuous mapping theorem.
Using this observation and Theorem 5 gives

THEOREM 8. If $Y_n$ satisfies the basic assumption and is R-mixing, and if $N_n/n \xrightarrow{P} N$ with $P[0 < N < \infty] = 1$, then $Y_n(n) \Rightarrow Y(1)$ and $Y_n(n) \Rightarrow \alpha(N) Y(1) + \beta(N)$.

The hypothesis of Theorem 8 contains more than is required to prove the result and we now discuss the following generalization of Theorem 8 first stated by Guiasu (1971).

THEOREM 9. If $Y_n(1) \Rightarrow Y$ and

1. $N_n/n \xrightarrow{P} N$ with $P(0 < N < \infty) = 1$

2. $Y_n(n)$ is R-mixing with respect to $\sigma(N)$, that is, for each $A$ such that $P(N \in A) > 0$

\[ P(Y_n(n) \epsilon \cdot |N \in A) \Rightarrow P(\epsilon) \]

3. for each $B$ such that $P(N \in B) > 0$,

\[ \lim_{c \to 0} \limsup_{n \to \infty} P\left( \max_{|m-n| < nc} |Y_m(n) - Y_n(n)| > \epsilon |N \in B\right) = 0 \]

Then $Y_n(n) \Rightarrow Y(1)$.

Note that we do not suppose the basic assumption holds so we cannot apply the continuous mapping theorem as we did in Theorems 7 and 8. We can prove Theorem 9 however by using computations from the proof of the continuous mapping theorem as a substitute for applying the result.
Proof. Given $\epsilon, \eta > 0$. Choose $s > 1$ so large that $P[N \notin [1/s, s]] < \epsilon$. Let $m$ be a positive integer and let $\delta = (s - 1/s)/m$ (so that $\delta \to 0$ if $m \to \infty$) and $t_k = 1/s \div k\delta$, $k = 0, \ldots, m$. If $Y_{N_n}(1) \leq x$, $N \in [1/s, s]$ and $|N_n/n - N| \leq \delta$ then $t_{k-1} \leq N < t_k$ for some $k$ and $Y_{[nt_k]}(1) \leq x + \eta$ unless the oscillation of $Y_m(1)$ when $nt_{k-2} \leq m \leq nt_{k+1}$ is $> \eta$.

Therefore

$$P[Y_{N_n}(1) \leq x] \leq \epsilon + P[|N_n/n - N| > \delta] \quad (1')$$

$$+ \sum_{k=1}^{m} P[t_{k-1} \leq N < t_k, Y_{[nt_k]}(1) \leq x + \eta] \quad (2')$$

$$+ \sum_{k=1}^{m} P[t_{k-1} \leq N < t_k, \max_{nt_{k-2} \leq m \leq nt_{k+1}} |Y_m(1) - Y_{[nt_k]}(1)| > \eta] \quad (3')$$

Using the three hypotheses to evaluate the limits of the correspondingly numbered terms above we obtain that

$$\lim_{s \to \infty} \lim_{n \to \infty} (1') = 0$$

$$\lim_{\delta \downarrow 0} \sup_{n \to \infty} (3') = 0$$

and

$$\lim_{n \to \infty} (2') = P[N \in [1/s, s]] P[Y \leq x + \eta]$$

if $x + \eta$ is a continuity point. Therefore

$$\lim_{n \to \infty} \sup_{n} P[Y_{N_n}(1) \leq x] \leq P[Y \leq x]$$

at continuity points. A related lower bound is derived similarly and the proof is complete.
Mogyoródi (1967) and Guiasu (1971) have conjectured that for Theorem 8 it was sufficient that (3) hold for $B = (0, \infty)$ (this is Anscombe's condition—see [16]). The following example shows that this is not true.

**EXAMPLE 4.** Let $U, Z_1, Z_2, \ldots$ be independent rv's such that $U$ has a uniform distribution on $(0, 1)$ and for each $n \geq 1$, $Z_n$ has an $N(0,1)$ distribution. Let $I_n = [2^nU] + 1$, $S_n^0 = \sum_{k=1}^{n} Z_k$, $X_1 = Z_1$. If $n \geq 1$ and $I_n = k$ with $1 \leq k < 2^n$ then for $1 \leq j < 2^n$ set

$$X_{2^n-1+j} = \begin{cases} 
Z_{2^n-1+j} & \text{j} \not\in \{k, k+1\} \\
-\delta^0 & \text{j} = k \\
\delta^0 & \text{j} = k+1
\end{cases}$$

Finally let $S_n = \sum_{k=1}^{n} X_k$ so that if $n \geq 1$, $k < 2^n$ we have

$$S_{2^n-1+k} = \begin{cases} 
\delta^0 & \text{if } I_n \neq k \\
2^n-k & \text{if } I_n = k
\end{cases}$$

To check that $S_n/\sigma \sqrt{n}$ is $R$-mixing and converges to $W(1) \overset{d}{=} Z_1$ note that for each $a < b$

$$|P(S_n^0/\sigma \sqrt{n} \in [a,b] \mid E) - P(S_n/\sigma \sqrt{n} \in [a,b] \mid E)| \leq P(S_n \neq S_n^0)/PE$$
and by construction the RHS → 0 as \( n \to \infty \). Let \( N_n = 2^n - 1 + I_n \) then \( \frac{N_n}{2^n} \xrightarrow{a.s.} 1 + U \) but \( S_{N_n} = 0 \) on \( \{ U < 1 - 2^{-n} \} \) and

\[
\limsup_{n \to \infty} P \left\{ \max_{|m-n| < nc} \left| \frac{S_m}{\sigma \sqrt{m}} - \frac{S_n}{\sigma \sqrt{n}} \right| \geq \epsilon \right\}
\]

\[
\leq \limsup_{n \to \infty} P \{ \text{for some } m, \, |m-n| < nc, \, S_m \neq S_n^0 \}
\]

\[
+ P \left\{ \max_{1-c \leq t \leq 1+c} \left| \frac{W(t)}{\sqrt{t}} - W(1) \right| \geq \epsilon \right\}.
\]

If \( c < 1/2 \) and \( \ell = [\log_2 n] \) then the first term of the RHS is

\[
\leq \limsup_{n \to \infty} (2[cn] + 1)/2^{\ell+1} < \infty.
\]

From the path continuity of Brownian motion the second term → 0 as \( c \to 0 \), so Anscombe's condition is satisfied. Of course, (2) is not satisfied since

\[
\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} P \left\{ \max_{|m-n| < nc} \left| \frac{S_m}{\sigma \sqrt{m}} - \frac{S_n}{\sigma \sqrt{n}} \right| \geq \epsilon \right\} = P(\{|W(1+U)| > 0\}) = 1.
\]

**REMARKS.** Theorem 9 has evolved slowly. Anscombe (1952) proved the result for random walks converging to Brownian motion in the case where \( P(N=1) = 1 \). His proof was extended to the case where \( N \) has a discrete distribution by Renyi (1965), and to the case \( P(0 < N < \infty) = 1 \) by Blum, Hanson, and Rosenblatt (1963). Later the result was shown by Wittenberg (1964) for convergence to stable processes and by Barndorf-Neilsen (1964) for convergence to extremal processes. After the work of Richter (1965) and Mogyoródi (1967), Guisau (1971) formulated and proved Theorem 9. The proof given here is new.
REFERENCES

I. General


II. Mixing and Random Index Limit Theorems


III. Limit Theorems for Partial Sums


IV. Limit Theorems for Maxima


