ON THE ASYMPTOTIC NORMALITY OF ONE-SIDED STOPPING RULES

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DAVID SIEGMUND

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PREFACE

This paper develops theoretical results concerning the approximate distribution of stopping times for certain statistical procedures involving sequential trials. In particular, these results are relevant to the problem of finding fixed-width sequential confidence intervals for the mean and to certain sequential hypothesis-testing problems. In designing sequential statistical procedures for such problems it is important to consider the probabilistic behavior of the stopping times. The present paper provides helpful insights into this matter.
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By
D. Siegmund

1. Introduction and Summary.

Assume that \( x_1, x_2, \ldots \) are independent random variables with expectation \( \mu > 0 \) and finite variance \( \sigma^2 \). Let \( s_k = x_1 + \cdots + x_k \) (\( k = 1, 2, \ldots \)). For any family of positive, non-decreasing, eventually concave functions \( f_c \) defined on the positive real numbers and indexed by \( c > 0 \) such that \( f_c \to \infty \) as \( c \to \infty \), define

\[
\tau = \tau(c) = \text{first } k \geq 1 \text{ such that } s_k > f_c(k)
\]

\( = \infty \text{ if no such } k \text{ exists.} \)

The stopping variable (s.v.) \( \tau \) arises in various problems in probability and statistics. For example, the sequential statistical procedures of [2], [3], and [8] involve s.v.'s similar to \( \tau \) (see also the next to last paragraph of this section).

Suppose that the family \( \{f_c : c > 0\} \) is such that we may define \( \lambda = \lambda(c) \) by

\[
\mu \lambda = f_c(\lambda).
\]

In [9] it is shown under conditions on the joint distribution of \( x_1, x_2, \ldots \) weaker than the above that

\[
\mathbb{E}\tau \sim \lambda \quad (c \to \infty)
\]
for a certain class of families \( \{ f_c \} \). In Section 2 of this note it is shown that if \( f_c(x) = cx^\alpha \) for some \( 0 \leq \alpha < 1 \), then \( \tau \) (suitably normalized) is asymptotically normally distributed whenever

\[
(s_n - nm)/\alpha n^{1/2}
\]

is. (See also remarks (a) and (b) in Section 4.) This extends Heyde's result \([5], [7]\), valid when the \( x_k \) have a common distribution and \( \alpha = 0 \).

Assume now for simplicity that \( f_c(x) = c \) and \( x_1, x_2, \ldots \) have a common distribution. If \( x_1 \geq 0 \) the random variable \( M(c) \) (\( N(c) \)) defined by

\[
M(c) = \sup \{ n: \inf_{k \geq n} s_k \leq c \}, \quad N(c) = \sum_{n=1}^{\infty} I[s_n \leq c]
\]

is of interest in renewal theory; and the observation that \( M(c) = N(c) = \tau(c) - 1 \) allows one to study \( M(N) \) by studying the stopping variable \( \tau \). In the general case it has been noted by several authors that the behavior of \( \tau \) and \( N \) may differ in important results. (For example, if \( E(x_1)^2 = \infty \), it is known that \( EN = \infty \) \([6]\), whereas it is easy to show that \( \tau \) is finite, \( E\tau ~ c/\mu \) (e.g. \([9]\)).) In Section 3 we point out that some knowledge of \( M(N) \) can be obtained in a direct fashion from relevant knowledge of \( \tau \).

Our methods throughout involve finding convenient estimates for the probability that \( s_k \) crosses the curve \( f_c \) for the first time at some index \( k < n \) and then falls back below the curve at time \( n \).
2. Asymptotic Normality of $\tau$

We use without comment the basic

**Lemma 1**: If $F$ is a distribution function and $(z_n)$ a sequence of random variables such that $\lim_{n \to \infty} P(z_n < x) = F(x)$ at all continuity points $x$ of $F$, then for any sequence $(\varepsilon_n)$ of random variables tending in probability to 0 as $n \to \infty$,

$$\lim_{n \to \infty} P(z_n + \varepsilon_n < x) = F(x)$$

at all continuity points $x$ of $F$.

**Theorem 1**: Let $(x_n)$, $f_c$, $\lambda$, $\tau$ be as above and suppose that for some $0 \leq \alpha < 1$

$$f_c(x) = cx^\alpha.$$ 

If

$$\lim_{n \to \infty} P\left(\frac{S_n - \mu}{\sigma n^{1/2}} \leq x\right) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy,$$

then

$$\lim_{c \to \infty} P\left(\frac{\tau - \lambda}{(1-\alpha)\lambda^{1/2} - \mu} \leq x\right) = \Phi(x).$$

**Proof**: We shall give a proof for the case $\alpha > 0$. The case $\alpha = 0$ is similar and somewhat easier.

From (1) we have $\lambda = (c/\mu)^{1/(1-\alpha)}$. Let $x$ be arbitrary and assume that $n$ is a function of $c$ such that

$$\frac{cn^\alpha \mu}{\sigma n^{1/2}} = -x.$$
so by inversion

\[
\frac{n-\lambda}{(1-\alpha)^{-\frac{1}{2}}\mu^{-1} \sigma} \to x.
\]

(Note that \( n \) is not, in general, an integer. We denote the largest integer \( \leq n \) by \([n]\).)

Now \( P(\tau \leq n) \geq P(s_{[n]} > cn^2) \), so by (4), (6), and (7)

\[
\lim \inf \ P\left( \frac{\tau - \lambda}{(1-\alpha)^{-\frac{1}{2}}\mu^{-1} \sigma} \leq x \right) \geq 1 - \Phi(-x) = \phi(x).
\]

Now suppose that \( x > 0 \). (The case \( x < 0 \) follows by a similar argument and the case \( x = 0 \) by continuity.) Let \( 0 < \epsilon, \delta < 1 \). Then from (6)

\[
P(\tau \leq n) \leq P(\tau \leq \epsilon n) + P(s_{[n]} > [n] \mu - (1+\delta)xn^{-\frac{1}{2}}) + P(\epsilon n < \tau < n, s_{[n]} \leq [n] \mu - (1+\delta)xn^{-\frac{1}{2}}) = p_1 + p_2 + p_3.
\]

By (4)

\[
\lim_{c \to \infty} p_2 = 1 - \phi(-x(1+\delta)) = \phi(x(1+\delta)).
\]

By (6) \( c\mu = (\alpha n^{-\frac{1}{2}} - k)\mu - x\alpha n^{-\frac{1}{2}} - \alpha \), which is easily seen to be increasing for \( k \leq \epsilon n \) provided that \( \epsilon \) is so small and \( n \) so large that

\[
(\alpha n^{-\frac{1}{2}} - \epsilon n^{-\frac{1}{2}} - 1)\mu - x\epsilon n^{-\frac{1}{2}} > 0.
\]

Thus we may apply the Hájek-Rényi inequality \([4]\) to obtain
\[ p_1 = P(s_{\mu} - k \mu > c \alpha - k \mu, \text{ some } k \leq \epsilon n) \]
\[ = P(s_{\mu} - k \mu > (k \alpha n^{-1-\alpha} - k) \mu - \epsilon \alpha n^{-1/2}, \text{ some } k \leq \epsilon n) \]
\[ \leq \sum_{\epsilon n}^{[\epsilon n]} \frac{\sigma^2}{n^2(1-\alpha) \alpha [((1-\frac{k}{n})^{1-\alpha}) \mu - \epsilon \alpha n^{-1/2}]^2} \]
\[ = 0(\frac{1}{n}) \rightarrow 0 \text{ as } c \rightarrow \infty . \]

From (6)
\[ p_j = \sum_{k=\epsilon n + 1}^{[\epsilon n]} P(\tau = k, s_{[\epsilon n]} \leq [n] \mu - (1+\delta) \epsilon \sigma n^{1/2}) \]
\[ \leq \sum_{k=\epsilon n + 1}^{[\epsilon n]} P(\tau = k) P(s_{[\epsilon n]} - s_k \leq [n] \mu - k \alpha n^{-1-\alpha} - \delta \epsilon \sigma n^{1/2}) \]
\[ = \sum_{k=\epsilon n + 1}^{[\epsilon n]} P(\tau = k) P(s_{[\epsilon n]} - s_k - ([n] - k) \mu \leq (k - k \alpha n^{-1-\alpha} - \delta \epsilon \sigma n^{1/2}) \].

If \( k \geq (1-\epsilon)n \), i.e., \( |k-n| \leq \epsilon n \), then by Chebyshev's inequality

\[ P(s_{[n]} - s_k - ([n] - k) \mu \leq (k - k \alpha n^{-1-\alpha} - \delta \epsilon \sigma n^{1/2}) \]
\[ \leq \frac{(n-k) \sigma^2}{2n} \leq \frac{\epsilon \sigma^2}{2n} \leq \frac{\sigma^2}{2n} . \]

If \( \epsilon n < k < (1-\epsilon)n \)

\[ \frac{\sigma^2}{2n} \leq \frac{\sigma^2}{2n} . \]
\[ P(s_n-s_k-(n-k)^{\mu} \leq (k-k^\alpha n^{1-\alpha})_{\mu-\delta n^{1/2}}) \]

\[(14) \quad \frac{\epsilon^2}{(k-k^\alpha n^{1-\alpha})_{\mu-\delta n^{1/2}}} \leq \frac{\epsilon^2}{(\frac{1-\alpha}{\alpha} n^{1-\alpha})_{\mu-\delta n^{1/2}}} \\
\rightarrow 0 \text{ uniformly in } k \text{ as } c \rightarrow \infty.\]

Thus from (9)-(14) we have

\[ \lim_{c \rightarrow \infty} \sup_{c} P\left(\frac{\epsilon-\lambda}{(1-\alpha)^{-1/2} n^{1-\alpha}} \leq x \right) \leq \Phi(x(1+\delta)) + \frac{\epsilon}{\delta x} \rightarrow \Phi(x) \]

as first \( \epsilon \rightarrow 0, \) then \( \delta \rightarrow 0, \) which in conjunction with (8) proves the theorem.

3. The Random Variable \( M = \sup \{m: \inf_{k \geq n} s_k \leq c\}. \)

Now suppose that \( x_1, x_2, \ldots \) are independent and identically distributed with positive mean \( \mu \) and that \( f_c(x) = c. \) It is known ([1] Theorem 3) that with \( M \) defined by (3) \( EM < \infty \) provided that \( E(x_1^-)^2 < \infty. \)

Theorem 2: If \( E(x_1^-)^2 < \infty, \) for any \( c > 0 \)

\[ 0 \leq E(M(c)-\tau(c)+1) \leq EM(0) < \infty. \]

Proof. The left hand inequality follows at once from the observation that

\[ M(c) \geq N(c) \geq \tau(c)-1. \]

To complete the proof we write
\[ E(M(c) - \tau(c) + 1) = \sum_{n=1}^{\infty} (P[M \geq n] - P[\tau > n]) \]

\[ = \sum_{n=1}^{\infty} P[\tau \leq n, M \geq n] = \sum_{n=1}^{\infty} \sum_{k=1}^{n} P[\tau = k, s_k > c, \inf_{j \geq n} s_j \leq c] \]

\[ \leq \sum_{k=1}^{\infty} P[\tau = k] \sum_{n=k}^{\infty} P[\inf_{j \geq n} (s_j - s_k) < 0] \]

\[ \leq \sum_{i=0}^{\infty} P[\inf_{j \geq i} s_j < 0] \leq EM(0) < \infty. \]

**Corollary 1:** If \( E(x_1^2) < \infty \), then \( EM(c) \sim EN(c) \sim ET(c) \sim \frac{c}{\mu} \) \((c \to \infty)\).

**Proof.** The corollary follows at once from the theorem and the result for \( \tau \), which is well-known.

**Corollary 2:** If \( E x_1^2 - \mu^2 = \sigma^2 < \infty \), then

\[ \lim_{c \to \infty} P \left[ M(c) - \frac{c - \mu - 1}{c^{1/2} \sigma \mu^{-3/2}} \leq x \right] = \Phi(x) \quad (\pm \infty < x < \infty). \]

**Proof.** By Theorem 2, \( (M(c) - \tau(c))/c^{1/2} \to 0 \) in probability, and hence the corollary follows from Lemma 1 and Theorem 1.

It is interesting to note that if \( \sigma^2 < \infty \) Heyde [5] has shown that \( \text{Var} \, \tau \sim c \sigma^2 \mu^{-3} \) \((c \to \infty)\). However, it is possible that \( EM^2 = \infty \).

In fact it is an easy consequence of results in [1] and [6] that

\[ \sum_{n=1}^{\infty} nP[M \geq n] \quad \text{(and thus EM}^2 \text{)} \quad \text{is finite if and only if } E(x_1^3) < \infty. \]

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4. **Remarks.**

The following comments suggest some straightforward generalizations of the results of this note.

(a) Theorem 1 remains true if the \(x_k\)'s do not have the same expectation but \(E S_n - n \mu = o(n^{1/2})\) \((n \to \infty)\). The case of non-constant variance can also be treated.

(b) In [2] stopping boundaries \(f(n)\) such that \(f(n) = 0((n \log \log n)^{1/2})\) \((n \to \infty)\) are discussed. A particularly simple parameterization of such boundaries giving an asymptotic result similar to Theorem 1 is

\[
f_c(n) = n^{1/2}(c + \log \log n)^{1/2}.
\]

The case of a more general slowly varying function \(L\) can be treated similarly.

(c) A version of Theorem 1 follows at once for

\[
\tau^* = \text{first } n \text{ such that } |S_n| > cn^\alpha.
\]

By the strong law of large numbers \(S_n/n \to \mu\) and hence \(S_n \to \infty\) a.s. It follows that \(P(\tau^*(c) \neq \tau(c)) \to 0\) as \(c \to \infty\). Lemma 1 and Theorem 1 now give the limiting distribution of \(\tau^*\).

(d) The method of proof of Theorem 2 may be used to relate

\[
EM^{r+1}, \sum_{n=1}^{\infty} n^r P(S_n \leq c), \text{ and } En^{r+1}
\]

for positive integral values of \(r\). It may also be adapted to the case of non-identically distributed variables.
REFERENCES


