A GENERALIZED VOTING-GAME

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MARCEL F. NEUTS

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1. Introduction.

A game over the square is characterized by a bounded, real-valued function \( K(\xi, \eta) \) defined over the square \( 0 \leq \xi \leq 1, 0 \leq \eta \leq 1 \). To an independent choice of respectively \( \xi \) and \( \eta \) by the players I and II corresponds a payment of \( K(\xi, \eta) \) by II to I.

A mixed strategy for player I can be represented by a probability distribution function \( F(\xi) \) over \( 0 \leq \xi \leq 1 \), which is non-decreasing, right-hand continuous and \( F(0) = 0, F(1) = 1 \).

Likewise a mixed strategy for player II is a function \( G(\eta) \) over \( 0 \leq \eta \leq 1 \), such that \( G(0) = 0, G(1) = 1 \) and \( G(\eta) \) is non-decreasing and right-hand continuous. The game is said to have a value \( v \) and a pair of optimal strategies \( F^O(\xi) \) and \( G^O(\eta) \) if it is possible to construct a distribution \( F^O(\xi) \) for player I and a distribution \( G^O(\eta) \) for player II, such that for some number \( v \):

\[
(1) \quad h(\eta) = \int_0^1 K(\xi, \eta) dF^O(\xi) \geq v \quad \text{for all} \quad 0 \leq \eta \leq 1
\]

\[
(2) \quad g(\xi) = \int_0^1 K(\xi, \eta) dG^O(\xi) \leq v \quad \text{for all} \quad 0 \leq \xi \leq 1
\]

(1) The author sincerely thanks Professor Samuel Karlin of Stanford University for his interest and suggestions in the solution of this problem. The author is an Aspirant of the Belgian National Fund for Scientific Research and a graduate Fellow of the Belgian American Educational Foundation.
In this case player I can secure himself at least the result \( v \) against all possible strategies of his opponent and player II can prevent I from getting more than \( v \), whatever strategy I may use.

A value and optimal strategies for a game over the square may not exist and even in cases, where general theorems assert their existence, it may not be possible to actually construct them.

In many cases however, when the game is derived from a realistic competitive situation, an insight in its nature can be of a great help in the discussion of the mathematical problem. In this paper, we will discuss a relatively simple game over the square, which can serve as a model for many competitive situations.

2. Some interpretations of the game.

A. Cumulative voting for two corporate directorships of possibly unequal value[1].

Cumulative voting is a voting procedure used in some corporations, under which every share gets as many votes as there are directors to be elected. We consider the case where only two directorships A and B are to be voted for. There is only one ballot, in which any vote may be cast on any candidate. We suppose that there are two opposing factions of stockholders I and II, respectively disposing of \( p \) and \( q \) votes \((p > q > 0)\). Let \( p\xi \leq \xi \leq 1 \) and \( p(1-\xi) \) be the portions of \( p \), which the faction I puts respectively on A and B and let \( q\eta \leq \eta \leq 1 \) and \( q(1-\eta) \) be the portions which the faction puts on A resp. B. Then faction I will get directorship A if:
(3) \[ p^\xi > q^\eta \]

and if:

(4) \[ p(l-\xi) > q(l-\eta) \]

Faction II will win these directorships is the reversed strict inequalities hold. We suppose that directorships \( A \) represents a value \( \alpha \), \( B \) a value \( \beta \) and both jointly a value \( \gamma \) \((\gamma > \alpha, \gamma > \beta)\) to both I and II. The ruling in case of ties does not seriously affect the solutions of the game, as will be shown subsequently. We define the outcome to be

(5) \[ \phi_1(\xi) \text{ for } p^\xi = q^\eta \text{ with } \beta \leq \phi_1(\xi) \leq \gamma \]

\[ \phi_2(\xi) \text{ for } p(l-\xi) = q(l-\eta) \text{ with } \alpha \leq \phi_2(\xi) \leq \gamma \]

The problem of finding optimal partitionings of the votes over the directorships \( A \) and \( B \) is now equivalent to solving the game over the unit-square with kernel \( K(\xi, \eta) \) defined as follows:
\[ \Phi(x,\eta) = \begin{cases} 
\alpha & \eta < \lambda \xi - \lambda + 1 \\
\phi_2(\xi) & \lambda \xi - \lambda + 1 < \eta < \lambda \xi \\
\gamma & \lambda \xi - \lambda + 1 < \eta < \lambda \xi \\
\phi_1(\xi) & \eta = \lambda \xi \\
\beta & \eta > \lambda \xi 
\end{cases} \]

**Figure I**

B. A Problem of Economic Competition:

Another interpretation of the game is the following. Let I and II be two competitions offering items A and B for sale. I puts a sum \( p \) in advertising (or some other form of competition) and II a sum \( q \) (\( p > q > 0 \)). I is sure to sell A if the amount \( p \xi \) (\( 0 \leq \xi \leq 1 \)), which he puts in advertising A, is strictly greater than the amount \( q \eta \) (\( 0 \leq \eta \leq 1 \)) which II puts in advertising A. Similar interpretations for the other cases.
We suppose that selling A is worth $\alpha$, B worth $\beta$ and selling both items is worth $\gamma$ to both players. ($\gamma > \alpha$ and $\gamma > \beta$)

The problem of partitioning $p$ and $q$ over the items A and B can again be reduced to solving the game with kernel(6).

3. **Reduction of the game.**

When $\lambda > 2$, the game has trivial solutions. Player I can always secure $v = \gamma$ by concentrating a mass 1 at any point $\xi_0$, such that

$$\frac{1}{\lambda} < \xi_0 < \frac{\lambda - 1}{\lambda}$$

All strategies for player II are trivially optimal. We now consider the case $\lambda \leq 2$ and in order to avoid further complications at this stage we suppose that:

$$(7) \quad \varphi_2 \left( \frac{\lambda - 1}{\lambda} \right) = \gamma$$

If now $G(\eta)$ is any mixed strategy for player II, we see that

$$\int_0^1 K(\xi, \eta) dG(\eta)$$

is a non-decreasing function of $\xi$ on the closed interval $[0, \frac{\lambda - 1}{\lambda}]$.

Player I, who wishes to maximize his minimum return, cannot put any weight on the half-open interval $\left[0, \frac{\lambda - 1}{\lambda} \right)$.
The only part of the unit-square, which hence contains relevant features for the solutions of the game is the region where:

\[ \frac{\lambda - 1}{\lambda} \leq \xi \leq 1 \quad \text{and} \quad 0 \leq \eta \leq 1 \]

To facilitate the sequel, we now map this rectangle on the unit-square \( 0 \leq \xi' \leq 1, 0 \leq \eta' \leq 1 \) by the mapping

\[ (8) \quad \xi' = \lambda \xi - \lambda + 1 \]

\[ \eta' = \eta \]

This mapping preserves all essential features of the game. The value is invariant and optimal strategies are mapped onto optimal strategies [2]. If we now set \( \lambda - 1 = \nu \) and drop the accents, it remains to study the game with kernel \( L(\xi, \eta) \).

\[ (9) \quad L(\xi, \eta) = \begin{cases} 
\beta & \xi < \eta - \gamma \\
\varphi_2(\xi) & \xi = \eta - \gamma \\
\gamma & \eta - \gamma > \xi < \eta \\
\varphi_1(\xi) & \xi = \eta \\
\alpha & \xi > \eta 
\end{cases} \]
As the case $\alpha = \beta = \frac{1}{2} \gamma$ is quite trivial - see [1] - we further simplify our notation by subtracting $\alpha$ from $L(\xi, \eta)$ - thus reducing $v$ to $v - \alpha$ and set:

\[ (10) \quad \beta - \alpha = a \quad \gamma - \alpha = b \]

We finally obtain the game-kernel $M(\xi, \eta)$:

![Diagram](image)

To simplify our notation throughout the discussion, we set

\[ (11) \quad \varphi_1(\xi) = \varphi_2(\xi) = b \]

This restriction can easily be replaced by very general conditions as we will subsequently show.
4. Discussion of the game with Kernel $M(\xi, \eta)$. (2)

If $v$ is the value and $F(\xi)$ and $G(\eta)$ are optimal strategies, the conditions (1) and (2) become here:

\[
\begin{align*}
(12) & \quad h(0) = b \ F(o^+) \geq v \\
& \quad h(\eta) = b \ F(\eta) \geq v \quad \quad 0 < \eta \leq v \\
& \quad h(\eta) = a \ F(\eta-v^+) + b[F(\eta) - F(\eta-v^-)] \geq v \quad v < \eta \leq 1
\end{align*}
\]

\[
\begin{align*}
(13) & \quad g(0) = a[1 - G(v)] + b \ G(v) \leq v \\
& \quad g(\xi) = a[1 - G(\xi+v^-)] + b[G(\xi+v^-) - G(\xi^-)] \leq v \quad 0 < \xi < 1-v \\
& \quad g(\xi) = b[1 - G(\xi^-)] \leq v \quad 1 - v \leq \xi \leq 1
\end{align*}
\]

In our discussion of the conditions (12) and (13) we must distinguish four cases: $1 > v > 1/2$, $v = 1/2$, $\frac{1}{n} > v > \frac{1}{n+1}$ and $v = \frac{1}{n}$ ($n = 2, 3, \ldots$).

4a. The case $1 > v > 1/2$.

In view of the interpretation of the game, we expect $v \geq 0$. In case

\[
\begin{align*}
(2) & \quad F(o^+) = \lim_{\xi \to 0} F(\xi) \quad F(\eta^-) = \lim_{\xi \to \eta} F(\xi) \\
& \quad G(\xi^-) = \lim_{\eta \to \xi} G(\eta)
\end{align*}
\]
v is positive, the first inequality in (12) implies that \( F(0^+) > 0 \). We know, that in the conditions (12) and (13) equality must hold at the points where the opposing player puts a positive mass. Hence \( g(0) = v \) or:

\[
(14) \quad G(v) = \frac{v-a}{b-a}
\]

In view of the simple character of the kernel, we search for optimal strategies, which are mixtures of a finite number of pure strategies. The only point, where another jump on \( F(\xi) \) can be expected is \( \xi = v \). Hence we conjecture that an optimal \( F(\xi) \) will exist of the form:

\[
F(\xi) = \mu I_o + (1-\mu)I_v \quad 0 \leq \mu \leq 1
\]

where \( I_c \) denotes the distribution, which is zero for \( x < c \) and one for \( x \geq c \) \( (0 < c \leq 1) \). \( I_o = 0 \) at the point 0 and one on \( (0,1) \).

The conditions (12) and (13) then yield:

\[
b \mu \geq v \quad b \geq v \quad a \mu + b(1-\mu) \geq v
\]

and \( G(v) = \frac{v-a}{b-a} \quad G(v^-) = \frac{b-v}{b} \). Since \( b > v \) we have \( h(v) > v \). This implies that no jump on \( G(\eta) \) is possible at \( \eta = v \).

Whence:

\[
G(v) = G(v^-)
\]
or:

\[
\frac{v-a}{b-a} = \frac{b-v}{b} \quad \text{or} \quad v = \frac{b^2}{2b-a}
\]

Substituting this value for \( v \), the first conditions yield

(16)

\[
\mu = \frac{b}{2b-a} = F(o+)
\]

\[
l - \mu = \frac{b-a}{2b-a} = F(v) - F(v-)
\]

The conditions (13) now become:

\[
G(v) = \frac{b-a}{2b-a}
\]

\[
a[1 - G(\xi+v)] + b[G(\xi+v) - G(\xi-)] \leq \frac{b^2}{2b-a} \quad 0 < \xi < 1-v
\]

\[
b[1 - G(\xi-)] \leq \frac{b^2}{2b-a} \quad 1-v \leq \xi \leq 1
\]

This last condition yields:

\[
G(\xi-) \geq \frac{b-a}{2b-a} \quad \text{for} \quad 1-v \leq \xi \leq 1
\]

This and (14) yields:
G(ξ) = \frac{b-a}{2b-a} \quad 1-v \leq ξ \leq v

It is to be expected, that again many distributions G(ξ) will satisfy these functional inequalities.

For instance any distribution of the form:

(17) \quad G(η) = \frac{b-a}{2b-a} I_0 + \frac{b}{2b-a} I_1 \quad 0 \leq η_0 < 1-v

\quad v < η_1 \leq 1

represents an optimal mixed strategy for player II. This will be verified if we show that the solutions (15), (16) and (17) satisfy the relations (12) and (13).

The functions h(η) and g(ξ) corresponding to (16) and (17) are

\[ h(η) = \frac{b^2}{2b-a} \quad \text{for} \quad 0 \leq η < v \quad \text{and} \quad v < η \leq 1 \]

\[ = b > v \quad \text{for} \quad η = v \]

In the case \( η = 0 \quad η_1 = 1 \) in (17),

\[ g(ξ) = \frac{b^2}{2b-a} \quad \text{for} \quad ξ = 0 \quad \text{and} \quad 1-v \leq ξ \leq 1 \]

\[ = \frac{ab}{2b-a} < v \quad \text{for} \quad 0 < ξ \leq 1-v \]
For other values of $\eta_0$ and $\eta_1$ the inequalities can easily be checked. The distribution given in (16) is by far not a unique optimal strategy. It is easily seen, that the weight $\frac{b-a}{2b-a}$ may be distributed in any arbitrary way over the interval $(1-\nu, \nu)$.

The great freedom in the construction of optimal mixed strategies will be used in 5 when the condition (11) will be considerably relaxed.

4b. The case $\nu = 1/2$.

It is easily seen, that the distributions

\begin{align*}
F(\xi) &= \frac{b}{2b-a} \ I_0 + \frac{b-a}{2b-a} \ I_{1/2} \\
G(\eta) &= \frac{b-a}{2b-a} \ I_0 + \frac{b}{2b-a} \ I_1 \quad 0 \leq \eta_1 \leq \frac{1}{2} \\
&\quad \frac{1}{2} \leq \eta_2 \leq 1
\end{align*}

will be optimal and the value remains $\frac{b^2}{2b-a}$. The solution for player I is unique in this case.

4c. The case $\nu < 1-\nu < (n+1)\nu$.

As in the case 4a we expect the value to be positive. This would imply that $F(0^+) > 0$ and hence:

\begin{align*}
G(\nu) &= \frac{\nu-a}{b-a}
\end{align*}

By analogy with the previous cases, we conjecture the existence of an optimal
strategy for \( I \), which is a mixture of the pure strategies \( \xi = 0 \), \( \xi = kv \)
\( k = 1, \ldots, n \) and \( \xi = 1-v \) or

\[
F(\xi) = \sum_{k=0}^{n} \alpha_k I_{kv} + \alpha_{n+1} I_{1-v}
\]

with \( \alpha_k > 0 \) \( k = 0, 1, \ldots, n+1 \) and \( \sum_{k=0}^{n+1} \alpha_k = 1 \)

The conditions (12) then become:

\[
b\alpha_0 \geq v
\]

\[
b\alpha_0 + b\alpha_1 \geq v \quad (*)
\]

\[
a \sum_{j=0}^{k-1} \alpha_j + b\alpha_k \geq v
\]

\( k = 1, \ldots, n \)

\[
a \sum_{j=0}^{k-1} \alpha_j + b(\alpha_k + \alpha_{k+1}) \geq v \quad (*)
\]

\[
a \sum_{j=0}^{n} \alpha_j + b\alpha_{n+1} \geq v
\]

\[
a \sum_{j=0}^{n-1} \alpha_j + b(\alpha_{n+1} + \alpha_n) \geq v \quad (*)
\]

The starred inequalities correspond to \( h(kv) \) \( k = 1, \ldots, n \) and \( h(1-v) \).
We notice that their left-hand sides are in any case strictly larger than the preceding expressions. From this we conclude that they are superfluous, and moreover if \( F(\xi) \) is indeed of the form (19) the value of \( h(\eta) \) is strictly larger than \( \nu \). This implies that no optimal strategy \( G(\eta) \) for player II can have jumps at the points \( \eta = kv \) \( k = 1, \ldots, n \) and \( \eta = 1-\nu \).

Or:

\[
(21) \quad G(kv) = G(kv) \quad k = 1, \ldots, n, \quad \text{for} \quad G(1-\nu) = G(1-\nu)
\]

Now if all the \( \alpha_i \) \( i = 0, \ldots, n+1 \) are strictly positive, the following equalities are implied by (13) and (21).

\[
(22) \quad g(c) = (b-a)G(\nu) + a = \nu
\]

\[
g(kv) = (b-a)G(kv+\nu) - bG(kv) + a = \nu \quad \quad k = 1, \ldots, n
\]

\[
g(1-\nu) = b - bG(1-\nu) = \nu
\]

If we set \( G(1-\nu) = G(n\nu+\nu) \) then the equations (22) can be reduced to:

\[
(23) \quad G(\nu) = \frac{\nu-a}{b-a}
\]
\[
G(kv) = \frac{v-a}{b-a} \sum_{j=0}^{k-1} \left( \frac{b}{b-a} \right)^j, \quad k = 1, \ldots, n
\]

\[
G(nv+v) = G(1-v) = \frac{v-a}{b-a} \sum_{j=0}^{n} \left( \frac{b}{b-a} \right)^j, \quad j = \frac{b-v}{b}
\]

Solving for \(v\) and substitution yields: (3).

\[
(24) \quad v = \frac{ab^{n+2}}{b^{n+2} - (b-a)^{n+2}}
\]

and

\[
(25) \quad G(kv) = (b-a)^{n-k+2} \frac{b^k - (b-a)^k}{b^{n+2} - (b-a)^{n+2}}
\]

\(k = 1, \ldots, n+1\)

This conditions are consistent with all our previous assumptions. We see immediately that \(v > 0\), \(G(kv) < G(kv + v)\) for \(k = 1, \ldots, n-1\).

This last inequality points out however that each of the intervals \([0,v), (v, 2v), \ldots (nv,1]\) must contain positive weight in every optimal \(G(v)\). Therefore the un-starred inequalities in (20) which precisely correspond to \(h(\eta)\) in those intervals must all be satisfied with equality.

Solving now the system (20) for the \(\alpha_i, 1 = 0, \ldots, n+1\), we find:

(3) The case \(a = 0\) can be solved easily - we suppose here \(a \neq 0\).
(26) \[ \alpha_0 = \frac{v}{b} \]
\[ \alpha_k = \frac{v}{b} \left( \frac{b-a}{b} \right)^k \quad k = 1, \ldots, n+1 \]

Now
\[ \sum_{j=0}^{n+1} \alpha_j = \frac{v}{b} \cdot \frac{\left( \frac{b-a}{b} \right)^{n+2} - 1}{\frac{b-a}{b} - 1} = 1 \]

which yields the same value for \( v \) as found in (24) and

(27) \[ \alpha_k = \frac{a b^{n-k+1} (b-a)^k}{b^{n+2} - (b-a)^{n+2}} \quad k = 0, 1, \ldots, n+1 \]

These values satisfy all the conditions sub (20), so the solution of the game will be complete if we exhibit a strategy \( G(\eta) \) for player II, which satisfies (25) and (13).

Such a strategy can be constructed in many ways. The following is also a mixture of \( n+2 \) pure strategies.

(28) \[ G(\eta) = \sum_{j=0}^{n+1} \beta_j \xi_j \]

with
\[ 0 < \xi_0 \leq 1 - (n+1)v \quad \beta_0 = G(v) \]
\[ \xi_0 + v < \xi_1 \leq 1 - nv \quad \beta_1 = G(2v) - G(v) \]
\[ \xi_{k-1} + \nu < \xi_k \leq 1 - (n-k+1)\nu \quad \beta_k = G(kv+\nu) - G(kv) \]

\[ k = 1, \ldots, n-1 \]

\[ \xi_{n-1} + \nu < \xi_n < 1 - \nu \quad \beta_n = G(n\nu+\nu) - G(n\nu) \]

\[ \xi_n + \nu < \xi_{n+1} \leq 1 \quad \beta_{n+1} = 1 - G(n\nu+\nu) \]

with the \( G(k\nu) \) \( k = 1, \ldots, n+1 \) as found in (25). To verify that these are indeed optimal strategies for player II, we calculate the corresponding function \( g(\xi) \).

\[
\begin{align*}
(29) \quad g(\xi) & = \nu & 0 \leq \xi \leq \xi_0 \\
g(\xi) & = a[1 - G(\nu)] < \nu & \xi_0 < \xi < \xi_{1-\nu} \\
g(\xi) & = \nu & \xi_{1-\nu} \leq \xi \leq \nu \\
\text{For } k = 1, \ldots, n-1 \\
g(\xi) & = a[1 - G(kv + \nu)] + b[G(kv + \nu) - G(kv)] = \nu & kv \leq \xi \leq \xi_k \\
g(\xi) & = a[1 - G(kv + \nu)] < \nu & \xi_k < \xi < \xi_{k+1-\nu} \\
g(\xi) & = a[1 - G(kv + 2\nu)] + b[G(kv + 2\nu) - G(kv + \nu)] = \nu & \\
& & \xi_{k+1-\nu} \leq \xi \leq kv + \nu \\
g(\xi) & = a[1 - G(n\nu + \nu)] + b[G(n\nu + \nu) - G(n\nu)] = \nu & n\nu \leq \xi \leq \xi_n
\end{align*}
\]
\[ g(\xi) = a[1 - G(nv + v)] < v \quad \xi_n < \xi \leq \xi_{n+1} - v \]
\[ g(\xi) = b[1 - G(nv + v)] = v \quad \xi_{n+1} - v \leq \xi \leq \xi_{n+1} \]
\[ g(\xi) = 0 < v \quad \xi_{n+1} < \xi \leq 1 \]

This proves that (28) is indeed an optimal strategy for player II.

**Some Remarks.**

a. The strategy \( F(\xi) \) given in (19) and (26) for player I is not unique. The weights, which we put at the points \( kv \quad k = 1, \ldots, n \) can be placed with some generality at other points in the intervals \( (1 - (n-k+1)v, kv)^k = 1, \ldots, n \). In our discussion of the strategy \( G(\eta) \) given in (28) for player II, we notice however that \( g(\xi) < v \) on certain intervals. This implies that no optimal strategy for player I can put a positive mass in these intervals.

By taking the \( \xi_k \quad k = 0, \ldots, n \) in (28) as close as possible to the points \( kv \), we see that no optimal strategy for I can put weight on the intervals \( (kv, 1 - (n-k+1)v)^k = 0, \ldots, n \). This fact will be used in the sequel.

b. Optimal strategies for the game with the condition (11) will in general not be optimal for the more general game. In the next paragraph we will exhibit distributions which are continuous, except at the points 0 and 1 possibly. They will be optimal in both games under very general conditions.
4d. The case \( 1 - \nu = (n+1)\nu \), or \( \nu = 1/n+2 \) \( n = 1, 2, \ldots \) 

The value of the game is the same as in previous case.

\[
v = \frac{ab^{n+2}}{b^{n+2} - (b-a)^{n+2}}
\]

The strategy

\[\tag{30} P(\xi) = \sum_{j=0}^{n+1} a_j \frac{I_j}{n+2}\]

\[
\alpha_j = \frac{ab^{n-j+1}(b-a)^j}{b^{n+2} - (b-a)^{n+2}} \quad j = 0, \ldots, n+1
\]

is now the unique optimal strategy for player I.

Optimal strategies for player II can be constructed with great generality as in the previous case. The analogue of (28) in this case will yield a wide class of optimal strategies, which are mixtures of \( n+2 \) pure strategies.

5. The general game.

We now drop the assumption (11) and allow \( \varphi_1(\xi) \) and \( \varphi_2(\xi) \) to be general bounded functions except for general conditions at some particular points. We maintain however the condition (7) i.e.,

\[\varphi_2(0) = b\]

which cannot be removed so easily.
If \( v \) is the value and \( F(\xi) \) and \( G(\eta) \) are optimal strategies, the conditions (1) and (2) become here:

(31) \( h(o) = bF(o+) \geq v \)

\[
h(\eta) = bF(\eta-) + \varphi_2(\eta)[F(\eta) - F(\eta-)] \geq v \quad 0 < \eta < v
\]

\[
h(v) = \varphi_1(o)F(o+) + b[F(v-) - F(o+)] + \varphi_2(v)[F(v) - F(v-)] \geq v
\]

\[
h(\eta) = aF(\eta-v-) + \varphi_1(\eta-v)[F(\eta-v) - F(\eta-v-)] + b[F(\eta-) - F(\eta-v)]
\]

\[
+ \varphi_2(\eta)[F(\eta) - F(\eta-)] \geq v \quad \text{for } v < \eta < 1
\]

\[
h(1) = aF(1-v-) + \varphi_1(1-v)[F(1-v) - F(1-v-)]
\]

\[
+ b[F(1-) - F(1-v)] + \varphi_2(1)[1 - F(1-)] \geq v
\]

(32) \( g(o) = bG(v-) + \varphi_1(o)[G(\nu) - G(v-)] + a[1 - G(\nu)] \leq v \)

\[
g(\xi) = \varphi_2(\xi)[G(\xi) - G(\xi-)] + b[G(\xi+v-) - G(\xi)]
\]

\[
+ \varphi_1(\xi)[G(\xi+v) - G(\xi+v-)] + a[1 - G(\xi+v)] \leq v
\]

for \( 0 < \xi < 1-v \)

\[
g(1-v) = \varphi_2(1-v)[G(1-v) - G(1-v-)] + b[G(1-) - G(1-v)]
\]

\[
+ \varphi_1(1-v)[1 - G(1-)] \leq v
\]
\[ g(\xi) = \varphi_2(\xi)[G(\xi) - G(\xi^-)] + b[1 - G(\xi)] \leq v \]

for \( 1 - v < \xi < 1 \)

\[ g(1) = \varphi_2(1)[1 - G(1^-)] \]

It will be shown that in general the value of the game will not depend on the values of \( \varphi_1(\xi) \) and \( \varphi_2(\xi) \). The optimal strategies which we have exhibited in the simplified version will not remain so, unless a weird set of conditions are satisfied. However we can construct strategies for the simplified game, which do not have jumps at points interior to the unit interval. Under very general conditions they will also be optimal in this case.

5a. The case \( 1 > v > 1/2 \).

A pair of optimal strategies for the general game can be obtained by spreading the weight at \( v \) in (16) uniformly over the interval \((1-v, v)\) and the weights at \( \eta_0 \) and \( \eta_1 \) in (17) over the intervals \((0, 1-v)\) and \((v, 1)\) respectively, i.e.

\[
F(\xi) = \frac{b}{2b-a} \quad 0 < \xi \leq 1-v
\]

\[
F(\xi) = \frac{1}{2b-a} \left[ b + (b-a) \frac{\xi - l + v}{2v-l} \right] \quad 1-v \leq \xi \leq v
\]

\[
F(\xi) = 1 \quad v \leq \xi \leq 1
\]
\[(34) \quad G(\eta) = \frac{b-a}{2b-a} \cdot \frac{\eta}{1-\nu} \quad 0 \leq \eta \leq 1-\nu \]
\[G(\eta) = \frac{b-a}{2b-a} \quad 1-\nu \leq \eta \leq \nu \]
\[G(\eta) = \frac{1}{2b-a} [(b-a) + b \frac{\eta-\nu}{1-\nu}] \quad \nu \leq \eta \leq 1 \]

The corresponding functions \( h(\eta) \) and \( g(\xi) \) are:

\[h(\eta) = \nu \quad 0 \leq \eta \leq 1-\nu \quad \text{and} \quad \nu < \eta \leq 1 \]
\[= \nu + \frac{b(b-a)}{2b-a} \cdot \frac{\eta-1+\nu}{2\nu-1} > \nu \quad 1-\nu < \eta < \nu \]
\[= \nu + \frac{b}{2b-a} [\varphi_1(0)-a] \quad \eta = \nu \]

\[g(\xi) = \nu \quad 0 \leq \xi \leq \nu \]
\[= \nu(1 - \frac{\xi-\nu}{1-\nu}) < \nu \quad \nu < \xi \leq 1 \]

This shows that the strategies (33) and (34) will be optimal under the general condition \( \varphi_1(0) \geq a \).

5b. The case \( \nu = \frac{1}{2} \)

The unique strategy \( F(\xi) \) for player I, found in (18) will be optimal in the general case, provided:
The strategy $G(\eta)$

$$G(\eta) = 2\eta \frac{b-a}{2b-a} \quad 0 \leq \eta \leq \frac{1}{2}$$

$$= (2\eta-1)\frac{b}{2b-a} + \frac{b}{2b-a} \quad \frac{1}{2} \leq \eta \leq 1$$

which is analogous to (34) is also optimal here.

5c. The case $\eta v < 1 - \nu < (n+1)\nu$

A pair of optimal strategies for the general game can easily be obtained by spreading the point-masses used in the case 4c over appropriate intervals. These are discussed in remark a in 4c.

$$F(\xi) = F(\xi) = \alpha_0 \quad 0 < \xi \leq 1 - (n+1)\nu$$

$$F(\xi) = \alpha_0 + \alpha_1 \frac{\xi + (n+1)\nu - 1}{(n+2)\nu - 1} \quad 1 - (n+1)\nu \leq \xi \leq \nu$$

$k = 1, \ldots, n-1$

$$F(\xi) = \sum_{j=0}^{k} \alpha_j \quad k\nu \leq \xi \leq 1 - (n-k+1)\nu$$

$$F(\xi) = \sum_{j=0}^{k} \alpha_j + \alpha_{k+1} \frac{\xi + (n-k+1)\nu - 1}{(n+2)\nu - 1}$$
\[ l - (n-k+1)\nu \leq \xi \leq (k+1)\nu \]

\[
F(\xi) = \sum_{j=0}^{n} \alpha_j \quad \text{for } n\nu \leq \xi \leq l-\nu
\]

\[
F(\xi) = \sum_{j=0}^{n} \alpha_j + \alpha_{n+1} \frac{\xi + \nu - 1}{(n+2)\nu - 1} \quad \text{for } l-\nu \leq \xi \leq (n+1)\nu
\]

\[
F(\xi) = 1 \quad \text{for } (n+1)\nu \leq \xi \leq l
\]

and

(36) \[ G(\eta) = G(\nu) + \frac{\eta - \nu}{l - (n+1)\nu} [G(\nu+\nu) - G(\nu)] \]

\[ \nu \leq \eta \leq l - (n-k+1)\nu \]

\[ k = 0, \ldots, n \]

\[ G(\eta) = G(\nu+\nu) \quad \text{for } l-(n-k+1)\nu \leq \xi \leq (k+1)\nu \]

\[ k = 0, 1, \ldots, n \]

\[ G(\eta) = G(\nu+\nu) + \frac{\eta - (n+1)\nu}{l -(n+1)\nu} [1 - G(\nu+\nu)] \]

\[ (n+1)\nu \leq \eta \leq l \]

The corresponding function \( g(\xi) \) is:

\[
g(\xi) = \frac{a b^{n+2}}{b^{n+2} - (b-a)^{n+2}} = \nu \quad 0 \leq \xi \leq (n+1)\nu
\]
\[ g(\xi) = v - b[1 - \frac{G(nv+v)}{(n+1)v} - \frac{\xi - (n+1)v}{1 - (n+1)v}] \]
\[ = v[1 - \frac{\xi - (n+1)v}{1 - (n+1)v}] < v \quad (n+1)v \leq \xi < 1 \]

The function \( h(\eta) \) corresponding to \( F(\xi) \) is:

\[ h(\eta) = v \quad 0 \leq \eta \leq 1 - (n+1)v \]

\[ h(\eta) = v + b\alpha_1 \frac{\eta + (n+1)v}{(n+2)v - 1} \quad 1 - (n+1)v \leq \eta < v \]

\[ h(v) = \varphi_1(0)\alpha_o + b\alpha_1 = v + \alpha_o[\varphi_1(0) - a] \]

\[ h(\eta) = v \quad v < \eta \leq 1 \]

The \( \alpha_i \) and the \( G(kv) \) are those given in formulae (25) and (26). We see that the strategies (35) and (36) are optimal under the general condition \( \varphi_1(0) \geq a \).

5d. The case \( v = \frac{1}{n+2} \)

The strategy for player I, given in (30) is the unique optimal strategy in the restricted game. The requirement

\[ h(kv) \geq v \quad k = 1, \ldots, n+1 \]

will only be satisfied in the general case if the following inequalities hold:
\[ b \varphi_1(0) + (b-a)\varphi_2(v) \geq b^2 \]

\[ \varphi_1(kv-v)_{k-1} + \varphi_2(kv)_{k} \geq v - a \sum_{j=0}^{k-1} \alpha_j \]

\[ k = 2, \ldots, n+1 \]

If these conditions are not satisfied, the value and optimal strategies of the game, if they exist, will depend upon the values of \( \varphi_1 \) and \( \varphi_2 \).

If these conditions are satisfied, an optimal strategy for player II is easily obtained by setting \( v = \frac{1}{n+2} \) in formula (36). We notice that the use of this strategy guarantees

\[ g(t) \leq v \]

against all strategies of I, whatever the values of \( \varphi_1 \) and \( \varphi_2 \).

6. The Condition (7).

Except in the rather pathological cases \( v = \frac{1}{n} \), \( n = 2, \ldots \), the condition (7) can also be removed. Let us consider the mapping (8) operating on the entire unit-square, which is then mapped onto the rectangle \(-v \leq \xi \leq 1, 0 \leq \eta \leq 1\). If \( \varphi_2(0) > b \), the solutions remain the same. If \( \varphi_2(0) < b \), an optimal strategy for player I can easily be constructed from those found earlier.

We consider first the case \( nv < 1-v < (n+1)v \), \( n = 1,2, \ldots \). We use the strategy in (35), but instead of concentrating the mass \( \alpha_0 \) at 0, we spread it uniformly over an interval \((-\epsilon, 0)\) with \( 0 < \epsilon \leq (n+2)v-1 \).
The corresponding function \( h(\eta) \) will remain unaltered, except in the interval \((v-\epsilon, v)\), where:

\[
h(\eta) = \alpha_o \left[ a \frac{\eta - V + \epsilon}{\epsilon} + b \frac{\eta - \eta_1}{\epsilon} \right] + \eta \alpha_1
\]

\[
= v + \eta \alpha_1 - (b-a)\alpha_o \frac{\eta - v + \epsilon}{\epsilon} > v
\]

\[
h(v) = v
\]

This construction hence yields an optimal strategy for player I. It cannot be applied in the case \( v = \frac{1}{n+2} \), since it makes \( h(\eta) \) smaller in the interval \((v-\epsilon, v)\).

In the case \( v > \frac{1}{2} \), we construct a new optimal strategy for player I from the strategy (16).

We do so by spreading the weight \( \frac{b}{2b-a} \) uniformly over an interval \((-\delta, 0)\); \( 0 < \delta < 2v - 1 \) and the weight \( \frac{b-a}{2b-a} \) uniformly over the interval \((v-\delta, v)\). The function \( h(\eta) \) remains unchanged, since the evaluation yields:

\[
h(\eta) = v + \frac{\eta - v + \delta}{\delta} [(a-b) \frac{b}{2b-a} + \frac{b(b-a)}{2b-a}] = v
\]

\[
v - \delta \leq \eta \leq v
\]

Again a construction of this kind is not possible in the case \( v = \frac{1}{2} \).

Since obviously \( g(\xi) \) corresponding to the optimal strategies for player II is not affected by dropping the condition (7), we suspect that if
\( \varphi_2(0) < b, \nu = \frac{1}{2} \quad n = 1, 2, \ldots \) the value, if it exists, is strictly less than the values found here. In this case, both the value and the optimal strategies must depend upon \( \varphi_2(0) \). In view of the very special character of these cases, we have not devoted further attention to them.

7. **The game with kernel (6).**

The value and the optimal strategies, which we have found can now easily be expressed in terms of the function \( K(\xi, \eta) \) given in (6). It suffices to substitute for \( a \) and \( b \) the expressions in (10) and to apply the inverse mapping of (8) to the distributions.
REFERENCES


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