ASYMPTOTIC EVALUATION OF THE NUMBER
OF LATIN RECTANGLES

by

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1. Introduction

An $l \times n$ Latin rectangle is an $l \times n$ rectangular array in which
each of the numbers $1, 2, \ldots, n$ occurs exactly once in each row and at
most once in each column. We shall denote by $p_{l,n}$ the probability
that a random $l \times n$ rectangular array in which the rows are independent
random permutations of $\{1, \ldots, n\}$ is a Latin rectangle. Erdős and
Kaplansky proved in 1946 that

$$p_{l,n} \sim e^{-\frac{l(l-1)}{2}}$$

as $n \to \infty$, uniformly for $l = O((\log n)^{3/2})$, and in 1951 Yamamoto proved
that (1) holds uniformly for $l = O(n^{1/3})$. Here the result of
Yamamoto will be improved to

$$p_{l,n} = e^{-\frac{l(l-1)}{2} - \frac{l^3}{6n} + O\left(\frac{l^2}{n}\right)^{1/2}}$$

uniformly for $l \leq \frac{n}{20}$, writing $a \vee b$ for $\max(a, b)$. In particular

$$p_{l,n} \sim e^{-\frac{l(l-1)}{2} - \frac{l^3}{6n}}$$
uniformly for $\ell = o(n^{1/2})$, and

$$\log p_{\ell, n} \sim -\frac{\ell(\ell-1)}{2}$$

uniformly for $\ell = o(n)$.

As in the work of Erdős and Kaplansky and in that of Yamamoto, the result is proved by adding one row at a time to the rectangular array. More precisely, the following result is proved in Section 2.

**Theorem:** Let $\pi_1, \ldots, \pi_\ell$ be permutations of the set $\{1, \ldots, n\}$ such that, for all $i \in \{1, \ldots, n\}$, the numbers $\pi_1(i), \ldots, \pi_\ell(i)$ are all different. Let $\mathcal{T}$ be a random permutation of $\{1, \ldots, n\}$, uniformly distributed over the set of all $n!$ such permutations, and let $W_\ell$ be the number of $i$ for which there exists $\alpha \in \{1, \ldots, \ell\}$ such that $\mathcal{T}(i) = \pi_\alpha(i)$. Then

$$P\{W_\ell = w\} = \left(\frac{n}{w}\right)^\ell \left(1 - \frac{\ell}{n}\right)^{n-w} e^{O(n^{-1/2} \ell^3 n^{-2})}$$

uniformly for $\ell \leq \frac{n}{20}$ and non-negative integers $w$ such that

$$w \leq w_0 = (\frac{\ell}{n^{1/4}})^{1/4} \ell^2.$$ 

The derivation of (2) from the Theorem is immediate:

Specializing (5) to $w = 0$, we have

$$P\{W_\ell = 0\} = (1 - \frac{\ell}{n})^n e^{O(n^{-1/2} \ell^3 n^{-2})}.$$

$$= e^{-k - \frac{k^2}{2n} + O(n^{-1/2} \ell^3 n^{-2}).$$
Now suppose $\prod_1^n, \ldots, \prod_\ell$ is a sequence of random permutations independently uniformly distributed, and let $W_k$ be the number of $i$ for which there exists $\alpha \in \{1, \ldots, k\}$ such that $\prod_{\alpha} (i) = \prod_{\alpha} (i)$. Then
\[
\begin{align*}
\mathbb{P}_{n, m} &= \mathbb{P}(W_1 = 0 \ldots W_{\ell-1} = 0) \\
&= \prod_{k=1}^{\ell-1} \mathbb{P}(W_k = 0|W_{\alpha} = 0 \text{ for all } \alpha \in \{1, \ldots, k-1\}) \\
&= e^{\frac{\ell-1}{2} \left( k + \frac{k^2}{2n} \right)} + \sum_{k=1}^{\ell-1} 0 \left( n^{-1/2} \right) \left( k^{3/2} n^{-2} \right) \\
&= e^{\frac{\ell-1}{2} \left( \frac{\ell}{6n} \right)} + 0 \left( n^{1/2} \right) \left( \frac{\ell^2}{n} \right)
\end{align*}
\]
\[
\text{by (5).}
\]

The method of proof of Theorem 2 is that used by Chen (1975) except that the elaborate formalism introduced there is not used here. It is possible that this formalism would be useful in the present problem. However, in order to obtain reasonably simple formulas, I found it convenient to use special features of the present problem, and I never returned to the general formalism.

I am indebted to Persi Diaconis for some helpful conversations.

2. Proof of the Theorem

Let us recall the problem and introduce some of the notation that will be needed. We are given $\ell$ permutations $\pi_1, \ldots, \pi_\ell$ of the set $\{1, \ldots, n\}$. For $i \in \{1, \ldots, n\}$ we define $S(i)$ to be the set whose
elements are $\pi_1(i), \ldots, \pi_\lambda(i)$. We assume that each $S(i)$ contains $\ell$ distinct elements. Let $\prod$ be a random permutation of $\{1, \ldots, n\}$ uniformly distributed over the set of all $n!$ such permutations, and let $W$ be the number of $i$ for which $\prod(i) \in S(i)$. Our aim is to study the distribution of $W$, and, in particular to show that, if $\ell = o(n^{2/3})$, this distribution is well approximated by a binomial distribution, the distribution of the number of successes in $n$ independent trials with probability $\ell/n$ of success on each trial.

In order to study the distribution of $W$, we introduce some additional randomness. Let the random ordered pair $(I,J)$ of distinct elements of $\{1, \ldots, n\}$ be uniformly distributed over the set of all $n(n-1)$ such ordered pairs, independent of $\prod$, and let the random permutation $\prod'$ of $\{1, \ldots, n\}$ be defined by

$$
\prod'(i) = \begin{cases} 
\prod(i) & \text{if } i \notin \{I,J\} \\
\prod(j) & \text{if } i = I \\
\prod(I) & \text{if } i = J
\end{cases}
$$

Clearly the pair $(\prod, \prod')$ is exchangeable, that is, it has the same distribution as $(\prod', \prod)$. Also let $W'$ be the number of $i$ for which $\prod'(i) \in S(i)$. Since $W'$ is related to $\prod'$ in the same way that $W$ is related to $\prod$, the pair $(W,W')$ is also exchangeable; in particular, for any $w \in \{0,1,2,\ldots\}$,

$$P(W = w \& W' = w+1) = P(W = w+1 \& W' = w).$$

The result of Yamamoto mentioned in the introduction can be obtained fairly easily by rewriting (2) in the form
(3) \[ \frac{P(W = w + 1)}{P(W = w)} = \frac{P(W' = w + 1|W = w)}{P(W' = w|W = w + 1)} \]

and showing that the conditional probabilities on the right hand side
are given approximately by

(4) \[ P(W' = w + 1|W = w) \approx \frac{2\ell}{n} \]

and

(5) \[ P(W' = w|W = w + 1) \approx \frac{2(w+1)}{n} \]

provided \( \ell \) and \( w \) are not too large compared to \( n \). In order to obtain
a better approximation, we shall proceed in a slightly different way.

Let the events \( C, D, N, C^*, D^* \), and \( N^* \) be defined by

(6) \[ C = \{ ||(I) \notin S(I) \& ||'(J) \in S(I) \} \]

(7) \[ D = \{ ||'(I) \in S(I) \& ||(J) \notin S(I) \} \]

(8) \[ N = (C \cup D)^C \]

(9) \[ C^* = \{ ||'(J) \notin S(J) \& ||(I) \in S(J) \} \]

(10) \[ D^* = \{ ||'(J) \in S(J) \& ||(I) \notin S(J) \} \]

and

(11) \[ N^* = (C^* \cup D^*)^C \]

More intuitively, \( C \) is the event that a coincidence is created in
column \( I \) when \( || \) is replaced by \( ||' \), and \( D \) the event that a coin-
cidence is destroyed. Similarly \( C^* \) and \( D^* \) refer to column \( J \) instead of
\( I \). The left hand side of (2) can be expressed as
P\{W = w & W' = w + 1\} = P(\{W = w\} (CN^* \cup C^*N))
= 2[P(\{W = w\}C) - P(\{W = w\}CC^*) - P(\{W = w\} CD^*)],
and the right hand side as
P\{W = w + 1 & W' = w\} = P(\{W = w + 1\}(DN^* \cup D^*N))
= 2[P(\{W = w + 1\}D) - P(\{W = w + 1\}DD^*) - P(\{W = w + 1\}DC^*)].

Equating (12) and (13), we obtain

\[
P\{W = w\}P(C|\{W = w\}) - P\{W = w + 1\}P(D|\{W = w + 1\})
= P(\{W = w\}C) - P(\{W = w + 1\}D)
= P(\{W = w\}CC^*) + P(\{W = w\}CD^*)
- P(\{W = w + 1\}DD^*) - P(\{W = w + 1\}DC^*).
\]

It is not difficult to compute the conditional probabilities occurring on the left hand side of (14). We have

\[
P^w(C) = (1 - \frac{w}{n}) \frac{\frac{a}{n-1}}
\]

since, in order for C to occur, I must be one of the n-W values for which \(\bigcap (i) \in S(I)\) and then J must be one of the \(l\) values for which \(\bigcap (j) \in S(I)\). Similarly

\[
P^w(D) = \frac{W}{n} (1 - \frac{\frac{a-1}{n-1}})
\]

In order to bound the right hand side of (14) we shall proceed differently, conditioning on events such as CC^* rather than \{W = w\}. Defining \(\bar{W}\) to be the number of \(i \notin \{I,J\}\) for which \(\bigcap (i) \in S(i)\), we can rewrite (14), using (15) and (16), as
\[
(1 - \frac{\lambda}{n-1}) P\{\bar{W} = w\} - \frac{w+1}{n} (1 - \frac{\lambda-1}{n-1}) P\{\bar{W} = w+1\}
= P(\{\bar{W} = w\} CC^*) + P(\{\bar{W} = w-1\} CD^*)
- P(\{\bar{W} = w-1\} DD^*) - P(\bar{W} = w) DC^*
= P(CC^*)P(\{\bar{W} = w\}| CC^*) + P(CD^*)P(\{\bar{W} = w-1\}| CD^*)
- P(DD^*)P(\{\bar{W} = w-1\}| DD^*) - P(DD^*)P(\{\bar{W} = w\}| DC^*).
\]

(17)

Let us first compute the unconditional probabilities such as \( P(CC^*) \) occurring on the right hand side of (17). Because the joint distribution of \((I,J,\bar{I},\bar{I}')\) is unchanged when I and J are exchanged or \(\bar{I}\) and \(\bar{I}'\) exchanged or both, while the first of these changes interchanges the starred and unstarred events while the second exchanges C with D and \(C^*\) with \(D^*\) we have

\[
P(CD^*) = P(DC^*)
\]
and

\[
P(CC^*) = P(DD^*).
\]

For the same reason, and because the value of \(\bar{W}\) is unaffected by these exchanges, we have

\[
P(\{\bar{W} = w\}| DC^*) = P(\{\bar{W} = w\}| CD^*),
\]
and

\[
P(\{\bar{W} = w-1\}| DD^*) = P(\{\bar{W} = w-1\}| CC^*).
\]

Thus (17) can be rewritten as
\[(1 - \frac{w}{n})^{\frac{\ell}{n-1}} P(\overline{W} = w) - \frac{w+1}{n} \left(1 - \frac{\ell-1}{n-1}\right) P(\overline{W} = w+1)\]

\[= P(\text{CC}^*) P(\{\overline{W} = w\} | \text{CC}^*) + P(\text{CD}^*) P(\{\overline{W} = w-1\} | \text{CD}^*)\]

\[- P(\text{CC}^*) P(\{\overline{W} = w-1\} | \text{CC}^*) - P(\text{CD}^*) P(\{\overline{W} = w\} | \text{CD}^*)\]

\[= P(\text{CD}^*) \left[ P(\{\overline{W} = w\} | \text{CC}^*) - P(\{\overline{W} = w\} | \text{CD}^*) \right]\]

\[- [P(\{\overline{W} = w-1\} | \text{CC}^*) - P(\{\overline{W} = w-1\} | \text{CD}^*)]\]

\[+ [P(\text{CC}^*) - P(\text{CD}^*)] [P(\{\overline{W} = w\} | \text{CC}^*) - P(\{\overline{W} = w-1\} | \text{CC}^*)].\]

Now let us compute the unconditional probabilities \(P(\text{CC}^*)\) and \(P(\text{CD}^*)\). By (6) and (9)

\[P(\text{CC}^*) = P(\overline{\{I\}} \in S^C(I) S(J) \& \overline{\{J\}} \in S(I) S^C(J))\]

\[= \frac{1}{[n(n-1)]^2} \sum_{1 \leq i, j \leq n} \left[ \ell - \#(S(i) S(j)) \right]^2\]

\[= \frac{1}{[n(n-1)]^2} \left[ \ell^2 n(n-1) - 2\ell n (\ell - 1) + \sum_{1 \leq i, j \leq n} \left(\#(S(i) S(j))\right)^2 \right]\]

\[= \frac{1}{[n(n-1)]^2} \left[ \ell^2 n(n-2\ell+1) + \sum_{1 \leq i, j \leq n} \left(\#(S(i) S(j))\right)^2 \right].\]

We have used the fact that

\[\sum_{1 \leq i, j \leq n} \left(\#(S(i) S(j))\right)^2 = n\ell (\ell - 1)\]

since each of the numbers in \([1, \ldots, n]\) occurs in \(\ell\) columns and thus in \(\ell(\ell - 1)\) ordered pairs of columns. Similarly
\[ P(\text{CD}^*) = \frac{1}{[n(n-1)]^2} \sum \#(S(i)S(j))[n-2\ell + \#(S(i)S(j))] \]

\[ = \frac{n\lambda(n-1)(n-2\ell) + \sum(\#(S(i)S(j)))^2}{[n(n-1)]^2} . \]

It follows that the first factor of the second of the two terms on the extreme right hand side of (22) is given by

\[ P(\text{CC}^*) - P(\text{CD}^*) = \frac{\lambda(n-\ell)}{n(n-1)} . \]

In order to bound the right hand side of (22) effectively, we need to show that the difference

\[ P(\{\overline{w} = w\} | \text{CC}^*) - P(\{\overline{w} = w\} | \text{CD}^*) \]

is small compared to either term. Roughly speaking, it will suffice to show that \(\overline{w}\) is nearly independent of the random variables

\[ Q = (I,J,\overline{\Pi}(I),\overline{\Pi}(J)) . \]

In order to prove this, it will be convenient to introduce additional randomness in the following way. Let \((\overline{I},\overline{J})\) be a random ordered pair of distinct elements of \(\{1,\ldots,n\}\), uniformly distributed over the set of all \(n(n-1)\) such ordered pairs, independent of \(\overline{\Pi}\), I, and J. Let the random permutation \(\overline{\Pi}^*\) be defined by

\[ \overline{\Pi}^*(i) = \begin{cases} \overline{\Pi}(i) & \text{if } i \in \{I,J,\overline{I},\overline{J}\} \\ \overline{\Pi}(\overline{I}) & \text{if } i = \overline{I} \\ \overline{\Pi}(J) & \text{if } i = \overline{J} \\ \end{cases} \]
Let $W^*$ be the number of $i$ for which $\prod^*(i) \in S(i)$. We must verify that

a) $\prod^*$ is independent of $Q$ and has the same distribution as $\prod$.

Thus the conditional distribution of $W^*$ given $Q$ is the same as the unconditional distribution of $W$.

b) Somewhat vaguely, $W^*$ has nearly the same conditional distribution given $Q$ as $\bar{W}$ because it is highly probable that they are equal.

Let us verify assertion a) above. For any permutation $\pi^*$ and $i, j, \bar{i}, \bar{j} \in \{1, \ldots, n\}$ with $i \neq j$, $\bar{i} \neq \bar{j}$, and $i', j' \in \{1, \ldots, n\}$ with $i' \neq j'$, we have

$$
\begin{align*}
P\{\prod^* = \pi^* & \text{ and } I = i \text{ and } J = j \text{ and } \bar{I} = \bar{i} \text{ and } \bar{J} = \bar{j} \\
& \text{ and } \prod(I) = i' \text{ and } \prod(J) = j'\} \\
& = P\{\prod(1') = \pi^*(1'') \text{ for all } i'' \in \{i, j, \bar{i}, \bar{j}\} \\
& \text{ and } \prod(i) = i' = \pi^*(\bar{i}) \text{ and } \prod(j) = j' = \pi^*(\bar{j}) \\
& \text{ and } I = i \text{ and } J = j \text{ and } \bar{I} = \bar{i} \text{ and } \bar{J} = \bar{j}\}
\end{align*}
$$

(30)

$$
= \begin{cases} 
\frac{1}{n! [n(n-1)]^2} & \text{if } i' = \pi^*(i) \text{ and } j' = \pi^*(j) \\
0 & \text{otherwise}
\end{cases}
$$

Summing over $i$ and $j$, we obtain

$$
P\{\prod^* = \pi^* \text{ and } I = i \text{ and } J = j \text{ and } \prod(I) = i' \text{ and } \prod(J) = j'\} \\
= \frac{1}{n!} \sum_{i' \neq j'} \frac{1}{[n(n-1)]^2}
$$

(31)

which proves assertion a).
In order to make assertion b) precise we must relate the conditional distribution of $\bar{w}$ given $Q$ to that of $\bar{w}^*$. Let

$$p(w) = P\{W = w\} = p^Q\{w^* = w\} \tag{32}$$

and

$$\frac{p^Q(\bar{w})}{p^Q(\bar{w})} = P^Q\{\bar{w} = \bar{w}\} \tag{33}$$

Then

$$p(\bar{w}) = P^Q\{w^* = \bar{w}\} \geq P^Q\{w^* = \bar{w} = \bar{w}\} \tag{34}$$

$$\geq p^Q(\bar{w}) \left(1 - O\left(\frac{\bar{w} + \bar{w}}{n}\right)\right),$$

since $w^*$ agrees with $\bar{w}$ unless one (or more) of the events

$$\{|I| \in S(\bar{I})\}, \{|I| \in S(\bar{I}) \cup S(I)\}, \{|I| \in S(J)\}, \{|I| \in S(\bar{J}) \cup S(\bar{J})\}$$

occurs. Similarly,

$$p(\bar{w}) \leq p^Q(\bar{w}) + \frac{\bar{w} + 1}{n} p^Q(\bar{w} + 1) + \frac{\bar{w} + 2}{n(n-1)} p^Q(\bar{w} + 2)$$

$$+ O\left(\frac{\bar{w}}{n}\right) \sum_{\alpha=1}^{4} p^Q(\bar{w} - \alpha)$$

$$\leq p^Q(\bar{w}) + \left[1 + O\left(\frac{\bar{w} + \bar{w}}{n}\right)\right] \frac{\bar{w} + 1}{n} p(\bar{w} + 1)$$

$$+ \frac{\bar{w} + 2}{n(n-1)} p(\bar{w} + 2) + O\left(\frac{\bar{w}}{n}\right) \sum_{\alpha=1}^{4} p(\bar{w} - \alpha) \right]. \tag{35}$$

The final inequality uses (34). Since $p(\bar{w})$, which occurs on the left hand side of (34) and (35) does not depend on the outcome of $Q$, and in particular does not depend on the occurrence or non-occurrence of $CC^*$ or $CD^*$, we have
\[ P(\overline{W} = w) | CC^* ) \] 
\[ = \frac{\ell + w}{n} p(w) + \frac{w+1}{n} p(w+1) \]
\[ + \frac{(w+2)(w+1)}{n(n-1)} p(w+2) \]
\[ + \frac{\ell}{n} \sum_{\alpha=1}^{4} p(w-\alpha) \]

uniformly for \( \ell + w < \varepsilon n \), where \( \varepsilon \) is an appropriately chosen absolute constant.

An outline of the remainder of the proof follows. We first prove a more precise version of (4) and (5):

\[ P(\overline{W}' = w | W = w) = \frac{2\ell}{n} \left( 1 + O\left( \frac{\ell+w}{n} \right) \right) \]

and

\[ P(\overline{W}' = w | W = w+1) = \frac{2(w+1)}{n} \left( 1 + O\left( \frac{w+2}{n} \right) \right) . \]

Then it follows from (3) that

\[ \frac{p(w+1)}{p(w)} = \frac{\ell}{w+1} \left( 1 + O\left( \frac{w+2}{n} \right) \right) \]

and then from (36) that

\[ |P(\overline{W} = w) | CC^* \) - p((\overline{W} = w) | CD^*) | \]

\[ = O\left( \frac{\ell+w}{n} \right) p(w) . \]

Using this and (25) and (26) we obtain from (22) the inequality

\[ \left| 1 - \frac{w}{n} \right| \frac{\ell}{n-1} p(w) - \frac{w+1}{n} \left( 1 - \frac{w-1}{n-1} \right) p(w+1) \]

\[ \leq K \left( \frac{\ell^2}{n^2} + \frac{\ell+w}{n^2} \right) (p(w) + p(w-1)) \]

12
where $K$ is an absolute constant. Using (39) to express $p(w-1)$ in terms of $p(w)$ and rearranging, we obtain

$$\frac{p(w+1)}{p(w)} = \frac{\lambda}{w+1} \frac{n-w}{n-w} \left[ 1 + \rho K \left( \frac{\lambda (\lambda + w)}{n^2} + \frac{1}{n} \left( 1 + \frac{w}{\lambda} \right) \right) \right]$$

$$= \frac{\lambda}{w+1} \frac{n-w}{n-w} \left[ 1 + \rho K \left( \left( \frac{\lambda + w}{n} \right)^2 + \frac{\lambda + w}{\lambda n} \right) \right]$$

with

$$|\rho| \leq 1.$$ 

Using the fact that

$$\Sigma p(w) = 1,$$

it will then be easy to prove the Theorem of Section 1.

From (12) and (13), we see that, in order to prove (37) and (38) it will suffice to bound $P^W(\text{CC}^*), P^W(\text{CD}^*), \text{and } P^W(\text{DD}^*)$ appropriately. We have

$$P^W(\text{CD}^*) \leq P^W\{ \square (J) \in S(I) | S(J) \}$$

$$= P^W\{ \square (J) \in S(J) \} P^W\{ \square (J) \in S(I) | \square (J) \in S(J) \}$$

$$= \frac{W(\lambda-1)}{(n-1)}.$$

and

$$P^W(\text{DD}^*) \leq P^W\{ \square (I) \in S(I) \& \square (J) \in S(J) \}$$

$$= \frac{W(W-1)}{n(n-1)}.$$

In order to bound $P^W(\text{CC}^*)$ we proceed differently. As in (3) we have
\[
\frac{p(w+2)}{p(w)} = \frac{P(W' = w+2 \mid W = w)}{P(W' = w \mid W = w+2)}
\]

\[
= \frac{P(CC^* \mid \{W = w\})}{P(DD^* \mid \{W = w+2\})}.
\]

If follows that

\[
P(CC^* \mid \{W = w\}) = P(DD^* \mid \{W = w+2\}) \frac{p(w+2)}{p(w+1)} \frac{p(w+1)}{p(w)}
\]

\[
\leq \frac{(w+2)(w+1)}{n(n-1)} \cdot \frac{P(C \mid \{W = w+1\})}{P(D \mid \{W = w+2\}) - P(DD^* \mid \{W = w+2\}) - P(DC^* \mid \{W = w+2\})}
\]

\[
\leq \frac{P(C \mid \{W = w\})}{P(D \mid \{W = w+1\}) - P(DD^* \mid \{W = w+1\}) - P(DC^* \mid \{W = w+1\})}
\]

\[
= 0 \left(\frac{\lambda^2}{n^2}\right).
\]

With the aid of (15), (16), (45), (46), and (48), formulas (37) and (38) follow from (12) and (13).

Since the derivation of (42) from (37) and (38) has already been adequately described, it remains only to derive the Theorem of Section 1 from (42). We have

\[
p(w) = p(0) \sum_{w' = 0}^{w-1} \frac{p(w' + 1)}{p(w')}
\]

\[
= p(0) \sum_{w' = 0}^{w-1} \left\{ \frac{\ell}{w' + 1} \frac{n - w'}{n - \ell} \left[ 1 + \mathcal{G}_K \left( \frac{(\ell + w')^2}{n^2} + \frac{\ell + w'}{\lambda n} \right) \right] \right\}
\]

\[
= p(0) \sum_{w' = 0}^{n-1} \frac{\ell}{n - \ell} \left[ 1 + \mathcal{G}_K \left( \frac{(\ell + w')^3}{3n^2} + \frac{(\ell + w')^2}{2\lambda n} \right) \right]
\]

for

\[
w \leq w_0,
\]
where \( w_0 \) will be chosen later, and \( \nu \), not necessarily the same at different occurrences, satisfies (43). It follows that

\[
1 - P\{W > w_0\} = \sum_{w=0}^{w_0} p(w)
\]

\[
= e^{\varphi K} \left( \frac{(\ell + w_0)^3}{3n^2} + \frac{(\ell + w_0)^2}{2\ell n} \right) p(0) \sum_{w=0}^{w_0} \binom{n}{w} \left( \frac{\ell}{n-\ell} \right)^w
\]

\[
= e^{\varphi K} \left( \frac{(\ell + w_0)^3}{3n^2} + \frac{(\ell + w_0)^2}{2\ell n} \right) p(0) \left[ 1 - p\{X > w_0\} \right] \left( \frac{n}{n-\ell} \right)^n
\]

where \( X \) is the number of successes in \( n \) independent trials with probability \( p = \ell/n \) of success on each trial.

It remains to bound \( P\{X > w_0\} \) and \( P\{W > w_0\} \) using Chebyshev's inequality and to choose \( w_0 \) appropriately. We have

\[
EX = \ell, \quad E(X-\ell)^2 = \ell \left( 1 - \frac{\ell}{n} \right) \leq \ell.
\]

It is also not difficult to compute the mean and variance of \( W \), obtaining

\[
EW = \ell, \quad E(W-\ell)^2 = \ell \left( \frac{n-\ell}{n-1} \right) < \ell.
\]

It follows from Chebyshev's inequality that

\[
P\{X > w_0\} \leq \frac{\ell}{(w_0 - \ell)^2}
\]

and
(55) \[ P\{w > w_0\} < \frac{\ell}{(w_0 - \ell)^2} \]

for \( w_0 > \ell \). Let

(56) \[ w_0 = \begin{cases} \frac{\ell^{1/2}}{n^{1/4}} & \text{if } \ell < \frac{1}{4} n^{1/2} \\ 2\ell & \text{if } \ell > \frac{1}{4} n^{1/2} \end{cases} \]

Then (51), (54), and (55) yield

(57) \[ p(0) = \begin{cases} \left(1 - \frac{\ell}{n}\right)^n e^{0(n^{-1/2})} & \text{if } \ell < \frac{1}{4} n^{1/2} \\ \left(1 - \frac{\ell}{n}\right)^n e^{0(\ell^3 n^{-2})} & \text{if } \ell \geq \frac{1}{4} n^{1/2} \end{cases} \]

It follows from (49) that

(58) \[ p(w) = \begin{cases} \left(n\frac{\ell}{w}\right)^w \left(1 - \frac{\ell}{n}\right)^{n-w} e^{0(n^{-1/2})} & \text{if } \ell < \frac{1}{4} n^{1/2} \\ \left(n\frac{\ell}{w}\right)^w \left(1 - \frac{\ell}{n}\right)^{n-w} e^{0(\ell^3 n^{-2})} & \text{if } \frac{1}{4} n^{1/2} \leq \ell \leq \varepsilon n, \end{cases} \]

uniformly for \( w \leq w_0 \) given by (56), and \( \ell \) in the range indicated in (58) as \( n \to \infty \). This completes the proof of the Theorem stated in the introduction.
References

