A TECHNICAL LEMMA IN THE STUDY OF
MAXIMA OF GAUSSIAN PROCESSES

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Introduction. Let \( \{X_t, \ t \geq 0\} \) be a stationary Gaussian process with mean zero, variance one and \( r(t) = \text{EX}_0 X_t \) as its covariance function. Assuming that the process is separable with continuous sample paths one can define \( M_T = \max_{0 \leq t \leq T} X_t \). Behavior of \( M_T \) is studied under two types of conditions on the covariance function viz "local" and "mixing". A fairly standard local condition assumed is

\[
(1) \quad 1-r(t) \sim C|t|^{\alpha}
\]

as \( t \to 0 \) where \( 0 < \alpha \leq 2 \) and \( C \) is some positive constant. (The constant sometimes is replaced by a slowly varying function which for all practical purposes behaves like \( (1) \). See for example Berman [3]). There is no such unanimity as far as mixing conditions are concerned. Two types of mixing conditions are in frequent use. The first type looks at the rate of decay of the covariance function. For example

\[
(2) \quad r(t)nt = o(1).
\]

(The symbol \( o(1) \) is used throughout when \( t \to \infty \). The other type is an integral condition. For example

\[
(3) \quad \int_0^\infty |r(t)|^p dt < \infty
\]

for some \( p > 0 \). The conditions \( (2) \) and \( (3) \) are not easily comparable and in general neither implies the other. Via an easy technical lemma a weaker condition \( (4) \) is proposed which may replace \( (2) \) or \( (3) \) in most of the proofs. This condition depends on "essential" rate of decay of \( r(t) \) to zero.
THE LEMMA. Suppose \{X(t)\} and \(r(t)\) are as before. For \(0 < \alpha \leq 2\) as in (1) and some constant \(k \geq 0\), define

\[ n = n(T) = T(\ell nT)^k \]

and

\[ u = u(T) = (2\ell nT)^{1/2} + ((k-1/2)\ell n\ell nT)/(2\ell nT)^{1/2} \]

Lemma 1. Let \(r(t)\) be \(o(1)\), satisfying (1) and

\[ \lambda\{t|0 \leq t \leq T ; r(t) > \frac{f(t)}{\ell nT}\} = o(T^\beta) \]

for some \(\beta\), \(0 \leq \beta < 1\) and some \(f(t) = o(1)\). The Lebesgue measure is denoted by \(\lambda\). Then

\[ n \sum_{j=\left[\frac{\ell nT}{n}\right]}^{n} |r(\frac{jT}{n})| \exp\left[ -\frac{u^2}{1+|r(\frac{jT}{n})|} \right] = o(1) \]

for every \(\varepsilon > 0\).

Proof. Define \(\delta = \sup_{s > \varepsilon} |r(s)|\) then \(0 \leq \delta < 1\). (Because \(r(t_0) = 1\) for \(t_0 \neq 0\) will imply that \(r(t) = 1\) i.o. which is contradictory to the assumption that \(r(t) = o(1)\)). Split the sum in (5) in two parts \([\delta \ell nT] \leq j \leq \left[\frac{\ell nT}{n}\right]\) and \(j > \left[\frac{\ell nT}{n}\right]\) for \(0 < \gamma < (1-\delta)/(1+\delta)\). The first part is at most

\[ n^{1+\gamma} \exp\left[ -\frac{u^2}{1+\delta} \right] \leq \exp((1+\gamma)(1+\delta)\ell nT + \frac{3(1+\gamma)}{\alpha} \ell n\ell nT) \]
having used the definitions of \( u \) and \( n \). The R.H.S. above is \( o(l) \) because of the choice of \( \gamma \). We now look at the second part of the sum in (5), i.e., when \( j > n^\gamma \). Let us define a set

\[
A_T = \left\{ T^\gamma \leq s \leq T \left| r(s) \right| > \frac{f(s)}{\ell ns} \right\}.
\]

Due to the continuity of \( r(t) \), \( A_T = \bigcup_{m=0}^{\infty} I_m \) where \( I_m \) is the union of all intervals in \( A_T \) of length \( \ell \), \( \frac{1}{m+1} < \ell < \frac{1}{m} \). Let \( m_0(T) = \left\lceil (\ell nT)^{2/\alpha} \right\rceil \).

The maximum number of intervals in \( \bigcup_{m=0}^{m_0} I_m \) and hence the maximum number of \( j \) such that \( \frac{jT}{n} \in \bigcup_{m=0}^{m_0} I_m \), is approximately \( T^{\frac{2}{\alpha}} (\ell nT)^{2/\alpha} \). Thus the sum in (5) when \( j > n^\gamma \) and \( \frac{jT}{n} \in \bigcup_{m=0}^{m_0} I_m \) is at most

\[
(6) \quad T^{1+\beta} (\ell nT)^{\frac{1}{\alpha}} \exp\{-u^2/(1+\delta(n^\gamma))\}
\]

where \( \delta(n^\gamma) = \sup_{j > n^\gamma} r(\frac{jT}{n}) \). Because \( r(t) = o(l) \), \( \delta(n^\gamma) = o(l) \).

Hence the expression in (6) tends to zero as \( T \to \infty \) \( \forall 0 < \beta < 1 \).

Because of (1) we know that \( r(t) \) satisfies a Lipschitz condition of order \( \alpha \) over the real line. (It could be deduced e.g. from Theorem 2.1.2 of Lukacs [4]). All the intervals in \( \bigcup_{m=0}^{\infty} I_m \) have length at most \( [(\ell nT)^{2/\alpha}]^{-1} \). Hence

\[
|r(s)-r(t)| \leq (\text{constant}) |s-t|^\alpha \leq (\text{const.})/(\ell nT)^2
\]

for all \( s \) and \( t \) in any one of the intervals belonging to \( \bigcup_{m=0}^{\infty} I_m \).
Thus we can see easily that

(7) \[ |r(t)| \leq f(t)/(\gamma nT) + \text{(const.)}/(\ell nT)^2 \]

for all \( t \in \bigcup_{m=0}^{\infty} I_m \). If \( t \in [0,T] \cap A_T^C \) then by definition of \( A_T \), we will have \( |r(t)| \leq f(t)/\ell nT \). Hence we can use (7) as the upper bound on \( r(t) \) for all \( t \in [0,T] \cap \bigcup_{m=0}^{\infty} I_m \). Thus the remaining sum in (5) when \( j > n^\gamma \) is at most

\[
\frac{n^2}{\ell \gamma nT} \left( \frac{f(T^\gamma)}{\ell nT} + \frac{\text{(const.)}}{\ell nT^2} \right) \exp\left[ -\frac{u^2}{(1 + \frac{f(T^\gamma)}{\gamma \ell nT} + \frac{\text{(const.)}}{\ell nT^2})} \right]
\]

\[
= \left( \frac{f(T^\gamma)}{\gamma} + \frac{\text{(const.)}}{\ell n} \right) \exp\left[ \frac{2f(T^\gamma)}{\gamma} + \frac{2\text{(const.)}}{\ell nT} \right] (1 + \frac{\ell n \ell nT}{\ell nT})
\]

which tends to zero as \( T \to \infty \).

Discussion. Remark 1: If (2) holds then choose \( f(t) = r(t)\ell nt \) and (4) is satisfied for \( \beta = 0 \). Suppose (3) holds then

\[
\int_0^T |r(s)|^P \, ds > \lambda(A_T) f(T^\gamma)/(\gamma \ell nT)
\]

implies that \( \lambda(A_T) = O(\ell nT/f(T^\gamma)) \). We can always choose \( f(t) \) so that it does not decrease too fast. (If (4) is valid for a fast decreasing \( f(t) \) then it is true for all slower decreasing \( f(t) \).) Obviously \( \lambda(E_T) = \lambda(t \mid 0 \leq t \leq T^\gamma; r(t) \geq f(t)/\ell nt) \) is at most of the order of \( T^\gamma \). The Lebesgue measure in (4) is \( \lambda(A_T) + \lambda(B_T) \) and hence (4) is satisfied.
Remark 2: The constant $k \geq 0$ is used in definition of $n(T)$ and $u(T)$ for generality. In fact we will be interested in only two values of $k$. For Gaussian sequences the proper normalizing constant for $\max_{1 \leq i \leq n} X_i$ is $(2 \ln n)^{1/2} = \frac{\ln n}{2(2 \ln n)^{1/2}}$. Hence we choose $k = 0$. For continuous time processes satisfying (1), the proper value of $k$ in $u(T)$ is $1/\alpha$. It should be noted that this value of $k$ is also appropriate in $n(T)$. The value of $n$ is used in comparing $M_T = \max_{0 \leq t \leq T} X_t$ with $M_n = \max_{0 \leq j \leq n} X(T/n)$ and hence should be large enough for the discretized version to approximate the process itself. Pickand's Lemma 4.2 in [7] shows that $(2 \ln T)^{1/2} (M_T - M_n) \to 0$ in probability if $n$ is chosen slightly larger than $T(\ln T)^{1/\alpha}$. It is easy to verify that we can choose $n(T) = T(\ln T)^{1/\alpha}/(f(t))^{1/2}$ for example and the proof of Lemma 1 will go through. Thus the choice of $n(T)$ for $k = 1/\alpha$ is in general fine enough for closeness of the maximum to that of the discretized version. (See also (2.6) of Mittal [5]). Berman [3] uses $n(T)$ for $k = 3/\alpha$ in (3.7). A close inspection shows that with the help of Lemma 1 we could change it to the above choice.

Remark 3: Condition (4) along with $r(t) = o(1)$ can replace for example Berman's conditions (0,5) in [3], (1,4), in [2], (3.15) of [1]. A number of other authors have used these conditions as well. Most of the proofs reduce to proving that a quantity similar to the one in (5) tends to zero.
This note is basically due to the encouragement and interest of Simeon Berman, Ross Leadbetter and Holger Rootzen. Berman's condition (0.5) in [3] is too weak for Theorem 3.1 to hold as is indicated by the following example. Define

\[ 1-r(t) = \begin{cases} \frac{1}{2e^t} |t| & \text{for } |t| \leq e^2 \\ 1 - \frac{1}{e^{nt}} & \text{for } |t| > e^2 \end{cases} \]

then \( r(t) \) satisfies the conditions of Theorem 3.1 [3] but the limiting distribution of the normalized maxima is a convolution of \( \exp(-e^{-X}) \) and the normal distribution as proved in [6] and not \( \exp(-e^{-X}) \) as indicated in Theorem 3.1 [3]. In private communication Prof. Berman had suggested an alternative condition

\[ \int_0^\infty \left( \sup_{x \geq t} |r(x)| \right)^p dt < \infty \]

to replace (0.5) of [3] and (1.4) of [2] since the later paper contains a unverified statement at the bottom of page 935. If one assumes the covariance function to be monotone then all the integral conditions are much stronger than (2). At the same time even the amount of smoothness induced by (2) is an artificial condition to assume when we are dealing with mixing conditions that let us view the process on \([0, T]\) as \( T \) independent chunks of unit length. All such mixing conditions attempt in a vague way to determine the "distance" between
a covariance that vanishes for \( t \geq t_0 \) and one that tends to zero as \( t \to \infty \). If one can talk about such a "distance" then it is clear that it does not depend on the area underneath the curve \( |r(T)|^p \). Whether it necessarily depends on the rate of decay of \( r(t) \) or not is an unanswered question. However the Lemma shows that all conditions of this type used so far depend "essentially" on the rate of decay of \( r(t) \).

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References


