LINEAR GROWTH, BIRTH AND DEATH PROCESSES

BY
SAMUEL KARLIN
JAMES McGREGOR

TECHNICAL REPORT NO. 3

PREPARED UNDER CONTRACT Nonr-225 (28)
(NR-047-019)
FOR
OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

JANUARY 31, 1958
LINEAR GROWTH, BIRTH AND DEATH PROCESSES

BY

SAMUEL KARLIN

AND

JAMES Mc Gregor

TECHNICAL REPORT NO. 3

PREPARED UNDER CONTRACT Nonr-225(28)

(NR-047-019)

FOR

OFFICE OF NAVAL RESEARCH

REPRODUCTION IN WHOLE OR IN PART IS PERMITTED FOR ANY
PURPOSE OF THE UNITED STATES GOVERNMENT

DEPARTMENT OF STATISTICS

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

JANUARY 31, 1958
LINEAR GROWTH, BIRTH AND DEATH PROCESSES

By

Samuel Karlin and James McGregor

A birth and death process is a stationary Markov process $x(t)$ whose state space is the non-negative integers and whose transition probability matrix

$$P_{ij}(t) = \Pr \{ x(t) = j \mid x(0) = i \}$$

satisfies the conditions (as $t \to 0$)

$$P_{ij}(t) = \begin{cases} 
\lambda_i t + o(t) & \text{if } j = i + 1 \\
\mu_i t + o(t) & \text{if } j = i - 1 \\
1 - (\lambda_i + \mu_i) t + o(t) & \text{if } j = i 
\end{cases}$$

(1)

where $\lambda_i > 0$ for $i \geq 0$, $\mu_i > 0$ for $i \geq 1$ and $\mu_0 \geq 0$. The process is called a linear growth process if $\lambda_n = \lambda n + a$ and $\mu_n = \mu n + b$ with $\lambda > 0$ and $\mu > 0$. Such processes occur naturally in the study of biological reproduction and population growth. If the state of the system $n$ describes the size of the population, then the average instantaneous rate of growth is $\lambda n + a$. Similarly, the probability of the state of the process decreasing by one after the elapse of a small period of time is $(\mu n + b) t + o(t)$. The factor $\lambda n$ represents the growth of the population due to the current size of the population. The second factor $a$ may be interpreted as the increase of the state of the system due to an external source. The two components $\mu n$ and $b$ which compose the death rate possess an analogous meaning.

These birth and death processes represent the most elementary formulations of probabilistic models describing biological growth. Various aspects of these
processes have received considerable attention in the recent literature [4] [see page 407]. Nevertheless, a thorough understanding of the nature of these simple models is still lacking. For a complete set of references and a lucid survey of the subject matter of linear growth processes, the reader should consult [11] and the extensive bibliography quoted therein.

Our approach to the study of linear growth models derives basically from our knowledge of the elaborate structure of birth and death processes as developed in [6] and [7]. The analysis relies decisively on the integral representation formulas and recognition of the orthogonal polynomial systems associated with linear growth processes. The orthogonal polynomials which are relevant for linear growth models are related to the classical Meixner polynomials and to the Laguerre polynomials. By virtue of the vast detailed information available concerning these polynomial systems, very precise statements can be made regarding these types of linear growth processes.

From the same point of view we studied extensively queueing processes for several servers with Poisson input and exponential service rate. For details we refer the reader to [8].

The infinitesimal matrix of the general birth and death process is of the form

\[
A = \begin{pmatrix}
-(\lambda_0 + \mu_0) & \lambda_0 & 0 \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 \\
\mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]
This matrix determines a system of polynomials by means of the recurrence relations

\[
\begin{cases}
Q_0(x) = 1, \\
-xQ_0(x) = -(\lambda_0 + \mu_0) Q_0(x) + \lambda_0 Q_1(x), \\
-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n) Q_n(x) + \lambda_n Q_{n+1}(x).
\end{cases}
\]

(3)

It is shown in [6] that there is a positive regular measure \( \psi \) on \( 0 \leq x < \infty \) for which the orthogonality relations

\[
\int_0^\infty Q_i(x) Q_j(x) \, d\psi(x) = \frac{\delta_{ij}}{\pi_j} \quad i, j = 0, 1, \ldots ,
\]

where \( \pi_0 = 1 \), \( \pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \), are valid. In the case of a linear growth process, the measure \( \psi \) is unique [6], and moreover the transition probability matrix \( P(t) = (P_{ij}(t)) \) of the process is uniquely determined by \( A \). It has the representation

\[
P_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) \, d\psi(x).
\]

(4)

This is an extremely useful formula for the transition probability function, and our first task will be to compute the polynomials \( \{Q_n(x)\} \) and the spectral measure \( \psi \) belonging to the various linear growth processes. Often we shall refer to the state of the process as the position of a diffusing particle subject to the transition laws defining the birth and death process.
Because of their probabilistic interpretations a distinction must be made between the two types of birth and death processes according as \( \mu_0 = 0 \) or \( \mu_0 > 0 \). In the former case the state zero is a reflecting barrier in the sense that whenever the particle reaches zero, a transition must occur in finite time which moves the particle back to state 1. But if \( \mu_0 \) is positive such a transition occurs only with probability \( \lambda_0/(\lambda_0 + \mu_0) \). One of the important structural questions regarding such processes is to determine whether recurrence to a prescribed state of the system is a certain event. Since all states communicate it is sufficient to resolve this problem for the state zero. An appeal to the general theory of birth and death processes shows that the states are recurrent if and only if \( \sum_k \frac{1}{\lambda_k \pi_k} \) diverges [7].

In that circumstance the recurrence time distribution has a finite mean value (the process is ergodic) if and only if \( \sum_k \pi_k < \infty \). Moreover, \( \frac{\pi_k}{\sum_{k=0}^{\infty} \pi_k} \) may then be interpreted as the limiting stationary probability that the state of the system is \( k \).

When the process is non recurrent or transient, then we distinguish two kinds of transient behavior which measure the rate at which the particle drifts to infinity.

We say that the process is weakly transient if the process is non recurrent and \( \sum_{j=0}^{\infty} P_{ij}(t) = 1 \) for all finite \( t \) and any \( i \). The occurrence of this phenomena is equivalent to

\[
\sum_{n=0}^{\infty} \pi_n \sum_{k=n}^{\infty} \frac{1}{\lambda_k \pi_k} = \infty \quad [7]
\]
The process is called strongly transient if for every $t > 0$ and all

$$\sum_{j=0}^{\infty} P_{ij}(t) < 1.$$ 

These two kinds of transient behavior describe all the possibilities of non-recurrence [7].

If $\mu_0 > 0$ then the process possesses an ignored absorbing state at $-1$, a state in which the system remains forever once it arrives there. When the system is in the zero state and a transition occurs, the system moves to state 1 with probability $\frac{\lambda_0}{\lambda_0 + \mu_0}$ and is absorbed with probability $\frac{\mu_0}{\lambda_0 + \mu_0}$. Starting in a given state $i$, the problem can be posed as to whether the particle will be absorbed with certainty into the $-1$ state.

The condition that this happens as one would expect is again equivalent to the divergence of the series $\sum_{k=0}^{\infty} \frac{1}{\lambda_k \pi_k}$. The distribution for the time of absorption for any prescribed initial state can be likewise computed [see 7].

In the same way the answers to many of the natural probabilistic queries may be obtained by applying the general structure theorems of birth and death processes [7]. Still more precise information about special processes can be gained by expressing the orthogonal polynomials and the spectral measure in workable form.

In section one we summarize some of the pertinent formulas connected with Meixner and Laguerre polynomials. Of particular use are the recurrence laws satisfied by these polynomial systems and their generating functions,
In the following section we determine the linear growth models of which the Meixner polynomials constitute the corresponding polynomial systems. There are four linear growth processes connected with Meixner polynomials. Two of these processes possess an ignored -1 state and the other two have zero as a reflecting barrier. These four processes are obtained by different types of normalization procedures applied to the classical Meixner polynomials. An enumeration of several of the quantities of interest for these processes is given in table 1.

Recurrence and absorption questions are discussed at the close of Section 2. Section 3 treats all linear processes connected with Laguerre polynomials. One special example was investigated earlier by Reuter and Lederman [12]. In Section 4 we utilize the generating function formulas displayed in Section 1 to derive explicit formulas for the transition probabilities. In Section 5 we restrict attention to special linear growth processes involving an absorbing state (-1). For these models we investigate the physically significant random variable which counts the number of transitions that transpire before absorption. The explicit determination of the distribution of this random variable derives from properties of systems of ultraspherical polynomials.

The concept of the "related process" where the zero state is converted into an absorbing state is introduced in Section 6. The formalism of the "related process" is exploited so as to characterize the form of the spectral measure corresponding to linear growth processes distinct from those expressed in terms of the Meixner and Laguerre polynomials.
Other quantities of interest related to linear growth processes such as coincidence probabilities and occupation time probabilities will be published separately. The final section discusses the orthogonal polynomial system and the spectral measure of the classical Ehrenfest model.

I. **Summary of formulas for Meixner polynomials and Laguerre systems.** [3], [13].

The classical hypergeometric function is given by

\begin{equation}
F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{n! (c)_n}, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.
\end{equation}

Define

\begin{equation}
\phi_n(x) = \phi_n(x; \beta, \gamma) = F(-n, -x, \beta, 1 - \frac{1}{\gamma})
\end{equation}

for \( \beta > 0 \) and \( 0 < \gamma < 1 \) and set \( \phi_0 \equiv 0 \) then \( \phi_n(x) \) constitute a system of polynomials and

\begin{equation}
m_n(x; \beta, \gamma) = (\beta)_n \phi_n(x; \beta, \gamma)
\end{equation}

are the Meixner polynomials [3] which are orthogonal with respect to a jump function with jumps at \( x = 0, 1, 2, \ldots \). Specifically

\begin{equation}
\sum_{x=0}^{\infty} \phi_m(x) \phi_n(x) \rho_x = \frac{\beta}{m+n} \frac{n!}{(\beta)_n \gamma^n},
\end{equation}

where

\begin{equation}
\rho_x = (1-\gamma)\beta \frac{(\beta)^x}{x!} \gamma^\alpha.
\end{equation}
From a contiguity relation satisfied by the hypergeometric function [see 3] it follows that

\[ -(x \frac{1-\gamma}{\gamma}) \phi_n(x) = \frac{n}{\gamma} \phi_{n-1} - (n + \frac{n}{\gamma} + \beta) \phi_n + (n + \beta) \phi_{n+1}. \]

An important generating function based on the polynomial \( \phi_n \) is

\[ \sum_{n=0}^{\infty} \phi_n(x) \frac{(\beta)_n}{n!} s^n = (1 - \frac{s}{\gamma})^x (1 - s)^{-x-\beta} \]

which converges for \(|s| < \gamma\), at least.

This last identity may be derived directly from the definition of \( \phi_n \). Alternately the reader is referred to [3].

Finally we summarize several formulae concerned with the Laguerre system of polynomials.

Let \( L_n^\alpha(x) \) denote the \( n \)th Laguerre polynomial normalized so that

\[ L_n^\alpha(0) = \left( \frac{n+\alpha}{n} \right) \]

and set

\[ \psi_n(x) = \frac{1}{\binom{n+\alpha}{n}} L_n^\alpha \left( \frac{x}{\kappa} \right) \]

so that \( \psi_n(0) = 1 \). These polynomials satisfy the recurrence relations

\[ -x\psi_n(x) = n\kappa \psi_{n-1} - (2n\kappa + (\alpha+1)\kappa) \psi_n + (n\kappa + (1+\alpha)\kappa) \psi_{n+1} \]

with \( \psi_{-1} = 0 \) and \( \psi_0 = 1 (\alpha > -1) \) \( n \geq 0 \) and are orthogonal with respect to a density function defined on the positive real axis given by

\[ \rho(x) = c e \left( -\frac{x}{\kappa} \right)^\alpha \]

where \( c \) is a normalizing constant.
Some pertinent generating functions associated with Laguerre systems of polynomials are

\( (14) \sum L_n^\alpha(x) s^n = (1-s)^{-\alpha-1} \frac{xs}{s-1} \)

\( (15) \sum \frac{L_n^\alpha(x) w^n}{\Gamma(n+\alpha+1)} = e^{w(xw)^{-\alpha/2}} J_\alpha(2\sqrt{xw}) \)

where \( J_\alpha \) stands for the usual Bessel function of order \( \alpha \). Other formulae as needed will be cited later in the text.

II. Linear growth processes connected with various kinds of Meixner polynomial systems

Starting with the relation (9), and employing appropriate renormalizations of \( \phi_n \) and affine transformations of \( x \), we may derive a series of 4 different recurrence laws which correspond to different kinds of processes.

Case I: For \( \lambda < \mu \), define

\( (16) \phi_n^{(1)}(x) = \phi_n^\prime \left( \frac{x}{\mu-\lambda}; \beta, \frac{\lambda}{\mu} \right) \)

The polynomials \( \phi_n^{(1)} \) constitute a polynomial system orthogonal with respect to the discrete mass measures \( \rho(x) \) which concentrates masses \( (1 - \frac{\lambda}{\mu};^\beta \frac{\mu}{n!} \left( \frac{\lambda}{\mu} \right)^n \)
at the points \((\mu - \lambda)n\) for \(n = 0, 1, 2, \ldots\) (see equation (8)). We see because of (9) that the process with birth rates \(\lambda_n = n \lambda + \beta \lambda\) and death rates \(\mu_n = n \mu\) for \(n \geq 0\) where \(\lambda < \mu\) is associated with the polynomial system \(\phi_n^{(1)}\) (where \(\beta\) is an arbitrary positive parameter.)

**Case II:** Multiplying \(\phi_n^{(1)}(x)\) by the factor \((\frac{\lambda}{\mu})^n\) and suitably translating the variable \(x\) we arrive at a set of polynomials

\[
\phi_n^{(2)}(x) = (\frac{\lambda}{\mu})^n \phi_n^{(1)}(\frac{x}{\mu - \lambda} - \beta \mu; \frac{\lambda}{\mu})
\]

which satisfy

\[
-x \phi_n^{(2)}(x) = n \lambda \phi_{n-1}^{(2)} - (n \mu + n \lambda + \beta \mu) \phi_n^{(2)} + (n + \beta) \mu \phi_{n+1}^{(2)}
\]

for \(n \geq 0\) with \(\lambda < \mu\).

The effect of the above transformation has in substance interchanged the parameters \(\lambda\) and \(\mu\). The birth rates are now identified with \((n + \beta)\mu\) and the death rate is seen to be \(n \lambda\). The corresponding polynomials \(\phi_n^{(2)}(x)\) are orthogonal with respect to the measure which consists of the discrete masses \((1 - \gamma)^{-\beta} \frac{(\beta)_n \gamma^n}{n!}\) located at the points

\[(n + \beta)(\mu - \lambda) \quad n = 0, 1, 2, \ldots\]

where

\[\gamma = \frac{\lambda}{\mu}\]

Cases I and II are related to growth processes for which the state 0 was a reflecting barrier. By means of further normalizations of the
Meixner polynomials we can obtain two additional classes of growth processes where there exists an absorbing state at -1.

With \( \lambda < \mu \), we have the following:

**Case III:** The polynomials

\[
\phi_n^{(3)}(x) = \frac{(\beta)_n}{n!} \phi_n \left( \frac{x}{\mu - \lambda} - \beta + 1; \frac{\lambda}{\mu} \right)
\]

satisfy

\[
-x\phi_n^{(3)}(x) = (n+\beta-1)\mu\phi_n^{(3)} - [(n+\beta-1)\mu + (n+1)\lambda]\phi_n^{(3)} + (n+1)\lambda\phi_{n+1}^{(3)}
\]

**Case IV:** The polynomials

\[
\phi_n^{(4)}(x) = \frac{(\beta)_n}{n!} \phi_n \left( \frac{x}{\mu - \lambda} - 1; \frac{\lambda}{\mu} \right)
\]

satisfy

\[
-x\phi_n^{(4)}(x) = (n+\beta-1)\lambda\phi_n^{(4)} - [(n+\beta-1)\lambda + (n+1)\mu]\phi_n^{(4)} + (n+1)\mu\phi_{n+1}^{(4)}
\]

The results are summarized in the following table. In order to avoid any ambiguities in interpretation of the various entries in the table, we review the meaning of some of the notation used.

(i) \( \gamma = \frac{\lambda}{\mu} \) and always \( \lambda < \mu \)

(ii) \( \phi_n^{(k)}(x; \beta, \gamma) \) is an \( n \)th degree polynomial defined by one of (16), (17), (16a), (17a).

(iii) \( \rho_n = (1-\gamma)^{-\beta} \frac{(\beta)_n}{n!} \gamma^n \)
<table>
<thead>
<tr>
<th>Process Label</th>
<th>Birth Rates</th>
<th>Death Rates</th>
<th>Character of Zero State</th>
<th>Polynomial System</th>
<th>Spectral Measure</th>
<th>$\pi_n$</th>
<th>$\frac{1}{\lambda_n} \pi_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C</strong></td>
<td>$(n+1)\lambda$</td>
<td>$(n+\beta-1)\mu$</td>
<td>If $\beta &gt; 1$ then a permanent absorbing state exists at $-1$ $\mu_0 = (\beta-1)\mu$</td>
<td>$\phi_n^{(3)}(x)$</td>
<td>Masses $\rho_n$ located at $(\mu-\lambda)(n+\beta-1)$ $n=0,1,...$</td>
<td>$\left(\frac{\lambda}{\mu}\right)^n \cdot \frac{n!}{(\beta)^n}$</td>
<td>$\left(\frac{\mu}{\lambda}\right)^n \cdot \frac{1}{(\beta)^n} \cdot \frac{1}{\lambda(n+1)!}$</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>$(n+1)\mu$</td>
<td>$(n+\beta-1)\lambda$</td>
<td>For $\beta &gt; 1$ a permanent absorbing state exists at $-1$ and $\mu_0 = (\beta-1)\lambda$</td>
<td>$\phi_n^{(4)}(x)$</td>
<td>Masses $\rho_n$ located at $(\mu-\lambda)(n+1)$ $n=0,1,2,...$</td>
<td>$\left(\frac{\mu}{\lambda}\right)^n \cdot \frac{n!}{(\beta)^n}$</td>
<td>$\left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{(\beta)^n} \cdot \frac{1}{\mu(n+\beta)!}$</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td>$(n+\beta)\mu$</td>
<td>$n\lambda$</td>
<td>The state 0 is a reflecting barrier $(\mu_0=0)$</td>
<td>$\phi_n^{(2)}(x)$</td>
<td>Masses $\rho_n$ located at $(n+\beta)(\mu-\lambda)$ $n=0,1,2,...$</td>
<td>$\left(\frac{\mu}{\lambda}\right)^n \cdot \frac{\beta_n}{n!}$</td>
<td>$\left(\frac{\lambda}{\mu}\right)^n \cdot \frac{n!}{\mu(\beta+1)!}$</td>
</tr>
<tr>
<td><strong>F</strong></td>
<td>$(n+\beta)\lambda$</td>
<td>$n\mu$</td>
<td>The state 0 is a reflecting barrier $(\mu_0=0)$</td>
<td>$\phi_n^{(1)}(x)$</td>
<td>Masses $\rho_n$ located at $(n+\beta)(\mu-\lambda)$ $n=0,1,2,...$</td>
<td>$\left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{\lambda} \cdot \frac{n!}{(\beta+1)!}$</td>
<td>$\left(\frac{\mu}{\lambda}\right)^n \cdot \frac{1}{\lambda} \cdot \frac{1}{(\beta)^n}$</td>
</tr>
</tbody>
</table>
All the results of the table are easily deduced by beginning with the recursion formula (9) and making the appropriate transformations. It is easy to show that in this manner we have obtained, in fact, all possible linear growth models connected with the classical Meixner polynomials.

It is worthwhile at this point to highlight some obvious qualitative consequences concerning the description of processes C through F. Several of these results depend on a knowledge of the asymptotic growth of \( \pi_n \) and \( \frac{1}{\lambda_n \pi_n} \). The familiar asymptotic relations \( \frac{\beta_n}{n!} \sim \Gamma(\beta) \frac{\beta-1}{n} \) is useful in this connection.

**Process C.**

Since \( \sum \frac{1}{\lambda_n \pi_n} = \infty \) we find that absorption into the -1 state takes place with certainty in finite time [7]. Moreover, the distribution of the time until absorption from an initial prescribed state has all moments finite. This follows because of the existence of the integrals \( \int_0^\infty \frac{d\psi(x)}{x^{n+1}} \) for each \( n \).

**Process D.**

In this case absorption into the -1 state is not a certain event as \( \sum \frac{1}{\lambda_n \pi_n} < \infty \). However, since \( \sum_{n=0}^\infty \pi_n \sum_{k=n}^\infty \frac{1}{\lambda_k \pi_k} > C \sum \frac{1}{n} = \infty \) we conclude that \( \lim_{t \to \infty} \sum_j P_{ij}(t) > 0 \). In more descriptive language this states that although the randomly moving particle is not with certainty absorbed into state -1, there is positive probability that the particle does not drift to infinity.[7].
Process E.

The state 0 is now a reflecting barrier. Since \( \sum \frac{1}{\lambda_n \pi_n} < \infty \), all states in the process are transient. However, since \( \sum_{n=0}^{\infty} \pi_n \sum_{k=0}^{\infty} \frac{1}{\lambda_k \pi_k} = \infty \), the process is weakly transient in the sense that \( \sum_{j=0}^{\infty} P_{1j}(t) \neq 1 \) for all \( t \geq 0 \).

Process F.

This process is strongly recurrent in the sense that all states have a recurrence time distribution possessing an infinite number of moments. Inspection of Table 1 shows that the rate of convergence of the transition probabilities, \( P_{ik}(t) \) to the limiting probabilities \( C\pi_k \) is of the order of magnitude \( e^{-\left(\mu-\lambda\right)t} \).

III. Linear Growth Processes associated with Laguerre polynomials.

Associated with the Laguerre system of polynomials are two processes. The process whose polynomials are given by (11) determines a linear growth process with no absorbing state. The process whose polynomial system is \( L_n^\alpha \left( \frac{x}{\lambda} \right) \) has a permanent absorbing state at -1. The descriptions of some of the meaningful quantities of these processes are summarized in the following table: (table 2).

We append some additional facts about processes A and B.

Process A. In view of the divergence of the series \( \sum \frac{1}{\lambda_k \pi_k} \) for all \( \alpha > 0 \),
we infer that \(-1\) is a certain absorbing state. We also observe that
\[
\int_{x}^{\infty} \frac{d\rho(x)}{x^{n+1}} < \infty \text{ is equivalent to the requirement } \alpha > n, \text{ so that the } n\text{th}
\]
moment of the absorption time distribution is finite if and only if \(\alpha > n\).
In particular if \(0 < \alpha \leq 1\) then absorption is certain but the expected
time of absorption is infinite, while the expected time of absorption is finite if \(\alpha > 1\).

Process B.

Since \(\sum \pi_n = \infty\) for all \(\alpha > -1\) and \(\sum \frac{1}{\lambda_n \pi_n}\) also diverges for
\(-1 < \alpha \leq 0\), we deduce that the process is null recurrent.

\(\alpha > 0\) implies that \(\sum \frac{1}{\lambda_n \pi_n} < \infty\) so in that circumstance the process
is transient. However \(\sum_{0}^{\infty} \pi_K \sum_{r=K}^{\infty} \frac{1}{\lambda_r \pi_r} = \infty\) and hence the process is
only weakly transient. We know this for another reason for if it were
strongly transient the spectral measure of the process would have to be
discrete which is evidently not true for the case of process B.

In dealing with transient processes one can investigate the random
variable \(N_{10}\) representing the first passage time from state 1 to state 0
even if the occurrence of first passage is not a certain event. The natural
concept to consider becomes the conditional first passage time given that
the event of passing from state 1 to state zero happened. We may then
analyze the properties of this conditional first passage random variable.
[7]. For example, the problem of the finiteness of \(E(N_{10} \mid N_{10} < \infty)\) is
<table>
<thead>
<tr>
<th>Process Label</th>
<th>Birth Rates $\lambda_n, n &gt; 0$</th>
<th>Death Rates $\mu_n, n \geq 0$</th>
<th>Polynomial System $L_n^\alpha \frac{x^\alpha}{\kappa}$</th>
<th>Spectral Measure $Ce^{\frac{x}{\kappa}} x^\alpha$</th>
<th>$\pi_n$</th>
<th>$\frac{1}{\lambda_n \pi_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A(-1 absorbing state exists $\alpha &gt; 0$ defined for all $\alpha &gt; 0$)</td>
<td>$(n + 1) \kappa$</td>
<td>$(n + \alpha) \kappa$</td>
<td>$L_n^\alpha \frac{x^\alpha}{\kappa}$</td>
<td>$Ce^{\frac{x}{\kappa}} x^\alpha$</td>
<td>$\frac{n!}{(\alpha+1)_n} \frac{\Gamma(\alpha+1)}{\alpha}$</td>
<td>$\frac{\lambda(\alpha+1)_n}{n+1} \frac{\lambda}{\Gamma(\alpha+1)}$</td>
</tr>
<tr>
<td>B(0 reflecting state) $\alpha &gt; -1$</td>
<td>$(n+1+\alpha)\kappa$</td>
<td>$n\kappa$</td>
<td>$L_n^\alpha \frac{x^\alpha}{(n+\alpha)_n}$</td>
<td>$Ce^{\frac{x}{\kappa}} x^\alpha$</td>
<td>$\frac{n!}{(\alpha+1)_n} \frac{\Gamma(\alpha+1)}{\alpha}$</td>
<td>$\frac{\lambda(\alpha+1)_n}{n+1} \frac{\lambda}{\Gamma(\alpha+1)}$</td>
</tr>
</tbody>
</table>
known to be equivalent to the convergence of the series \([7]\)

\[
\sum_{n=0}^{\infty} \pi_r \left( \sum_{\ell=r}^{\infty} \frac{1}{\ell^\alpha} \right)^2 \sim \sum \frac{1}{n^\alpha}.
\]

Specifically for \(\alpha > 1\), the series converges to a finite value. The higher order moments may also be thus evaluated. Other points of interest for these processes may be analyzed in terms of the elaborate structure theorems developed for birth and death processes \([6]\).

**IV. Generating functions and explicit representations of probabilities**

In this section we obtain explicit expressions for the transition probabilities of the various processes introduced previously. The classical approach in securing these formulae was to derive a first order partial differential equation for the generating function of the transition probabilities. This differential equation was solved whenever possible. By our methods the sought for representations and explicit formulae for the transition probabilities emerge as a result of simple manipulations of the corresponding generating function for the appropriate polynomials.

We illustrate the methods for the case of process \((F)\). We first compute \(P_{0j}(t)\).

From the general theory

\[
P_{0j}(t) = \pi_j \int_0^\infty e^{-xt} Q_j(x) \, d\rho(x)
\]

where \(Q_n\) is the polynomial system of the process \((F)\) and \(\rho(x)\) is the spectral measure. These quantities are explicitly given in Table 1. Writing
out (18) we have
\[ P_{0,j}(t) = (1 - \frac{\lambda}{\mu})^{-\beta} \pi_j \sum_{n=0}^{\infty} e^{-(\mu - \lambda)nt} \phi_n(n; \beta, \gamma) \frac{(\beta)_n}{n!} \left( \frac{\lambda}{\mu} \right)^n. \]

With the aid of identities \( \phi_j(n; \beta, \gamma) = \phi_n(j; \beta, \gamma) \) eqs. (6) and (10), we have
\[ P_{0,j} = (1 - \frac{\lambda}{\mu})^{-\beta} \pi_j \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \phi_n(j; \beta, \frac{\lambda}{\mu}) \sigma^n \]
\[ = \pi_j (1 - \frac{\lambda}{\mu})^{-\beta} (1 - \frac{\mu \sigma}{\lambda})^j (1 - \sigma)^{-j - \beta} \]
where
\[ \sigma = \frac{\lambda}{\mu} e^{-(\mu - \lambda)t}. \]

Next we seek to obtain an explicit formula for \( P_{ij}(t) \). This will depend upon evaluation of a series of the form
\[ M_{ij} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} \phi_n(i) \phi_n(j) s^n. \]

To this end, we form
\[ \sum_{j=0}^{\infty} \frac{(\beta)_j}{j!} z^j M_{ij} \]
which after using (10) twice and interchanging orders of summation reduces to the expression
\[ \sum_{j=0}^{\infty} \frac{(\beta)_j}{j!} z^j \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} \phi_n(i) \phi_n(j) s^n \]
\[ = (1 - \frac{s}{\gamma})(1 - s)^{-1 - \beta} \left[ \frac{1 - \frac{s}{\gamma}}{1 - \frac{s}{\gamma} z} \right]^i \left[ \frac{1 - \frac{s}{\gamma}}{1 - \frac{s}{\gamma} z} \right]^j \]
Expanding in powers of $z$ and reading off the coefficient of $z^j$ (where for definiteness we have assumed that $i \leq j$) we get

$$
\frac{j!}{(\beta)_j} \sum_{k=0}^i \binom{i}{k} (-1)^k \left( \frac{1 - s}{\gamma^2} \right)^k \left( \frac{i+\beta}{j-k} \frac{1 - s}{\gamma} \right)^{(j-k)!} \left( \frac{1 - s}{1 - s} \right)^{(j-k)}
$$

except for a factor $(1 - \frac{s}{\gamma})^i (1 - s)^{-i-\beta}$.

In order to apply this to the process $(\mathcal{F})$ we observe that

$$
P_{ij}(t) = (1 - \gamma)^{-\beta} \pi_j \sum_{n=0}^\infty \sigma^n \frac{(\beta)_n}{n!} \phi_n(i) \phi_n(j)
$$

where

$$
\sigma = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}
$$

Identifying (22) with (20) we obtain from (21) for $i \leq j$ that

$$
P_{ij}(t) = (1 - \gamma)^{-\beta} \pi_j \frac{j!}{(\beta)_j} (1 - \frac{s}{\gamma})^i (1 - \sigma)^{-i-\beta}
$$

$$
\cdot \sum_{k=0}^i \binom{i}{k} (-1)^k \left( \frac{1 - s}{\gamma^2} \right)^k \left( \frac{1 - s}{1 - \sigma} \right)^{(j-k)!} \left( \frac{1 - s}{1 - s} \right)^{(j-k)}
$$

where

$$
\sigma = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}
$$

and

$$
\gamma = \frac{\lambda}{\mu}
$$

The formula for $P_{ij}$ when $i > j$ is easily obtained from the relation

$$
P_{ij} = \frac{\pi_j}{\pi_i} P_{ji}.
$$

In the following tables we summarize the formulae for the
transition probabilities for the process indicated.

Table 3. \[ i \leq j, \sigma = \frac{\lambda}{\mu} \cdot e^{-(\mu-\lambda)t}, \gamma = \frac{\lambda}{\mu} \]

**Process C.**

\[ P_{ij} = (1-\gamma)^{-\mu} \pi_j \left( 1 - \frac{\sigma}{\gamma} \right)^i \left( 1 - \beta \right)^{-1-\beta} e^{-(\mu-\lambda)(\beta-1)t} \frac{(\beta)_i}{i!} \]

\[ \times \sum_{k=0}^{i} \binom{i}{k} (-1)^k \left( \frac{1 - \frac{\sigma}{\gamma^2}}{1 - \frac{\sigma}{\gamma}} \right)^k \left( \frac{1 - \frac{\sigma}{\gamma}}{1 - \sigma} \right)^{j-k} \frac{(i+\beta)^{j-k}}{(j-k)!} \]

**Process D.**

\[ P_{ij}(t) = (1-\gamma)^{-\mu} \pi_j \frac{(\beta)_i}{i!} \left( \frac{\lambda}{\mu} \right)^{i+j} \left( 1 - \frac{\sigma}{\gamma} \right)^i \left( 1 - \sigma \right)^{-1-\beta} e^{-(\mu-\lambda)t} \]

\[ \times \sum_{k=0}^{i} \binom{i}{k} (-1)^k \left( \frac{1 - \frac{\sigma}{\gamma^2}}{1 - \frac{\sigma}{\gamma}} \right)^k \left( \frac{1 - \frac{\sigma}{\gamma}}{1 - \sigma} \right)^{j-k} \frac{(i+\beta)^{j-k}}{(j-k)!} \]

**Process E.**

\[ P_{ij}(t) = \left( \frac{\lambda}{\mu} \right)^{i+j} (1-\gamma)^{-\mu} \pi_j \frac{j!}{(\beta)_j} \left( 1 - \frac{\sigma}{\gamma} \right)^i \left( 1 - \sigma \right)^{-1-\beta} e^{-(\mu-\lambda)\beta t} \]

\[ \times \sum_{k=0}^{i} \binom{j}{k} (-1)^k \left( \frac{1 - \frac{\sigma}{\gamma^2}}{1 - \frac{\sigma}{\gamma}} \right)^k \left( \frac{1 - \frac{\sigma}{\gamma}}{1 - \sigma} \right)^{j-k} \frac{(i+\beta)^{j-k}}{(j-k)!} \]

**Process F.**

\[ P_{ij}(t) = (1-\gamma)^{-\mu} \pi_j \frac{j!}{(\beta)_j} \left( 1 - \frac{\sigma}{\gamma} \right)^i \left( 1 - \sigma \right)^{-1-\beta} \]

\[ \times \sum_{k=0}^{i} \binom{j}{k} (-1)^k \left( \frac{1 - \frac{\sigma}{\gamma^2}}{1 - \frac{\sigma}{\gamma}} \right)^k \left( \frac{1 - \frac{\sigma}{\gamma}}{1 - \sigma} \right)^{j-k} \frac{(i+\beta)^{j-k}}{(j-k)!} \]
We next deduce the explicit expressions for the transition probabilities of processes A and B. For process A

\begin{equation}
P_{i,j}(t) = \kappa^{\alpha+1} \frac{\pi_i^j}{\Gamma(\alpha+1)} \int_0^\infty e^{-xt} x^\alpha \frac{x}{\kappa} e^{-\frac{x}{\kappa}} \, dx
\end{equation}

\begin{equation}
= \frac{\pi_i^j}{\Gamma(\alpha+1)} \mathcal{L} \left( \alpha \mathcal{L}_j(y) \right)
\end{equation}

evaluated at \( s = \kappa t + 1 \) (where \( \mathcal{L} \) denotes the Laplace transform of)

\begin{equation}
= \frac{\pi \alpha}{\Gamma(\alpha+1)} \left( \frac{\Gamma(\alpha+j+1)}{j!} \right) \frac{(\kappa t)^j}{(1+\kappa t)^{\alpha+j+1}} = \frac{(\kappa t)^j}{(1+\kappa t)^{\alpha+j+1}} \quad [\text{see 2}].
\end{equation}

Also,

\begin{equation}
\sum_i P_{i,j} \frac{s^i}{(1-s)^{\alpha+1}} = \frac{\pi_i^j}{\Gamma(\alpha+1)} \frac{(1-s)^{-\alpha-1}}{(1-\kappa t)^{\alpha+j+1}} \int_0^\infty e^{-x t} x^{\alpha+1} / (1-s) \, dx
\end{equation}

and by virtue of (14) this reduces to

\begin{equation}
= \frac{\pi_i^j}{\Gamma(\alpha+1)} \frac{(1-s)^{-\alpha-1}}{(1-\kappa t)^{\alpha+j+1}} \int_0^\infty e^{-(\kappa t + 1 - \frac{s}{s-1}) x} x^{\alpha+1} \mathcal{L}_j(x) \, dx
\end{equation}

\begin{equation}
= \frac{(\kappa t)^j}{(1+\kappa t)^{\alpha+j+1}} \left[ 1 + \frac{1-\kappa t}{\kappa t} s \right]^{-\alpha-1} \left[ 1 - \frac{\kappa t}{1+\kappa t} \right]^{-\alpha-1}
\end{equation}

Hence for \( i \leq j \)

\begin{equation}
P_{i,j}(t) = \frac{(\kappa t)^j}{(1+\kappa t)^{\alpha+j+1}} \sum_{k=0}^{j} \binom{j}{k} \left( \frac{1-\kappa t}{\kappa t} \right)^k \frac{(\kappa t)^{i-k}}{(1+\kappa t)^{i-k}} \frac{(\alpha+j+1)_{i-k}}{(i-k)!}
\end{equation}
For Process B similar methods apply and we get

\[
\sum_{j=0}^{\infty} P_{ij}(t)s^j = \frac{(kt)^i}{(1+kt)^{\alpha+i+1}} \frac{(1 - \frac{1-kt}{kt}s)^i}{(1 - \frac{kt}{1+kt}s)^{\alpha+i+1}}
\]

and so for \( i \leq j \)

\[
P_{ij}(t) = \frac{(kt)^i}{(1+kt)^{\alpha+i+1}} \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} \left( \frac{1-kt}{kt} \right)^k \left( \frac{kt}{1+kt} \right)^{j-k} \frac{\alpha+i+1}{j-k} !
\]

V. Number of transitions until absorption

A very interesting concept that was introduced by D. Kendall in connection with linear growth models is the number of transitions to the right which the diffusing particle undergoes prior to ultimate absorption [11]. This measures, for example, the intensity of the epidemic if the state of a system describes the number of people currently infected. The relevant probabilities of this random variable are readily evaluated once \( R^n_1 \)

(26) \( (R^n_1 = \text{probability of being absorbed on the } n\text{th transition given that initially the particle is in state } i.) \)

is completely determined. The quantities \( R^n_1 \) evidently satisfy the recurrence relations

\[
R^n_0 = p_0 R^n_1
\]

\[
R^n_1 = q_1 R_{i-1}^n + p_1 R_{i+1}^{n-1}
\]

where \( p_i = \frac{\lambda_i}{\lambda_i + \mu_i} \) and \( q_i = \frac{\mu_i}{\lambda_i + \mu_i} \). The particular values of \( p_n \) and \( q_n \)
corresponding to process (A) are

\[ p_n = \frac{n+1}{2n+\alpha+1} \quad \text{and} \quad q_n = \frac{n+\alpha}{2n+\alpha+1} . \]

Consider the polynomial system

\[ xT_n^{\alpha}(x) = \frac{n+\alpha}{2n+\alpha+1} T_n^{\alpha}(x) + \frac{n+1}{2n+\alpha+1} T_{n+1}^{\alpha}(x) \]

and \( T_0 = 1 \) \( T_1 = (1+\alpha)x \). These are recognized as the ultraspherical polynomials \( C_n^\lambda \) of order \( \lambda = \frac{1+\alpha}{2} \) normalized so that \( C_n^\lambda(1) = \binom{n+2\lambda-1}{n} \)[13 pg80],

\[ T_n^{\alpha}(x) = C_n^{\frac{1+\alpha}{2}}(x) . \]

It follows immediately from the theory of random walks that

\[ R_n = \frac{\Gamma(\alpha+3)}{\Gamma^2(\frac{\alpha}{2}+1)} \frac{\alpha}{(1+\alpha)} \frac{1}{2^\alpha+1} \int_{-1}^{1} x^{M-1} \frac{1+\alpha}{2^\alpha(x-1)^{\frac{\alpha}{2}}} \, dx \]

[see 9].

Another linear growth process to which a modification of formula (29) may be applied is the case where \( \lambda_n = (n+1)\lambda \) and \( \mu_n = (n+1)\mu, \ n \geq 0 \). We have \( p_n = \frac{\lambda}{\lambda+\mu} = p, q_n = \frac{\mu}{\lambda+\mu} = q \).

The associated polynomials \( U_n(x) = \left( \frac{P}{q} \right)^n T_n\left( \frac{x}{4pq} \right) \) satisfy

\[ xU_n(x) = pU_{n+1}(x) + qU_{n-1}(x) \quad n \geq 0 \]

where \( U_0 = 1 \quad U_{-1} = 0 \).
Hence, corresponding to $\lambda_n = (n+1)\lambda$ and $\mu_n = (n+1)\mu$ we have

\[
R_n = \frac{\mu}{\lambda + \mu} - \frac{1}{2\pi^2 (3/2)} \int_{-1}^{1} x^{n-1} \left( \sqrt{\frac{\mu}{\lambda}} \right)^i \left( \frac{x}{\sqrt{4pq}} \right) (1-x^2)^{1/2} \, dx.
\]

VI. Recurrence distribution and related processes.

Inspection of Tables 1 and 2 indicates that not all linear growth processes are directly related to the classical Meixner and Laguerre polynomial systems. However, from a study of the concept of the "related process" it is possible to compute the spectral measure of the remaining linear models. Based on the same analysis we can derive approximate expressions for the recurrence time distributions connected with processes A through F.

From a given birth and death process with infinitesimal matrix (2) a new process is obtained by stopping the given process whenever the state 0 is reached. For this new process the state 0 is transformed into an absorbing state, and if we ignore this state the process is a birth and death process for which the parameter $\mu_0$ is positive. The waiting time in any state $i \geq 1$ has the same distribution for both the original and the new process, and moreover both processes have the same post exit distributions for each state $i \geq 1$. Consequently the infinitesimal matrix of the new process (with the state 0 ignored) is
\[
\begin{pmatrix}
-\lambda_1 + \mu_1 & \lambda_1 & 0 & \cdots \\
\mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots \\
0 & \mu_3 & -(\lambda_3 + \mu_3) & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

(31)

which is obtained from (1) by removing the zero row and zero column.

The polynomials defined by

\[
\begin{align*}
Q_0^{(0)}(x) &= 0 , \\
Q_1^{(0)}(x) &= -\frac{1}{\lambda_0} , \\
-xQ_n^{(0)}(x) &= \mu_n Q_{n-1}^{(0)}(x) - (\lambda_n + \mu_n) Q_n^{(0)}(x) + \lambda_n Q_{n+1}^{(0)}(x), \quad n \geq 1
\end{align*}
\]

are called the associated polynomials of the system \( \{Q_n(x)\} \). It is seen that, except for the constant factor \(-\frac{1}{\lambda_0}\), they are the polynomials belonging to the new birth and death process. Consequently the transition probability matrix \( (\tilde{P}_{i,j}(t)), \ i, j \geq 1 \), of the new process is given by

\[
\tilde{P}_{i,j}(t) = \frac{\mu_i}{\lambda_0} \pi_j \int_0^\infty e^{-xt} [\frac{Q_1^{(0)}(x)}{-\lambda_0}] [-\lambda_0 Q_j^{(0)}(x)] \, d\alpha(x)
\]

where \( \alpha \) is the spectral measure of the "related process." In [7] it is shown that Stieltjes transforms of the spectral measures \( \psi \) and \( \alpha \) of the two processes,

\[
B(s) = \int_0^\infty \frac{d\psi}{x-s}, \quad C(s) = \int_0^\infty \frac{d\alpha}{x-s},
\]

(32)
are related by the identity

(33) \[ B(s) = \frac{1}{\lambda_0 + \mu_0 - s - \lambda_0 \mu_1 C(s)} \]

Once the function \( B(s) \) or \( C(s) \) has been found, the measures \( \psi \) and \( \alpha \) can be computed by means of known formulae for inverting the Stieltjes transform. See [7] for a discussion of this inversion relative to the identity (33) and [14], [15] for the general inversion problem. If \( \psi \) is a discrete distribution with masses \( \rho_0, \rho_1, \rho_2, \ldots \) located at the points \( 0 \leq x_0 < x_1 < x_2 < \ldots \ x_n \to \infty \), then

(34) \[ B(s) = \sum_{n=0}^{\infty} \frac{\rho_n}{x_n - s} \]

is a meromorphic function whose only poles are simple poles located at the points \( x_n \). In each of the open intervals \( (x_n, x_{n+1}) \) \( B(s) \) strictly increases from \( -\infty \) to \( +\infty \) and consequently has precisely a simple zero \( y_n \) in this interval. \( C(s) \) is therefore a meromorphic function whose only poles are at the zeros of \( B(s) \). The measure \( \alpha \) is also a discrete distribution with jumps of magnitude

(35) \[ \gamma_n = -\frac{1}{B'(y_n)} = \frac{1}{\lambda_0 \mu_1} \]

located at \( y_n \) which strictly interlock with \( x_n \);

(36) \[ 0 \leq x_0 < y_0 < x_1 < y_1 < x_2 < y_2 < \ldots \]

We tabulate \( B(s) \) for each of the processes (C) - (F)
\[ B(s) = (1 - \gamma)^{-\beta} \sum_{n=0}^{\infty} \frac{(\beta)^n}{n!} \frac{\gamma^n}{(n+\beta-1)(\mu-\lambda)^s} \quad \text{process (C)} \]

\[ B(s) = (1 - \gamma)^{-\beta} \sum_{n=0}^{\infty} \frac{(\beta)^n}{n!} \frac{\gamma^n}{(n+1)(\mu-\lambda)^s} \quad \text{process (D)} \]

\[ B(s) = (1 - \gamma)^{-\beta} \sum_{n=0}^{\infty} \frac{(\beta)^n}{n!} \frac{\gamma^n}{(n+\beta)(\mu-\lambda)^s} \quad \text{process (E)} \]

\[ B(s) = (1 - \gamma)^{-\beta} \sum_{n=0}^{\infty} \frac{(\beta)^n}{n!} \frac{\gamma^n}{n(\mu-\lambda)^s} \quad \text{Process (F)} \]

In any practical situation the zeros \( y_n \) of \( B(s) \) may be readily computed with the aid of high speed computers. Only one of the four functions \( B(s) \) need be dealt with in any detail. In fact, examination of (37) shows that once the zeros of \( B(s) \) for any one of the four processes is found then the zeros of the other \( B(s) \) are obtained by translation. Some special properties shared by the zeros \( y_n \) which is an aid to their calculation may be cited. One such characteristic is the inequality \( y_{n-1} < y_n - (\mu-\lambda) \).

The proof of this fact may be found in [8, section 3].

We now indicate how knowledge regarding the spectral measure \( \alpha \) enables us to compute the recurrence time distribution. For definiteness we concentrate attention on process (F). Suppose we seek to evaluate \( F_{10}(t) \) which represents the distribution function of the first passage time for the transition of the particle from the state one to the state zero. The solution of the problem is given in terms of the related spectral measure, [7], as follows:
(38) \[ F_{10}(t) = \mu_1 \int_0^t \int_0^\infty \frac{1}{\gamma_n} e^{-\gamma_n \xi} d\alpha(x) \]

\[ = \mu \sum_{n=0}^{\infty} \left[ \frac{1-e^{-\gamma_n \xi}}{\gamma_n} \right] \gamma_n \]

Since \( \gamma_n \) tends to \( \infty \) and \( 0 \leq \gamma_n \sum \gamma_n = 1 \), a good approximation may be obtained by taking an appropriate finite number of terms.

We next show how knowledge of \( \alpha \) can be used to generate other cases of linear growth not encompassed by those reviewed in tables 1 and 2. Restricting again attention to process \( F \) we know that the associated spectral measure \( \alpha \) has birth and death rates given by \( \lambda_n = (n+1) \lambda, \mu_n = (n+1) \mu, \ n \geq 0 \). We construct the measure

(39) \[ \Theta(x) = \begin{cases} \mu \frac{d\alpha(x)}{x} & x > 0 \\ 1-\mu \int_0^\infty \frac{d\alpha(x)}{x} & x = 0+ \end{cases} \]

whose orthogonal polynomials are \( \frac{\lambda_n \pi_n}{\lambda_0} [Q_{n+1}(x) - Q_n(x)] \) where \( Q_n(x) \) is the polynomial system attached to \( \alpha(x) \).

The birth and death rates corresponding to the spectral measure \( \Theta(x) \) can be shown to be respectively \( \lambda_n = (n+1) \mu, \ n \geq 0 \), \( \mu_n = (n+1) \mu, \ n > 1 \), \( \mu_0 = 0 \) [6].

By applying similar transformations on the "related spectral measure" of the processes \( C \) through \( F \) we obtain all linear growth processes with \( \lambda \) distinct from \( \mu \). For further details concerning these transformations the reader is referred to [see 6] and [7]. It should be
emphasized that none of the four additional processes so obtained may be represented simply in terms of known elementary functions or recognizable classical systems of function. The series representations in terms of the zeros \( y_n \), should be regarded as in essence the simplest expression of the formulae linked to the related linear growth process.

We close this section with a brief discussion of some of the features of the "related process" of process (B) of Table 2. The fundamental equation (32) which relates the Stieltjes transform of \( \psi \) and the spectral measure \( \alpha \) of the related process must be inverted. The mechanism of the inversion when \( \psi \) has a positive continuous density is discussed in proposition A of [7]. It is shown that \( \alpha \) has a density

\[
(40) \quad \alpha'(\xi) = \frac{\psi'(\xi)}{(P.V.) \int_0^\infty \frac{\psi'(x)}{x-\xi} dx + \psi'(\xi)^2} \quad \xi \geq 0
\]

\( \text{(P.V. denotes principal value).} \)

For the special case \( d\psi(\xi) = \frac{1}{\Gamma(\beta+1)} e^{-\xi} \xi^\beta d\xi \)

\[
P.V. \int_0^\infty \frac{d\psi(x)}{x-y} = -\frac{\zeta}{\Gamma(\beta+1)} e^{-y} \frac{\beta}{e^{-y} - 1} e^{y t - t \beta - 1 dt} \text{ for } -1 < \beta < 0 \text{ and (40) becomes}
\]

\[
(41) \quad \alpha'(\xi) = \frac{1}{\pi^2} \frac{1}{\Gamma(\beta+1)} e^{-\xi} \frac{1}{\xi^\beta} \frac{1}{2} e^{\xi t - \beta - 1 dt} \text{ for } -1 < \beta < 0
\]
With the aid of the explicit evaluation of $\alpha(\xi)$ as given by (41) we can now write out the first passage distribution $F_{10}(t)$ of process B. In fact,

$$F_{10}(t) = \int_0^\infty \left[ \frac{1-e^{-xt}}{x} \right] \alpha'(x) \, dx$$

With (41) and use of the device suggested in (39) (see lemma A and (B) of [7]) we can introduce the two remaining types of linear growth processes for which $\lambda = \mu$. The analysis has already been described in connection with the "related processes" to the Meixner system and so we do not spell out the corresponding details for the Laguerre polynomial systems. In any practical case of linear growth the methodology of this paper may be routinely applied.

VII. Continuous Ehrenfest Model and The Krawtchouk Polynomials.

We discuss briefly the classical continuous Ehrenfest process and determine its orthogonal polynomial system and the spectral measure. For a description of the physical background and the analysis of certain limit the laws attached to special random variables of the Ehrenfest process we refer the reader to [1], [5], and [16]. The Ehrenfest model is a birth and death process for which $\lambda_n = (N-n) p$, $\mu_n = n q$, $0 \leq n \leq N$, $0 < p < 1, l = 1-p$ such that the state -1 and the state $N+1$ constitute reflecting barriers. We recognize the polynomial system as the classical Krawtchouk polynomials

$$Q_n(x) = \binom{N}{n}^{-1} p^n \sum_{v=0}^{n} \binom{N-x}{n-v} \binom{x}{v} p^{n-v} q^v$$
where $x$ is a discrete variable ranging over the integers 0, 1, 2, ... $N$. The spectral measure $\psi$ is the familiar binomial distribution which places mass $\binom{N}{x} p^x q^{N-x}$ at $x = 0, 1, 2, ... N$. Because of the generating function relationship

$$\sum_{k=0}^{N} \pi_k Q_k(x) w^k = (1-w)^x (1 + \frac{p}{q} w)^{N-x},$$

$$\pi_k = \binom{N}{k} \left( \frac{p}{q} \right)^k$$

we find easily

$$\sum_{s=0}^{N} P_{r,s}(t) w^s = \left[ pe^{-t}(1-w) + (q+pw) \right]^{N-r} \left[ (q+pw) - q(1-w)e^{-t} \right]^r$$

where $P_{r,s}(t)$ is the transition function.

Another form of $P_{r,s}(t)$ is

$$P_{r,s}(t) = \binom{N}{s} \left( \frac{p}{q} \right)^s \int_0^\infty e^{-xt} Q_r(x) Q_s(x) d\psi(x)$$

which is in essence a finite series. Recurrence, and absorption probabilities may be computed by utilizing the technique of the "related process" in a standard fashion.
REFERENCES


STANFORD UNIVERSITY

Technical Report Distribution List

CONTRACT Nonr-225(28)
(NR 047-019)

Armed Services
Technical Information Agency
Knott Building
Dayton 2, Ohio

Commanding Officer
Office of Naval Research
Branch Office
The John Crerar Library Building
86 East Randolph Street
Chicago 1, Illinois

Commanding Officer
Office of Naval Research
Branch Office
346 Broadway
New York 13, New York

Commanding Officer
Office of Naval Research
Branch Office
Navy No. 100,
c/o Fleet Post Office
New York, New York

Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco 9, California

Commanding Officer
Office of Naval Research
Branch Office
1030 E. Green Street
Pasadena 1, California

Logistics Branch
Room 2718, Code 436
Office of Naval Research
Washington 25, D. C.

Contract Administrator,
Southeastern Area
Office of Naval Research
2110 "G" Street, N.W.
Washington 7, D. C.

Industrial College of the Armed Forces
Fort Lesley J. McNair
Washington 25, D. C.
Attn: Mr. L. L. Henkel

Director
National Science Foundation
Washington 25, D. C.

Director
National Security Agency
Washington 25, D. C.

Director
Naval Research Laboratory
Washington 25, D. C.
Attn: Technical Information Officer

Naval War College
Logistics Department, Luce Hall
Newport, Rhode Island

Office of Technical Services
Department of Commerce
Washington 25, D. C.

Professor Kenneth J. Arrow
Department of Economics
Stanford University
Stanford, California

Dr. Edward Barankin
University of California
Berkeley, California

Dr. Richard Bellman
The RAND Corporation
1700 Main Street
Santa Monica, California

Dean L.M.K. Boelter
Department of Engineering
University of California
Los Angeles 24, California
Prof. S. S. Cairns, Head
Department of Mathematics
University of Illinois
Urbana, Illinois

Dr. E. W. Cannon
Applied Mathematics Division
National Bureau of Standards
Washington 25, D. C.

Dr. A. Charnes
School of Industrial Engineering
and Management
Purdue University
Lafayette, Indiana

Dr. Randolph Church, Chairman
Department of Mathematics
and Mechanics
U. S. Naval Postgraduate School
Monterey, California

Prof. William W. Cooper
Graduate School of Industrial
Administration
Carnegie Institute of Technology
Pittsburgh 13, Pennsylvania

Dr. George B. Dantzig
The RAND Corporation
1700 Main Street
Santa Monica, California

Rear Admiral H. E. Eccles, USN, (RET)
101 Washington Street
Newport, Rhode Island

Prof. Eberhard Fels
Dept. of Economics
University of California
Berkeley 4, California

Prof. David Gale
Department of Mathematics
Brown University
Providence, Rhode Island

Mr. Murray A. Geisler
The RAND Corporation
1700 Main Street
Santa Monica, California

Mrs. Dorothy M. Gilford
Office of Naval Research, Code 433
Room 2709, T-3 Bldg.
Washington 25, D. C.

Dr. Theodore E. Harris
The RAND Corporation
1700 Main Street
Santa Monica, California

Dr. J. Heller
Navy Management Office
Washington 25, D. C.

Dr. C. C. Holt
Graduate School of Industrial
Engineering
Carnegie Institute of Technology
Schenley Park
Pittsburgh 13, Pennsylvania

Dr. Walter Jacobs
Hqds. USAF, DCS/Comptroller
AFADA-3D, The Pentagon
Washington 25, C. C.

Prof. J. R. Jackson
Management Sciences Research
Project
University of California
Los Angeles 24, California

Capt. W. H. Keen
Staff, Commander Fleet Air Wing
Western Pacific
C/o Fleet Post Office
San Francisco, California

Prof. Harold Kuhn,
Dept. of Mathematics
Bryn Mawr College
Bryn Mawr 6, Pennsylvania

Prof. S. B. Littauer
Dept. of Industrial Engineering
Columbia University
409 Engineering Building
New York 27, New York
Dr. William Madow  
Center for Advanced Study in Behavioral Sciences  
202 Junipero Serra Blvd.  
Stanford, California  

Dr. W. H. Marlow  
The George Washington University  
Logistics Research Project  
707 22nd Street  
Washington 7, D. C.  

Prof. Jacob Marschak  
Cowles Commission  
Yale University  
New Haven, Connecticut  

Prof. Oskar Morgenstern  
Economics Research Project  
Princeton University  
9-11 Lower Pyne Building  
92 A Nassau Street  
Princeton, New Jersey  

Prof. R. R. O'Neill  
Department of Engineering  
University of California  
Los Angeles, California  

Dr. I. Richard Savage  
School of Business, Vincent Hall  
University of Minnesota  
Minneapolis, Minn.  

Prof. H. N. Shapiro  
New York University  
Institute of Mathematical Sciences  
New York, New York  

Prof. H. A. Simon  
Head, Dept. of Industrial Administration  
Carnegie Institute of Technology  
Schenley Park  
Pittsburgh 13, Pennsylvania  

Mr. J. R. Simpson  
Bureau of Supplies and Accounts, Code SS  
Arlington Annex  
U. S. Department of the Navy  
Washington 25, D. C.  

Prof. R. M. Thrall  
Department of Mathematics  
University of Michigan  
Ann Arbor, Michigan  

Prof. L. M. Tichvinsky  
Department of Engineering  
University of California  
Berkeley 4, California  

Prof. James Tobin  
Cowles Foundation for Research in Economics  
Box 2125, Yale Station  
New Haven, Connecticut  

Dr. C. B. Tompkins, Director  
Numerical Analysis Research  
University of California  
405 Hilgard Avenue  
Los Angeles 24, California  

Prof. A. W. Tucker  
Department of Mathematics  
Fine Hall, Box 708  
Princeton University  
Princeton, New Jersey  

Prof. Jacob Wolfowitz  
Department of Mathematics  
Cornell University  
Ithaca, New York  

Dr. Max A. Woodbury  
Department of Mathematics  
New York University  
University Heights  
New York 53, New York  

Additional copies for project leader and assistants and reserve for future requirements  