EXAMPLES FOR THE THEORY OF
INFINITE ITERATION OF SUMMABILITY METHODS

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PERSI DIACONIS

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1. Introduction

Gärtner and Knopp [8] introduced the notion of infinite iteration of Cesaro ($C_1$) averages, which they called $H_\infty$ summability. Fleischer [7] (apparently unaware of [8]) produced the first nontrivial example of an $H_\infty$ summable sequence: the sequence $\{a_i\}_{i=1}^\infty$ where $a_i$ is 1 or 0 as the lead digit of the integer $i$ is one or not. Duran [3] has provided an elegant treatment of $H_\infty$ summability as a special case of summability with respect to an ergodic semi-group of transformations. Duran showed that logarithmic summability contains $H_\infty$ summability and that, for bounded sequences, the $H_\infty$ method was equivalent to Banach-Hausdorff summability introduced by Eberlein [4].

In Section 2 of this paper it is shown that a bounded sequence can be assigned a limit by a finite number of iterations of $C_1$ density if and only if the sequence is $C_1$ summable to the same limit. The logarithmic method is introduced and shown to be equivalent to the more widely used zeta (or Dirichlet) density. An elementary proof of the inclusion of the $H_\infty$ method in the logarithmic method is given. Similar results are stated for iterates of the logarithmic method.

In Section 3 examples are given of subsets of the integers which differentiate between the summability methods of Section 2. Roughly stated, any set of integers with polynomial gaps has $C_1$ density; if the gaps are exponential then the set of integers has log density (but not $C_1$ density). The set of integers will have $H_\infty$ density (but not $C_1$ density) if and only if the gaps are linear exponential.
2. Definitions and Basic Theorems

Let $M$ be the Banach space of all bounded sequences of real numbers $(x_1, x_2, \ldots)$ with norm $\|x\| = \sup_n |x_n|$. For $x \in M$ write

$$d(x, n, l) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

and inductively define $d(x, n, k) = \frac{1}{n} \sum_{i=1}^{n} d(x, n, k-1)$ for $k > 1$. Clearly $\liminf \frac{d(x, n, k)}{n} \geq \liminf \frac{d(x, n, k-1)}{n}$. Similarly the upper limits are decreasing in $k$.

Definition

$x \in M$ is said to be $H_k$ summable to $c$ if $\lim_n d(x, n, k) = c$ ($H_1$ summability is usually called $C_1$ summability). $x$ is said to be $H_\infty$ summable to $c$ if $\lim \frac{\liminf_n d(x, n, i)}{i} = \lim \frac{\limsup_n d(x, n, i)}{i}$.

Lemma 1

For $x \in M$, $k < \infty$, $x$ is $H_k$ summable to $c$ if and only if $x$ is $C_1$ summable to $c$.

Proof

It is elementary that if $x$ is $C_1$ summable to $c$ then $x$ is $H_k$ summable to $c$ for all $k$. For the converse, Bromwich ([1], pg. 423) gives the following Tauberian theorem: If $b_n$ is a real sequence satisfying $n(b_n - b_{n+1}) < M$ then $\lim \frac{1}{n} \sum_{i=1}^{n} b_i = c$ implies $\lim b_n = c$. Assume that $x$ is $H_k$ summable to $c$; take $b_i = \frac{1}{i} \sum_{j=1}^{i} d(x, j, k-1)$. An easy computation gives

$$n(b_n - b_{n+1}) = \left\{ \frac{1}{n+1} \sum_{i=1}^{n} d(x, i, k-1) \right\} - \frac{n}{n+1} d(x, n+1, k-1).$$
The numbers \( d(x,j,m) \) are uniformly bounded for all \( j \) and \( m \) so that 
\( n(b_n - b_{n+1}) \) stays bounded. This shows that \( x \) is \( H_{k-1} \) summable to \( c \) 
and inductively \( C_1 \) summable to \( c \).

\( H_k \) summability is discussed at some length in Hardy [10]. A consequence of Lemma 1 is that \( C_1 \) summability to \( c \) implies \( H_\infty \) summability to \( c \).

**Definition**

\( x \in M \) is said to be log summable to \( c \) if 
\[
\lim_{n} \frac{1}{\log n} \sum_{i=1}^{n} \frac{x_i}{i} = c.
\]

\( x \) is zeta summable to \( c \) if 
\[
\lim_{s \to 1^+} (s-1)^{\infty} \sum_{i=1}^{\infty} \frac{x_i}{i^s} = c.
\]

Zeta summability is used fairly regularly in analytic number theory 
where it is also known as Dirichlet density or analytic density (see Hasse [11], pg. 223-226, Serre [13], pg. 125, or Golumb [9]). Ishiguro [12] and 
Hardy ([10], pg. 87) discuss other summability methods equivalent to log 
summability.

**Theorem 1**

Log and zeta summability are equivalent on \( M \).

**Proof**

Let \( x \in M \). Since the sequence \( x_i \) is bounded, there is no loss 
in generality in assuming \( x_i > 0 \). Define a measure on the positive real 
numbers with mass \( \frac{x_i}{i} \) at the points \( \log i \). Let the distribution function 
of the measure be 
\[
U(x) = \sum_{\log i < x} \frac{x_i}{i}.
\]
The Laplace transform of \( U \) is
\[
\omega(t) = \sum_{i=1}^{\infty} \frac{x_i}{i} e^{-t \log i} = \sum_{i=1}^{\infty} \frac{x_i}{i^{t+1}}.
\]
Theorem 2 in Feller [6], pg. 445, implies that \( \lim_{t \to 0} \omega(t) = \ell \) if and only if \( \lim_{x \to \infty} \frac{U(x)}{x} = \ell \). Let \( s = t + 1 \), \( x = \log y \); this becomes

\[
\lim_{s \to \infty} (s-1) \sum_{i=1}^{\infty} \frac{x}{s} = \ell \text{ if and only if } \lim_{y \to \infty} \frac{1}{\log y} \sum_{i \leq y} \frac{x}{i} = \ell .
\]

Duran [3] gives a useful necessary and sufficient condition for a matrix method to dominate \( H_\infty \) summability. The proof depends on a theorem announced by Eberlein [5]. As a special case, Duran showed that if \( x \in M \) has \( H_\infty \) limit \( c \) then \( x \) is log summable to \( c \). The next theorem is an elementary proof of this result.

**Theorem 2**

If \( x \in M \) is \( H_\infty \) summable to \( \ell \) then \( x \) is log summable to \( \ell \).

**Proof**

Writing \( S_n = \sum_{i=1}^{n} x_i \), summation by parts shows that for \( m < n \),

\[
(2-1) \quad \sum_{i=m}^{n} \frac{x_i}{i} = \sum_{i=m}^{n} \frac{S_i}{i(i+1)} + O(1) = \sum_{i=m}^{n} \left( \frac{S_i}{i} \right) \frac{1}{i} + O(1).
\]

Inductively from (2-1), for each fixed \( k \),

\[
(2-2) \quad \sum_{i=m}^{n} \frac{x_i}{i} = \left\{ \sum_{i=m}^{n} d(x,i,k) \frac{1}{i} \right\} + O_k(1)
\]

\[
> \inf_{i > m} d(x,i,k) \{ \log n - \log m + O(1) \} + O_k(1).
\]

Divide both sides of (2-2) by \( \log n \) and let \( n \) go to \( \infty \) to get

\[
(2-3) \quad \lim \inf_{n} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{i} = \lim \inf_{n} \frac{1}{n} \sum_{i=m}^{n} \frac{x_i}{i} > \inf_{i > m} d(x,i,k).
\]
Letting \( m \) go to \( \infty \) in (2-3) yields, for each \( k \),

\[
\lim \inf_n \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{i} \geq \lim \inf_k d(x,i,k) .
\]

A similar inequality holds for the upper limits; thus

\[
\lim \inf_k d(x,n,k) \leq \lim \inf_n \frac{1}{\log n} \sum_{i=1}^{n} \frac{x_i}{i} \\
\leq \lim \sup_n \frac{1}{\log n} \sum_{i=1}^{n} \frac{x_i}{i} \leq \lim \sup_k d(x,n,k) .
\]

This proves the theorem.

Entirely similar results can be derived for iterates of log density. It can be shown that \( k \)th log summability (defined in the obvious way) is equivalent to log summability, and that infinite iteration of log summability (\( L_{\infty} \) summability) is dominated by the matrix method with

\[
(i,j) \text{ entry } \begin{cases} 
\frac{1}{(\log \log i)j \log j} , & 2 \leq j \leq i \\
0 , & \text{elsewhere}.
\end{cases}
\]

Details may be found in Diaconis [2]. Duran [3] discusses other related results.
3. Examples

Let $A$ be a subset of the integers $\{1, 2, 3, \ldots\} = \mathbb{N}$. Let $a_1$ be the indicator function of the set $A$. The convergence properties associated with the vector $a = (a_1, a_2, a_3, \ldots) \in \mathbb{M}$ allow a natural definition of various notions of density of the set $A$. Thus $A$ is said to have $C_1$ density $\lambda$ if $a$ is $C_1$ summable to $\lambda$. Similar conventions will be used for $\log$ and $H_\infty$ summability.

In this section $[x]$ denotes the greatest integer less than or equal to $x$, $(x) = x - [x]$ denotes the fractional part of $x$, and for real numbers $s$ and $t$, $\langle s, t \rangle = \{i \in \mathbb{N}: s \leq i \leq t\}$. In what follows, $f$ and $g$ will denote polynomials written

$$f(x) = ax^n + bx^{n-1} + d_{n-2}x^{n-2} + \cdots + d_0,$$

$$g(x) = cx^n + cx^{n-1} + e_{n-2}x^{n-2} + \cdots + e_0.$$

To rule out trivial cases, assume that

$$\deg f = \deg g,$$

(3-1) both leading coefficients are positive and equal, and

$$0 < \frac{c-b}{na} < 1.$$

In all cases where one of the assumptions in (3-1) is violated it is straightforward to check that the set $\bigcup_{i=1}^{\infty} \langle g(i), f(i) \rangle$ is either finite or has finite complement.
Theorem 3

Let \( f \) and \( g \) be polynomials satisfying the assumptions (3-1). Let \( A = \bigcup_{i=1}^{\infty} \langle f(i), g(i) \rangle \).

Case 1: If \( n = 1 \) and \( a \) is irrational, then \( A \) has \( C_1 \) density \( \frac{c-b}{a} \).

Case 2: If \( n \geq 2 \), then \( A \) has \( C_1 \) density \( \frac{c-b}{na} \).

Proof

Case 1. The set \( \langle ai+b, ai+c \rangle \) contains either \( [c-b] \) or \( [c-b] + 1 \) points. It contains \( [c-b] + 1 \) points if and only if \( 1-|c-b| \leq |ai+b| \leq 1 \). Since \( a \) is irrational, the number, \( \gamma(k) \), of sets \( \langle ai+b, ai+c \rangle, 1 \leq i \leq k \), which contain \( [c-b] + 1 \) points is \( k[c-b] + o(k) \). Thus

\[
\limsup_n \frac{1}{n} \sum_{i=1}^{n} a_i = \limsup_k \frac{1}{g(k)} \sum_{i \leq g(k)} a_i
\]

\[
= \limsup_k \frac{1}{ak+c} \{ k[c-b] + \gamma(k) \} = \frac{c-b}{a}.
\]

A similar argument yields the same lower limit, concluding the proof of Case 1.

Case 2. The number of integers in the set \( \langle g(i), f(i) \rangle \) is \( f(i) - g(i) + o(1) \) as \( i \to \infty \). Thus

\[
\limsup_n \frac{1}{n} \sum_{i=1}^{n} a_i = \limsup_k \frac{1}{g(k)} \sum_{i \leq g(k)} a_i
\]

\[
= \limsup_k \frac{1}{g(k)} \sum_{i=1}^{k} \{ g(i) - f(i) + o(1) \}
\]

\[
= \limsup_k \frac{1}{g(k)} \left\{ \frac{(c-b)k^n}{n} + o(k^{n-1}) + o(k) \right\} = \frac{c-b}{na}.
\]

A parallel argument for the lower limit concludes the proof.
Theorem 4

Let \( f \) and \( g \) satisfy (3-1). Let \( A = \bigcup_{k=0}^{\infty} \left\{ 10^{f(k)}, 10^{g(k)} \right\} \). Then \( A \) has log density \( \frac{c-b}{na} \) but not \( C_1 \) density.

Proof

To simplify notation, write \( t(x) = 10^x \). Standard bounds for the sum \( \sum_{m=p}^{q} \frac{1}{m} \) yield

\[
\limsup_n \frac{1}{\log n} \sum_{i=1}^{n} \frac{a_i}{i} = \limsup_k \frac{1}{\log 10} \sum_{i \leq t(g(k))} \frac{a_i}{i}.
\]

The last sum is

\[
\sum_{i=1}^{k} \text{meas}\{t(f(i), t(g(i))\} \frac{1}{m} = \sum_{i=1}^{k} \left( \log \frac{t(g(i))}{t(f(i))} + O\left( \frac{1}{t(f(i))} \right) \right)
\]

\[
= \log 10 \sum_{i=1}^{k} \{g(i) - f(i)\} + O(1).
\]

Making the substitution leads to

\[
\limsup_n \frac{1}{\log n} \sum_{i=1}^{n} \frac{a_i}{i} = \frac{c-b}{na}
\]

as required. Again, the lower limit follows from similar arguments.

The proof of Theorem 5 below shows that \( A \) does not have \( C_1 \) density in the case that \( f \) and \( g \) are linear. In fact, the limit points of the sequence \( \frac{1}{n} \sum_{i=1}^{n} a_i \) form the interval

\[
\left[ \frac{10^{c-b}-1}{10^a-1}, \frac{10^{a+(b-c)}(10^{c-b}-1)}{10^a-1} \right]
\]
in the linear case. If \( f \) and \( g \) are quadratic or higher degree polynomials, arguments similar to that of Theorem 6 show that \( A \) does not have \( H_\infty \) density. Thus \( A \) does not have \( C_1 \) density. Detailed proofs for the nonexistence of \( C_1 \) density in those cases are recorded in Diaconis [2].

**Theorem 5**

Let \( 0 < \frac{c-b}{a} < 1 \). Then \( A = \bigcup_{k=0}^{\infty} \left\{ 10^{ak+b}, 10^{ak+c} \right\} \) has \( H_\infty \) density \( \frac{c-b}{a} \).

**Proof**

Flehinger [7] proved this in the special case \( a=1, b=0, c=\log_{10} 2 \). Flehinger's proof generalizes in a straightforward if somewhat longwinded way to yield the results stated. Further details can be found in Diaconis [2].

**Theorem 6**

The set \( A = \bigcup_{k=0}^{\infty} \left\{ 10^{k}, 10^{k+\frac{1}{2}} \right\} \) does not have \( H_\infty \) density. Rather, 

\[
\lim \inf_n d(a,n,k) = 0, \quad \lim \sup_n d(a,n,k) = 1 \quad \text{for every } k .
\]

**Proof**

Writing \( 10^y = t(y) \) and \( d(x,k) \) for \( d(a,x,k) \), consider \( x \) of the form \( x = t(n+\frac{s}{2}) \) where \( s \) is a real variable, \( 0 < \gamma_1 < s < 1 \), for \( \gamma_1 \) to be chosen later.

\[
d(x,1) = \frac{1}{x} \left\{ \sum_{k=1}^{n-1} \left\{ t\left(\left(k+\frac{1}{2}\right)^2 - t(k) + 0(1)\right) + t\left(\left(n+\frac{s}{2}\right)^2 - t(n^2)\right) \right\} \right\}.
\]

The largest term in the sum, when divided by \( x \), is

\[
t \left\{ (n-\frac{1}{2})^2 - \left(n+\frac{s}{2}\right)^2 \right\} = 0(t(-n)) ,
\]
where the implied constant is independent of \( s \) and \( n \). Thus, for \( \gamma_1 \leq s \leq 1 \),

\[
d(x,1) = 1 + O(nt(-n)) + o(t(-n)) = 1 + o(1)
\]

where the implied constant may depend on \( \gamma_1 \), but is independent of \( s \) and \( n \). This proves \( \lim \sup_x d(x,1) = 1 \). Assume inductively \( \gamma_1 \),

\[
0 \leq \gamma_1 \leq \gamma_2 \cdots \leq \gamma_j < 1,
\]

have been found such that for \( \gamma_1 < s \leq 1 \),

\[
d(t((n+\frac{s}{2})^2),i) = 1 + o(1) \quad \text{as} \quad n \to \infty.\]

We now shown for any \( \epsilon > 0 \),

\[
\gamma_j + \epsilon < s \leq 1 \quad \text{implies} \quad d(t((n+\frac{s}{2})^2),j+1) = 1 + o(1), \quad \text{as} \quad n \to \infty.
\]

\[
d(t((n+\frac{s}{2})^2),j+1) = \frac{1}{t((n+\frac{s}{2})^2)} \left[ t((n+\frac{\gamma_j+\epsilon}{2})^2) \sum_{k=1}^{\gamma_j+\epsilon} d(k,j) + \sum_{k=1}^{\gamma_j} d(k,j) \right].
\]

In the first sum, since \( \gamma_j + \epsilon - s \leq 1 \), dividing by \( t((n+\frac{s}{2})^2) \) shows

the first sum is \( O(t((\gamma_j+\epsilon-s)n)) = o(1) \), where the implied constant may depend on \( \epsilon \) but not on \( n \), \( s \). All terms in the second sum are

\( 1 + o(1) \). Making this substitution, the second sum is \( Y + o(Y) \) where

\[
Y = \frac{t((n+\frac{s}{2})^2) - t((n+\frac{1}{2})^2)}{t((n+\frac{s}{2})^2)} = 1 + o(1).
\]

Combining these estimates gives
\[ d(t((n + \frac{s}{2})^2), j+1) = 1 + o(1) + o(1 + o(1)) + o(1) = 1 + o(1), \]

as was to be shown. The result for upper limits now follows by taking
\[ \gamma_1 = \epsilon, \quad \gamma_2 = \gamma_1 + \frac{\epsilon}{2}, \quad \gamma_3 = \gamma_2 + \frac{\epsilon}{4}, \ldots \]

Essentially, the same estimates give the same results for the lower limits. Again \( x \) is chosen of the form \( x = t((n + \frac{s}{2})^2) \), but now \( 1 < \gamma_1 \leq s \leq 2 \). Then
\[ d(x, l) = O(t((l-s)\sqrt{n})) = o(1), \]
the rest of the proof following similarly.

**Remarks**

1. In the linear case of Theorem 3 the set \( A \) always has \( C_1 \) density even if the leading coefficient \( a \) is rational. Simple examples show that the density need not be equal to \( \frac{c-b}{a} \).

2. Theorem 4 shows that the set of integers with an even number of digits has log density \( \frac{1}{2} \).

3. It is possible, but notationally awkward, to extend Theorem 6 to any polynomials \( f(x), \ g(x) \) of degree greater than 1. The set \( A \) of Theorem 6 is \( \bigcup_{k=0}^{\infty} \left( 10^{k^2}, 10^{k^2 + k + \frac{1}{4}} \right) \). Theorem 4 shows that \( A \) has log density \( \frac{1}{2} \).

3. A computation similar to Theorem 5 shows that sets of the form \( \bigcup_{k=0}^{\infty} \left( 10^{10^k f(k)}, 10^{10^k g(k)} \right) \), where \( f \) and \( g \) are linear, have \( L_{\infty} \) density but not log density.

5. Theorem 6 together with the results of Section 6 of Duran [3] show that the logarithmic method is not a Hausdorff method.
References


