A CLASS OF DYNAMIC GAMES

BY

WILLIAM E. PRUITT

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Introduction.

There is a large class of business and military situations in which decisions must be made on the basis of certain observed random variables. When these situations are described as games they often bear a striking resemblance to poker. A fixed-point method for the solutions of games of this type was proposed by Karlin and Restrepo in [1]. Several illustrations of its application to poker models were given in [1] and [2]. One of their examples, known as "le Her", indicated the applicability of the method to games in which there is more than one relevant random variable observed by each player. It is the purpose of this paper to examine some further games in which more than one random variable is observed by each player and some of the random variables may be observed by both players. Since the primary interest is in the method of solution, poker models have been chosen to illustrate the method because of their familiarity. It is hoped that solutions of some tactical problems of interest will be forthcoming.

In the games considered here the strategies for Players I and II will be represented by the vectors

\[ \phi(x^1, y^2) = \left( \phi_1(x^1, y^2), \cdots, \phi_m(x^1, y^2) \right) \]
and

\[ \psi(y^1, x^2) = \left\{ \psi_1(y^1, x^2), \ldots, \psi_n(y^1, x^2) \right\}, \]

respectively, where \( x^1, x^2, y^1, \) and \( y^2 \) are random vectors. \( x^1 \) denotes Player I's hand at the time he must make his decision as to his course of action; \( y^2 \) is the part of Player II's hand that I is allowed to see at the time I must make this decision; \( y^1 \) is Player II's hand at the time he must decide on his course of action and \( x^2 \) the part of I's hand II is allowed to see at this time.

Let \( x \) denote Player I's entire hand, and \( y \) Player II's entire hand. Then the pay-off in these games will have the form

\[ K(\varphi, \psi) = \iint P(x, y; \varphi(x^1, y^2), \psi(y^1, x^2)) dF(x) dG(y). \]

It will be possible to represent this in two different forms as follows:

\[ K(\varphi, \psi) = c_1(\varphi) + \sum_{j=1}^{\infty} \iint c_{\psi_j}(\varphi, y^1, x^2) \psi_j(y^1, x^2) dG_1(y^1) dF_2(x^2) \]

and

\[ K(\varphi, \psi) = c_2(\psi) + \sum_{i=1}^{m} \iint c_{\varphi_i}(\psi, x^1, y^2) \varphi_i(x^1, y^2) dF_1(x^1) dG_2(y^2), \]

where \( c_1 \) is independent of \( \psi \), the \( c_{\psi_j} \) are also independent of \( \psi \) but are the coefficients of the \( \psi_j \), and analogous remarks apply to \( c_2 \) and the \( c_{\varphi_i} \). \( F_1 \) is the marginal distribution of \( x^1 \), \( F_2 \) the marginal distribution of \( x^2 \), etc. In this paper, in the interest of
simplicity, the cards are represented by values in the unit interval, each such value being considered equally likely. Also the various cards are independent and thus \( F_1, F_2, G_1, \) and \( G_2 \) are uniform distributions over unit cubes of the appropriate dimension.

In order for a pair of strategies \( \phi^*, \psi^* \) to be optimal, it is necessary and sufficient that

\[
K(\phi^*, \psi^*) = \max_{\varphi} \left\{ c_2(\psi^*) + \sum_{i=1}^{m} \int c_{\varphi_i}(\psi^*, x^1, y^2) \phi_i(x^1, y^2) \, df_1(x^1) \, dg_2(y^2) \right\}
\]

and

\[
K(\phi^*, \psi^*) = \min_{\psi} \left\{ c_1(\phi^*) + \sum_{j=1}^{n} \int c_{\psi_j}(\phi^*, y^1, x^2) \psi_j(y^1, x^2) \, df_1(y^1) \, dg_2(x^2) \right\}
\]

where the maximum and minimum are taken over all possible strategies for the first and second players respectively, which will be all possible \( \varphi \) and \( \psi \) vectors subject to certain constraints.

The basic technique in the fixed-point method is to use any information available as to the form of one of the optimal strategies (which may come from intuitive considerations, knowledge of the solution of similar games, examination of the \( c_{\varphi_i} \) and \( c_{\psi_j} \) functions, or possibly from experience in playing the game) to obtain some information about the \( c_{\psi_j} \) or \( c_{\varphi_i} \), as the case may be, which will then give some information about the form of the other optimal strategy. This process is then iterated until the optimal strategies are obtained.

In Section 1, three continuous modifications of the two-person version of Baccarat are considered. The discrete version has been
solved by Kemeny and Snell in [4]. Their description of the game is as follows:

"The game to be considered is a card game with the special feature that hands are evaluated modulo 10. A hand will consist of two or three cards. Each card from ace through 9 is worth its face value, while each card from 10 through king is worth 10 points (and hence 0 modulo 10). Thus a hand consisting of an 8 and a 3 gives a count of 1. A count of 9 is the best possible hand.

The banker serves as dealer, dealing his opponent and himself two cards each, face down. If either man has a count of 8 or 9, he announces this fact, and hands are compared at once.

If neither player has a count of 8 or 9, then the player has the option of taking an additional card. If he elects to do so, this card is dealt him face up. The banker may then decide to take an additional card, if he wishes one. The hands are then compared.

When hands are compared, the higher count (modulo 10) wins. If the two men have equal counts, the game is declared a draw.

Since nonplayers may bet on the player's hand, the player's strategy is restricted by the rules of the game. He must draw if he has a count of 4 or less, and is not allowed to draw on a count of 6 or more. His only free choice arises when he has a count 5. The banker is free to make his own decisions."

In the versions considered here the restrictions on 8 and 9 are removed, and the player is allowed complete freedom in his choice of strategy. Furthermore the first two cards are replaced by one since this only changes the distribution involved. In the first model the player's additional card (if any) is dealt face down. This is done to give a very simple example as an introduction to the method. The second model is the same with this card dealt face up. In the third model betting is introduced and again the player's additional card is dealt face down for simplicity. In the first two models there is a
single critical number for each player, which is explicitly obtained in the analysis, such that his optimal strategy is to draw if his card is smaller than the critical number and refuse to draw if his card is larger than the critical number. Of course, in the case of the banker (Player II) the critical number is a function of whether or not the player (Player I) drew and, in the second version, on what card he drew. The optimal strategies in the third model depend on the ratio of the size of bet to the size of ante and bring in the concepts of folding and bluffing in addition to the above.

In Section 2, a continuous version of stud poker is analyzed. The description is as follows: After an ante each player is dealt two cards, one face up and one face down. Player I then has the option of betting or folding. If he bets, Player II then has the option of folding or seeing the bet. In the latter case, each player is dealt one additional card and the player with the higher maximum card wins the pot. The form of the optimal strategies depends on the ratio of the size of bet to the size of ante. If this is less than 2, for example, there is a critical number $k$, again explicitly obtained, such that:

If II's "up card" is both larger than $k$ and larger than I's maximum card, then Player I should bet with some probability between zero and one which depends on the size of II's "up card;" otherwise he should bet with probability one. Player II should fold if his maximum card is smaller than I's "up card" and I's "up card" is larger than $k$; he should bet with some probability between zero and one, depending on
the size of his "up card," if his "up card" is greater than his "down card," greater than $k$, and greater than I's "up card"; in all other cases he should bet with probability one.

An interesting feature of the optimal strategies of this game in all cases is that I's strategy depends only on the maximum of his two cards, while II's strategy is a function of his two cards individually.

In Section 3, a poker model with simultaneous betting at three bet levels is considered. This model was solved for the case of two bet levels by Von Neumann and Morgenstern [5]; this solution was also obtained by Karlin [2] using the fixed-point technique. In this game each player is dealt only one card, but there are more actions available to the players than in the previous games. It is also of interest because of the complexity of the solution compared to that of the case with two bet levels. With two bet levels, there is a single critical number such that both players should bluff on hands smaller than this value and bet high on hands larger than this value. In the present case, in contrast, there are as many as six critical numbers depending on the bet sizes.

1. **Baccarat.**

   **Model I.** This game is played as follows: Each player receives a card. After seeing his card, Player I has the option of drawing another card. Then Player II, knowing the value of his card and whether I has drawn or not, has the option of drawing another card. The player who has the greater sum, reduced modulo one, wins one unit from the other.
The hands are \( <x_1, x_2> \) for Player I and \( <y_1, y_2> \) for Player II, \( x_1, x_2, y_1, y_2 \) being independent uniformly distributed random variables. Of course, \( x_2 \) and \( y_2 \) will not necessarily come into the picture every time the game is played and in any case are unknown initially. The strategies are completely described by:

\[
\begin{align*}
\varphi(x_1) &= \text{probability I will draw if his card is } x_1 \\
1 - \varphi(x_1) &= \text{probability I will not draw if his card is } x_1 \\
\psi_1(y_1) &= \text{probability II will draw if his card is } y_1 \text{ and I drew} \\
1 - \psi_1(y_1) &= \text{probability II will not draw if his card is } y_1 \text{ and I drew} \\
\psi_2(y_1) &= \text{probability II will draw if his card is } y_1 \text{ and I did not draw} \\
1 - \psi_2(y_1) &= \text{probability II will not draw if his card is } y_1 \text{ and I did not draw},
\end{align*}
\]

where \( 0 \leq \varphi(x_1) \leq 1, 0 \leq \psi_1(y_1) \leq 1, \) and \( 0 \leq \psi_2(y_1) \leq 1 \).

Let

\[
L(r,s) = \begin{cases} 
1 & \text{if } r - [r] > s - [s] \\
0 & \text{if } r - [r] = s - [s] \\
-1 & \text{if } r - [r] < s - [s],
\end{cases}
\]

where \([r]\) denotes the greatest integer contained in \( r \). Then the pay-off is given by
\[ K(\varphi, \psi) = \int \int \int [\varphi(x_1)\psi(y_1)I(x_1 + x_2, y_1 + y_2) + \varphi(x_1)[1-\psi(y_1)]I(x_1 + x_2, y_1) \\
+ \{1-\varphi(x_1)\}\psi_2(y_1)I(x_1, y_1 + y_2) + \{1-\varphi(x_1)\}[1-\psi_2(y_1)]I(x_1, y_1)]dx_1 dx_2 dy_1 dy_2 \\
= \int \int [\varphi(x_1)[1-\psi(y_1)]I(x_1, y_1 + y_2) + \{1-\varphi(x_1)\}\psi(y_1)(2x_1 - 1) \\
+ \{1-\varphi(x_1)\}[1-\psi_2(y_1)]I(x_1, y_1)]dx_1 dy_1 \\
= c_1(\varphi) + \int_0^1 c_\psi(\varphi, y_1)\psi_1(y_1)dy_1 + \int_0^1 c_\psi(\varphi, y_1)\psi_2(y_1)dy_1 \\
= c_2(\psi) + \int_0^1 c_\varphi(\psi, x_1)\varphi(x_1)dx_1. \]

In order to minimize \( K(\varphi^*, \psi) \), I should choose \( \psi_1(y_1) \) to be one or zero according as \( c_\psi(\varphi^*, y_1) \) is negative or positive and similarly for \( \psi_2(y_1) \). In the same way I will choose \( \varphi(x_1) \) to be one or zero according as \( c_\varphi(\psi^*, x_1) \) is positive or negative.

Since
\[ c_\psi(\varphi^*, y_1) = (2y_1 - 1) \int_0^1 \varphi^*(x_1)dx_1, \]

\[ \psi^*_1(y_1) = \begin{cases} 
1 & y_1 < \frac{1}{2} \\
0 & y_1 > \frac{1}{2} 
\end{cases}. \]

This part of I's strategy is intuitively clear, since if I draws his sum (mod 1) will be uniformly distributed regardless of the first card.
Now
\[
c_{\psi_2}(\psi^*,y_1) = \int_0^1 (2x_1 - 1)(1 - \psi^*(x_1))dx_1 + \frac{y_1}{\psi_2} \int_0^1 (1 - \psi^*(x_1))dx_1 - \frac{1}{\psi_2}
\]
and
\[
c_{\psi}(\psi^*,x_1) = -\frac{1}{4} + (1 - 2x_1)\int_0^1 \psi^*(y_1)dy_1 = \int_0^1 (1 - \psi^*(y_1))dy_1 + \frac{1}{\psi_2}
\]

The first is monotonic non-decreasing in \(y_1\) and the second monotonic decreasing in \(x_1\). Therefore
\[
\psi^*(x_1) = \begin{cases} 
1 & x_1 < \alpha \\
0 & x_1 > \alpha 
\end{cases}
\]

where \(0 < \alpha < 1\). Substituting this in \(c_{\psi_2}(\psi^*,y_1)\) yields
\[
c_{\psi_2}(\psi^*,y_1) = \begin{cases} 
(1 - \alpha)^2 & y_1 < \alpha \\
-1 - \alpha^2 + 2y_1 & y_1 > \alpha 
\end{cases}
\]

and thus
\[
\psi_2^*(y_1) = \begin{cases} 
1 & y_1 < \frac{1 + \alpha^2}{2} \\
0 & y_1 > \frac{1 + \alpha^2}{2}
\end{cases}
\]

There remains only the determination of \(\alpha\) so that \(c_{\psi}(\psi^*,\alpha) = 0\) or
\[-\frac{1}{4} + (1-2\alpha)\left(1+\frac{\alpha^2}{2}\right) + 1 = \frac{1+\alpha^2}{2} = \frac{3}{4} - \alpha = \alpha^3 = 0\]

which is satisfied for \(\alpha = 0.567\).

The value of the game is given by \(K(\psi^*, \psi^*) = -0.0112\).

In summary, the optimal strategies are: I should draw whenever his card is less than 0.567. Then II should draw whenever his card is less than 0.5 if I drew, and whenever his card is less than 0.661 if I did not draw.

**Model II.** This game is the same as the first one with the single modification that if I draws an additional card, this card is dealt face up so that II knows the size of it before deciding whether or not to draw.

The structure of the hands will be the same as above and the only difference in strategies is that now \(\psi_1(y_1)\) is replaced by

\[\psi_1(y_1, x_2) = \text{probability II will draw if his card is } y_1 \text{ and I drew } x_2.\]

The pay-off is given by

\[
K(\psi, \psi) = \int \int \int \int [\phi(x_1, y_1, x_2)\lambda(x_1 + x_2, y_1 + y_2) + \phi(x_1)[1-\psi_1(y_1, x_2)]\lambda(x_1 + x_2, y_1) + (1-\phi(x_1))\psi_1(y_1)\lambda(x_1, y_1) + (1-\phi(x_1))[1-\psi_2(y_1)]\lambda(x_1, y_1)]dx_1dx_2dy_1dy_2
\]

\[
= \int \int \int \phi(x_1, y_1, x_2)[2x_1 + 2x_2 - 1]dx_1dx_2dy_1 + \int \int \int \phi(x_1, y_1, x_2)[2x_1 + 2x_2 - 3]dx_1dx_2dy_1
\]
It will now be assumed that \( \psi^*(x_1) \) has the same form as in the first model, i.e.,

\[
\psi^*(x_1) = \begin{cases} 
1 & x_1 < \alpha \\
0 & x_1 > \alpha 
\end{cases}
\]

where \( 0 < \alpha < 1 \). After obtaining \( \psi^* \), it will be clear that \( c_{\psi}(\psi^*, x_1) \) is monotonic decreasing and this will justify the assumption about \( \psi^* \).

Now

\[
c_{\psi}(\psi^*, y_1, x_2) = \begin{cases} 
\alpha^2 + 2\alpha x_2 - 4\alpha + 2 + 2y_1 - 2x_2 & y_1 - x_2 < \alpha - 1 \\
\alpha^2 + 2\alpha x_2 - 2\alpha & \alpha - 1 < y_1 - x_2 < 0 \\
\alpha^2 + 2\alpha x_2 - 2\alpha + 2y_1 - 2x_2 & 0 < y_1 - x_2 < \alpha \\
\alpha^2 + 2\alpha x_2 & \alpha < y_1 - x_2 
\end{cases}
\]

and thus \( \psi^*(y_1, x_2) \) has the form
where the shaded area is the region where \( \psi_1(y_1, x_2) = 1 \), the non-shaded area the region where \( \psi_1(y_1, x_2) = 0 \). It is interesting to note here that 
\[ \int_0^1 \int_0^1 \psi_1(y_1, x_2) dy_1 dx_2 = \frac{1}{2} \]
which indicates that the probability of II drawing when I has drawn is \( \frac{1}{2} \) as it was in the original model.

Since \( c_{\psi_2}(\phi^*, y_1) \) is the same as in the previous case

\[
\psi_2(y_1) = \begin{cases} 
1 & y_1 < \frac{1 + \alpha^2}{2} \\
0 & y_1 > \frac{1 + \alpha^2}{2} \end{cases}
\]

Substituting \( \psi^* \) into \( K(\phi, \psi) \) yields

\[
c(\psi^*, x_1) = \begin{cases} 
- x_1^2 + \frac{x_1^2}{\alpha} - 2x_1 + \alpha x_1 - \alpha^2 x_1 - \frac{\alpha^2}{4} + \frac{\alpha}{3} + \frac{2}{3} & x_1 < \alpha \\
- x_1^2 + \alpha x_1 - \alpha^2 x_1 - \frac{\alpha^2}{4} = \frac{2\alpha}{3} + \frac{2}{3} & \alpha < x_1 < \frac{1 + \alpha^2}{2} \\
- x_1^2 - 2x_1 + \alpha x_1 - \alpha^2 x_1 + \frac{3\alpha^2}{4} - \frac{2\alpha}{3} + \frac{5}{3} & \frac{1 + \alpha^2}{2} < x_1 \end{cases}
\]

This is monotone decreasing and thus \( \phi^* \) does have the assumed form. Now \( \alpha \) must be determined so that \( c(\psi^*, \alpha) = 0 \), or

\[- \alpha^3 - \frac{\alpha^2}{4} + \frac{2\alpha}{3} + \frac{2}{3} = 0\]

which is satisfied for \( \alpha = 0.580 \). In this case the critical number for II, i.e., \( \frac{1 + \alpha^2}{2} \), is 0.668. The value of the game is \(-0.0200\).
Model III. This is the first game with the addition of betting. Both players initially ante $b$ units. After seeing his first card, I may either fold (forfeiting his ante to II), or bet $a-b$ units. If he bets, he may draw an additional card (face down) without further payment if he so desires. Then II has the option of folding or seeing the bet, and of drawing an additional card if he sees the bet. In the event that II sees the bet, the hands are compared as in the first game, the player with the higher hand winning the pot.

The hands have the same structure as before, but now the strategies have additional components. The strategies in this case are described by:

\[ \varphi_1(x_1) = \text{probability I will fold if his card is } x_1 \]
\[ \varphi_2(x_1) = \text{probability I will bet and draw if his card is } x_1 \]
\[ \varphi_3(x_1) = \text{probability I will bet and not draw if his card is } x_1 \]
\[ \psi_{11}(y_1) = \text{probability II will fold if his card is } y_1 \text{ and I drew} \]
\[ \psi_{21}(y_1) = \text{probability II will see and draw if his card is } y_1 \text{ and I drew} \]
\[ \psi_{31}(y_1) = \text{probability II will see and not draw if his card is } y_1 \text{ and I drew} \]
\[ \psi_{12}(y_1) = \text{probability II will fold if his card is } y_1 \text{ and I did not draw} \]
\[ \psi_{22}(y_1) = \text{probability II will see and draw if his card is } y_1 \text{ and I did not draw} \]
\[ \psi_{32}(y_1) = \text{probability II will see and not draw if his card is } y_1 \text{ and I did not draw} \]

where these are subject to the restrictions \( \varphi_i(x_1) \geq 0, \psi_{ij}(y_1) \geq 0, \)

\[ \sum_i \varphi_i(x_1) = 1, \quad \sum_i \psi_{ij}(y_1) = 1, \quad j = 1,2. \]
The pay-off is given by

\[ K(\varphi, \psi) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 [- b \varphi_1(x_1) + b \varphi_2(x_1) \psi_{11}(y_1) + a \varphi_2(x_1) \psi_{21}(y_1) L(x_1 + x_2, y_1 + y_2) + a \varphi_2(x_1) \psi_{31}(y_1) L(x_1 + x_2, y_1) + b \varphi_3(x_1) \psi_{12}(y_1) + a \varphi_3(x_1) \psi_{22}(y_1) L(x_1, y_1 + y_2) + a \varphi_3(x_1) \psi_{32}(y_1) L(x_1, y_1)] dx_1 dx_2 dy_1 dy_2 \]

\[ = \int_0^1 \int_0^1 [- b \varphi_1(x_1) + b \varphi_2(x_1) \psi_{11}(y_1) + a \varphi_2(x_1) \psi_{31}(y_1)(1 - 2y_1) + b \varphi_3(x_1) \psi_{12}(y_1) + a \varphi_3(x_1) \psi_{22}(y_1)(2x_1 - 1) + a \varphi_3(x_1) \psi_{32}(y_1) L(x_1, y_1)] dx_1 dy_1. \]

In this case, in order for \( I \) to maximize \( K(\varphi, \psi^*) \), he should choose \( \varphi^*_1(x_1) = 1 \) for all \( x_1 \) such that \( c_{\varphi_1}(\psi^*, x_1) > c_{\varphi_j}(\psi^*, x_1) \), \( j \neq 1 \).

Player II will achieve his minimization of \( K(\varphi^*, \psi) \) in a similar way except that he will be interested in the smallest \( c_{\psi_{ij}}(\varphi^*, y_1) \) for fixed \( j \), and he must do this separately for each \( j \).

The optimal strategies will, of course, depend on \( a \) and \( b \) and even the form of the optimal strategies depends on the ratio \( \frac{a}{b} \). With the exception of small values of \( \frac{a}{b} \) (when the optimal strategies are the same as in Model I), the set of optimal strategies for at least one of the two players contains more than one member and the entire set of optimal strategies will be given here.
Since \( c_{\psi_1}(\varphi^*, y_1) = b \int \varphi_2^*(x_1) \, dx_1, \) \( c_{\psi_2}(\varphi^*, y_1) = 0, \) and 
\[ c_{\psi_3}(\varphi^*, y_1) = s(1-2y_1) \int \varphi_2^*(x_1) \, dx_1, \]

\[ \psi_{11}(y_1) = 0; \quad \psi_{21}(y_1) = \begin{cases} 1 & y_1 < \frac{1}{2} \\ 0 & y_1 > \frac{1}{2} \end{cases}; \quad \psi_{31}(y_1) = \begin{cases} 0 & y_1 < \frac{1}{2} \\ 1 & y_1 > \frac{1}{2} \end{cases}. \]

Now \( c_{\varphi_1}(\psi^*, x_1) = -b, \) \( c_{\varphi_2}(\psi^*, x_1) = -\frac{a}{4}, \) and \( c_{\varphi_3}(\psi^*, x_1) = b \int \psi_2^*(y_1) \, dy_1 + a(2x_1-1) \int \psi_2^*(y_1) \, dy_1 + a \int \psi_3^*(y_1) \, L(x_1, y_1) \, dy_1. \) This indicates that the first range to consider for \( \frac{a}{b} \) is \( 1 < \frac{a}{b} < 4. \) Then, since \( c_{\varphi_3}(\psi^*, x_1) \) is non-decreasing

\[ \varphi_1^*(x_1) = 0; \quad \varphi_2^*(x_1) = \begin{cases} 1 & x_1 < \alpha \\ 0 & x_1 > \alpha \end{cases}; \quad \varphi_3^*(x_1) = \begin{cases} 0 & x_1 < \alpha \\ 1 & x_1 > \alpha \end{cases}. \]

Substituting this into \( K(\varphi, \psi) \) yields

\[ c_{\psi_1}(\varphi^*, y_1) = b(1-\alpha); \quad c_{\psi_2}(\varphi^*, y_1) = a \alpha(1-\alpha); \quad c_{\psi_3}(\varphi^*, y_1) = \begin{cases} a(1-\alpha) & y_1 < \alpha \\ a(1-2y_1+\alpha) & y_1 > \alpha \end{cases}. \]

If \( \alpha \) is such that \( a\alpha < b, \) then

\[ \psi_{12}^*(y_1) = 0; \quad \psi_{22}^*(y_1) = \begin{cases} 1 & y_1 < \frac{1+\alpha^2}{2} \\ 0 & y_1 > \frac{1+\alpha^2}{2} \end{cases}; \quad \psi_{32}^*(y_1) = \begin{cases} 0 & y_1 < \frac{1+\alpha^2}{2} \\ 1 & y_1 > \frac{1+\alpha^2}{2} \end{cases}. \]
There remains the requirement that \( c_{\varphi_3}(\psi^*, \alpha) = -\frac{a}{4}, \) or

\[
a \frac{1 + \alpha^2}{2} (2\alpha - 1) - a \left(1 - \frac{1 + \alpha^2}{2}\right) = -\frac{a}{4}
\]

or

\[
\alpha + \alpha^3 - \frac{3}{4} = 0
\]

which is satisfied by \( \alpha = 0.567 \) as indicated above. This then is the solution for \( 1 < \frac{a}{b} < \frac{1}{0.567} = 1.764. \)

If \( \alpha \) is such that \( a\alpha = b, \) then

\[
\psi_{12}^*(y_1) + \psi_{22}^*(y_1) = \begin{cases} 
1 & y_1 < \frac{1 + \alpha^2}{2} \\
0 & y_1 \geq \frac{1 + \alpha^2}{2}
\end{cases} \quad \quad \psi_{32}^*(y_1) = \begin{cases} 
0 & y_1 < \frac{1 + \alpha^2}{2} \\
1 & y_1 \geq \frac{1 + \alpha^2}{2}
\end{cases}
\]

and \( \psi_{12}^*(y_1), \psi_{22}^*(y_1) \) are arbitrary for \( y_1 < \frac{1 + \alpha^2}{2} \) except that they add to one. There is yet the requirement \( c_{\varphi_3}(\psi^*, \frac{b}{a}) = -\frac{a}{4}, \) or

\[
b \int \psi_{12}^*(y_1) dy_1 + a(2 \frac{b}{a} - 1) \left(\frac{1 + \frac{b^2}{a^2}}{2} - \int \psi_{12}^*(y_1) dy_1\right) = a \left(1 - \frac{1 + \frac{b^2}{a^2}}{2}\right) = -\frac{a}{4}
\]

\[
\Rightarrow (a - b) \int \psi_{12}^*(y_1) dy_1 = \frac{3a}{4} - b - \frac{b^3}{a^2}
\]

\[
\Rightarrow \int \psi_{12}^*(y_1) dy_1 = \frac{3a^3 - 4a^2b - 4b^3}{4a^2(a - b)}
\]

and this restriction must then be put on \( \psi_{12}^*(y_1) \) (and therefore on \( \psi_{22}^*(y_1) \)).

It is possible to do this if and only if \( \frac{a}{b} \) is such that
\( 0 \leq \frac{3a^3 - 4a^2 b - 4b^3}{4a^2 (a-b)} \leq \frac{1 + \frac{b^2}{a^2}}{2} \)

which is true for \( 1.764 \leq \frac{a}{b} \leq 2.920 \).

If \( \alpha \) is such that \( a \alpha > b \), then

\[
\psi_{12}^*(y_1) = \begin{cases} 
1 & y_1 < \beta \\
0 & y_1 > \beta 
\end{cases};
\psi_{22}^*(y_1) = 0; \quad \psi_{32}^*(y_1) = \begin{cases} 
0 & y_1 < \beta \\
1 & y_1 > \beta 
\end{cases}.
\]

This implies that \( c_{\psi_3}(\psi^*, x_1) \) is constant for \( x_1 < \beta \). But comparison of \( c_{\psi_{12}}(\psi^*, y_1) \) and \( c_{\psi_{32}}(\psi^*, y_1) \) shows that \( \alpha < \beta \) and thus \( c_{\phi_3}(\psi^*, x_1) = c_{\phi_2}(\psi^*, x_1) \) for \( 0 \leq x_1 \leq \beta \). This will be true if and only if

\( b \beta - a (1-\beta) = -\frac{a}{4} \) or \( \beta = \frac{3a}{4(a+b)} \).

The equality of \( c_{\phi_3}(\psi^*, x_1) \) and \( c_{\phi_2}(\psi^*, x_1) \) for \( 0 \leq x_1 \leq \beta \) gives more freedom in the choice of \( \phi^* \) than was originally supposed, i.e.,

\[
\phi_1^*(x_1) = 0; \quad \phi_2^*(x_1) = \begin{cases} 
\text{arbitrary} & x_1 < \beta \\
0 & x_1 > \beta 
\end{cases}; \quad \phi_3^*(x_1) = \begin{cases} 
\text{arbitrary} & x_1 < \beta \\
1 & x_1 > \beta 
\end{cases}.
\]

In order to insure that these strategies are optimal, however, it is necessary to require \( c_{\psi_{12}}(\psi^*, \beta) = c_{\psi_{32}}(\psi^*, \beta) \) and \( c_{\psi_{12}}(\psi^*, 0) \leq c_{\psi_{22}}(\psi^*, 0) \). These conditions are that
\[ \int \phi_2^*(x_1)dx_1 = \frac{a^2 + 2b^2}{2(a+b)^2} \quad \text{and} \quad \int \phi_2^*(x_1)(2x_1-1)dx_1 \leq -\frac{b(a+4b)}{2(a+b)^2}, \]

and this is always possible if \( \frac{a}{b} > 2.920 \), at least by taking \( \phi_2^*(x_1) = 1 \) for \( 0 \leq x_1 \leq \frac{a^2 + 2b^2}{2(a+b)^2} \) and zero elsewhere.

In case \( \frac{a}{b} = 4 \), the analysis is very similar to the last case above except that now \( \phi_1^*(x_1) \) is also arbitrary for \( 0 \leq x_1 \leq \beta = \frac{3}{5} \), and now the conditions on \( \phi^* \) are

\[ \int (\phi_1^*(x_1) + \phi_2^*(x_1))dx_1 = \frac{9}{25}, \quad \text{and} \quad \int (\phi_1^*(x_1) + \phi_2^*(x_1))(2x_1-1)dx_1 \leq -\frac{4}{25} \]

which are easily satisfied.

In case \( \frac{a}{b} > 4 \), the analysis is once again similar except that now \( \phi_2^*(x_1) \equiv 0 \) and only \( \phi_1^*(x_1) \) and \( \phi_3^*(x_1) \) will be arbitrary on \( 0 \leq x_1 \leq \beta \). Also \( \beta \) has a different form since now the condition is \( c_{\phi_3}(\psi^*,x_1) = c_{\phi_1}(\psi^*,x_1) \) for \( 0 \leq x_1 \leq \beta \) which yields \( b\beta - a(1-\beta) = -b \) or \( \beta = \frac{a-b}{a+b} \). The conditions \( c_{\psi_{12}}(\psi^*,\beta) = c_{\psi_{32}}(\psi^*,\beta) \) and \( c_{\psi_{12}}(\psi^*,0) = c_{\psi_{22}}(\psi^*,0) \) are now

\[ \int \phi_1^*(x_1)dx_1 = \left(\frac{a-b}{a+b}\right)^2 \quad \text{and} \quad \int \phi_1^*(x_1)(2x_1-1)dx_1 \leq -\frac{b}{a}(1-\left(\frac{a-b}{a+b}\right)^2) \]

which are easily satisfied.

In this case, since \( \phi_2^* \equiv 0 \), \( c_{\psi_{11}}(\psi^*,y_1) = c_{\psi_{21}}(\psi^*,y_1) = c_{\psi_{31}}(\psi^*,y_1) = 0 \). Hence \( \psi_{11}^*(y_1), \psi_{21}^*(y_1), \) and \( \psi_{31}^*(y_1) \) can be arbitrary in this case except for the condition \( c_{\phi_2}(\psi^*,x_1) \leq c_{\phi_1}(\psi^*,x_1) \) or

\[ b \int \psi_{11}^*(y_1)dy_1 + a \int \psi_{31}^*(y_1)(1-2y_1)dy_1 \leq -b. \]
A summary of the optimal strategies in all cases follows. In cases I - IV,

\[
\psi^*_{11}(y_1) = 0; \quad \psi^*_{21}(y_1) = \begin{cases} 
1 & y_1 < \frac{1}{2} \\
0 & y_1 > \frac{1}{2}
\end{cases}; \quad \psi^*_{31}(y_1) = \begin{cases} 
0 & y_1 < \frac{1}{2} \\
1 & y_1 > \frac{1}{2}
\end{cases}
\]

**Case I.** \(1 < \frac{a}{b} < 1.764\).

\[
\psi^*_{12}(y_1) = 0; \quad \psi^*_{22}(y_1) = \begin{cases} 
1 & y_1 < 0.661 \\
0 & y_1 > 0.661
\end{cases}; \quad \psi^*_{32}(y_1) = \begin{cases} 
0 & y_1 < 0.661 \\
1 & y_1 > 0.661
\end{cases}
\]

\[
\varphi^*_{1}(x_1) = 0; \quad \varphi^*_{2}(x_1) = \begin{cases} 
1 & x_1 < 0.567 \\
0 & x_1 > 0.567
\end{cases}; \quad \varphi^*_{3}(x_1) = \begin{cases} 
0 & x_1 < 0.567 \\
1 & x_1 > 0.567
\end{cases}
\]

**Case II.** \(1.764 \leq \frac{a}{b} \leq 2.920\).

\[
\psi^*_{12}(y_1) + \psi^*_{22}(y_1) = \begin{cases} 
1 & y_1 < \frac{a^2 + b^2}{2a^2} \\
0 & y_1 > \frac{a^2 + b^2}{2a^2}
\end{cases}; \quad \psi^*_{32}(y_1) = \begin{cases} 
0 & y_1 < \frac{a^2 + b^2}{2a^2} \\
1 & y_1 > \frac{a^2 + b^2}{2a^2}
\end{cases}
\]

and \(\int \psi^*_{12}(y_1)dy_1 = \frac{3a^3 - 4a^2b - 4b^3}{4a^2(a+b)}\).

\[
\varphi^*_{1}(x_1) = 0; \quad \varphi^*_{2}(x_1) = \begin{cases} 
1 & x_1 < \frac{b}{a} \\
0 & x_1 > \frac{b}{a}
\end{cases}; \quad \varphi^*_{3}(x_1) = \begin{cases} 
0 & x_1 < \frac{b}{a} \\
1 & x_1 > \frac{b}{a}
\end{cases}
\]
Case III. $2.920 \leq \frac{a}{b} < 4$.

$$\psi_{12}^*(y_1) = \begin{cases} 
1 & y_1 < \frac{3a}{4(a+b)} \\
0 & y_1 > \frac{3a}{4(a+b)}
\end{cases} \quad \psi_{22}^*(y_1) = 0 \quad \psi_{32}^*(y_1) = \begin{cases} 
0 & y_1 < \frac{3a}{4(a+b)} \\
1 & y_1 > \frac{3a}{4(a+b)}
\end{cases}$$

$$\varphi_1^*(x_1) = \begin{cases} 
\text{arb} & x_1 < \frac{3a}{4(a+b)} \\
0 & x_1 > \frac{3a}{4(a+b)}
\end{cases} \quad \varphi_2^*(x_1) = \begin{cases} 
\text{arb} & x_1 < \frac{3a}{4(a+b)} \\
0 & x_1 > \frac{3a}{4(a+b)}
\end{cases} \quad \varphi_3^*(x_1) = \begin{cases} 
1 & x_1 > \frac{3a}{4(a+b)}
\end{cases}$$

and satisfying $\int \varphi_2^*(x_1)dx_1 = \frac{a^2 + 2b^2}{2(a+b)^2}$ and $\int \varphi_2^*(x_1)(2x_1-1)dx_1 \leq -\frac{b(a+4b)}{2(a+b)^2}$.

Case IV. $\frac{a}{b} = 4$.

$$\psi_{12}^*(y_1) = \begin{cases} 
1 & y_1 < \frac{3}{5} \\
0 & y_1 > \frac{3}{5}
\end{cases} \quad \psi_{22}^*(y_1) = 0 \quad \psi_{32}^*(y_1) = \begin{cases} 
0 & y_1 < \frac{3}{5} \\
1 & y_1 > \frac{3}{5}
\end{cases}$$

$$\varphi_1^*(x_1) = \begin{cases} 
\text{arb} & x_1 < \frac{3}{5} \\
0 & x_1 > \frac{3}{5}
\end{cases} \quad \varphi_2^*(x_1) = \begin{cases} 
\text{arb} & x_1 < \frac{3}{5} \\
0 & x_1 > \frac{3}{5}
\end{cases} \quad \varphi_3^*(x_1) = \begin{cases} 
1 & x_1 > \frac{3}{5}
\end{cases}$$

and satisfying $\int \varphi_1^*(x_1)dx_1 + \int \varphi_2^*(x_1)dx_1 = \frac{9}{25}$ and

$$\int \varphi_1^*(x_1)(2x_1-1)dx_1 + \int \varphi_2^*(x_1)(2x_1-1)dx_1 \leq -\frac{4}{25}.$$
Case V. \( \frac{a}{b} > 4 \).

\( \psi^*_{11}(y_1), \psi^*_{21}(y_1), \) and \( \psi^*_{31}(y_1) \) are arbitrary except for satisfying

\[
b \int \psi^*_{11}(y_1)dy_1 + a \int \psi^*_{31}(y_1)(1-2y_1)dy_1 \leq -b.
\]

\[
\psi^*_{12}(y_1) = \begin{cases} 
1 & y_1 < \frac{a-b}{a+b} \\
0 & y_1 > \frac{a-b}{a+b}
\end{cases}; \quad \psi^*_{22}(y_1) = 0; \quad \psi^*_{32}(y_1) = \begin{cases} 
0 & y_1 < \frac{a-b}{a+b} \\
1 & y_1 > \frac{a-b}{a+b}
\end{cases}
\]

\[
\phi^*_{11}(x_1) = \begin{cases} 
\text{arb} & x_1 < \frac{a-b}{a+b} \\
0 & x_1 > \frac{a-b}{a+b}
\end{cases}; \quad \phi^*_{22}(x_1) = 0; \quad \phi^*_{32}(x_1) = \begin{cases} 
\text{arb} & x_1 < \frac{a-b}{a+b} \\
1 & x_1 > \frac{a-b}{a+b}
\end{cases}
\]

and satisfying \( \int \phi^*_{11}(x_1)dx_1 = \left(\frac{a-b}{a+b}\right)^2 \) and \( \int \phi^*_{11}(x_1)(2x_1-1)dx_1 \leq -\frac{b}{a}(1-\left(\frac{a-b}{a+b}\right)^2). \)

2. Stud Poker.

In this game both players are dealt two cards, one face up and one face down, after an ante of \( b \) units. Then I has the option of betting \( a-b \) units or folding. Following a bet by I, Player II has the option of seeing the bet or folding. If II sees, both players receive a third card and the player with the higher maximum card wins the pot.

The hands are \( < x_1, x_2, x_3 > \) for Player I and \( < y_1, y_2, y_3 > \) for Player II \( (x_1, y_1 \) are face down; \( x_2, y_2 \) face up; \( x_3, y_3 \) the third cards, if any). The strategies are completely described by:
\[ \phi(x_1, x_2, y_2) = \text{probability I will bet holding } \langle x_1, x_2 \rangle \text{ if II's "up card" is } y_2 \]
\[ 1 - \phi(x_1, x_2, y_2) = \text{probability I will fold holding } \langle x_1, x_2 \rangle \text{ if II's "up card" is } y_2 \]
\[ \psi(y_1, y_2, x_2) = \text{probability II will bet holding } \langle y_1, y_2 \rangle \text{ if I's "up card" is } x_2 \]
\[ 1 - \psi(y_1, y_2, x_2) = \text{probability II will fold holding } \langle y_1, y_2 \rangle \text{ if I's "up card" is } x_2 . \]

Let

\[ L(r, s) = \begin{cases} 
1 & r > s \\
0 & r = s \\
-1 & r < s 
\end{cases} \]

Then the pay-off is given by

\[ K(\phi, \psi) = \int \int \int \int \int \int \int [-b(1 - \phi(x_1, x_2, y_2)) + b\phi(x_1, x_2, y_2)(1 - \psi(y_1, y_2, x_2)) \\
+ a \phi(x_1, x_2, y_2) \psi(y_1, y_2, x_2) L(\max(x_1, x_2, x_3), \max(y_1, y_2, y_3))] d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3. \]

Let \( \xi = \max(x_1, x_2) \), \( \eta = \max(y_1, y_2) \) and denote \( c_\phi(\phi^*, y_1, y_2, x_2) \) by \( c_\phi \)
and \( c_\psi(\psi^*, y_1, y_2, x_2) \) by \( c_\psi \). Then

\[
c_\psi = \begin{cases} 
-b \phi^*(x_1, x_2, y_2) dx_1 - a \eta^2 \int_0^\eta \phi^*(x_1, x_2, y_2) dx_1 + a \int_0^1 x_1^2 \phi^*(x_1, x_2, y_2) dx_1 & x_2 < \eta \\
-b \phi^*(x_1, x_2, y_2) dx_1 + a x_2^2 \int_0^x \phi^*(x_1, x_2, y_2) dx_1 + a \int_0^1 x_1^2 \phi^*(x_1, x_2, y_2) dx_1 & x_2 > \eta 
\end{cases} \]

and
\[
\begin{aligned}
\psi = \left\{ \begin{array}{ll}
\frac{2b-b}{\xi} \int \psi^*(y_1, y_2, x_2) \, dy_1 + a \frac{2}{\xi} \int \psi^*(y_1, y_2, x_2) \, dy_1 - \frac{1}{\xi} \int \psi^*(y_1, y_2, x_2) \, dy_1 & \quad y_2 < \xi \\
\frac{2b-b}{\xi} \int \psi^*(y_1, y_2, x_2) \, dy_1 - a y_2^2 \int \psi^*(y_1, y_2, x_2) \, dy_1 - \frac{1}{\xi} \int \psi^*(y_1, y_2, x_2) \, dy_1 & \quad y_2 > \xi 
\end{array} \right.
\end{aligned}
\]

If \(a-b\) is sufficiently small, it seems reasonable that I should bet whenever \(\xi > y_2\) or whenever \(y_2 < k\) where \(k\) is a parameter between zero and one. When \(\xi < y_2\) and \(y_2 > k\), I should bluff, i.e., bet with some probability between zero and one. With this in mind, let

\[
\psi^*(x_1, x_2, y_2) = \begin{cases} 
1 & \text{if } y_2 < k \text{ or } y_2 < \xi \\
? & \text{if } y_2 > k \text{ and } y_2 > \xi 
\end{cases}
\]

Then

\[
\begin{aligned}
\psi = \left\{ \begin{array}{ll}
b + \frac{a}{3} - \frac{4}{3} a \eta^3, & \quad y_2 < k, \ x_2 < \eta \\
b + \frac{a}{3} - \frac{4}{3} a \eta^3, & \quad y_2 > k, \ y_2 < x_2 < y_1 \\
b \int \psi^*(x_1, x_2, y_2) \, dx_1 - b y_2^2 \int \psi^*(x_1, x_2, y_2) \, dx_1 + \frac{a}{3} - \frac{a}{3} y_2^3, & \quad y_2 > k, \ y_2 > x_2, \ y_2 > y_1 \\
b \int \psi^*(x_1, x_2, y_2) \, dx_1 - b y_2^2 \int \psi^*(x_1, x_2, y_2) \, dx_1 - a y_2^2 (y_1 - y_2) + \frac{a}{3} = \frac{a}{3} y_1^3, & \quad y_2 > k, \ y_2 > x_2, \ y_2 < y_1 \\
b + \frac{a}{3} + \frac{2}{3} ax_2^3, & \quad x_2 > \eta 
\end{array} \right.
\end{aligned}
\]
The requirement on the size of \( \frac{a}{b} \) for this case now clearly stands out as \( \frac{a}{b} < 3 \). Then

\[
\psi^*(y_1, y_2, x_2) = \begin{cases} 
1 & y_2 < k, \ x_2 < \eta \\
1 & y_2 > k, \ y_2 < x_2 < y_1 \\
1 & x_2 > \eta, \ x_2 < \alpha \\
0 & x_2 > \eta, \ x_2 > \alpha
\end{cases}
\]

where \(-b + \frac{3a}{3} + \frac{2a}{3}a^2 = 0\). The form of \( \psi^*(y_1, y_2, x_2) \) in the remaining regions is not yet clear.

Since bluffing by \( I \) is expected when \( y_2 > \frac{s}{3} \), \( y_2 > k \), \( c_\psi \) must be zero in this region, i.e.,

\[
2b-b \int \psi^*(y_1, y_2, x_2)dy_1 = 2y_2 \int_0^{y_2} \psi^*(y_1, y_2, x_2)dy_1 = -a \int_0^{1/y_2} \psi^*(y_1, y_2, x_2)dy_1 = 0
\]

when \( y_2 > k, \ y_2 > x_2 \). Examination of \( c_\psi \) in this region shows that it is a function of \( y_2 \) and \( x_2 \) only provided that \( y_1 < y_2 \). If for some \( y_2, x_2 \) this were positive, then \( \psi^*(y_1, y_2, x_2) = 0 \) for \( y_2 > k, \ y_2 > x_2, \ y_1 < y_2 \) and

\[
c_\psi = 2b-b \int_0^{1/y_2} \psi^*(y_1, y_2, x_2)dy_1 - a \int_0^{1/y_2} \psi^*(y_1, y_2, x_2)dy_1 > b+by_2 - \frac{a}{3} + \frac{a}{3}y_2^3 > 0
\]

a contradiction. Thus \( c_\psi \leq 0 \) for \( y_2 > k, \ y_2 > x_2, \ y_1 < y_2 \). But \( c_\psi \) for \( y_2 > k, \ y_2 > x_2, \ y_1 > y_2 \) is strictly less than this (holding \( y_2 \) and \( x_2 \) fixed) and hence strictly less than zero. Thus
\begin{align*}
\psi^*(y_1, y_2, x_2) &= 1 \\ y_2 > k, y_2 > x_2, y_1 > y_2.
\end{align*}

Substituting this back into \( c_{\phi} \) yields

\[
\int_0^{y_2} \psi^*(y_1, y_2, x_2) dy_1 = \frac{b - \frac{a}{3}y_2 + \frac{a}{3}y_2^3}{y_2 + ay_2^2}, \quad y_2 > k, y_2 > x_2.
\]

Any \( \psi^* \) satisfying this would give an optimal strategy, but the simplest is

\[
\psi^*(y_1, y_2, x_2) = \frac{b - \frac{a}{3}y_2 + \frac{a}{3}y_2^3}{by_2 + ay_2^3}, \quad y_2 > k, y_2 > x_2, y_2 > y_1.
\]

To insure that this is less than one it is sufficient to require

\[
b - \frac{a}{3} - \frac{2}{3}ak^3 \leq 0. \tag{1}
\]

The next step is to determine the \( \phi^* \) that maximizes against this \( \psi^* \), hoping that it will be the original \( \phi^* \). Now

\[
c_{\phi} = \begin{cases}
b - \frac{a}{3} + \frac{4}{3}ak^3, & y_2 < x_2, x_2 < a \\
b + bx_2 - \frac{a}{3} + \frac{a}{3}x_2^3, & y_2 < x_2, x_2 > a, x_2 > x_1 \\
b + b(x_2 + ax_1^2(x_1 - x_2)) - \frac{a}{3} + \frac{a}{3}x_2^3, & y_2 < x_2, x_2 > a, x_1 > x_2 \\
b - \frac{a}{3} + \frac{4}{3}ak^3, & x_2 < y_2 < x_1, y_2 < k \\
b + by_2 - b \int_0^{y_2} \psi^*(y_1, y_2, x_2) dy_1 + ax_1^2 \int_0^{y_2} \psi^*(y_1y_2, x_2) dy_1 + ax_1^2(x_1 - y_2) - \frac{a}{3} + \frac{a}{3}x_2^3, & x_2 < y_2 < x_1, y_2 > k \\
b - \frac{a}{3} - \frac{2}{3}ay_2^3, & y_2 > x_1, y_2 < k \\
0, & y_2 > x_1, y_2 > k.
\end{cases}
\]
The first five expressions are positive and to insure the positivity of the sixth it is sufficient to require

\[ b - \frac{a}{3} - \frac{2}{3}ak^3 \geq 0. \]  

(2)

Then the original \( \Phi^* \) maximizes against \( \Psi^* \). (1) and (2) imply \( k \) is the solution of

\[ b - \frac{a}{3} = \frac{2}{3}ak^3 = 0. \]

The condition \( c_\Psi = 0 \) for \( y_2 > k, y_2 > x_2, y_2 > y_1 \) yields

\[ \Phi^*(x_1, x_2, y_2) = \frac{by_2 - \frac{a}{3}y_2^3 + \frac{a}{3} - b}{by_2 + ay_2^3}, \quad y_2 > \xi, y_2 > k, \]

where the simplest version has been chosen as was the case with \( \Psi^* \). This adds the condition \( by_2 - \frac{a}{3}y_2^3 + \frac{a}{3} - b \geq 0 \) for \( k \leq y_2 \leq 1 \). Let

\[ f(y_2) = by_2 - \frac{a}{3}y_2^3 + \frac{a}{3} - b. \]

Then \( f(1) = 0, f(0) < 0, f'(1) < 0 \) imply \( f \) has exactly one root between zero and one. Thus \( f(y_2) \geq 0 \) for

\[ k \leq y_2 \leq 1 \iff f(k) \geq 0 \iff bk - ak^3 \geq 0 \iff \frac{b}{a} \geq k^2 \iff \left( \frac{b}{a} \right)^3 \geq k^6 \iff \left( \frac{b}{a} \right)^3 - \frac{9}{4} \left( \frac{b}{a} \right)^2 + \frac{3}{2} \frac{b}{a} - \frac{1}{4} \geq 0. \]

Let \( g(x) = x^3 - \frac{9}{4}x^2 + \frac{3}{2}x - \frac{1}{4}. \) Then \( g(1) = 0, g'(1) = 0, g''(1) > 0, g(0) < 0 \) imply \( g(x) > 0 \) for

\[ \frac{1}{3} < x < 1 \iff g\left( \frac{1}{3} \right) \geq 0 \quad \text{and} \quad g\left( \frac{1}{3} \right) = \frac{1}{27} > 0. \]

Hence the condition is satisfied for \( \frac{1}{3} < \frac{b}{a} < 1 \), the range under consideration.

In summary, if \( \frac{a}{b} < 3 \)

\[ \Phi^*(x_1, x_2, y_2) = \begin{cases} 
1 & \text{if } y_2 < k \text{ or } y_2 < \xi \\
\frac{by_2 - \frac{a}{3}y_2^3 + \frac{a}{3} - b}{by_2 + ay_2^3} & \text{if } y_2 > k \text{ and } y_2 > \xi 
\end{cases} \]
\[ \psi^*(y_1, y_2, x_2) = \begin{cases} 
0 & \text{if } x_2 > \eta, \ x_2 > k \\
\frac{b - \frac{a}{3} + by_2 + \frac{a}{3} y_2^3}{by_2 + ay_2^3} & \text{if } y_2 > k, \ y_2 > x_2, \ y_2 > y_1 \\
1 & \text{otherwise} 
\end{cases} \]

where \( k = \left( \frac{b - \frac{a}{3}}{\frac{2}{3} a} \right)^{1/3} \).

When \( \frac{a}{b} \geq 3 \), it seems reasonable that I should bet less often, namely when \( \xi > y_2 \) and \( \xi > k \). Accordingly let

\[ \phi^*(x_1, x_2, y_2) = \begin{cases} 
1 & y_2 < \xi, \ \xi > k \\
? & y_2 > \xi \text{ or } \xi < k 
\end{cases} \]

Then

\[ c_\psi = \begin{cases} 
-b + by_2 - (b + a \eta^2) \int_0^{y_2} \phi^*(x_1, x_2, y_2) dx_1 - a \eta^2 (\eta - y_2) + \frac{a}{3} - \frac{a}{3} \eta^3 & x_2 < y_2, \ y_2 > k \\
-b + \frac{a}{3} - \frac{4}{3} a \eta^3 & y_2 < x_2 < y_1, \ x_2 > k \\
\int_0^{x_2-k} \phi^*(x_1, x_2, y_2) dx_1 - b + b - a \eta^2 \int_0^{y_2-k} \phi^*(x_1, x_2, y_2) dx_1 + a \int_0^{y_2-k} \phi^*(x_1, x_2, y_2) dx_1 + \frac{a}{3} - \frac{a}{3} k^3 & x_2 < \eta < k \\
\int_0^{x_2-k} \phi^*(x_1, x_2, y_2) dx_1 - b + b - a \eta^2 \int_0^{y_2-k} \phi^*(x_1, x_2, y_2) dx_1 + a \int_0^{y_2-k} \phi^*(x_1, x_2, y_2) dx_1 + \frac{a}{3} - \frac{a}{3} k^3 & x_2 < k < y_1, \ y_2 < k \\
\int_0^{x_2} \phi^*(x_1, x_2, y_2) dx_1 - b + b - a \eta^2 \int_0^{y_2} \phi^*(x_1, x_2, y_2) dx_1 + a \int_0^{y_2} \phi^*(x_1, x_2, y_2) dx_1 + \frac{a}{3} - \frac{a}{3} k^3 & x_2 > \eta, \ x_2 < k \\
-b + \frac{a}{3} + \frac{2}{3} a x_2^3 & x_2 > \eta, \ x_2 > k 
\end{cases} \]
Suppose that \( c_\psi = 0 \) when \( y_2 > x_2, y_2 > k, y_2 > y_1 \) as was the case when \( \frac{a}{b} < 3 \). Then

\[
\psi^*(x_1, x_2, y_2) = \frac{\frac{a}{3} - b + by_2 - \frac{a}{3} y_2^3}{by_2 + ay_2^3}, \quad x_2 < y_2, y_2 > k, y_2 > x_1,
\]

where \( \psi^* \) again is taken as a function of \( y_2 \) alone for simplicity. The numerator may be written as \( (\frac{a}{3} - b)(1 - y_2) + \frac{a}{3} (y_2 - y_2^3) \geq 0 \) and the fraction will be less than one provided

\[
\frac{a}{3} - b - \frac{a}{3} k^3 \leq 0.
\]

In the region \( x_2 < k, y_2 < k \), it is reasonable to suppose that \( \psi^* \) will be one or zero according as \( \eta > k \) or \( \eta < k \), this corresponding to the behavior of \( \psi^* \) and also being a natural break point in \( c_\psi \) as given above. This will be the case if

\[
f(\psi^*, k) = -(b + ak^2) \int_0^k \psi^*(x_1, x_2, y_2) dx_1 - b + bk + \frac{a}{3} - \frac{a}{3} k^3 = 0 \quad (x_2 < k, y_2 < k).
\]

This can be solved if \( k \) satisfies (3), for \( \psi^* \equiv 0 \) on this region makes \( f(\psi^*, k) < 0 \) and \( \psi^* \equiv 1 \) on this region makes \( f(\psi^*, k) \leq 0 \). Supposing \( \psi^* \) to be chosen on this region so that \( f(\psi^*, k) = 0 \) yields

\[
\psi^*(y_1, y_2, x_2) = \begin{cases} 
0 & \text{if } x_2 > \eta \text{ or } \eta < k \\
1 & \text{if } y_2 > x_2, y_2 > k, y_2 > y_1 \\
? & \text{otherwise}.
\end{cases}
\]
Substituting this into $c_\phi$ gives

$$c_\phi = \begin{cases} 
  b + bk - \frac{a}{3} + \frac{a}{3} k^3, & y_2 < \xi < k \\
  b + bk + a \xi^2 (\xi - k) - \frac{a}{3} + \frac{a}{3} \xi^3, & y_2 < k < x_1, x_2 < k \\
  b + b y_2 - b y_2^2 \int_0^{y_2} \psi(y_1, y_2, x_2) dy_1 + a \xi^2 \int_0^{y_2} \psi(y_1, y_2, x_2) dy_1 - a \xi^2 (\xi - y_2) - \frac{a}{3} + \frac{a}{3} \xi^3, & x_2 < y_2 < x_1, y_2 > k \\
  b + b x_2 + a \xi^2 (\xi - x_2) - \frac{a}{3} + \frac{a}{3} \xi^3, & x_2 > y_2, x_2 > k \\
  b + bk - \frac{a}{3} + \frac{a}{3} k^3, & \xi < y_2 < k \\
  b + b y_2 - (b + ay_2) \int_0^{y_2} \psi(y_1, y_2, x_2) dy_1 - \frac{a}{3} + \frac{a}{3} y_2^3, & y_2 > \xi, y_2 > k.
\end{cases}$$

Since $\psi$ is between zero and one when $y_2 > \xi, x_2 > k$, $c_\phi$ must be zero in this region which implies

$$\psi(y_1, y_2, x_2) = \frac{b + by_2 - \frac{a}{3} + \frac{a}{3} y_2^3}{by_2 + ay_2^3}, \quad y_2 > x_2, y_2 > y_1, y_2 > k.$$ 

To make this positive, it is sufficient to require

$$b + bk - \frac{a}{3} + \frac{a}{3} k^3 > 0. \quad (4)$$

This determines $\psi$ except in the region $y_2 < k, \xi < k$ and it agrees with the original choice of $\psi^*$. 

The range of $\frac{a}{b}$ must now be divided once more. Let $\alpha$ satisfy

$$\frac{a}{3} - b - \frac{h}{3} \alpha^3 = 0 \quad \text{and} \quad \beta \quad \text{satisfy} \quad b + b \beta - \frac{a}{3} + \frac{a}{3} \beta^3 = 0.$$ 

Then
\[
\beta \leq \alpha \iff b + b\beta - \frac{a}{3} + \frac{a}{3} \beta^2 \geq 0 \iff b\alpha - a\alpha^2 \geq 0 \iff \frac{b}{a} \geq \alpha \iff \left(\frac{b}{a}\right)^{3/2} \geq \alpha^3
\]
\[
= \frac{1}{4} - \frac{3}{4} \frac{b}{a} \iff \frac{1}{4} \left(\frac{b}{a}\right)^{3/2} + 3 \frac{b}{a} - 1 \geq 0 \iff \frac{a}{b} \leq 4.822.
\]

Consider first the range \(3 \leq \frac{a}{b} \leq 4.822\). To satisfy (3), it is sufficient to require \(k \geq \alpha\). If \(k > \alpha\), then \(b + bk - \frac{a}{3} + \frac{a}{3} k^3 > b + b\alpha\)
\[-\frac{a}{3} + \frac{a}{3} \alpha^3 \geq b + b\beta - \frac{a}{3} + \frac{a}{3} \beta^3 = 0\]
which implies \(\phi^*(x_1, x_2, y_2) = 1\) when \(y_2 < k, \xi < k\) which in turn implies \(f(\phi^*, k) = -b + \frac{a}{3} - \frac{4}{3} ak^3 < 0\), a contradiction. Thus \(k = \alpha\) and \(b + bk - \frac{a}{3} + \frac{a}{3} k^3 \geq b + b\beta - \frac{a}{3} + \frac{a}{3} \beta^3 = 0\)
with strict inequality holding except for \(\frac{a}{b} = 4.822\) and \(\frac{a}{b} = 3\). Thus (4) is satisfied and
\[
\phi^*(x_1, x_2, y_2) = 1, \quad y_2 < k, \xi < k.
\]

This in turn implies that \(f(\phi^*, k) = 0\). (When \(\frac{a}{b} = 3\), \(k\) is zero and this region doesn't exist anyway. When \(\frac{a}{b} = 4.822\), \(\phi^*\) must be one in this region in order to satisfy \(f(\phi^*, k) = 0\).)

If \(\frac{a}{b} > 4.822\), (4) \(\iff k > \beta\). If \(k > \beta\), then \(\phi^*(x_1, x_2, y_2) = 1\) when \(y_2 < k, \xi < k \Rightarrow f(\phi^*, k) = -b + \frac{a}{3} - \frac{4}{3} ak^3 < 0\), a contradiction. Hence \(k = \beta\), (3) is satisfied, and in order to satisfy \(r(\phi^*, k) = 0\)
\[
\phi^*(x_1', x_2, y_2) = \frac{\frac{a}{3} - b + bk - \frac{a}{3} k^3}{bk + ak^3}, \quad y_2 < k, \xi < k.
\]

This is legitimate since \(c_\phi\) is zero for \(y_2 < k, \xi < k\) by the choice of \(k\).
In summary:

\[ 3 \leq \frac{a}{b} \leq 4.822 \]

\[
\varphi^*(x_1, x_2, y_2) = \begin{cases} 
1 & \text{if } y_2 < k \text{ or } y_2 < \xi \\
\frac{by_2 - \frac{a}{3} y_2^3 + \frac{a}{3} - b}{by_2 + ay_2^3} & \text{if } y_2 > k, y_2 > \xi
\end{cases}
\]

\[
\psi^*(y_1, y_2, x_2) = \begin{cases} 
0 & \text{if } x_2 > \eta \text{ or } \eta < k \\
\frac{b + by_2 - \frac{a}{3} + \frac{a}{3} y_2^3}{by_2 + ay_2^3} & \text{if } y_2 > x_2, y_2 > y_1, y_2 > k \\
1 & \text{otherwise}
\end{cases}
\]

where \( k = \left( \frac{\frac{a}{3} - b}{\frac{4}{3} a} \right)^{1/3} \).

\[ \frac{a}{b} > 4.822 \]

\[
\varphi^*(x_1, x_2, y_2) = \begin{cases} 
1 & \text{if } y_2 < \xi \text{ and } \xi > k \\
\frac{\frac{a}{3} - b + bk - \frac{a}{3} k^3}{bk + ak^3} & \text{if } y_2 < k \text{ and } \xi < k \\
\frac{\frac{a}{3} - b + by_2 - \frac{a}{3} y_2^3}{by_2 + ay_2^3} & \text{if } y_2 > \xi \text{ and } y_2 > k
\end{cases}
\]
\[ \psi^*(y_1, y_2, x_2) = \begin{cases} 
0 & \text{if } x_2 > \eta \text{ or } \eta < k \\
\frac{b + by_2 - \frac{a}{3} + \frac{a}{3} y_2^3}{by_2 + ay_2^3} & \text{if } y_2 > x_2, y_2 > y_1, y_2 > k \\
1 & \text{otherwise}
\end{cases} \]

where \( k \) satisfies \( b + bk = \frac{a}{3} + \frac{a}{3} k^3 = 0 \).

3. **Poker with Simultaneous Betting.**

The distinctive feature of this game is that the two players bet simultaneously. After receiving his card, each player has the choice of three bets, \( a, b, \) or \( c \), with \( c < b < a \). In the event that the bets made by the two players are equal, the player with the higher card wins. If they are unequal, the player making the lower bet has the option of folding (losing his initial bet) or seeing by bringing the size of his bet up to that made by the other player. In the latter case the player with the higher card then wins the pot.

Since the game is symmetric, it suffices to describe the strategies of Player I. They are given by:

- \( \varphi_1(x) = \) probability that I bets \( c \) and folds if II bets \( b \) or \( a \)
- \( \varphi_2(x) = \) probability that I bets \( c \) and sees if II bets \( b \) but folds if II bets \( a \)
- \( \varphi_3(x) = \) probability that I bets \( c \) and folds if II bets \( b \) but sees if II bets \( a \)
- \( \varphi_4(x) = \) probability that I bets \( c \) and sees if II bets \( b \) or \( a \)
- \( \varphi_5(x) = \) probability that I bets \( b \) and folds if II bets \( a \)
- \( \varphi_6(x) = \) probability that I bets \( b \) and sees if II bets \( a \)
- \( \varphi_7(x) = \) probability that I bets \( a \)
where these must satisfy $\phi_1(x) \geq 0$, $\sum_{i=1}^{7} \phi_i(x) = 1$.

The pay-off is given by

$$K(\psi, \hat{\psi}) = c \int \int \int \int \int \int \int \phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x) \{ \psi_1(y) + \psi_2(y) + \psi_3(y) + \psi_4(y) \} L(x, y) dxdy$$

$$- c \int \int \int \int \int \int \phi_5(x) \{ \psi_5(y) + \psi_6(y) \} dxdy$$

$$+ b \int \int \int \int \int \int \phi_7(x) \{ \psi_7(y) \} dxdy$$

$$- c \int \int \int \int \int \int \phi_8(x) \{ \psi_8(y) \} dxdy$$

$$- c \int \int \int \int \int \int \phi_9(x) \{ \psi_9(y) \} dxdy$$

$$+ c \int \int \int \int \int \int \phi_{10}(x) \{ \psi_{10}(y) \} dxdy$$

$$- c \int \int \int \int \int \int \phi_{11}(x) \{ \psi_{11}(y) \} dxdy$$

$$+ c \int \int \int \int \int \int \phi_{12}(x) \{ \psi_{12}(y) \} dxdy$$

$$+ c \int \int \int \int \int \int \phi_{13}(x) \{ \psi_{13}(y) \} dxdy$$

Denote the function $c_\phi \psi^*, x$ by $c_\phi(x)$ and $\int_0^x \phi^*(y) dy - \int_0^1 \phi^*(y) dy$ by $\alpha_\phi(x)$.

Then

$$c_1(x) = c(\alpha_1(x) + \alpha_2(x) + \alpha_3(x) + \alpha_4(x)) + c(\alpha_5(x) + \alpha_6(x) + \alpha_7(x))$$

$$c_2(x) = c(\alpha_1(x) + \alpha_2(x) + \alpha_3(x) + \alpha_4(x)) + b(\alpha_5(x) + \alpha_6(x)) + c(\alpha_7(x))$$

$$c_3(x) = c(\alpha_1(x) + \alpha_2(x) + \alpha_3(x) + \alpha_4(x)) + b(\alpha_5(x) + \alpha_6(x)) - a\alpha_7(x)$$

$$c_4(x) = c(\alpha_1(x) + \alpha_2(x) + \alpha_3(x) + \alpha_4(x)) + b(\alpha_5(x) + \alpha_6(x)) + a\alpha_7(x)$$

$$c_5(x) = c(\alpha_1(x) + \alpha_2(x) + \alpha_3(x) + \alpha_4(x)) + b(\alpha_5(x) + \alpha_6(x)) - b\alpha_7(x)$$

$$c_6(x) = c(\alpha_1(x) + \alpha_2(x) + \alpha_3(x) + \alpha_4(x)) + b(\alpha_5(x) + \alpha_6(x)) + a\alpha_7(x)$$

$$c_7(x) = c(\alpha_1(x) + \alpha_2(x) + \alpha_3(x) + \alpha_4(x) + \alpha_6(x) + \alpha_7(x))$$.
The form of the optimal strategy for this game depends on the ratios \(\frac{a}{c}\) and \(\frac{b}{c}\). The various forms are fairly complex and were discovered by using a combination of intuitive considerations and manipulations of the functions \(c_1(x)\). The form of the optimal strategy for small values of \(\frac{a}{c}\) and \(\frac{b}{c}\), i.e., such that \(1 < \frac{b}{c} < \frac{a}{c} < 1 + c\), is as follows:

\[
\begin{array}{cccccccc}
1,5,7 & 1,7 & 2,7 & 2 & 6 & 7 \\
0 & x_1 & x_2 & x_3 & x_4 & x_5 & 1 & x
\end{array}
\]

The meaning of the diagram is that \(\Phi_1, \Phi_5,\) and \(\Phi_7\) are the only positive components of the optimal strategy on the interval \(0 < x < x_1\); \(\Phi_1\) and \(\Phi_7\) are the only positive components on \(x_1 < x < x_2\); etc.

The parameters involved in this strategy and the conditions on \(\frac{a}{c}\) and \(\frac{b}{c}\) under which it is optimal will now be determined. It is somewhat easier to work with \(\xi_1 = x_1, \xi_2 = x_2 - x_1, \ldots, \xi_6 = 1 - x_5\) than to work directly with the \(x_i\)'s.

Now \(c_1(x) = c_5(x) = c_7(x)\) for \(0 < x < x_1 \implies c_1(x) = c_5(x) = c_7(x)\) for \(0 < x < x_1\), i.e.,

\[
c(\alpha_1'(x) + \alpha_2'(x) + \alpha_3'(x) + \alpha_4'(x)) = b(\alpha_2'(x) + \alpha_4'(x) + \alpha_5'(x) + \alpha_6'(x))
\]

\[
= a(\alpha_3'(x) + \alpha_4'(x) + \alpha_6'(x) + \alpha_7'(x)).
\]
But $\alpha'_1(x) = 2\phi'_1(x) \implies$

$$2c \phi'_1(x) = 2b \phi'_2(x) = 2a \phi'_4(x), \quad 0 < x < x_1.$$  

Since $\phi'_1(x) + \phi'_2(x) + \phi'_4(x) = 1, \quad 0 < x < x_1$, this gives

$$\phi'_1(x) = \frac{ab}{ab + ac + bc}; \quad \phi'_2(x) = \frac{ac}{ab + ac + bc}; \quad \phi'_4(x) = \frac{bc}{ab + ac + bc}, \quad 0 < x < x_1.$$  

Similarly

$$\phi'_1(x) = \frac{a}{a + c}; \quad \phi'_2(x) = \frac{c}{a + c}, \quad x_1 < x < x_2,$$

$$\phi'_2(x) = \frac{a}{a + c}; \quad \phi'_4(x) = \frac{c}{a + c}, \quad x_2 < x < x_3.$$  

Now the following conditions on the $c_i(x)$ are clearly necessary:

$$c_1(0) = c_2(0) = c_4(0); \quad c_1(x_2) = c_2(x_2); \quad c_2(x_4) = c_6(x_4); \quad c_6(x_5) = c_7(x_5).$$  

Since $\alpha_1(x) = -\alpha_1(0)$ and $\alpha_3(x) = \alpha_4(x) = 0$, the first condition yields

$$2c \alpha_1(0) = (b - c)(\alpha_2(0) + \alpha_4(0) + \alpha_6(0) + \alpha_7(0))$$  

or

$$2c \int_0^1 \phi'_1(y)dy = (b - c)(\int_0^1 \phi'_2(y)dy + \int_0^1 \phi'_4(y)dy + \int_0^1 \phi'_6(y)dy + \int_0^1 \phi'_7(y)dy)$$  

or

$$2c \int_0^1 \phi'_1(y)dy = (b - c)(1 - \int_0^1 \phi'_1(y)dy)$$  

or

$$\int_0^1 \phi'_1(y)dy = \frac{b - c}{b + c}. \quad (5)$$
The next condition \((c_5(0) = c_7(0))\) is

\[(b+c)\alpha_2(0) + 2b\alpha_3(0) = (a-b)(\alpha_6(0) + \alpha_7(0))\]

or

\[(b+c)\int_0^1 \varphi_2^*(y)dy + 2b\int_0^1 \varphi_3^*(y)dy = (a-b)(1-\int_0^1 \varphi_1^*(y)dy - \int_0^1 \varphi_2^*(y)dy - \int_0^1 \varphi_5^*(y)dy)\]

which with (5) implies

\[(a+c)\int_0^1 \varphi_2^*(y)dy + (a+b)\int_0^1 \varphi_5^*(y)dy = (a-b)\frac{2c}{b+c} \quad \text{(6)}\]

The other three conditions are

\[(b+c)\int_0^1 \varphi_5^*(y)dy = (b-c)\int_0^1 \varphi_6^*(y)dy \quad \text{(7)}\]

\[(a-c)\int_0^1 \varphi_7^*(y)dy = (b-c)\int_0^1 \varphi_2^*(y)dy + (a+c)\int_0^1 \varphi_7^*(y)dy \quad \text{(8)}\]

and

\[(b-c)\int_0^1 \varphi_2^*(y)dy = (a-b)\int_0^1 \varphi_6^*(y)dy \quad \text{(9)}\]

Now (9) and (7) give

\[\int_0^1 \varphi_6^*(y)dy = \frac{b+c}{b-c} \int_0^1 \varphi_5^*(y)dy = \frac{b-c}{a-b} \int_0^1 \varphi_2^*(y)dy \]

Substituting into (6) then yields
\[ \int_0^1 \varphi_2^*(y)dy = \frac{2c(a-b)^2}{a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c} \] (10)

\[ \int_0^1 \varphi_6^*(y)dy = \frac{2c(a-b)(b-c)}{a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c} \] (11)

\[ \int_0^1 \varphi_5^*(y)dy = \frac{2c(a-b)(b-c)^2}{(b+c)(a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c)} \] (12)

Subtracting from one leaves

\[ \int_0^1 \varphi_7^*(y)dy = \frac{4c(ab^3 + ac^2 - 3b^2c)}{(b+c)(a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c)} \] (13)

Now (12) \Rightarrow

\[ \xi_1 = \frac{2(a-b)(b-c)^2(ab + ac + bc)}{a(b+c)(a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c)} \] (14)

(11) \Rightarrow

\[ \xi_5 = \frac{2c(a-b)(b-c)}{a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c} \] (15)

(14) and (5) \Rightarrow

\[ \xi_2 = \frac{(a+c)(b-c)(a^2b + a^2c + 2ac^2 + 3b^3 - 5b^2c - 2ab^2)}{a(b+c)(a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c)} \]
(13) and (8) \[ \Rightarrow \]

\[ \xi_6 = \frac{c(a^2b^2 + a^2c^2 - 6abc^2 + 2a^2bc + 2b^3c + b^4 + 2ac^3 - 7b^2c^2 + 4abc^2)}{a(b+c)(a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c)} \tag{16} \]

(14), (15), (16), and (13) \[ \Rightarrow \]

\[ \xi_3 = \frac{2(a+c)(a-b)^2(2c-b)}{a(a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c)} \tag{17} \]

and finally (17) and (10) \[ \Rightarrow \]

\[ \xi_4 = \frac{2(a-b)^2(b-c)}{a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c} \]

These add to one, of course, since this condition was used in determining (13). Thus the only conditions that must be imposed on the \( \xi_i \) are those necessary to insure that \( \xi_i \geq 0 \) for all \( i \). Now

\[ a^2b + a^2c + 2ac^2 + 3b^3 - 5b^2c - 2ab^2 = a^2b^2 + b^3 + 2bc^2 + 4b^2c + 2b^3 + a^2c - 2c^2 + 2ac^2 - 2bc^2 \]

\[ = b(a-b)^2 + 2b(b-c)^2 + c(a^2 - b^2) + 2c^2(a-b) > 0 ; \]

\[ a^2b + a^2c + 2ac^2 + b^3 - 2abc - 3b^2c = (a^2b + a^2c + 2ac^2 + 3b^3 - 5b^2c - 2ab^2) - 2b^3 + 2b^2c - 2abc + 2ab^2 \]

\[ = (a^2b + a^2c + 2ac^2 + 3b^3 - 5b^2c - 2ab^2) + 2b(a-b)(b-c) > 0 \]

\[ a^2b^2 + a^2c^2 - 6abc^2 + 2a^2bc + 2b^3c + b^4 + 2ac^3 - 7b^2c^2 + 4abc^2 = (b+c)(a^2b + a^2c + 2ac^2 + 3b^3 - 5b^2c - 2ab^2) \]

\[ + 2abc^2 - 2b^3 + 2b^2c^2 + 2abc - 4ab^2c \]

\[ = (b+c)(a^2b + a^2c + 2ac^2 + 3b^3 - 5b^2c - 2ab^2) + 2(b-a)(b-c) > 0. \]
Thus the only condition needed to insure that all \( \xi_i \geq 0 \) is \( 2c-b \geq 0 \).

Let

\[
\beta(x) = \begin{cases} 
    c_1(x) & 0 < x < x_3 \\
    c_2(x) & x_3 < x < x_4 \\
    c_6(x) & x_4 < x < x_5 \\
    c_7(x) & x_5 < x < 1 
\end{cases}
\]

Then to insure that this strategy is optimal it is sufficient to check that \( c_i(x) \leq \beta(x) \) for all \( i \) and \( x \). Notice that \( \beta(x) \) increases strictly.

First \( c_1(x) = \beta(x) \) for \( 0 < x < x_2 \); and \( c_i'(x) = c_1'(x) \) for \( x_2 < x < x_4 \)

\[ c_1(x) = \beta(x) \text{ for } x_2 < x < x_4 \quad c_i'(x) = 0 \text{ for } x > x_4 \Rightarrow c_1(x) < \beta(x) \]

for \( x > x_4 \). Similarly \( c_2(x) = \beta(x) \) for \( x_1 < x < x_5 \). For \( x < x_1 \), \( c_2(x) = 2\beta'(x) \Rightarrow c_2(x) < \beta(x) \) on this interval; also \( c_2'(x) = 0 \) for \( x > x_5 \Rightarrow c_2(x) < \beta(x) \) on this interval. It is easy to check \( c_6(x) \) and \( c_7(x) \) in a similar way. In checking \( c_3, c_4, \text{ and } c_5 \) the following conditions are found to be necessary and sufficient for \( c_3(x) \leq \beta(x) \), \( c_4(x) \leq \beta(x) \), and \( c_5(x) \leq \beta(x) \) for all \( x \):

\[
c_3(x_4) \leq c_2(x_4); \quad c_4(x_4) \leq c_2(x_4); \quad c_5(x_4) \leq c_6(x_4)
\]

which are equivalent to

\[
(a-c)\int_{x_4}^{1} \phi_i^*(y)dy + (b+c)\int_0^{x_4} \phi_i^*(y)dy \geq (b-c)\int_0^{x_4} \phi_i^*(y)dy + (a+c)\int_{x_4}^{1} \phi_i^*(y)dy
\]

\[
(a-c)\int_{x_4}^{x_4} \phi_i^*(y)dy \geq (a+c)\int_{x_4}^{x_4} \phi_i^*(y)dy
\]

and

\[
(a+b)\int_0^{x_4} \phi_i^*(y)dy \geq (a-b)\int_{x_4}^{x_4} \phi_i^*(y)dy
\]
The first two are seen to be true by comparison with (7) and (8). The third is equivalent to

\[(a+b) \int_0^1 \phi_7^*(y) dy \geq 2a \int_{x_4}^1 \phi_7^*(y) dy = 2a \phi_6^* \]

\[\iff 2(a+b)(abc+b^3+ac^2-3b^2c) \geq a^2b^2+a^2c^2-6ab^2c+2a^2bc+2b^3c+b^4+2bc^2-7b^2c^2+4abc^2 \]

\[\iff (a^2b+a^2c-2ac^2-4abc-2ab^2-b^3+7b^2c)(b-c) \leq 0 \]

\[\iff a^2b+a^2c-2ac^2-4abc-2ab^2-b^3+7b^2c \leq 0 \]

which is an added condition on \(\frac{a}{c}\) and \(\frac{b}{c}\) for this strategy to be optimal.

A summary of the optimal strategies in the five cases that arise follows. The derivation in the last four cases will not be given since they are very similar to the derivation in the first case.

**Case I.** \(a^2b+a^2c-2ac^2-4abc-2ab^2-b^3+7b^2c \leq 0\); \(2c-b \geq 0\).

<table>
<thead>
<tr>
<th></th>
<th>(0)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\phi_1^*(y))</td>
<td>(\frac{ab}{ab+ac+bc})</td>
<td>(\phi_2^*(y))</td>
<td>(\frac{a}{a+c})</td>
<td>(\phi_3^*(y))</td>
<td>(\frac{a}{a+c})</td>
<td>(\phi_4^*(y))</td>
<td>(1)</td>
</tr>
<tr>
<td>(\phi_5^*(y))</td>
<td>(\frac{ac}{ab+ac+bc})</td>
<td>(\phi_7^*(y))</td>
<td>(\frac{c}{a+c})</td>
<td>(\phi_6^*(y))</td>
<td>(\frac{c}{a+c})</td>
<td>(\phi_7^*(y))</td>
<td>(\frac{bc}{ab+ac+bc})</td>
</tr>
</tbody>
</table>
where

\[ x_1 = \frac{2(a-b)(b-c)^2(ab+ac+bc)}{a(b+c)(a^2b+b^2c+2ac^2+b^3-2abc-3b^2c)} \]

\[ x_2 - x_1 = \frac{(a+c)(b-c)(a^2b+ac^2+2ac^2+b^3-5b^2c-2ab^2)}{a(b+c)(a^2b+b^2c+2ac^2+b^3-2abc-3b^2c)} \]

\[ x_3 - x_2 = \frac{2(a+c)(a-b)^2(2c-b)}{a^2b+a^2c+2ac^2+b^3-2abc-3b^2c} \]

\[ x_4 - x_3 = \frac{2(a-b)^2(b-c)}{a^2b+a^2c+2ac^2+b^3-2abc-3b^2c} \]

\[ x_5 - x_4 = \frac{2c(a-b)(b-c)}{a^2b+a^2c+2ac^2+b^3-2abc-3b^2c} \]

\[ 1 - x_5 = \frac{c(a^2b^2+ac^2-6ab^2+2ac^2+2b^3c+2bc^2+b^4+2ac^2-7b^2c^2+4abc^2)}{a(b+c)(a^2b+b^2c+2ac^2+b^3-2abc-3b^2c)} \]

**Case II.**

\[ a^2b + a^2c - 4ac^2 + 4abc - 2ab^2 + b^3 + 7b^2c \leq 0; \quad 2c-b \leq 0. \]

\[
\begin{array}{cccccc}
0 & x_1 & x_2 & x_3 & x_4 & 1 \\
\hline
\phi_1^*(y) = \frac{ab}{ab+ac+bc} & \phi_2^*(y) = \frac{a}{a+c} & \phi_3^*(y) = \frac{b-2c}{b-c} & \phi_4^*(y) = 1 & \phi_5^*(y) = 1 \\
\phi_6^*(y) = \frac{ac}{ab+ac+bc} & \phi_7^*(y) = \frac{c}{a+c} & \phi_8^*(y) = \frac{c}{b-c} \\
\phi_9^*(y) = \frac{bc}{ab+ac+bc}
\end{array}
\]
where

\[ x_1 = \frac{2(a-b)(b-c)^2(ab+ac+bc)}{a(b+c)a^2(b^2c+2ac^2+b^3-2abc-3b^2c)} \]

\[ x_2 - x_1 = \frac{(a+c)(3a^2c^2-a^2b^2-2ab^2c+2a^2bc-6b^2c^2+b^4+2ab^3-2ac^3+9b^2c^2-6abc^2)}{a(b+c)(a^2b+a^2c+2ac^2+b^3-2abc-3b^2c)} \]

\[ x_3 - x_2 = \frac{2(a-b)^2(b-c)}{a^2b+a^2c+2ac^2+b^3-2abc-3b^2c} \]

\[ x_4 - x_3 = \frac{2c(a-b)(b-c)}{a^2b+a^2c+2ac^2+b^3-2abc-3b^2c} \]

\[ x_5 = 1 - x_4 = \frac{c(a^2b+a^2c-6ab^2c+2a^2bc+2b^3c+b^4+2ac^3-7b^2c^2+4abc^2)}{a(b+c)(a^2b+a^2c+2ac^2+b^3-2abc-3b^2c)} \]

\[ x_6 = 1 \]

\[ \phi_1^*(y) = \frac{ab}{ab+ac+bc} \quad \phi_2^*(y) = \frac{a}{a+c} \quad \phi_3^*(y) = \frac{a}{a+c} \quad \phi_4^*(y) = 1 \quad \phi_5^*(y) = 1 \quad \phi_6^*(y) = 1 \quad \phi_7^*(y) = 1 \]

\[ \phi_2^*(y) = \frac{ac}{ab+ac+bc} \quad \phi_3^*(y) = \frac{c}{a+c} \quad \phi_4^*(y) = \frac{c}{a+c} \]

\[ \phi_7^*(y) = \frac{bc}{ab+ac+bc} \]
where

\[ \xi_1 = x_1 = \frac{(b-c)(a^2+3b^2-2ab-2ac)(ab+ac+bc)}{ab(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_2 = x_2 - x_1 = \frac{4b(a-b)(a+c)(b-c)}{a(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_3 = x_3 - x_2 = \frac{(a+c)(a^2+b^2-2ac^2-2ab^2+6ab^2+3b^3-b^2c)}{a(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_4 = x_4 - x_3 = \frac{-a^2b-a^2c+2ac^2+6ab^2-3b^3-3b^2c}{(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_5 = x_5 - x_4 = \frac{c(a^2+b^2-2ac^2-4abc-2ab^2-b^3+7b^2c)}{b(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_6 = x_6 - x_5 = \frac{4bc(b-c)}{(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_7 = 1 - x_6 = \frac{2bc(a-b)(a+b)}{a(b+c)(a^2-b^2+2ab-2ac)} \cdot \]

**Case IV.**

\[ a^2b + a^2c - 2ac^2 - 4abc - 2ab^2 - b^3 + 7b^2c \geq 0; \]

\[ a^2b + a^2c - 2ac^2 + 4abc - 6ab^2 + 3b^3 - b^2c \leq 0. \]

<table>
<thead>
<tr>
<th>0</th>
<th>x_1</th>
<th>x_2</th>
<th>x_3</th>
<th>x_4</th>
<th>x_5</th>
<th>1</th>
</tr>
</thead>
</table>

\[ \phi_1^*(y) = \frac{ab}{ab+ac+bc} \quad \phi_2^*(y) = \frac{a}{a+c} \quad \phi_3^*(y) = k \quad \phi_4^*(y) = 1 \quad \phi_5^*(y) = 1 \quad \phi_6^*(y) = 1 \]

\[ \phi_5^*(y) = \frac{ac}{ab+ac+bc} \quad \phi_6^*(y) = \frac{c}{a+c} \quad \phi_7^*(y) = 1-k \]

\[ \phi_7^*(y) = \frac{bc}{ab+ac+bc} \]
where

\[ k = \frac{-a^2b-a^2c+2ac^2-4abc+6ab^2-3b^3+b^2c}{-a^2b-a^2c+2ac^2+6ab^2-3b^3-3b^2c} \]

\[ \xi_1 = x_1 = \frac{(b-c)(a^2+3b^2-2ab-2ac)(ab+ac+bc)}{ab(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_2 = x_2-x_1 = \frac{(a+c)(a^2+b^2c-2ac^2-2ab^2-b^3+3b^2c)}{a(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_3 = x_3-x_2 = \frac{-a^2b-a^2c+2ac^2+6ab^2-3b^3-3b^2c}{(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_4 = x_4-x_3 = \frac{c(a^2+b^2c-2ac^2-4abc+2ab^2-b^3+7b^2c)}{b(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_5 = x_5-x_4 = \frac{4bc(b-c)}{(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_6 = 1-x_5 = \frac{2bc(a-b)(a+b)}{a(b+c)(a^2-b^2+2ab-2ac)} \]

Case V. \[ a^2b+a^2c-2ac^2-6ab^2+3b^3+3b^2c \geq 0 \]

\[
\begin{array}{cccccccc}
0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \\
\hline
\phi_1^*(y) = \frac{ab}{ab+ac+bc} & \phi_1^*(y) = \frac{a}{a+c} & \phi_2^*(y) = \frac{a}{a+c} & \phi_5^*(y) = \frac{a}{a+b} & \phi_5^*(y) = 1 & \phi_6^*(y) = 1 & \phi_7^*(y) = 1 \\
\phi_5^*(y) = \frac{ac}{ab+ac+bc} & \phi_7^*(y) = \frac{c}{a+c} & \phi_7^*(y) = \frac{c}{a+c} & \phi_7^*(y) = \frac{b}{a+b} \\
\phi_7^*(y) = \frac{bc}{ab+ac+bc} \\
\end{array}
\]
where

\[ \xi_1 = x_1 = \frac{(b-c)(a^2+3b^2-2ab-2ac)(ab+ac+bc)}{ab(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_2 = x_2 - x_1 = \frac{4b(a-b)(a+c)(b-c)}{a(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_3 = x_3 - x_2 = \frac{4bc(a-b)(a+c)}{a(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_4 = x_4 - x_3 = \frac{c(a+b)(a^2b+a^2c-2ac^2-6ab^2+3b^3+3b^2c)}{a(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_5 = x_5 - x_4 = \frac{4c(a-b)(b-c)}{(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_6 = x_6 - x_5 = \frac{4bc(b-c)}{(b+c)(a^2-b^2+2ab-2ac)} \]

\[ \xi_7 = 1 - x_6 = \frac{2bc(a-b)(a+b)}{a(b+c)(a^2-b^2+2ab-2ac)} \]
References


