STEADY STATE PROPERTIES OF SELECTED INVENTORY MODELS

BY
RICHARD C. SINGLETON

TECHNICAL REPORT NO. 23
JULY 21, 1960

PREPARED UNDER CONTRACT Nonr-225 (28)
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OFFICE OF NAVAL RESEARCH

APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY
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1. INTRODUCTION

In this report, known techniques of analysis are applied to the investigation of a group of mathematical models related to the problem of inventory control. These models are felt to be of considerable practical interest, and have not been treated previously in the literature in the present generality.

A characteristic common to these models is the assumption that the quantity of resupply available for delivery to a stocking point in any one period is limited, in general random, and independent of demands served by the stocking point. It is assumed that a stocking point may accept all, part, or none of an available delivery, but does not have control over the availability of deliveries. An exception to this is the model studied in the final section, where the stocking policy allows the placing of orders for emergency delivery to supplement the basic random supply process.

In all of the models treated here, time is regarded as divided into periods of equal length. During each period, a demand on the stocking point occurs; demands in successive time periods are assumed independent and identically distributed with known continuous probability density function. It is assumed that demands not fulfilled in a period are backlogged, and satisfied later when additional supplies become available.

The stocking policies considered are all of simple form, action being taken after comparison of the existing stock level with a single fixed number. In some cases the quantity delivered is limited by a second
number, corresponding to the storage capacity of the stocking point. These simple forms of policy are often used in actual practice. However, no attempt is made to determine conditions under which they are optimum with respect to a larger class of possible policies.

The analysis is directed toward obtaining the steady state properties of each model. Although there are variations from model to model in the methods used, a basic pattern of analysis is repeated. First, considering the sequence of stock levels in successive periods as a sequence of random variables, a subsequence forming a Markov chain is identified. For the models in which delivery takes place each period, the full sequence of stock levels has the Markov property. For the models in which the time between deliveries is a random variable, the subsequence of stock levels at delivery times forms an embedded Markov chain; in this analysis, the methods used by Karlin and Fabens in [5] are followed. Once having identified a sequence of Markov random variables, the one-step transition relations are determined, and from these a one-step transition operator $T$ is formulated. If $F_{\eta_j}$ and $F_{\eta_{j+1}}$ represent the distributions of stock levels at the successive delivery times $\eta_j$ and $\eta_{j+1}$, then $T$ transforms $F_{\eta_j}$ into $F_{\eta_{j+1}}$. This transformation can be written as $TF_{\eta_j} = F_{\eta_{j+1}}$. Then the stationary distribution $F$ of stock level at delivery times, if it exists, is a fixed point of $T$, i.e. $TF = F$. This relation is formulated for each model as an integral equation. In the examples worked, it is possible in each case to reduce this integral equation to a differential equation which can be solved.

Next, the limiting distribution of stock level in an arbitrary period is expressed as a transformation of the stationary distribution of stock
level at delivery times. To do this, one needs to find the limiting distribution of time since the last delivery. The renewal theory methods applied by Karlin in [4] are used to obtain this distribution.

Once the limiting distribution of stock level in an arbitrary period has been evaluated, an appropriate cost structure can be assumed, and limiting one-period expected costs evaluated. The optimum values of the decision parameters of the stocking policy are then determined by finding those values which minimize expected costs. These methods are standard. It should be noted that the limiting distribution of stock level is independent of the cost structure. Thus, alternative cost structures can be investigated using the same stock level distribution.

The problem of the existence of a unique stationary probability distribution of stock level for a given model is not treated here. In general, such proofs are difficult to carry out. A condition that the mean quantity of goods available for resupply per period exceed the mean demand per period is required to prevent the mean stock level from drifting off to minus infinity. This condition is necessary for the existence of a stationary distribution of stock level. The uniqueness of the solutions in the examples is conjectured, but not proved. Karlin in [3] shows, under mild conditions on the distribution of delivery size, the uniqueness of the solution to one of the systems of differential equations which appears in the examples below. It appears likely that the uniqueness of the other solutions could also be established under similar conditions.

The models in sections 2, 3, and 4 are based on a common demand and resupply structure, but differ in their stock policies. A single stocking point is assumed to have random quantities of resupply available to it,
with random numbers of periods between availability of resupply. The distributions of demand, available resupply quantity, and time between resupply are assumed to be mutually independent.

Under policy I, discussed in section 2, the stocking point accepts delivery whenever available, subject to a fixed storage capacity limitation of S. (Cf. Karlin and Fabens [5] for a related model.) Under policy II, defined in section 3, the stocking point accepts the full available delivery if the stock level is below a fixed level s, and otherwise rejects delivery. In section 4, policy III is studied; in following this policy, the stocking point accepts an available delivery subject to a storage capacity limitation S when the stock is below s, where s ≤ S, and otherwise rejects delivery. This policy can be considered to contain policies I and II as limiting cases.

Inventory systems with these characteristics occur in a number of cases of practical interest. Canners and packers of agricultural commodities or fish, for example, frequently have little control over the times and quantities in which commodities are offered to them for purchase. Another case of interest arises in the resupply of an unattended stocking point, such as a coin-operated vending machine; the stocking policies studied here could be applied by a deliveryman without advance knowledge of the existing stock level. A closely allied situation arises when communication is relatively costly. The Army, for example, is investigating the possibility of developing systems for automatic resupply of combat units, with reduction of communication requirements as one objective.

In section 5, the models of the three previous sections are extended to the case where the distribution of available delivery quantity depends
on the number of periods since the last delivery. This model prepares the way for analysis of a class of multiple stocking point policies, but is also of interest in its own right.

The model in section 6 deals with the problem of controlling stock levels at n stocking points. It is assumed that a random delivery quantity is available each period, and is offered to the stocking points in the fixed priority order 1, 2, ..., n. Each stocking point is assumed to be following one of the policies I, II, or III, making its stocking decisions independently of the stock level at all other stocking points. However, the stocking points need not all follow the same policy. An extension of this model is presented in section 7. The analysis of this process requires study of the joint probability distribution of stock levels at the n stocking points.

In section 8, a single stocking point model is considered. In this model, a random quantity of resupply is assumed to be available for regular delivery each period. Two versions of this model are studied. In the first, the regular delivery is accepted in full if the stock level is below a fixed level S, and otherwise rejected. Then if the stock level after the regular delivery is below a fixed emergency level s, where s < S, an additional order is placed to bring the stock level up to S. In the second version, the regular delivery is again accepted if the stock level is below S, and otherwise rejected. Then without taking into account the amount of the regular delivery, an additional order is placed to bring the stock level up to S if the stock level is below s. This second version is formulated with the possibility in mind of a delay of up to one period in receiving deliveries.
2. POLICY I--ACCEPT DELIVERY WHEN AVAILABLE, LIMITED STORAGE CAPACITY

This model deals with the problem of controlling the inventory level for a single item at a single stocking point. It is assumed that a random number of periods elapse between opportunities for resupply, and that the quantities available for delivery are also random variables. Furthermore, the distributions of demand, quantity available for delivery, and time between deliveries are assumed mutually independent. These conditions hold also for the models discussed in sections 3 and 4.

In this section, it is assumed that the stocking policy followed is to accept delivery whenever available, up to the storage capacity $S$ of the stocking point. This storage capacity will be considered as an initial decision parameter, but once determined will be regarded as fixed.

An example may be helpful in visualizing possible applications of this model. Suppose that a gasoline tank truck is making routine deliveries to an army unit in a combat area, following a plan of automatic resupply of the unit's normal requirements. The combat unit may be assumed to have a fixed storage capacity for gasoline, and it would seem reasonable for it to follow a policy of accepting any available resupply, up to the limit imposed by this capacity. The times between deliveries would be subject to delays from many causes, and might be considered as random variables. Furthermore, the quantity of gasoline in the truck when it reaches the unit may vary
for a number of reasons, and might be considered a random variable.

The related model in which the stock level is always
brought up to \( S \) when a delivery takes place, in other words the case
in which the delivery capacity is infinite, has been discussed by Karlin
and Fabens in [5]. Their treatment has the added generality of
allowing the demand distribution to change from period to period.

Below, the model of this section is investigated by first
considering the sequence of stock levels prior to deliveries. This
sequence can be analyzed as an embedded Markov chain contained in the
sequence of stock levels at the close of successive periods. A stability
condition necessary for the existence of a stationary distribution
of stock level prior to delivery is formulated, and functional equations
satisfied by this distribution are derived.

Next, the limiting distribution of stock level at the close of
an arbitrary period is expressed as a transformation of the above
stationary distribution. A specific cost structure is then assumed, and
the limiting one-period expected costs formulated. Specific examples
of the computation of the stationary distribution of stock level prior
to deliveries, the limiting distributions of stock level at the close of
an arbitrary period, and the limiting one-period expected costs are given.
In these examples are illustrated the selection of an optimum policy by
the determination of the value of \( S \) which minimizes expected costs.

At the end of this section, an expression is derived for the
stationary distribution of delivery size.
Stationary Distribution of Stock Level Prior to Delivery

Here, functional equations for the stationary probability distribution of stock level just prior to delivery are derived for the model described above. It is assumed that the stocking point has a storage capacity \( S \). Deliveries take place at the beginning of a period, and the number of periods between deliveries is a random variable \( D \), with

\[
\text{Pr}(D = i) = d_i \quad \text{for} \quad i = 1, 2, \ldots,
\]

and

\[
\sum_{i=1}^{\infty} d_i = 1.
\]

When a delivery takes place a random quantity \( R \) is available for delivery, where \( R \) has the distribution function \( H(r) \). The stocking policy followed is to always accept delivery, the quantity delivered being the minimum of \( R \) and the quantity necessary to satisfy backlogged demand and bring the stock level back up to \( S \). The demand \( \xi_i \) in the \( i^{\text{th}} \) period is assumed to be a positive-valued random variable with known continuous density function \( \varphi(\xi) \); furthermore, the demands \( \xi_1, \xi_2, \ldots \) are independent and identically distributed.

The random variable \( X_i \) will represent the stock level at the close of the \( i^{\text{th}} \) period, in other words at time \( i \). The stock level is measured before adding the delivery, if any, at time \( i \). Then if \( \{\eta_j\} \) is the sequence of random variables denoting delivery times, with \( \eta_j \) the time of the \( j^{\text{th}} \) delivery (where \( \eta_1 = 0 \) with
probability one), the sequence of stock levels \( \{X_{\eta_j}\} \) prior to
delivery form an embedded Markov chain with one-step transitions
following the relations

\[
X_{\eta_j+1} = \begin{cases} 
S - \sum_{i=\eta_{j+1}}^{\eta_j} \xi_i & \text{if } S \geq X_{\eta_j} \geq S - R_{\eta_j} \\
X_{\eta_j} + R_{\eta_j} - \sum_{i=\eta_{j+1}}^{\eta_j} \xi_i & \text{if } X_{\eta_j} < S - R_{\eta_j}
\end{cases}
\]

The distribution function \( F_{\eta_{j+1}}(x) \) for \( X_{\eta_{j+1}} \) may be expressed as

\[
F_{\eta_{j+1}}(x) = \Pr(X_{\eta_{j+1}} < x) = \begin{cases} 
\int_0^x \int_0^S \int_{S-r} \psi(\xi) \, d\xi \, dF_{\eta_j}(t) \, dH(r) \\
\quad + \int_0^x \int_{S-r}^S \int_{S-r-t-x} \psi(\xi) \, d\xi \, dF_{\eta_j}(t) \, dH(r) \\
\quad + \int_0^x \int_{S-r}^x \int_{S-r-t-x} \psi(\xi) \, d\xi \, dF_{\eta_j}(t) \, dH(r) & \text{for } x \leq S \\
1 & \text{for } x > S
\end{cases}
\]

where

\[
\psi(\xi) = \sum_{i=1}^{\infty} d_i \varphi^{(i)}(\xi)
\]
and \( \phi^{(1)}(\xi) \) is the density function for the distribution of the
sum of \( i \) identically and independently distributed random variables,
each with density function \( \phi(\xi) \). Throughout this report, distribution
functions will be defined as left continuous. The right side of
equations (3) is obtained by noting first that if \( S - R_{\eta_j} \leq X_{\eta_j} < S \),
then the demand up to time \( \eta_{j+1} \) must be \( \geq S - x \) for \( X_{\eta_{j+1}} \) to
be \( < x \). Second, if \( x - R_{\eta_j} \leq X_{\eta_j} < S - R_{\eta_j} \), then the demand up to
time \( \eta_{j+1} \) must be \( \geq R_{\eta_j} + X_{\eta_j} - x \) for \( X_{\eta_{j+1}} \) to be \( < x \). And
finally, if \( X_{\eta_j} < x - R_{\eta_j} \), then \( X_{\eta_{j+1}} < x \) regardless of demand.

In equations (3) the lower limits are assumed to be included in each
integration, but not the upper limits; this convention will be followed
throughout this report. However, it should be noted that the fact
that the demand distribution has a continuous density function ensures
in the present case that \( F_{\eta_{j+1}}(x) \) is continuous for all \( x \), and in
particular at \( S \). Equations (3) can also be written in the form

\[
\int_{\xi}^{\infty} \int_{R_{\eta_j}}^{S-x} \int_{-\infty}^{x+R_{\eta_j} - R} \psi(t) \, dF_{\eta_j}(t) \, d\xi \, dH(r) + \int_{S-x}^{\infty} \psi(t) \, dt
\]

(5) \( F_{\eta_{j+1}}(x) = \left\{ \begin{array}{ll}
1 & \text{for } x \leq S \\
\int_{\xi}^{\infty} \int_{R_{\eta_j}}^{S-x} \int_{-\infty}^{x+R_{\eta_j} - R} \psi(t) \, dF_{\eta_j}(t) \, d\xi \, dH(r) + \int_{S-x}^{\infty} \psi(t) \, dt & \text{for } x > S
\end{array} \right. \)

by interchanging the order of integration and re-grouping terms.

Another way of expressing equations (3) is in the abstract
operator form
\[ (6) \quad F_{\eta_{j+1}} = T^j_{\eta_j} \]

or, in terms of the initial distribution \( F_{\eta_1} \),

\[ F_{\eta_{j+1}} = T^j_{F_{\eta_1}} \]

where \( T^j \) represents the operation \( T \) repeated \( j \) times. A necessary condition for the existence of a stationary distribution \( F \) of stock level prior to delivery is that the expected demand per cycle be less than the delivery capacity, i.e. that

\[ (7) \quad E(D) \int_0^\infty \xi \phi(\xi) \, d\xi < E_H(r). \]

It is assumed here that, with possible additional restriction on \( H(r) \), this condition is also sufficient to ensure that \( T^j_{F_{\eta_1}} \) converges in the sense of distributions to a unique stationary probability distribution \( F \), independent of the initial distribution \( F_{\eta_1} \). This stationary distribution is then a fixed point of the transformation \( T \), i.e. the unique probability distribution which satisfies the equation

\[ (8) \quad F = T F \]

or

\[ \left\{ \begin{array}{l}
\int_0^\infty \int_{-\infty}^{x+r} \psi(\xi) \, dF(t) \, d\xi \, dH(r) + \int_0^\infty \psi(\xi) \, d\xi \\
\end{array} \right. \]

\[ (9) \quad F(x) = \left\{ \begin{array}{ll}
1 & \text{for } x \leq S \\
 & \\
 & \text{for } x > S.
\end{array} \right. \]
Since \( \phi(\xi) \) is assumed continuous, \( \psi(\xi) \) is also continuous, and the stationary distribution has a continuous density function \( f(x) \) for \( x \leq S \), given by the solution to

\[
(10) \quad f(x) = \int_0^\infty \int_{S-r}^S \psi(S-x,t) f(t) \, dt \, d\mathcal{H}(r)
\]

\[
+ \int_0^\infty \int_{x-r}^{S-r} \psi(r+t-x,t) f(t) \, dt \, d\mathcal{H}(r) \quad \text{for} \quad x \leq S;
\]

this result is obtained by differentiation of equation (9) with respect to \( x \).

**Limiting Distribution of Stock Level at End of Arbitrary Period**

Above, the analysis was directed toward finding the stationary probability distribution of stock level at the ends of periods just prior to those in which deliveries take place. Here, relationships for the stationary distribution of stock level at the end of an arbitrary period are derived.

It should first be noted that the sequence \( \{X_\eta\} \) of random variables, where \( X_\eta \) represents the stock level at the close of the \( \eta \)th period, is not a Markov chain. In general, the distribution of \( X_{\eta+1} \) is conditioned by the number of periods since the last delivery, as well as by the value of \( X_\eta \). An exception to this is the special
case in which the probability of a delivery in a given period is independent of the number of periods since the last delivery, as for example if the time between deliveries has a geometric distribution.

Assuming that a stationary distribution \( F(x) \) of the stock level prior to delivery exists and has been determined, it is possible to express the limiting distribution \( G(x) \) of the stock level \( X \) at the end of an arbitrary period in terms of \( F(x) \), the demand distribution \( \phi(t) \), and the limit distribution of the number of periods \( \Delta \) since the last delivery took place. Thus

\[
G(x) = \Pr[X < x] = \begin{cases} 
\int_0^\infty \int_0^{S-x} \int_{-\infty}^{x+t-r} \pi(\xi) \, dF(t) \, d\xi \, dH(r) + \int_{S-x}^\infty \pi(\xi) \, d\xi \\
1 & \text{for } x \leq S \\
& \text{for } x > S,
\end{cases}
\]

where

\[
\pi(\xi) = \sum_{k=1}^{\infty} \delta_k \phi^{(k)}(\xi) \quad \text{for } \xi \geq 0,
\]

and

\[
\delta_k = \Pr[\Delta = k] \quad \text{for } k = 1, 2, \ldots,
\]
This relation is derived by applying a line of reasoning similar to that used to derive equations (3). The close similarity of equations (9) and (11) should be noted.

It is a well-known result in renewal theory [4] that the limiting distribution of the number of periods since the last delivery is given in terms of the distribution of times between deliveries by

$$
\delta_k = \frac{1}{E(D)} \sum_{i=k}^{\infty} d_i \quad \text{for } k = 1,2, \ldots
$$

A simple heuristic proof of this result is as follows:

$$
\delta_1 = \lim_{n \to \infty} \Pr\{\text{delivery at } n-1\} = \frac{1}{E(D)}
$$

$$
\delta_2 = \lim_{n \to \infty} \Pr\{\text{delivery at } n-2, \text{ no delivery at } n-1\}
= \lim_{n \to \infty} \Pr\{\text{no delivery at } n-1 | \text{delivery at } n-2\} \Pr\{\text{delivery at } n-2\}
= (1 - d_1) \frac{1}{E(D)} = \frac{1}{E(D)} \sum_{i=2}^{\infty} d_i
$$

$$
\vdots
$$

$$
\delta_k = \lim_{n \to \infty} \Pr\{\text{delivery at } n-k, \text{ no delivery at } n-k+1, \ldots \ n-1\}
= \lim_{n \to \infty} \Pr\{\text{delivery at } n-k+1, \ldots , n-1 | \text{delivery at } n-k\} \Pr\{\text{delivery at } n-k\}
= (1 - d_1 - \cdots - d_{k-1}) \frac{1}{E(D)} = \frac{1}{E(D)} \sum_{i=k}^{\infty} d_i
$$
Since $\phi(t)$ is assumed continuous, $\pi(t)$ is also continuous, and the limiting distribution $G(x)$ has a continuous density function given by

$$
(15) \quad g(x) = \int_0^\infty \int_{S-r}^S \pi(s-x) f(t) \, dt \, dh(r)
+ \int_0^\infty \int_{x-r}^{S-r} \pi(r+t-x) f(t) \, dt \, dh(r) \quad \text{for} \quad x \leq S.
$$

The similarity of this result with equation (10) should be noted.

After obtaining the limiting probability distribution $G(x)$ for stock level at the end of an arbitrary period, it can be utilized to compute limiting one-period expected costs; the optimum size $S$ of storage capacity can then be determined by finding the value of $S$ which minimizes these costs.

In formulating a cost structure, it is first noted that if the purchase cost of additional supply is assumed to be linear, then expected purchase costs can be omitted from the analysis since they will be independent of $S$. Then if the holding cost $c_1$ and the shortage cost $c_2$ are assumed linear and are assessed at the end of each period, the total limiting one-period expected costs are

$$
(16) \quad L_1(s) = c_1 \int_0^S x \, dG(x) - c_2 \int_{-\infty}^0 x \, dG(x) \quad \text{for} \quad S \geq 0.
$$
Or if an additional factor $\gamma \xi$ is added for amortization of investment in storage facilities,

\begin{equation}
L_2(S) = \gamma S + c_1 \int_0^S x \, dG(x) - c_2 \int_{-\infty}^0 x \, dG(x) \quad \text{for} \quad S \geq 0.
\end{equation}

In either of these formulations, if it is possible to also vary the mean delivery capacity $E_H(R)$ or the mean time between deliveries $E(D)$ for a given delivery time distribution, it is clear that costs will be minimized by making $E_H(R)$ as large as possible and $E(D)$ as small as possible; i.e., the ideal resupply system would replenish stock to the level $S$ each period.

Below, examples are given to illustrate the computation of the distributions derived above, and the corresponding limiting one-period expected costs.

**Example 1.** Suppose the distribution of times between deliveries is given by

\begin{equation}
\begin{aligned}
\delta_k &= pq^{k-1} \quad \text{for} \quad k = 1, 2, \ldots, \\
\end{aligned}
\end{equation}

where

\[ 0 < p = 1 - q < 1, \]

and the demand distribution is given by

\begin{equation}
\phi(\xi) = \lambda e^{-\lambda \xi} \quad \text{for} \quad \xi \geq 0.
\end{equation}
Then

\[
\phi(k)(\xi) = \frac{k^{k-1} e^{-\lambda \xi}}{(k-1)!} \quad \text{for} \quad \xi \geq 0, \quad k = 1, 2, \ldots,
\]

and

\[
\psi(\xi) = \sum_{k=1}^{\infty} pq \frac{k^{k-1} e^{-\lambda \xi}}{(k-1)!} = p \lambda e^{-p \lambda \xi}
\]

for \( \xi \geq 0 \).

Substituting (21) into equation (10), one obtains

\[
f(x) e^{-p \lambda x} = \int_{0}^{\infty} \int_{x-r}^{S} p \lambda e^{-p \lambda S} f(t) \, dt \, dH(r)
\]

\[
+ \int_{0}^{\infty} \int_{x-r}^{S-r} p \lambda e^{-p \lambda (r+t)} f(t) \, dt \, dH(r) \quad \text{for} \quad x \leq S,
\]

and differentiating with respect to \( x \), one obtains

\[
\{f'(x) - p \lambda f(x)\} e^{-p \lambda x} = -\int_{0}^{\infty} p \lambda e^{-p \lambda x} f(x-r) \, dH(r) \quad \text{for} \quad x < S,
\]

or

\[
f'(x) - p \lambda f(x) = -p \lambda E_H[f(x-R)] \quad \text{for} \quad x < S,
\]

where \( E_H(\cdot) \) represents the expected value taken with respect to the probability distribution \( H(r) \). Trying a solution of the form
(23) \[
    f(x) = \begin{cases} 
    K e^{-K(S-x)} & \text{for } x \leq S \\
    0 & \text{otherwise},
    \end{cases}
\]

where the requirement that the integral of \( f(x) \) over the region \(-\infty \leq x \leq S\) be finite imposes the condition \( K > 0 \), one finds after substituting in equation (22) that \( K \) must also satisfy the equation

(24) \[
    \frac{p\lambda - K}{pA} = E_H(e^{-KR}).
\]

If the stability condition of equation (7) is satisfied, i.e.

(25) \[
    \frac{1}{p\lambda} < E_H(R),
\]

then equation (24) has exactly one positive solution, which will be denoted by \( \alpha \). Furthermore,

(26) \[
    0 < \alpha < p\lambda.
\]

(This solution is illustrated in Figure 1.) These results follow from a careful examination of

(27) \[
    A(K) = E_H(e^{-KR}) - \frac{p\lambda - K}{p\lambda}.
\]
Figure 1.

Solution to Equation (24)
One first notes that

\[ A(0) = 0 \]

and

\[ A(K) > 0 \quad \text{for} \quad K \geq p\lambda. \]

Thus since

\[ A'(0) < 0 \]

if condition (25) is satisfied, and since \( A(K) \) is continuous for \( K \geq 0 \), \( A(K) \) must have at least one zero crossing from below in the region \( 0 < K < p\lambda \). However since

\[ A''(K) = E_H(K^2 e^{-KR}) > 0 \]

and is continuous for \( K \geq 0 \), \( A(K) \) is convex and can have at most one zero crossing from below for positive \( K \). This proves the desired result.\(^\dagger\) To illustrate the above results, if \( R \) has the density

\[^\dagger\] In [3], Karlin proves a more general lemma which states that if

\[ \frac{n}{p\lambda} < E_H(R), \]

then

\[ \left[ \frac{p\lambda - K}{p\lambda} \right]^n = E_H(e^{-KR}) \]

has exactly \( n \) roots, counting multiplicities, located interior to the right half plane \( \text{Re}(K) > 0 \). Furthermore, these roots occur in conjugate pairs, and there exists one positive real root \( \alpha_0 \) such that for any other root \( \alpha \) with \( \text{Re}(\alpha) > 0 \), \( \text{Re}(\alpha) > \alpha_0 \). These results applied to the present case, i.e., for \( n = 1 \), establish the existence of a single positive real solution to equation (23) if condition (24) is satisfied.
function

\[ h(r) = \frac{k^r r^{k-1} e^{-kr/r_0}}{r_0^k (k-1)!} \quad \text{for} \quad r > 0, \ k > 0, \ r_0 > \frac{1}{p\lambda}, \]

a gamma distribution with mean \( r_0 \), then \( \alpha \) is the unique positive solution to

\[ \frac{p\lambda - K}{p\lambda} = \left( 1 + \frac{Kr_0}{k} \right)^{-k}. \]

Or if \( R = r_0 > 1/p\lambda \) with probability one, then \( \alpha \) is the unique positive solution to

\[ \frac{p\lambda - K}{p\lambda} = e^{-Kr_0}. \]

However, the explicit distribution of \( R \) will not enter into the calculations which follow, since the results will all hold true regardless of the form of the distribution of \( R \).

The probability density function for the stationary distribution of stock level prior to delivery in this example is thus

\[
(28) \quad f(x) = \begin{cases} 
\alpha e^{-\alpha(S-x)} & \text{for} \quad x \leq S \\
0 & \text{otherwise},
\end{cases}
\]

and the distribution function is given by
(29) \[ F(x) = \begin{cases} 
  e^{-\alpha(S-x)} & \text{for } x \leq S \\
  1 & \text{for } x > S.
\end{cases} \]

As a simple consequence of this result, one can obtain the stationary probability distribution function \( F^*(y) \) for the stock level immediately after delivery; it is

(30) \[ F^*(y) = \begin{cases} 
  \int_0^\infty \int_{-\infty}^y e^{t} \ dF(t) \ dH(r) & \text{for } y < S \\
  1 & \text{for } y > S.
\end{cases} \]

\[ = \begin{cases} 
  \int_0^\infty e^{-(S-y+r)} \ dH(r) & \text{for } y < S \\
  1 & \text{for } y > S.
\end{cases} \]

\[ = \begin{cases} 
  (1 - \frac{\alpha}{p\lambda}) e^{-\alpha(S-y)} & \text{for } y < S \\
  1 & \text{for } y > S.
\end{cases} \]

It should be noted that this distribution has a jump of \( \alpha/p\lambda \) at \( y = S \); in other words, the stationary probability of the stock being at level \( S \) after delivery is \( \alpha/p\lambda \).

Next the problem of obtaining the limiting distribution of stock level at the end of an arbitrary period is considered. Since
in the present example the distribution of times between availabilities of deliveries (18) is geometric, the limiting distribution of time since the last delivery is also geometric with the same parameter. That is,

\[ \delta_k = \frac{1}{E(D)} \sum_{i=k}^{\infty} d_i = p \sum_{i=k}^{\infty} p^{i-1} = p q^{k-1} = d_k \quad \text{for} \quad k = 1, 2, \ldots . \]

Thus \( G(x) = F(x) \).

This result holds regardless of the demand distribution.

In other words, if the distribution of times between availabilities of deliveries is geometric, the limiting probability distribution of stock level at the close of an arbitrary period is the same as that for stock level at the close of a period just prior to a delivery. Thus since the demand distribution \( \phi(\xi) \) has a continuous density function, the limiting distribution of stock level at the end of an arbitrary period has a continuous density function which satisfies equation (10), i.e.,

\[
(31) \quad g(x) = \int_{0}^{\infty} \int_{S-r}^{S} \psi(S-x) g(t) \, dt \, dH(r) \\
\quad + \int_{0}^{\infty} \int_{x-r}^{S-r} \psi(r+t-x) g(t) \, dt \, dH(r) \quad \text{for} \quad x \leq S,
\]

where

\[
(32) \quad \psi(\xi) = \sum_{k=1}^{\infty} p q^{k-1} \phi^{(k)}(\xi) \quad \text{for} \quad \xi \geq S.
\]
However, since the sequence \( \{X_n\} \) of random variables form a Markov chain in this case, the limiting stationary distribution of stock level at the end of a period can be computed directly from

\[
(33) \quad g(x) = p \int_0^\infty \int_{S-r}^S \phi(s-x) \, g(t) \, dt \, dH(r)
\]

\[
+ p \int_{x-r}^{S-r} \phi(r+t-x) \, g(t) \, dt + q \int_x^S \phi(t-x) \, g(t) \, dt
\]

for \( x \leq S \).

In deriving this equation, use is made of the fact that the probability of a delivery occurring at the beginning of the following period is \( p \), and the probability of a delivery not occurring is \( q = 1 - p \). The equivalence of equations (31) and (33) can be shown by observing that they have identical Fourier transforms.

Since in the present example the demand has an exponential distribution, the density function for the stationary distribution of stock level is

\[
(34) \quad g(x) = \begin{cases} 
\alpha e^{-\alpha(s-x)} & \text{for } x \leq S \\
0 & \text{otherwise},
\end{cases}
\]

where \( \alpha \) is defined as in equation (28) above. Then evaluating equation (16), the stationary one-period expected costs are
(35) \[ L_1(S) = \frac{c_1}{\alpha} (\alpha S - 1) + \frac{c_1 + c_2}{\alpha} e^{-\alpha S} \quad \text{for } S \geq 0. \]

Since
\[ L_1''(S) = \alpha (c_1 + c_2) e^{-\alpha S} > 0 \quad \text{for } S \geq 0 \]
and is continuous, and
\[ L_1'(0) = -c_2 < 0, \]
the function \( L_1(S) \) takes on its minimum value for
\[ (36) \quad S = \frac{1}{\alpha} \log \frac{c_1 + c_2}{c_1}. \]

If a factor \( \gamma S \) for amortization of investment in storage capacity is included in the costs, then the expected cost becomes

(37) \[ L_2(S) = \gamma S + \frac{c_1}{\alpha} (\alpha S - 1) + \frac{c_1 + c_2}{\alpha} e^{-\alpha S} \quad \text{for } S \geq 0. \]

Since
\[ L_2''(S) = \alpha (c_1 + c_2) e^{-\alpha S} > 0 \quad \text{for } S \geq 0 \]
and is continuous, \( L_2(S) \) has a single minimum for \( S \geq 0 \). Observing that
\[ L_2'(0) = \gamma - c_2, \]
on one finds that \( L_2(S) \) takes on its minimum value for
\begin{align}
S &= \begin{cases} 
\frac{1}{\alpha} \log \frac{c_1 + c_2}{c_1 + \gamma} & \text{if } \gamma < c_2 \\
0 & \text{if } \gamma \geq c_2 
\end{cases} 
\tag{38}
\end{align}

**Example 2.** Suppose the distribution of times between deliveries is a member of the family of negative binomial or Pascal distributions. A representation for the general form of this distribution is

\begin{align}
\bar{d}(k; n, p) &= \binom{k+n-1}{n} p^{n+1} q^{k-1} & \text{for } k = 1, 2, \ldots ,
\tag{39}
\end{align}

where

\[ n = 0, 1, \ldots \]

and

\[ 0 < p = 1 - q < 1. \]

It can readily be shown that

\[ \sum_{k=1}^\infty \bar{d}(k; n, p) = 1. \]

If \( n = 0 \), this distribution becomes the geometric distribution of example 1.

Again assuming an exponential distribution

\[ \phi(\xi) = \lambda e^{-\lambda \xi} \]

for \( \xi \geq 0 \)

for demand, one obtains
(40) \[ \psi(\xi) = \sum_{k=1}^{\infty} \frac{n}{k!} \binom{n}{k} \frac{(p\lambda)^{k+1}}{k!} \frac{e^{-p\lambda \xi}}{k!} \]

\[ \psi(\xi) = \sum_{k=0}^{n} \binom{n}{k} \frac{n-k}{k!} \frac{(p\lambda)^{k+1}}{k!} \frac{e^{-p\lambda \xi}}{k!} \]

\[ = \sum_{k=0}^{n} \binom{n}{k} \frac{n-k}{k!} \frac{(p\lambda)^{k+1}}{k!} \frac{e^{-p\lambda \xi}}{k!} \]

This result follows from the identity

(41) \[ \sum_{k=1}^{\infty} \binom{k+n-1}{n} \frac{x^{k-1}}{(k-1)!} = e^x \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{k!} \]

which can be shown by integrating the left side \(n\) times, summing, then differentiating \(n\) times with respect to \(x\).

One can then substitute (25) into equation (10) and obtain

(42) \[ f(x) \frac{e^{-p\lambda x}}{s} \int_{0}^{\infty} \int_{s-r}^{s} \sum_{k=0}^{n} \binom{n}{k} \frac{n-k}{k!} \frac{(p\lambda)^{k+1}(S-x)^{k}e^{-p\lambda S}}{k!} \]

\[ \times f(x) \, dt \, dH(r) \]

\[ + \int_{0}^{\infty} \int_{s-r}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{n-k}{k!} \frac{(p\lambda)^{k+1}(r+t-x)^{k}e^{-p\lambda (r+t)}}{k!} \]

\[ \times f(t) \, dt \, dH(r) \quad \text{for} \quad x \leq S. \]
Then differentiating with respect to \( x \), and denoting the differentiation operator by \( D \), one obtains the result

\[
(D - p\lambda)^{n+1} f(x) = -p\lambda \int_{0}^{\infty} (pD - p\lambda)^n f(x-r) \, d\mathcal{H}(r)
\]

for \( x < S \).

Trying a solution of the form

\[
f(x) = \begin{cases} 
K e^{-K(S-x)} & \text{for } x < S \\
0 & \text{otherwise},
\end{cases}
\]

where the requirement that the integral of \( f(x) \) over the region \(-\infty \leq x \leq S\) be finite imposes the condition \( K > 0 \), one finds after substitution in equation (43) that \( K \) must also satisfy the equation

\[
\frac{-(K-p\lambda)^{n+1}}{p\lambda^{n+1}} = E_{\mathcal{H}}(e^{-Kx}).
\]

Thus, in order to construct the general solution to (43) one must find the locations of the zeros of the function

\[
A(K) = \left(\frac{K}{\lambda} - 1\right)^n \int_{0}^{\infty} e^{-Kr} \, d\mathcal{H}(r) + \left(\frac{K}{p\lambda - 1}\right)^{n+1},
\]

where \( K \) is considered as a complex variable. The following lemma is needed.
**Lemma:** If the stability condition (7) is satisfied, i.e.

\[ \frac{1 + n \alpha}{p \lambda} < E_H(R), \]

then the equation \( A(K) = 0 \) has exactly \( n+1 \) roots, counting multiplicities, located interior to the right half plane \( (\text{Re}(K) > 0) \).

**Proof:** (Patterned on a proof by Karlin in [3]). Let

\[ B(K) = \left( \frac{K}{\lambda} - 1 \right)^n \int_0^\infty e^{-Kr} dH(r) \]

and

\[ C(K) = \left( \frac{K}{p \lambda} - 1 \right)^{n+1}. \]

One notes that \( C(K) \) has a zero of multiplicity \( n+1 \) at \((p \lambda, 0)\).

If there exists an open simply connected region \( R \) in the right half plane such that

\begin{enumerate}
  \item \((p \lambda, 0) \in R,\)
  \item \(B(K)\) and \(C(K)\) are analytic in \( R \),
\end{enumerate}

and

\begin{enumerate}
  \item \(|B(K)| < |C(K)|\) on the boundary of \( R \),
\end{enumerate}

then by a theorem due to Rouché (see [6]), \( A(K) \) has \( n+1 \) zeros in \( R \), counting multiplicities. Letting \( K = re^{i\theta} \),
\[ |B(r, \theta)| \leq \frac{r e^{i \theta}}{\lambda} - 1 \left| \int_0^\infty e^{-r \rho e^{i \theta}} \, d\rho \right|^n \]

\[ \leq \left\{ \left( \frac{r \cos \theta}{\lambda} - 1 \right)^2 + \left( \frac{r \sin \theta}{\lambda} \right)^2 \right\}^{n/2} \int_0^\infty e^{-r \rho \cos \theta} \, d\rho \]

\[ = |B^*(r, \theta)| \]

\[ \leq \left\{ \left( \frac{r \cos \theta}{\lambda} - 1 \right)^2 + \left( \frac{r \sin \theta}{\lambda} \right)^2 \right\}^{n/2} \]

for \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\),

and

\[ |C(r, \theta)| = \left\{ \left( \frac{r \cos \theta}{\rho \lambda} - 1 \right)^2 + \left( \frac{r \sin \theta}{\rho \lambda} \right)^2 \right\}^{(n+1)/2} \]

Then along the imaginary axis, for \( r \neq 0 \),

\[ |B(r, \pm \frac{\pi}{2})| \leq \left\{ 1 + \left( \frac{r}{\lambda} \right)^2 \right\}^{n/2} \]

\[ < \left\{ 1 + \left( \frac{r}{\rho \lambda} \right)^2 \right\}^{(n+1)/2} = |C(r, \pm \frac{\pi}{2})| \]

since

\[ \frac{r}{\lambda} < \frac{r}{\rho \lambda} \]

At the origin, \( |B^*(0)| = |C(0)| \), where \( B^*(K) \) is defined by equation (47). Evaluating the derivatives with respect to \( r \) of \( |B^*(r, \theta)| \)
and \( |C(r, \theta)| \) at \( r = 0 \), one obtains

\[ \frac{\partial |B^*(r, \theta)|}{\partial r} \bigg|_{r=0} = - \left\{ \frac{n}{\lambda} + E_H(R) \right\} \cos \theta \]

and
From condition (46), it follows that $B^*(r, \theta)$ decreases in magnitude faster than $C(r, \theta)$ in any fixed direction $\theta$ from the origin, where $-\pi/2 < \theta < \pi/2$. Then since

$$|B^*(r, \pm \frac{\pi}{2})| < |C(r, \pm \frac{\pi}{2})|$$

for $r \neq 0$,

and since $B^*(K)$ and $C(K)$ are analytic in the right half plane, there exists an $\varepsilon_o > 0$ such that

$$|B(\varepsilon, \theta)| \leq |B^*(\varepsilon, \theta)| < |C(\varepsilon, \theta)|$$

for any $0 < \varepsilon \leq \varepsilon_o$,

and for all $-\pi/2 \leq \theta \leq \pi/2$. Letting $r = 2\lambda$, it is seen that

$$|B(2\lambda, \theta)| \leq (5 - 4 \cos \theta)^{n/2}$$

$$< \left(\frac{\lambda}{p^2} + 1 - \frac{\lambda}{p} \cos \theta\right)^{n+1/2} = |C(2\lambda, \theta)|$$

for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Thus if $R$ is the region in the right half plane such that

$\varepsilon_o < r < 2\lambda$ and $-\pi/2 < \theta < \pi/2$, the desired result is established.

Now if equation (45) has $k$ distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_k$, with multiplicities $n_1, n_2, \ldots, n_k$ respectively, where

$$\sum_{i=1}^{k} n_i = n + 1,$$
then the general solution to equation (43) can be represented as

\[
\begin{align*}
\sum_{i=1}^{k} \sum_{j=1}^{n_i} c_{ij} x^{i-1} e^{\alpha_i x} & \quad \text{for } x \leq S \\
0 & \quad \text{otherwise,}
\end{align*}
\]

where the \( c_{ij} \) are constants. In the usual case, the roots of (45) will be distinct, i.e., of multiplicity one.

Considering the case \( n = 1 \), equation (39) becomes

\[
d_k = q^2 k p^{k-1} \quad \text{for } k = 1, 2, \ldots
\]

where

\[0 < p = 1 - q < 1.\]

If the stability condition of equation (7) is satisfied, i.e.

\[
\frac{1}{p} < \frac{\alpha}{\beta} < E_{\bar{R}}(R),
\]

then the general solution to equation (43) can be written as

\[
\begin{align*}
f(x) = \begin{cases} 
  c_1 e^{-\alpha(S-x)} + c_2 e^{-\beta(S-x)} & \text{for } x \leq S \\
  0 & \text{otherwise,}
\end{cases}
\end{align*}
\]

where \( \alpha \) and \( \beta \) are the two solutions in the interior of the right half plane to the equation.
\[
\frac{- (K - p \lambda)^2}{p^2 \lambda (K - \lambda)} = E_H(e^{-Kt}) .
\]

(This solution is illustrated in figure 2.) It can be readily shown that \( \alpha \) and \( \beta \) are both real, and that

\[
0 < \alpha < p \lambda < \beta < \lambda .
\]

The condition that

\[
\int_{-\infty}^{S} f(x) \, dx = 1
\]

gives

\[
C_1 + C_2 = 1 .
\]

Then substituting the general solution (55) into equation (10) and setting \( x = S \), one obtains

\[
C_1 \alpha + C_2 \beta = \psi(0) \int_{0}^{\infty} \int_{S-r}^{S} \left[ C_1 \alpha e^{-\alpha(S-t)} + C_2 \beta e^{-\beta(S-t)} \right] \, dt \, dH(r)
\]

\[
= p^2 \lambda \left[ 1 - C_1 E_H(e^{-\alpha R}) - C_2 E_H(e^{-\beta R}) \right]
\]

\[
= p^2 \lambda + \frac{C_1 (\alpha - p \lambda)^2}{(\alpha - \lambda)} + \frac{C_2 (\beta - p \lambda)^2}{(\beta - \lambda)} .
\]

Solving for \( C_1 \) and \( C_2 \) gives

\[
C_1 = \frac{\beta(\lambda - \alpha)}{\lambda(\beta - \alpha)}
\]
Figure 2

Solutions to Equation (56)
and

$$c_2 = - \frac{\alpha(\lambda - \beta)}{\lambda(\beta - \alpha)}.$$

Thus the density function for the stationary distribution of stock level prior to delivery is

$$f(x) = \begin{cases} \frac{\alpha \beta}{\lambda(\beta - \alpha)} \left[ (\lambda - \alpha) e^{-\alpha(S-x)} - (\lambda - \beta) e^{-\beta(S-x)} \right] & \text{for } x \leq S \\ 0 & \text{otherwise.} \end{cases}$$

The limiting distribution of time since the last delivery is

$$\delta_k = \frac{p}{l+q} \sum_{i=k}^{\infty} q^2 \lambda \lambda^{i-1} = \frac{p}{l+q} \left[ kpq^{k-1} + q^k \right] \text{ for } k = 1, 2, \ldots,$$

and thus

$$\pi(x) = \sum_{k=1}^{\infty} \delta_k \theta(k)(x) = \sum_{k=1}^{\infty} \frac{p}{l+q} \left[ kpq^{k-1} + q^k \right] \frac{\lambda^k \lambda^{k-1} e^{-\lambda x}}{(k-1)!}$$

$$= \frac{p\lambda}{l+q} e^{-p\lambda x} \left[ pq\lambda x + 1 \right] \text{ for } x \leq S.$$
Using this result to compute the limiting one-period expected costs according to equation (16),

\[ L_1(S) = \frac{\alpha \beta}{p\lambda(\beta-\alpha)(1+q)} \left[ \frac{1}{\alpha} \left[ p\lambda(1+q)-\alpha \right] \left[ c_1(\alpha S) + (c_1 + c_2) e^{-\alpha S} \right] - \frac{1}{\beta} \left[ p\lambda(1+q) - \beta \right] \left[ c_1(\beta S) + (c_1 + c_2) e^{-\beta S} \right] \right] \]

for \( S \geq 0 \).

Differentiating this expression with respect to \( S \), one obtains

\[ L_1'(S) = \frac{\alpha \beta}{p\lambda(\beta-\alpha)(1+q)} \left[ \frac{1}{\alpha} \left[ p\lambda(1+q)-\alpha \right] \left[ -c_1 - (c_1 + c_2) e^{-\alpha S} \right] \right. \]

\[ - \frac{1}{\beta} \left[ p\lambda(1+q) - \beta \right] \left[ -c_1 - (c_1 + c_2) e^{-\beta S} \right] \]

and

\[ L_1''(S) = \frac{\alpha \beta(c_1 + c_2)}{p\lambda(\beta-\alpha)(1+q)} \left[ \left( p\lambda(1+q) - \alpha \right) e^{-\alpha S} - \left( p\lambda(1+q) - \beta \right) e^{-\beta S} \right] \]

\[ = \frac{\alpha \beta(c_1 + c_2)}{p\lambda(\beta-\alpha)(1+q)} \left[ p\lambda e^{-\alpha S} - e^{-\beta S} \right] + (p\lambda - \alpha) e^{-\alpha S} + (\beta - p\lambda) e^{-\beta S} > 0 \]

for \( S \geq 0 \),

since

\[ 0 < \alpha < p\lambda < \beta < \lambda \]

Thus since

\[ L_1'(0) = -c_2 \]

the value of \( S \) which minimizes \( L_1(S) \) is that value which satisfies the equation
\[ L_1'(S) = 0, \]
in other words the solution to
\[(62) \quad \beta e^{-\alpha S} [p \lambda (1+q) - \alpha] - \alpha e^{-\beta S} [p \lambda (1+q) - \beta] = \frac{c_1}{c_1 + c_2} (\beta - \alpha) p \lambda (1+q). \]

If a factor \(\gamma S\) for amortization of investment in storage capacity is included, then the limiting one-period expected costs become
\[(63) \quad L_2(S) = \gamma S + L_1(S), \]
where \(L_1(S)\) is given by equation (61) above. Since
\[ L_2''(S) > 0 \quad \text{for} \quad S > 0 \]
and
\[ L_2'(0) = \gamma - c_2, \]
the non-negative value of \(S\) which minimizes \(L_2(S)\) is given by the solution to
\[ L_2'(S) = 0, \]
i.e., the solution to
\[ (64) \quad \beta e^{-\alpha S} [p \lambda (1+q) - \alpha] - \alpha e^{-\beta S} [p \lambda (1+q) - \beta] \\
= \frac{c_1}{c_1 + c_2} (\beta - \alpha) p \lambda (1+q) \quad \text{if} \quad \gamma < c_2. \]
or by

\[ S = 0 \quad \text{if} \quad \gamma \geq c_2. \]

Example 3. Suppose that the delivery time distribution is given by

\[ d_k = pq^{k-1} \quad \text{for} \quad k = 1, 2, \ldots, \]

where

\[ 0 < p = 1 - q < 1, \]

and the demand distribution is given by

(65) \[ \phi(\xi) = \lambda^2 \xi e^{-\lambda \xi} \quad \text{for} \quad \xi \geq 0. \]

Then

(66) \[ \psi(\xi) = \sum_{k=1}^{\infty} pq^{k-1} \frac{\lambda^{2k} \xi^{2k-1} e^{-\lambda \xi}}{(2k-1)!} \]

\[ = \frac{p\lambda}{\sqrt{q}} e^{-\lambda \xi} \sinh(\sqrt{q} \lambda \xi) \quad \text{for} \quad \xi \geq 0. \]

Substituting (66) into equation (10), one obtains

(67) \[ f(x) e^{-\lambda x} = \int_0^\infty \int_{S-r}^S \frac{p\lambda}{\sqrt{q}} e^{-\lambda S \sinh[\sqrt{q} \lambda (S-x)]} f(t) \, dt \, dH(r) \]

\[ + \int_0^\infty \int_{x-r}^{S-r} \frac{p\lambda}{\sqrt{q}} e^{-\lambda (r+t)} \sinh[\sqrt{q} \lambda (r+t-x)] f(t) \, dt \, dH(r) \]

for \( x \leq S. \)
Then differentiating twice with respect to \( x \) gives

\[
(68) \quad f''(x) - 2\lambda f'(x) + \lambda^2 f(x) = q\lambda^2 f(x) + \lambda^2 \int_0^\infty f(x-r) \, dH(r)
\]

for \( x < S \).

Trying a solution of the form

\[
(69) \quad f(x) = \begin{cases} 
K e^{-K(S-x)} & \text{for } x \leq S \\
0 & \text{otherwise},
\end{cases}
\]

where the requirement that the integral over the real line of the solution be finite imposes the condition \( K > 0 \), one finds after substituting in equation (68) that \( K \) must also satisfy the equation

\[
(70) \quad \frac{K^2 + 2\lambda K + \mu^2}{\mu^2} = E_H(e^{-KR}).
\]

If the stability condition (7) is satisfied, i.e.

\[
(71) \quad \frac{2}{\mu^2} < E_H(R),
\]

then equation (70) has exactly two real positive solutions, which will be denoted by \( \alpha \) and \( \beta \). Furthermore,

\[
(72) \quad 0 < \alpha < \lambda(1 - \sqrt{q})
\]
and

\[(73) \quad \lambda(1 + \sqrt{q}) < \beta < 2\lambda.\]

(These solutions are illustrated in figure 3.) These results follow from an investigation of the roots of the equation

\[(74) \quad A(K) = 0,\]

where

\[(75) \quad A(K) = E_{\alpha}(e^{-KR}) - \frac{K^2 - 2\lambda K + \frac{p\lambda^2}{p\lambda}}{p\lambda}.\]

Considering \(K\) as a complex variable, and letting

\[(76) \quad B(K) = E_{\alpha}(e^{-KR}),\]

and

\[(77) \quad C(K) = -\frac{K^2 - 2\lambda K + \frac{p\lambda^2}{p\lambda}}{p\lambda},\]

one notes that \(C(K)\) has two zeros, located at \((\lambda(1 - \sqrt{q}), 0)\) and \((\lambda(1 + \sqrt{q}), 0)\). If it can be shown that there exists an open simply connected region \(R\) in the right half plane \((\text{Re}(K) > 0)\) such that \((\lambda(1 - \sqrt{q}), 0) \in R\) and \((\lambda(1 + \sqrt{q}), 0) \in R\), \(B(K)\) and \(C(K)\) are analytic in \(R\), and \(|B(K)| < |C(K)|\) on the boundary of \(R\), then by the theorem of Rouché used in example 2 above, equation \((75)\) has exactly two zeros in \(R\). One observes first that if \(K = r e^{i\theta}\),

\[(78) \quad |B(r, \theta)| \leq \int_{0}^{\infty} e^{-r \rho} \cos \theta \ d\rho \leq 1 \quad \text{for} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.\]
Figure 3.

Solutions to Equation (70)
Then forming
\[|C(r, \theta)| = \frac{1}{2\pi \lambda^2} \left( (r^2 \cos 2\theta - 2\lambda r \cos \theta + \lambda^2)^2 + (r^2 \sin 2\theta - 2\lambda r \sin \theta)^2 \right)^{1/2},\]
one finds that along the imaginary axis
\[|C(r, \pm \frac{\pi}{2})| > 1 \quad \text{for all} \quad r \neq 0.
\]
Furthermore, one observes that
\[
\frac{\partial}{\partial r} \left| \frac{|C(r, \theta)|}{r} \right| \bigg|_{r=0} = \frac{2}{\pi \lambda} \cos \theta
\]
and
\[
\frac{\partial}{\partial r} \int_0^\infty e^{-r \rho} \cos \theta \ d\rho \bigg|_{r=0} = -E_H(R) \cos \theta.
\]
Since condition (71) holds, and since \( B(K) \) and \( C(K) \) are analytic in the right half plane, there exists an \( \epsilon_0 > 0 \) such that
\[|B(\epsilon, \theta)| < |C(\epsilon, \theta)| \quad \text{for any} \quad 0 < \epsilon \leq \epsilon_0,
\]
and for all \(-\pi/2 \leq \theta \leq \pi/2\). Then clearly there exists an \( M \) such that
\[|C(r, \theta)| > 1 \quad \text{for} \quad r \geq M,
\]
and for all \(-\pi/2 \leq \theta \leq \pi/2\). Letting \( R \) be the region in the right half plane such that \( \epsilon_0 < r < M \) and \(-\pi/2 \leq \theta \leq \pi/2\), the conditions of Rouché's theorem are met, and the existence of exactly two roots
of equation (75) in the right half plane is proved. The inequalities (72) and (73) are established readily by examining the behavior of $A(k)$ along the positive real axis. Since

$$A(e_0, 0) < 0,$$

$$A(\lambda, 0) > 0,$$

and

$$A(2\lambda, 0) < 0,$$

there exist at least two real positive roots of equation (75); by the above analysis, these are the only roots of equation (75) in the right half plane.

The general solution of (68) can now be written as

$$f(x) = \begin{cases} 
  c_1 e^{-\alpha(S-x)} + c_2 e^{-\beta(S-x)} & \text{for } x \leq S \\
  0 & \text{otherwise}
\end{cases}$$

(79)

The condition that

$$\int_{-\infty}^{S} f(x) \, dx = 1$$

gives

$$c_1 + c_2 = 1.$$
and
\[ c_2 = \frac{-\alpha}{\beta - \alpha}, \]

and thus the density function for the stationary probability distribution of stock level prior to delivery is
\[
f(x) = \begin{cases} \frac{\alpha \beta}{\beta - \alpha} \{e^{-\alpha(S-x)} - e^{-\beta(S-x)}\} & \text{for } x \leq S \\ 0 & \text{otherwise}. \end{cases}
\]

Since the times between delivery availabilities have a geometric distribution in this example, \( G(x) = F(x) \), as was shown in example 1 above. Thus the density function for the stationary distribution of stock level is given by
\[
g(x) = \frac{\alpha \beta}{\beta - \alpha} \{e^{-\alpha(S-x)} - e^{-\beta(S-x)}\} \quad \text{for } x \leq S,
\]
where \( \alpha \) and \( \beta \) are defined as in equation (80). Then the limiting one-period expected costs are
\[
L_1'(S) = \frac{1}{\alpha \beta (\beta - \alpha)} \left( \beta^2 c_1 (\alpha S - 1) + \beta^2 (c_1 + c_2) e^{-\alpha S} - \alpha^2 c_1 (\beta S - 1) + \beta^2 c_1 (\beta S - 1) e^{-\beta S}\right) \quad \text{for } S \geq 0,
\]
Noting that
\[
L_1''(S) = \frac{\alpha \beta}{\beta - \alpha} (c_1 + c_2) (e^{-\alpha S} - e^{-\beta S}) > 0 \quad \text{for } S \geq 0
\]
since \( \beta > \alpha \), and that
\[
L'_1(0) = -c_2,
\]

one finds that \( L_1(S) \) takes on its minimum for that value of \( S \) which satisfies the equation
\[
L'_1(S) = 0,
\]
i.e. for the solution to
\[
(83) \quad \beta e^{-\alpha S} - \alpha e^{-\beta S} = (\beta - \alpha) \frac{c_1}{c_1 + c_2}.
\]

If a factor \( \gamma S \) for amortization of investment in storage capacity is included in the costs, then the expected cost becomes
\[
(84) \quad L_2(S) = \gamma S + \frac{1}{\alpha \beta (\beta - \alpha)} \left( \beta^2 c_1(\alpha S - 1) + \beta^2 (c_1 + c_2) e^{-\alpha S} \right.
\]
\[
- \beta^2 c_1(\beta S - 1) - \alpha^2 (c_1 + c_2) e^{-\beta S})
\]
\[\text{for } S > 0.\]

Noting that
\[
L''_2(S) = \frac{\alpha \beta}{\beta - \alpha} (c_1 + c_2) (e^{-\alpha S} - e^{-\beta S}) > 0 \quad \text{for } S > 0
\]
since \( \beta > \alpha \), and that
\[
L'_2(0) = \gamma - c_2,
\]
one finds that \( L_1(S) \) takes on its minimum for that value of \( S \) which satisfies the equation

\[
(85) \quad \beta e^{-\alpha S} - \alpha e^{-\beta S} = (\beta - \alpha) \frac{c_1 + \gamma}{c_1 + c_2} \quad \text{if} \quad \gamma < c_2,
\]

or for

\[
S = 0 \quad \text{if} \quad \gamma \geq c_2.
\]

**Stationary Distribution of Size of Delivery**

The delivery of goods to a stocking point may in turn represent a demand on a higher level stocking point. In this case, it will be of interest to know the stationary probability distribution of size of delivery.

The stationary distribution function \( H^*(x) \) for the size of a delivery to the stocking point during a period, given that a delivery has taken place, is

\[
(86) \quad H^*(x) = \Pr[R < x] + \Pr[R \geq x] \Pr[X > S-x]
\]

\[
= \begin{cases} 
H(x) + [1-H(x)][1-F(S-x) - \Pr[X=S-x]] & \text{if } x \geq 0 \\
0 & \text{if } x < 0,
\end{cases}
\]

where \( F(x) \) is the stationary distribution function of stock level at the stocking point just prior to delivery (the solution to equation (9)) and \( H(x) \) is the distribution function of the quantity \( R \) available for delivery.
3. POLICY II--FIXED REORDER LEVEL, UNLIMITED STORAGE CAPACITY

This model deals with the problem of controlling the inventory level for a single item at a single stocking point, under the same conditions of demand and resupply availability as in section 2 above. However, in this section it is assumed that the stocking policy followed is to accept the full amount of an available delivery if the stock level is below a fixed level $s$, and otherwise to reject the delivery. The storage capacity of the stocking point is not considered as a limitation.

One possible application of this model is to the case of a fish cannery which is visited from time to time by fishing boats. It might be reasonable to assume that a boat would require that the canner purchase all or none of its catch, so as to minimize unloading costs.

The special case of the model of this section in which delivery is available each period has been studied in considerable detail by Karlin in [3]; there, and in [2], he mentions the possible applications of his model to the control of inventory of items which are produced in random-sized lots, such as agricultural crops and complex electronic components subject to a high rejection rate.

Below, the model of this section is investigated in a manner parallel to that followed in section 2 above. A necessary condition for the existence of a stationary distribution of stock level prior to delivery is formulated, and functional equations satisfied by this stationary distribution derived. Next, expressions for the limiting distribution of stock level at the close of an arbitrary point are derived.
Then assuming a cost structure, expressions for limiting one-period expected costs are formulated. An example is given to illustrate the computation of these distributions and costs, and to demonstrate the selection of an optimum policy by determining the value of \( s \) which minimizes expected costs.

**Stationary Distribution of Stock Level Prior to Delivery Availability**

Here, functional equations for the stationary distribution of stock level at the close of periods just prior to those in which delivery is available are derived for the model described above.

The number of periods between deliveries is a random variable \( D \), with distribution

\[
\Pr(D = i) = d_i \quad \text{for} \quad i = 1, 2, \ldots ,
\]

and

\[
\sum_{i=1}^{\infty} d_i = 1.
\]

The random sequence of times at which delivery is available will be denoted by \( \{\eta_j\} \), where \( \eta_1 = 0 \) with probability one.

The random variable \( X_j \) will represent the stock level at the close of the \( j \)th period, i.e. at time \( j \); negative values of \( X_j \) represent amounts owed to consumption (backlogged demand). The sequence \( \{X_j\} \) of stock levels at the close of periods prior to delivery availability form an embedded Markov chain with one-step transitions.
\[
X_{\eta_j+1} = \begin{cases} 
X_{\eta_j} + R_{\eta_j} - \sum_{i=\eta_j+1}^{\eta_{j+1}} \xi_i & \text{if } X_{\eta_j} < s \\
X_{\eta_j} - \sum_{i=\eta_j+1}^{\eta_{j+1}} \xi_i & \text{if } X_{\eta_j} \geq s,
\end{cases}
\]

where \( R_{\eta_j} \) is the random variable representing the quantity of goods available at time \( \eta_j \). The distribution of \( R_{\eta_j} \) will be denoted by

\[
H(r) = \Pr(R_{\eta_j} < r).
\]

Then if \( F_{\eta_j}(x) \) is the distribution function for \( X_{\eta_j} \), one obtains as the relation, in differential form, between the distributions of \( X_{\eta_j} \) and \( X_{\eta_j+1} \):

\[
dx\left( \int_s^\infty \psi(t-x) \, dF_{\eta_j}(t) + \int_0^s \psi(t-\tau-x) \, dF_{\eta_j}(t) \, dH(\tau) \right)
\]

for \( x \leq s \)

\[
dF_{\eta_j}(x) = \begin{cases} 
\int_x^\infty \psi(t-x) \, dF_{\eta_j}(t) + \int_{x-s}^s \psi(t+\tau-x) \, dF_{\eta_j}(t) \, dH(\tau) & \text{for } x \geq s,
\end{cases}
\]

where

\[
\psi(\xi) = \sum_{k=1}^{\infty} \frac{d_k}{\xi} \phi^{(k)}(\xi),
\]

and \( \phi^{(k)}(\xi) \) is the density function for the sum of demands in \( k \) periods.

This relation is obtained by enumerating the ways in which \( X_{\eta_j+1} \) can equal \( x \), and summing the probabilities of these events. Thus if \( x < s \),

\[
X_{\eta_j+1} = x \quad \text{if } X_{\eta_j} > s \quad \text{and the demand } \sum_{i=1}^{\eta_j} x_i = X_{\eta_j} - x,
\]

or if
\( x - R \leq X_{\eta_j} < s \) and the demand \( \Sigma_{\xi_i} = X_{\eta_j} + R_{\eta_j} - x \); if \( X_{\eta_j} < x - R_{\eta_j} \), then clearly there is no possibility that \( X_{\eta_{j+1}} = x \). Similarly, if \( x \geq s \), then \( X_{\eta_{j+1}} = x \) if \( X_{\eta_j} > x \) and the demand \( \Sigma_{\xi_i} = X_{\eta_j} - x \), or if \( x - R_{\eta_j} \leq X_{\eta_j} < s \) (which implies also \( x - s < R_{\eta_j} \)) and the demand \( \Sigma_{\xi_i} = X_{\eta_j} + R_{\eta_j} - x \); if \( s < X_{\eta_j} < x \) or \( X_{\eta_j} < x - R_{\eta_j} \), then clearly there is no possibility that \( X_{\eta_{j+1}} = x \). One notes further that due to the stocking policy and the assumption that the demand has a continuous density function \( \phi(\xi) \), \( F_{\eta_{j+1}}(x) \) is continuous for all \( x \), and has a continuous first derivative \( \frac{dF_{\eta_{j+1}}(x)}{dx} \) for all \( x \).

Equations (4) may be represented in cumulative distribution function form by

\[
\begin{align*}
(6) \quad F_{\eta_{j+1}}(x) &= \begin{cases} \\
\int_{-\infty}^{\xi - R} \int_{-\infty}^{x - \xi} \psi(t-\xi)dF_{\eta_j}(t) \, d\xi + \int_{-\infty}^{x} \int_{-\infty}^{s} \psi(t+\xi) \, dF_{\eta_j}(t) \, dH(r) \, d\xi \\
\int_{-\infty}^{s} \int_{s}^{\infty} \psi(t-\xi) \, dF_{\eta_j}(t) \, d\xi + \int_{-\infty}^{s} \int_{-\infty}^{s} \psi(t+\xi) \, dF_{\eta_j}(t) \, dH(r) \, d\xi \\
+ \int_{s}^{x} \int_{s}^{\infty} \psi(t-x) \, dF_{\eta_j}(t) \, d\xi \\
+ \int_{s}^{x} \int_{s}^{\xi - R} \int_{s}^{s} \psi(t+\xi) \, dF_{\eta_j}(t) \, dH(r) \, d\xi
\end{cases}
\end{align*}
\]

for \( x \leq s \).
If the stability condition

\[ E(D) \int_0^\infty \xi \phi(\xi) \, d\xi < E_H(R) \]

is satisfied, it is assumed that, subject to possible additional restrictions on \( H(r) \), each demand is ultimately satisfied with probability one, and that the distributions \( F_{\eta_j}(x) \) tend to a unique stationary limit distribution \( F(x) \) as \( j \to \infty \). Representing equations (6) in the abstract operator form

\[ F_{\eta_{j+1}} = T F_{\eta_j}, \]

then \( F \) is the probability distribution which satisfies the equation

\[ F = T F. \]

Since \( \phi(\xi) \) is continuous, \( F(x) \) has a continuous density function \( f(x) \) which satisfies

\[
\begin{cases} 
\int_s^\infty \psi(t-x) f(t) \, dt + \int_0^s \int_{x-r}^s \psi(t+r-x) f(t) \, dt \, dH(r) \\
\text{for } x \leq s \\
\int_x^\infty \psi(t-x) f(t) \, dt + \int_{x-s}^s \int_{x-r}^s \psi(t+r-x) f(t) \, dt \, dH(r) \\
\text{for } x \geq s.
\end{cases}
\]

Karlin in [3] has solved this equation for the important cases in which \( \psi(\xi) \) is the density of a sum of \( k \) exponentials, each exponential with a different parameter.
Limiting Distribution of Stock Level at End of Arbitrary Period.

Above, the functional equation relationships were derived for the stationary distribution of stock level at the end of periods just prior to those in which delivery is available. Here it is assumed that this stationary distribution exists and has been determined, and the limiting distribution of stock level at the end of an arbitrary period is derived. The distribution function for this new distribution will be denoted by $G(x)$.

Using the fact that the limiting distribution of the number of periods since the last delivery availability is given by

$$\delta_k = \frac{1}{E(D)} \sum_{i=k}^{\infty} d_i \quad \text{for} \quad k = 1, 2, \ldots,$$

and letting

$$\pi(\xi) = \sum_{k=1}^{\infty} \delta_k \phi^{(k)}(\xi),$$

one obtains as the relations, in differential form, between the distributions $G(x)$ and $F(x)$

$$dx\left( \int_s^{\infty} \pi(t-x) \, dF(t) + \int_0^{\infty} \int_{x-r}^{s} \pi(t+r-x) \, dF(t) \, dH(r) \right)$$

for $x \leq s$.

(13) $dG(x) =$

$$dx\left( \int_x^{\infty} \pi(t-x) \, dF(t) + \int_{x-s}^{\infty} \int_{x-r}^{s} \pi(t+r-x) \, dF(t) \, dH(r) \right)$$

for $x \geq s$.
these relations are obtained by following the same line of reasoning used to derive equations (4) above. In abstract operator form, the relations (13) define a transformation

\begin{equation}
G = S F .
\end{equation}

The operator $S$ differs from the operator $T$ defined above only by the substitution of the function $\pi(\xi)$ for $\psi(\xi)$. Thus in the special case in which the time $D$ between deliveries has the geometric distribution

\[ d_k = p q^{k-1} \quad \text{for} \quad k = 1, 2, \ldots , \]

one obtains $\phi_k = d_k$ for all $k$, which implies $\pi(\xi) = \psi(\xi)$, which in turn implies $S = T$; since then $TF = F$ and $TF = G$, one obtains the result $G(x) = F(x)$ for all $x$.

Since $\phi(\xi)$ is continuous, $\pi(\xi)$ is also continuous, and the limiting distribution of stock level at the close of an arbitrary period has a continuous density function given by

\begin{equation}
g(x) = \begin{cases} 
\int_{\pi(t-x)}^{\infty} f(t) \, dt + \int_{x}^{\infty} \int_{x-r}^{s} \pi(t+\xi-x) \, f(t) \, dt \, dH(r) \\
\text{for} \quad x \leq s \\
\int_{x}^{\infty} \int_{x-r}^{\infty} \pi(t+\xi-x) \, f(t) \, dt \, dH(r) \\
\text{for} \quad x \geq s .
\end{cases}
\end{equation}
The above limiting distribution can be used to compute the limiting one-periodic expected costs as a function of the reorder level \( s \). Then the optimum value of \( s \) can be determined by locating that value which minimizes costs. For example, if a linear holding cost \( c_1 \) and a linear shortage cost \( c_2 \) are assumed, then the total limiting expected one-period costs are

\[
L(s) = c_1 \int_0^\infty x \, dG(x) - c_2 \int_{-\infty}^0 x \, dG(x).
\]

**Example:**

If the mean time between deliveries has the geometric distribution

\[
d_k = pq^{k-1} \quad \text{for} \quad k = 1, 2, \ldots,
\]

then the distribution of time since the last delivery availability will be geometric with the same parameter. Thus

\[
\pi(\xi) = \psi(\xi),
\]

and

\[
G(x) = F(x)
\]

regardless of the demand distribution \( \phi(\xi) \).

If the demand distribution is given by

\[
\phi(\xi) = \lambda e^{-\lambda \xi} \quad \text{for} \quad \xi \geq 0,
\]
then

\[(19) \quad \psi(\xi) = \lambda e^{-\lambda \xi} \quad \text{for} \quad \xi \geq 0,\]

as was seen in example 1 of section 2. Substituting this result in equation (10), one obtains

\[
\begin{align*}
(20) \quad f(x) e^{-\lambda x} &= \left\{ \begin{array}{ll}
\int_{-\infty}^{\infty} \lambda e^{-\lambda t} f(t) dt + \int_{0}^{\infty} \int_{x-r}^{\infty} \lambda e^{-\lambda (t+r)} f(t) dt dH(r) & \quad \text{for} \quad x \leq s \\
\int_{x}^{\infty} \lambda e^{-\lambda t} f(t) dt + \int_{x-s}^{\infty} \int_{x-r}^{\infty} \lambda e^{-\lambda (t+r)} f(t) dt dH(r) & \quad \text{for} \quad x \geq s.
\end{array} \right.
\]

Differentiating with respect to \( x \), one obtains

\[
(21) \quad f'(x) - \lambda f(x) = \left\{ \begin{array}{ll}
-\lambda \int_{0}^{\infty} f(x-r) dH(r) & \quad \text{for} \quad x < s \\
-\lambda f(x) - \lambda \int_{x-s}^{\infty} f(x-r) dH(r) & \quad \text{for} \quad x > s.
\end{array} \right.
\]

Trying a solution for the region \( x < s \) of the form

\[
(22) \quad f(x) = C e^{-K(s-x)},
\]

where \( K \) must be positive to ensure that the integral of this function from \( -\infty \) to \( s \) is finite, one finds after substituting in the first line of (21) that \( K \) must satisfy the equation
\[
\frac{p\lambda - K}{p\lambda} = E_H(e^{-KR}).
\]

It was seen in example 1 of section 2 that if the stability condition (7) is satisfied, i.e. if

\[
\frac{1}{p\lambda} < E_H(R),
\]

then equation (23) has exactly one positive solution \( \alpha \), where \( \alpha < p\lambda \). Then substituting the solution for \( x < s \) into the second line of (21) and integrating from \( x \) to \( \infty \), and using the fact that \( f(\infty) = 0 \), one obtains

\[
f(x) = \begin{cases} 
C e^{-\alpha(s-x)} & \text{for } x \leq s \\
p\lambda C \int_x^\infty \int_{t-s}^\infty e^{-\alpha(s-t-r)} \, dH(r) \, dt & \text{for } x \geq s.
\end{cases}
\]

The continuity of these equations at \( x = s \) is easily verified by carrying out the integration in the second line and using the identity (23). From the condition that \( f(x) \) integrates to one over the entire real line, the result

\[
C = \frac{\alpha}{p\lambda E_H(R)}
\]

is obtained. Using this, and making a change of variable, one obtains the result
\[ f(x) = \begin{cases} \frac{\alpha}{\lambda E_H(R)} e^{-\alpha(x-s)} & \text{for } x < s \\ \frac{\alpha}{E_H(R)} \int_{x-s}^{x} \int_{t}^{\infty} e^{-\alpha(r-t)} dH(r) \, dt & \text{for } x \geq s. \end{cases} \]

In [3] Karlin shows that, under mild conditions on $H(r)$, this solution is unique.

Since in the present case $G(x) = F(x)$, the result

\[ g(x) = \begin{cases} \frac{\alpha}{\lambda E_H(R)} e^{-\alpha(s-x)} & \text{for } x < s \\ \frac{\alpha}{E_H(R)} \int_{x-s}^{x} \int_{t}^{\infty} e^{-\alpha(r-t)} dH(r) \, dt & \text{for } x \geq s. \end{cases} \]

follows immediately. Then the limiting one-period expected costs calculated from equation (16) are, assuming consideration is restricted to policies such that $s \geq 0$,

\[ L(s) = \frac{\alpha}{\lambda E_H(R)} \left[ c_1 \int_{s}^{\infty} \int_{x-s}^{\infty} \int_{t}^{\infty} \lambda e^{-\alpha(r-t)} dH(r) \, dt \, dx \right. \\
\left. + c_1 \int_{0}^{s} x e^{-\alpha(s-x)} \, dx - c_2 \int_{-\infty}^{0} x e^{-\alpha(x-s)} \, dx \right] \\
= c_1 \left\{ \frac{E_H(R^2)}{2E_H(R)} + s - \frac{1}{\alpha} \right\} + \frac{(c_1 + c_2) e^{-\alpha s}}{\alpha \lambda E_H(R)}.
\]

Since $L''(s) > 0$, $L(s)$ is a convex function of $s$. Furthermore since $L'(s)$ is a strictly increasing continuous function of $s$, $L(s)$ will have a single minimum given by the solution to
\[ L'(s) = 0, \]
i.e. for
\[
(30) \quad s = \frac{1}{\alpha} \log \left( \frac{c_1 + c_2}{c_1 \lambda E_H(R)} \right).
\]
The condition that this solution be non-negative is that
\[
(31) \quad 1 \leq \frac{c_1 + c_2}{c_1 \lambda E_H(R)}.
\]
If this condition is not satisfied, then \( L'(s) > 0 \) for all \( s \geq 0 \), and the minimum of \( L(s) \) for \( s \geq 0 \) is at \( s = 0 \). Thus the non-negative value of \( s \) which minimizes \( L(s) \) is
\[
(32) \quad s = \max \left\{ 0, \frac{1}{\alpha} \log \left( \frac{c_1 + c_2}{c_1 \lambda E_H(R)} \right) \right\}.
\]
This result is very similar to that obtained by Karlin in [3] for a parallel example for the case in which delivery is available every period.
4. POLICY III--FIXED REORDER LEVEL, LIMITED STORAGE CAPACITY

This model deals with the problem of controlling the inventory level for a single item at a single stocking point, under the same conditions of demand and resupply availability as in sections 2 and 3 above. However, in this section it is assumed that the stocking policy followed is to accept delivery only if the stock level is below a fixed level \( s \); the amount accepted is the quantity necessary to raise the stock level to the storage capacity \( S \) (where \( S \geq s \)), or the quantity available, whichever is smaller. The limiting case of this policy as \( S \to \infty \) is policy II discussed in the previous section. On the other hand if \( S = s \), the policy reduces to policy I discussed in section 2.

Although this model resembles what is commonly referred to as an \((s,S)\) policy model, it is not of that type. In the \((s,S)\) model, an order is placed when the stock level at the end of a period drops below \( s \), and the waiting period for delivery begins with the placing of the order (see [1] and [5]). In the present model, the probability distribution of the time between the end of the period in which the stock level first drops below \( s \) and the time when delivery takes place depends in general on the number of periods since the last availability of delivery. (Of course in the case of a geometric distribution of waiting time, the results are the same in both cases.) A further difference between the present model and the \((s,S)\) model is that here a random quantity is available for delivery, while
in the \((s,S)\) model it is assumed that the stock level is always restored to \(S\) at the time of delivery. This difference is equivalent to letting the delivery capacity be infinite in the present model.

In investigating the model of this section, a stability condition necessary for the existence of a stationary distribution of stock level prior to delivery is first formulated, then functional equations satisfied by this stationary distribution are derived. Next, expressions for the limiting distribution of stock level at the close of an arbitrary period are derived. Then assuming a cost structure, expressions for limiting one-period expected costs are formulated. An example is given to illustrate the computation of these distributions and costs.

**Stationary Distribution of Stock Level Prior to Delivery Availability**

Here, functional equations for the stationary distribution of stock level at the close of periods just prior to those in which delivery is available are derived for the model described above.

The number of periods between delivery availabilities is a random variable \(D\), with distribution

\[
\Pr(D = i) = d_i \quad \text{for } i = 1, 2, \ldots
\]

with

\[
\sum_{i=1}^{\infty} d_i = 1.
\]
The random sequence of times at which delivery is available will be denoted by \( \{ \eta_j \} \), where \( \eta_1 = 0 \) with probability one.

The random variable \( X_j \) will represent the stock level at the close of the \( j \)th period, i.e. at time \( j \); negative values of \( X_j \) represent amounts owed to consumption (backlogged demand). Then the sequence \( \{ X_{\eta_j} \} \) of stock levels at the close of periods prior to delivery availability form an embedded Markov chain with one-step transitions

\[
X_{\eta_j+1} = \begin{cases} 
S - \sum_{i=\eta_j+1}^{\eta_j+1} \xi_i & \text{if } X < s \text{ and } X + R > S \\
X_j + R - \sum_{i=\eta_j+1}^{\eta_j+1} \xi_i & \text{if } X < s \text{ and } X + R < S \\
X_j - \sum_{i=\eta_j+1}^{\eta_j+1} \xi_i & \text{if } X_j \geq s
\end{cases}
\]

where \( R_{\eta_j} \) is the random variable representing the quantity of goods available at time \( \eta_j \). The distribution of \( R_{\eta_j} \) will be denoted by

\[
H(r) = \Pr(R_{\eta_j} < r).
\]

Then if \( F_{X_{\eta_j}}(x) \) is the distribution function for \( X_{\eta_j} \), one obtains
\begin{align*}
\text{(4) } \frac{d^2 \eta_j}{dx^2} &= \begin{cases}
\frac{dx}{\int_{x}^{S} \frac{\psi(t-x)}{\eta_j} \, dF_\eta_j(t) + \int_{-\infty}^{x} \int_{x}^{S} \frac{\psi(r-x)}{\eta_j} \, dH(r) \, dF_\eta_j(t) + \int_{-\infty}^{x} \int_{x}^{S-t} \frac{\psi(r_{t-x})}{\eta_j} \, dH(r) \, dF_\eta_j(t)}{\text{for } x \leq S,} \\
\frac{dx}{\int_{x}^{S} \frac{\psi(t-x)}{\eta_j} \, dF_\eta_j(t) + \int_{-\infty}^{s} \int_{s}^{S} \frac{\psi(r-x)}{\eta_j} \, dH(r) \, dF_\eta_j(t) + \int_{-\infty}^{s} \int_{s}^{S-t} \frac{\psi(r_{t-x})}{\eta_j} \, dH(r) \, dF_\eta_j(t)}{\text{for } s \leq x < S,}
\end{cases}
\end{align*}

where

\begin{align*}
\psi(\xi) &= \sum_{k=1}^{\infty} d_k \phi^{(k)}(\xi) \quad \text{for } \xi \geq 0
\end{align*}

and \( \phi^{(k)}(\xi) \) is the density function for the demand summed over \( k \) periods. This result follows from an enumeration of the possible ways in which a stock level of \( x \) can occur at time \( \eta_{j+1} \). If \( x < s \), then \( X = x \) if \( s \leq X < S \) and demand is \( X - x \), or if \( X < s \), \( R - X - x \) and demand is \( S - x \), or if \( X < s \), \( 0 < R < S - X \) and demand is \( R + X - x \), or if \( X < s \), \( x - X < R < S - X \) and demand is \( R + X - x \). Similarly, if \( s \leq x < S \), then \( X = x \) if \( x \leq X < S \) and demand is \( X - x \), or if \( X < s \), \( R > S - t \) and demand is \( S - x \), or if \( X < s \), \( x - X < R - S - X \) and demand is \( S - x \), or if \( X < s \), \( x - X < R - S - X \) and demand is \( S - x \).
\( R_j + X_j = x \). It will be seen that these events are disjoint and exhaust the possible ways in which \( X_j = x \) can occur.

If the stability condition

\[
E(D) \int_0^\infty \phi(\xi) \, d\xi < E_H(R)
\]

is satisfied, it is assumed that, subject to possible additional restrictions on \( H(r) \), each demand is ultimately satisfied with probability one, and that the distributions \( F_j(x) \) tend in the limit as \( j \to \infty \) to a unique stationary limit distribution \( F(x) \). This condition implies that the average rate of delivery exceeds the average rate of demand. Thus if the stock level at a given time is below \( s \), with probability one the stock level will rise above \( s \) again in a finite number of periods under the assumed stocking policy.

The limiting stationary probability distribution function \( F(x) \) is then a fixed point of the transformation defined by equations (4); thus \( F(x) \) must satisfy the function equation

\[
\begin{aligned}
\frac{dx}{x} \left[ \int_x^S \psi(t-x) \, dF(t) + \int_{-\infty}^S \int_{S-t}^\infty \psi(s-x) \, dH(r) \, dF(t) \\
+ \int_x^S \int_0^{S-t} \psi(r+t-x) \, dH(r) \, dF(t) + \int_{-\infty}^x \int_{x-t}^{S-t} \psi(r+t-x) \, dH(r) \, dF(t) \right]
\end{aligned}
\]

(7) \( dF(x) = \)

\[
\begin{aligned}
\frac{dx}{x} \left[ \int_x^S \psi(t-x) \, dF(t) + \int_{-\infty}^S \int_{S-t}^\infty \psi(s-x) \, dH(r) \, dF(t) \\
+ \int_{-\infty}^S \int_{x-t}^{S-t} \psi(r+t-x) \, dH(r) \, dF(t) \right] \quad \text{for } x < s
\end{aligned}
\]

\[
\begin{aligned}
\frac{dx}{x} \left[ \int_x^S \psi(t-x) \, dF(t) + \int_{-\infty}^S \int_{S-t}^\infty \psi(s-x) \, dH(r) \, dF(t) \\
+ \int_{-\infty}^S \int_{x-t}^{S-t} \psi(r+t-x) \, dH(r) \, dF(t) \right] \quad \text{for } s \leq x < S
\end{aligned}
\]
\[
\begin{align*}
\text{for } x < s & \quad \text{for } s \leq x < S.
\end{align*}
\]

The second form is obtained from the first by interchanging the orders of integration. Since the demand distribution is assumed to have a continuous density function \( \phi(t) \), \( \psi(t) \) is a continuous density function and the stationary distribution of stock level will have a continuous density function given by

\[
\begin{align*}
\text{for } x \leq s & \quad \text{for } s \leq x < S.
\end{align*}
\]
Limiting Distribution of Stock Level at End of Arbitrary Period

After obtaining the stationary distribution $F(x)$ of stock level at the end of periods just before those in which a delivery is available, it is possible then to compute from this distribution the limiting distribution $G(x)$ of stock level at the close of an arbitrary period. The method of doing so will be shown in this section.

Using the fact that the limiting distribution of the number of periods since the last delivery availability is given by

\begin{equation}
\delta_k = \frac{1}{E(D)} \sum_{i=1}^{\infty} d_i \quad \text{for } k = 1, 2, \ldots ,
\end{equation}

and letting

\begin{equation}
\pi(\xi) = \sum_{i=1}^{\infty} \delta_k \phi_{(k)}(\xi) \quad \text{for } \xi \geq 0,
\end{equation}

one obtains the result

\begin{align}
(10) \quad dG(x) &= \begin{cases} 
\int^S_x \pi(t-x)dF(t) + \int_{S-s}^{\infty} \int_{S-r}^{S} \pi(S-x)dF(t)dH(r) \\
+ \int_{S-s}^{\infty} \int_{x-r}^{S-r} \pi(r+t-x)dF(t)dH(r) + \int_{s-r}^{S-r} \int_{s-r}^{S} \pi(r+t-x)dF(t)dH(r) \\
\quad \text{for } x < s
\end{cases}
\end{align}
by following the same line of reasoning used to derive equations (4). Since the demand distribution is assumed to have a continuous density function \( \phi(x) \), \( \pi(x) \) is a continuous density function and the stationary distribution of stock level will have a continuous density function given by

\[
\int_{S}^{S} \pi(t-x)f(t)dt + \int_{S-s}^{\infty} \int_{S-r}^{S} \pi(s-x)f(t)dt dH(r)
+ \int_{S-s}^{\infty} \int_{x-r}^{S-r} \pi(r+t-x)f(t)dt dH(r)
+ \int_{0}^{S-s} \int_{x-r}^{S} \pi(r+t-x)f(t)dt dH(r) \quad \text{for} \quad x \leq s
\]

(11) \( g(x) = \)

\[
\int_{x}^{S} \pi(t-x)f(t)dt + \int_{S-s}^{\infty} \int_{S-r}^{S} \pi(s-x)f(t)dt dH(r)
+ \int_{S-s}^{\infty} \int_{x-r}^{S-r} \pi(r+t-x)f(t)dt dH(r)
+ \int_{x-s}^{S-s} \int_{x-r}^{S} \pi(r+t-x)f(t)dt dH(r) \quad \text{for} \quad s \leq x < S.
\]

The above distribution can be used to compute limiting one-period expected costs as a function of \( s \) and \( \Delta = S-s \). Then the optimum values of \( s \) and \( \Delta \) can be determined by finding those values which minimize expected costs. For example, if a linear holding cost \( c_1 \), a linear shortage cost \( c_2 \), and a linear reorder cost \( K + c_3 \) are assumed, where the cost \( K \) is incurred only if an order is placed, the limiting one-period expected costs are
\( L_1(s, \Delta) = c_1 \int_0^{s+\Delta} x \, dG(x) - c_2 \int_{-\infty}^{0} x \, dG(x) + \frac{K_F(s)}{E(D)} \),

the final term on the right side of this equation is the product of the fixed order cost \( K \) and the limiting probability of placing an order during a period. Since the expected cost of goods purchased is a constant independent of \( s \) and \( \Delta \) in the present case, it is omitted.

If an additional factor \( \gamma S \) is included in costs to allow for amortization of the investment in storage capacity, then the above expression for stationary one-period expected costs becomes

\( L_2(s, \Delta) = (s+\Delta)\gamma + c_1 \int_0^{s+\Delta} x \, dG(x) - c_2 \int_{-\infty}^{0} x \, dG(x) + \frac{K_F(s)}{E(D)} \).

Example.

Suppose the time between deliveries has the geometric distribution

\( d_k = p \, q^{k-1} \quad \text{for} \quad k = 1, 2, \ldots \),

where

\( 0 < p = 1 - q < 1 \),

and the demand distribution

\( \phi(\xi) = \lambda \, e^{-\xi \lambda} \quad \text{for} \quad \xi \geq 0 \)

is exponential. Then
\begin{align*}
\psi(\xi) &= p \lambda \ e^{-p \lambda \xi} \quad \text{for } \xi \geq 0
\end{align*}

is also an exponential distribution.

Substituting equation (16) into equations (7), one obtains

\begin{align*}
\psi(t)f(t)dt + \int_{S-s}^{S} \psi(t)f(t)dt \Delta H(r)
\begin{cases}
+ \int_{S-s}^{S} \int_{x-r}^{S-r} \psi(r+t)f(t)dt \Delta H(r) \\
+ \int_{S-s}^{S} \int_{x-r}^{S} \psi(r+t)f(t)dt \Delta H(r)
\end{cases} 
\text{for } x \leq s
\end{align*}

(17) \quad e^{-p \lambda x} f(x) = \begin{cases}
\psi(t)f(t)dt + \int_{S-s}^{S} \psi(t)f(t)dt \Delta H(r)
\begin{cases}
+ \int_{S-s}^{S} \int_{x-r}^{S-r} \psi(r+t)f(t)dt \Delta H(r) \\
+ \int_{S-s}^{S} \int_{x-r}^{S} \psi(r+t)f(t)dt \Delta H(r)
\end{cases} 
\text{for } s \leq x < S.
\end{cases}

Then differentiating with respect to \( x \), one obtains

\begin{align*}
(18) \quad f'(x) - p \lambda f(x) &= \begin{cases}
- p \lambda \int_{0}^{x-r} f(x-r) \ dH(r) \quad \text{for } x < s \\
- p \lambda f(x) - p \lambda \int_{x-s}^{\infty} f(x-r) \ dH(r) \quad \text{for } s < x < S.
\end{cases}
\end{align*}
In example 1 of section 2, it was found that if the stability condition (6) is satisfied, i.e. if

\[ \frac{1}{p \lambda} < E_H(R), \]

the first part of equation (18) is satisfied by the solution

\[ f(x) = C \, e^{-\alpha(s-x)} \quad \text{for } x \leq s, \]

where \( \alpha \) is the unique positive solution to

\[ \frac{p \lambda - \alpha}{p \lambda} = E_H(e^{-\alpha R}). \]

Substituting this solution into the second part of equation (18), one obtains after integrating from \( s \) to \( x \) and using the fact that by continuity \( f(s) = C \),

\[ f(x) = C - p \lambda \int_s^x \int_{t-s}^\infty C \, e^{-\alpha(s+r-t)} \, dH(r) \, dt \]

\[ = \frac{p \lambda C}{\alpha} \int_{x-s}^\infty (1 - e^{-\alpha(s+r-x)}) \, dH(r) \quad \text{for } s \leq x < S. \]

Then using the condition that the density function integrates to 1, one obtains the result

\[ C = \frac{\alpha^2}{p \lambda} \left( \alpha E_H(R) + \int_\Delta [1 + \alpha(\Delta-r) - e^{\alpha(\Delta-r)}] \, dH(r) \right)^{-1} \]

where

\[ \Delta = S - s. \]
Thus the stationary distribution of stock level at the ends of period prior to delivery availability is

\[
(27) \quad f(x) = \begin{cases} 
C \ e^{-\alpha (s-x)} & \text{for } x \leq s \\
\frac{C_{pk}}{\alpha} \int_{x-s}^{\infty} \left[ 1 - e^{-\alpha (s+r-x)} \right] \, dH(r) & \text{for } s \leq x < s + \Delta,
\end{cases}
\]

where \( C \) is given by equation (26) and is a function only of \( \Delta \).

Since in the case of a geometric distribution of time between delivery availability \( \delta_k = d_k \), which implies \( \pi(\xi) = \psi(\xi) \), the limiting distribution of stock level at the end of an arbitrary period is the same as that at the ends of periods prior to those in which deliveries are available. Thus in the present example, \( g(x) = f(x) \) and is given by equation (27).

The stationary one-period expected costs can then be computed from equation (12) as

\[
(28) \quad L_1(s, \Delta) = C \left\{ \frac{\frac{p_{c_1}}{\alpha}}{\int_{s}^{\infty} \int_{x-s}^{\infty} x (1 - e^{-\alpha (s+r-x)}) \, dH(r) \, dx} \\
+ \ c_1 \int_{0}^{s} x e^{-\alpha (s-x)} \, dx - c_2 \int_{-\infty}^{0} x e^{-\alpha (s-x)} \, dx + \frac{p_{k}}{\alpha} \right\}.
\]

The optimum values of \( s \) and \( \Delta \) are then assumed to be those which minimize \( L_1(s, \Delta) \), subject to the restrictions that \( s \) and \( \Delta \) be non-negative. This calculation can be performed for any specific choice of distribution \( H(r) \) of size of delivery.
in the special case in which $\Delta$ is fixed, a simple general solution for the optimum level of $s$ exists. In this case,

$$
\frac{\partial L_1(s, \Delta)}{\partial s} = c_1 - \frac{C}{\alpha} (c_1 + c_2) e^{-\alpha s}
$$

and

$$
\frac{\partial^2 L_1(s, \Delta)}{\partial s^2} = C(c_1 + c_2) e^{-\alpha s} > 0 \quad \text{for all } s.
$$

Thus $L_1(s, \Delta)$ is a convex function of $s$ for fixed $\Delta$, and takes on its minimum value for

$$
s = \begin{cases}
\frac{1}{\alpha} \log \frac{(c_1 + c_2) \, C}{\alpha c_1} & \text{if } \frac{(c_1 + c_2) \, C}{\alpha c_1} > 1 \\
0 & \text{otherwise}.
\end{cases}
$$
5. EXTENSION TO A MORE GENERAL DELIVERY SIZE DISTRIBUTION

In formulating the models in sections 2, 3, and 4 above, the distribution \( H(r) \) of quantity of stock available for delivery was assumed independent of the distribution of time between deliveries. In the present section, this assumption of independence is relaxed; it is assumed that there exists a sequence \( \{H_j(r)\} \) of conditional delivery quantity distributions, where \( j \) indicates the number of periods which have elapsed since the time of the last delivery. For example, the mean of the delivery quantity distribution may be an increasing function of the number of periods \( j \). All other conditions of demand and resupply availability remain as before.

This extension of the earlier models is motivated by the need for this added generality in considering the multiple stocking point model in section 7. Direct applications for this model to single stocking point problems can also be visualized.

The model in this section is formulated in three versions, using the stocking policies I, II, and III defined above. In formulating these extensions of the earlier models, it is more convenient to consider the sequence of stock levels at the beginning of each period, after delivery (if any) has taken place, rather than at the end of each period as before. However, once the limiting distribution of stock level at the beginning of an arbitrary period is obtained, the limiting distribution of stock level at the end of an arbitrary period may be obtained, if desired, by a simple transformation.
Below, stability conditions necessary for the existence of stationary distributions of stock level after delivery are formulated for the model of this section, and functional equations satisfied by these stationary distributions are derived for the three versions. Relations for the corresponding limiting distributions of stock level at the end of an arbitrary period are then derived. An example is given to illustrate the application of these results.

**Stationary Distribution of Stock Level After Delivery**

First, functional equations for the stationary distribution of stock level after delivery are derived for the extensions of the earlier models.

The three extended models share the same probability structure of supply availability and demand, but differ in their reorder policies. It is assumed that deliveries of a random size \( R_j \) become available to the stocking point at the beginnings of a random sequence \( \{ \eta_j \} \) of periods; furthermore, \( \text{Pr}(\eta_1 = 0) = 1 \), and the sequence \( \{ D_j \} = \{ \eta_{j+1} - \eta_j \} \) of times between successive delivery availabilities is a sequence of independent, identically distributed random variables. Letting the random variable \( D \) represent the time between any two successive delivery availabilities, the probability distribution of \( D \) will be denoted by
(1) \[ \Pr[D = i] = d_i \quad \text{for} \quad i = 1, 2, \ldots, \]

where

\[ \sum_{i=1}^{\infty} d_i = 1. \]

It is assumed that the conditional distribution \( H_i(r) \) of available delivery quantity, given that \( i \) periods have elapsed since the last delivery, is otherwise independent of time; thus

(2) \[ H_i(r) = \Pr[R_{\eta_{j+1}} < r | D_j = i] \quad \text{for} \quad i = 1, 2, \ldots, \]

and for all \( j \). The sequence \( \{\xi_i\} \) of demands, where \( \xi_i \) is the demand in the \( i \)th period, is assumed to be a sequence of independent, identically distributed random variables. Furthermore, the demand, delivery time, and delivery quantity distributions are assumed to be mutually independent, except for the dependence noted above of the delivery quantity distribution on the time between deliveries.

The random variable \( Y_i \), with probability distribution function \( F_i(y) \), will represent the stock level at time \( i \), after taking into account any delivery made to the stocking point at time \( i \). The sequence \( \{Y_i\} \) is not in general a Markov process. However, under the stocking policies introduced in sections 2, 3, and 4 above, the subsequence \( \{Y_{\eta_j}\} \) forms a Markov process with constant transition operator.
In section 4, the stocking policy was to accept an available delivery, subject to a storage capacity limitation of $S$, if the stock level at the close of the previous period was below $s$; if this stock level was not below $s$, the policy called for rejecting delivery. In the present case, this policy can be described by the one-step transition relations

\[
\begin{align*}
\eta_{j+1} &\begin{cases} 
\eta_j - \Sigma_i \xi_i & \text{if } s \leq \eta_j - \Sigma_i \xi_i < S \\
\eta_j - \Sigma_i \xi_i + \eta_{j+1} & \text{if } \eta_j - \Sigma_i \xi_i < s \text{ and } \eta_j - \Sigma_i \xi_i + \eta_{j+1} < S \\
S & \text{if } \eta_j - \Sigma_i \xi_i < s \text{ and } \eta_j - \Sigma_i \xi_i + \eta_{j+1} \geq S,
\end{cases}
\end{align*}
\]

where the summations all run from $\eta_j + 1$ to $\eta_{j+1}$. The transition operator $T$ is then given by

\[
\begin{align*}
F_{\eta_{j+1}}(y) = T F_{\eta_j}(y) = \sum_{i=1}^{\infty} d_i \left\{ \int_0^\infty \int_{-\infty}^{y+\xi} \int_0^{y+\xi-t} \phi(i)(\xi) dH_i(r) dF_j(t) d\xi \right\} & \text{ for } y \leq s \\
= \sum_{i=1}^{\infty} d_i \left\{ \int_0^{s+\xi} \int_{y+\xi}^{s+\xi-t} \phi(i)(\xi) dF_j(t) d\xi \\
+ \int_0^\infty \int_{-\infty}^{y+\xi} \int_0^{y+\xi-t} \phi(i)(\xi) dH_i(r) dF_j(t) d\xi \right\} & \text{ for } s < y \leq S \\
1 & \text{ for } y > S,
\end{align*}
\]
where \( \phi^{(1)}(\xi) \) is the density function for the sum of \( i \) random variables, independent and identically distributed with density function \( \phi(\xi) \), and \( F_{\eta_y^{(i)}}(y) \) is the initial distribution. These relations follow from a careful consideration of the possible ways in which the event \( Y_{\eta_j^{(i)}} < y \) can occur. First suppose that a total demand of \( \xi \) occurs during the \( i \) periods between deliveries. In this case if \( y \leq s \), then \( Y_{\eta_j^{(i)}} < s \) if \( Y_{\eta_j^{(i)}} - \xi < y \) and \( R_{\eta_j^{(i)}} < y + \xi - Y_{\eta_j^{(i)}} \). Similarly, if \( s \leq y < S \), then \( Y_{\eta_j^{(i)}} < y \) if \( s \leq Y_{\eta_j^{(i)}} - \xi < y \), or if \( Y_{\eta_j^{(i)}} - \xi < s \) and \( R_{\eta_j^{(i)}} < y + \xi - Y_{\eta_j^{(i)}} \). Then summing over all \( \xi \geq 0 \) and all \( i \geq 1 \), equations (4) are obtained.

Since \( \phi(\xi) \) is assumed continuous, it may be noted that \( F_{\eta_j^{(i)}}(y) \) is a continuous function of \( y \), except for a simple discontinuity at \( y = S \). The relations of (4) can also be expressed in differential form as

\[
\begin{align*}
\frac{dy}{\eta_j^{(i)}} &= \sum_{i=1}^{\infty} \int_{0}^{\infty} \int_{y-r}^{\infty} \phi^{(i)}(r+t-y) dF_{\eta_j^{(i)}}(t) dH_{\eta_j^{(i)}}(r) \quad \text{for } y < s \\
\frac{dy}{\eta_j^{(i)}} &= \sum_{i=1}^{\infty} \left( \int_{y}^{\infty} \phi^{(i)}(t-y) dF_{\eta_j^{(i)}}(t) \\
&\quad + \int_{y-s}^{\infty} \int_{y-r}^{\infty} \phi^{(i)}(r+t-y) dF_{\eta_j^{(i)}}(t) dH_{\eta_j^{(i)}}(r) \right) \quad \text{for } s \leq y < S \\
\frac{dF_{\eta_j^{(i)}}(y)}{\eta_j^{(i)}} &= \begin{cases} 
1 - F_{\eta_j^{(i)}}(S) & \text{for } y = S \\
1 & \text{for } y > S.
\end{cases}
\end{align*}
\]
In the present case it is assumed that, subject to possible additional restrictions on \( H(r) \), the condition that \( \mathcal{T}^n F_{\eta_1}(y) \) converge to a unique limiting distribution \( F(y) \), independent of \( F_{\eta_1}(y) \), is that the expected quantity of goods available for delivery each delivery cycle exceed the expected demand; this condition can be expressed as

\[
E(D) \int_0^\infty \xi \phi(\xi) \, d\xi < \sum_{i=1}^\infty d_i E_{H_1}(R).
\]

The stationary probability distribution \( F(y) \) is then a fixed point of the operator \( \mathcal{T} \), and is given by the solution to the equation

\[
F(y) = \begin{cases} 
\sum_{i=1}^\infty d_i \int_0^\infty \int_{-\infty}^{y+\xi} \int_0^{y+\xi-t} \phi(1)(\xi) \, dH_1(r) \, dF(t) \, d\xi & \text{for } y \leq s \\
\sum_{i=1}^\infty d_i \left( \int_0^\infty \int_{s+\xi}^{y+\xi} \phi(1)(\xi) \, dF(t) \, d\xi \\
+ \int_0^\infty \int_{-\infty}^{s+\xi} \int_0^{y+\xi-t} \phi(1)(\xi) \, dH_1(r) \, dF(t) \, d\xi \right) & \text{for } s < y \leq S \\
1 & \text{for } y > S.
\end{cases}
\]

Considering stocking policy I of section 2, which is to accept delivery whenever available, subject to a storage capacity limitation of \( S \), and assuming that the stability condition (6) holds, one obtains the relations
\[
F(y) = \begin{cases} 
\sum_{i=1}^{\infty} d_i \int_{0}^{\infty} \int_{y+\xi}^{\infty} \int_{0}^{y+\xi-t} \phi^{(i)}(\xi) \, dH_i(r) \, dF(t) \, d\xi & \text{for } y \leq S \\
1 & \text{for } y > S
\end{cases}
\]

for the stationary distribution of stock level after delivery. This result can be obtained directly by an analysis similar to that performed above. However, it also follows by letting \( S = s \) in equations (7).

Similarly, for stocking policy II of section 3, which is to accept the full available delivery if the stock level is below \( s \), and reject delivery if the stock level is not below \( s \), one obtains the relations

\[
f(y) = \begin{cases} 
\sum_{i=1}^{\infty} d_i \int_{0}^{\infty} \int_{0}^{y+\xi} \int_{0}^{y+\xi-t} \phi^{(i)}(\xi) \, dH_i(r) \, dF(t) \, d\xi & \text{for } y \leq s \\
\sum_{i=1}^{\infty} d_i \left[ \int_{0}^{c} \int_{y+\xi}^{\infty} \phi^{(i)}(\xi) \, dF(t) \, d\xi \\
+ \int_{0}^{\infty} \int_{-\infty}^{y+\xi} \int_{0}^{y+\xi-t} \phi^{(i)}(\xi) \, dH_i(r) \, dF(t) \, d\xi \right] & \text{for } y > s
\end{cases}
\]

for the stationary distribution of stock level after delivery. This result can also be obtained directly by an analysis similar to that performed above, although it may be seen to be the limiting case of relations (7) as \( S \to \infty \).
Stationary Distribution of Stock Level at Beginning of Arbitrary Period

Here, it is assumed that a unique stationary distribution \( F(y) \) of stock level after delivery exists, and that one wishes to obtain the limiting distribution \( G(y) \) of stock level at the beginning of an arbitrary period. Relationships between \( F(y) \) and \( G(y) \) will be formulated.

In the present case, the limiting distribution of the number of periods since the last delivery is given in terms of the distribution of times between deliveries by

\[
\delta_k = \frac{1}{E(D)} \sum_{i=k+1}^{\infty} d_i \quad \text{for } k = 0, 1, \ldots
\]

This is a slightly modified version of the distribution which was introduced in section 2. Then given that the last delivery took place \( k \) periods earlier, the stock level will be below \( x \) if and only if the stock level at the time of the last delivery less the demand during the intervening \( k \) periods is less than \( x \). Multiplying the conditional probability of this event by \( \delta_k \), the probability that the last delivery took place \( k \) periods earlier, and summing over all \( k \), one obtains the result

\[
G(y) = \sum_{i=0}^{\infty} \delta_i \int_0^{\infty} \int_{-\infty}^{y+\xi} g^{(1)}(\xi) dF(t) d\xi \quad \text{for all } y,
\]
where

\[ \int_{-\infty}^{x} \phi(0)(\xi) \, d\xi = \begin{cases} 
0 & \text{if } x \leq 0 \\
1 & \text{if } x > 0
\end{cases} \]

This result holds under all three stocking policies discussed above, if \( F(y) \) is defined by equations (7), (8), or (9), depending on the stocking policy.

If one wishes to find the limiting distribution of stock level at the end of an arbitrary period, rather than the beginning, the above computation must be modified to include the demand during one additional period. Thus letting \( G^*(y) \) be the distribution function for the limiting distribution of stock level at the end of an arbitrary period, one has

\[ G^*(y) = \sum_{i=0}^{\infty} \delta_i \int_{0}^{\infty} \int_{-\infty}^{y+\xi} \phi(i+1)(\xi) \, dF(t) \, d\xi \quad \text{for all } y. \]

Thus the examples in sections 2, 3, and 4 above could have been worked out using the approach of this part, and the same limiting distributions obtained for the stock level at the end of an arbitrary period.

**Example:**

Suppose that the distribution of time between delivery availabilities is given by
\[ d_i = \begin{cases} 
  p & \text{for } i = 1 \\
  q = 1 - p & \text{for } i = 2 
\end{cases} \]

and the density function for the demand distribution is

\[ \phi(\xi) = \lambda e^{-\lambda \xi} \quad \text{for } \xi \geq 0. \]

Furthermore, suppose the stocking point is following policy I.

Then the stationary distribution of stock level after delivery has a continuous density function \( f(y) \) for \( y < S \) which satisfies the relation

\[ f(y) = \sum_{i=1}^{2} d_i \left( \int_{0}^{\infty} \int_{y-r}^{S} \phi^{(i)}(r+t-y) f(t) \, dt \, dH_i(r) \right. \]

\[ + dF(S) \int_{0}^{\infty} \phi^{(i)}(r+S-y) \, dH_i(r) \left. \right) , \]

where the jump in the distribution function at \( S \) is given by

\[ dF(S) = 1 - F(S). \]

After substituting (13) and (14) into equation (15) and differentiating twice with respect to \( y \), one obtains

\[ (\lambda - D)^2 f(y) = p\lambda \int_{0}^{\infty} (\lambda - D) f(y-r) dH_1(r) + q\lambda^2 \int_{0}^{\infty} f(y-r) dH_2(r) \]

\[ \quad \text{for } x < S , \]
where $D$ represents the operation of differentiation with respect to $y$. Trying a solution of the form

$$f(y) = C e^{Ky} \quad \text{for} \quad y < S,$$

where the requirement that the integral of $f(y)$ over the region $-\infty < y < S$ be finite imposes the condition $K > 0$, one finds after substitution into equation (17) that $K$ must also satisfy the equation

$$\left(\lambda - K\right)^2 = p\lambda \left(\lambda - K\right) E_{H_1} \left(e^{-KR}\right) + q\lambda^2 E_{H_2} \left(e^{-KR}\right).$$

It will be shown that under the stability condition (6), this equation has two positive solutions.

\textbf{Lemma:} If the stability condition (6) is satisfied, i.e.

$$\frac{D+2q}{\lambda} < p E_{H_1} (R) + q E_{H_2} (R),$$

then the equation

$$A(K) = p(1-K/\lambda) E_{H_1} \left(e^{-KR}\right) + q E_{H_2} \left(e^{-KR}\right) - (1-K/\lambda)^2$$

has exactly two roots $\alpha$ and $\beta$ interior to the right half plane ($\text{Re}(K) > 0$). Furthermore, both roots are real, and

$$0 < \alpha < \lambda < \beta < 2\lambda$$

(These solutions are illustrated in figure 4.)
Figure 4.
Solutions to Equation (19)
Proof: (Patterned on a proof by Karlin in [3].) Let

\[(23) \quad B(K) = p(1-K/\lambda) E_{H_1}^{\lambda_1}(e^{-KR}) + q E_{H_2}^{\lambda_2}(e^{-KR})\]

and

\[(24) \quad C(K) = (1-K/\lambda)^2.\]

One notes that \(C(K)\) has a zero of multiplicity two at \((\lambda, 0)\). If there exists an open simply connected region \(R\) in the right half plane such that

(i) \((\lambda, 0) \in R,\)

(ii) \(B(K)\) and \(C(K)\) are analytic in \(R\), and

(iii) \(|B(K)| < |C(K)|\) on the boundary of \(R,\)

then by a theorem due to Rouché (see [6]), \(A(K)\) has exactly two zeros in \(R\), counting multiplicities. Letting \(K = re^{i\theta},\)

\[(25) \quad |B(r,\theta)| \leq \rho \left\{ \left(1 - \frac{r \cos \theta}{\lambda} \right)^2 + \left(\frac{r \sin \theta}{\lambda}\right)^2 \right\}^{1/2} \int_0^\infty e^{-rp \cos \theta} dH_1(\rho)\]

\[\quad + q \int_0^\infty e^{-rp \cos \theta} dH_2(\rho) = |B^*(r,\theta)|\]

\[\quad \leq \rho \left\{ \left(1 - \frac{r \cos \theta}{\lambda} \right)^2 + \left(\frac{r \sin \theta}{\lambda}\right)^2 \right\}^{1/2} \quad \text{for} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\]

and
(26) \[ |C(r, \theta)| = (1 - r \cos \frac{\theta}{\lambda})^2 + (r \sin \frac{\theta}{\lambda})^2. \]

Then along the imaginary axis for \( r \neq 0, \)

(27) \[ \left| B(r, \pm \frac{\pi}{2}) \right| \leq \sqrt{p \left( 1 + (r/\lambda)^2 \right)} \]

\[ < 1 + (r/\lambda)^2 = |C(r, \pm \frac{\pi}{2})|, \]

since \( 0 < p < 1. \) At the origin, \( |B^*(0)| = |C(0)|, \) where \( B^*(K) \) is defined by equation (25). Evaluating the derivatives with respect to \( r \) of \( |B^*(r, \theta)| \) and \( |C(r, \theta)| \) at \( r = 0, \) one obtains

(28) \[ \left. \frac{\partial |B^*(r, \theta)|}{\partial r} \right|_{r=0} = -\left[ \frac{p}{\lambda} + p \frac{E_{n_1}(R)}{R_1} + q \frac{E_{n_2}(R)}{R_2} \right] \cos \theta \]

and

(29) \[ \left. \frac{\partial |C(r, \theta)|}{\partial r} \right|_{r=0} = \frac{2 \cos \theta}{\lambda}. \]

From condition (20), it follows that \( B^*(r, \theta) \) decreases in magnitude faster than \( C(r, \theta) \) in any fixed direction \( \theta \) from the origin, where \( -\pi/2 < \theta < \pi/2. \) Then since

\[ |B^*(r, \pm \frac{\pi}{2})| < |C(r, \pm \frac{\pi}{2})| \quad \text{for} \quad r \neq 0, \]

and since \( B^*(K) \) and \( C(K) \) are analytic in the right half plane, there exists an \( \epsilon_o > 0 \) such that

\[ |B(\epsilon, \theta)| \leq |B^*(\epsilon, \theta)| < |C(\epsilon, \theta)| \quad \text{for any} \quad 0 < \epsilon \leq \epsilon_o, \]
and for all $-\pi/2 \leq \theta \leq \pi/2$. Letting $r = 2\lambda$, it is seen that

$$|B(2\lambda, \theta)| \leq p \left(5 - 4 \cos \theta\right)^{1/2}$$

$$< 5 - 4 \cos \theta = |C(2\lambda, \theta)| \quad \text{for} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$ 

Thus if $R$ is the region in the right half plane such that $\varepsilon_0 < r < 2\lambda$ and $-\pi/2 < \theta < \pi/2$, the first part of the lemma is established. The existence of two real roots, and condition (22), are readily shown by examining the behavior of $A(K)$ along the positive real axis. Since $A(K)$ is continuous for $K \geq 0$ and since

$$A(0) = 0,$$

$$A'(0) < 0,$$

$$A(\lambda) > 0,$$

and

$$A(K) < 0 \quad \text{for} \quad K \geq 2\lambda,$$

there is one zero crossing from below in the interval $0 < K < \lambda$ and one zero crossing from above in the interval $\lambda < K < 2\lambda$.

The general solution to equation (17) can now be written as

$$f(y) = C_1 \alpha e^{\alpha(y-S)} + C_2 \beta e^{\beta(y-S)} \quad \text{for} \quad y < S.$$ 

The constants $C_1$ and $C_2$ are determined by substituting this solution into equations (8) and (15), and making use of (16) and (19) to simplify the resulting equations. The expressions obtained are
\[ C_1 = \frac{2(\lambda-\alpha)^2(\beta-\lambda)}{q_\lambda^2(\beta-\alpha)E_{H_2}(e^{-\lambda R})} \left( pE_{H_1}(e^{-\lambda R}) + q_\lambda E_{H_2}(\text{Re}^{-\lambda R}) \right) - \frac{(\lambda-\alpha)^2}{\lambda(\beta-\alpha)} \]

and

\[ C_2 = \frac{2(\lambda-\alpha)(\beta-\lambda)^2}{q_\lambda^2(\beta-\alpha)E_{H_2}(e^{-\lambda R})} \left( pE_{H_1}(e^{-\lambda R}) + q_\lambda E_{H_2}(\text{Re}^{-\lambda R}) \right) + \frac{(\beta-\lambda)^2}{\lambda(\beta-\alpha)} . \]

The jump at \( S \) is evaluated as

\[ dF(S) = 1 - C_1 - C_2 \]

\[ = 3 - \frac{\beta+\alpha}{\lambda} - \frac{2(\lambda-\alpha)(\beta-\lambda)}{q_\lambda^2E_{H_1}(e^{-\lambda R})} \left( pE_{H_1}(e^{-\lambda R}) + q_\lambda E_{H_2}(\text{Re}^{-\lambda R}) \right) . \]

Now substituting the stationary distribution \( F(y) \) for the stock level after delivery into equation (11), one obtains as the limiting distribution \( G(y) \) for the stock level at the beginning of an arbitrary period

\[ G(y) = \begin{cases} \frac{C_1}{(1+q)(\lambda-\alpha)} \left[ ((\lambda-\alpha) + q_\lambda) e^{\alpha(y-S)} - q_\lambda e^{\lambda(y-S)} \right] \\ + \frac{C_2}{(1+q)(\beta-\lambda)} \left[ ((\beta-\lambda) - q_\lambda) e^{\beta(y-S)} + q_\lambda e^{\lambda(y-S)} \right] + \frac{q}{1+q} e^{\lambda(y-S)} \\ \text{for } y \leq S \\ 1 \text{ for } y > S \end{cases} \]

Similarly, substituting into equation (12), one obtains as the limiting distribution \( G^*(y) \) for the stock level at the end of an arbitrary period
\[ G^*(y) = \begin{cases} 
\lambda C_1 \left\{ \frac{\alpha - \lambda (1+q)}{(1+q)(\lambda - \alpha)} \left[ e^{\lambda(y-S)} - e^{\alpha(y-S)} \right] - \frac{q \lambda (S-y)}{(1+q)(\lambda - \alpha)} e^{\lambda(y-S)} \right\} \\
+ \lambda C_2 \left\{ \frac{\beta - \lambda (1+q)}{(1+q)(\beta - \lambda)} \left[ e^{\lambda(y-S)} - e^{\beta(y-S)} \right] + \frac{q \lambda (S-y)}{(1+q)(\beta - \lambda)} e^{\lambda(y-S)} \right\} \\
+ \left\{ 1 + \frac{q \lambda (S-y)}{(1+q)} \right\} e^{\lambda(y-S)} \quad \text{for } y \leq S \\
1 \quad \text{for } y > S. 
\end{cases} \]
6. MULTIPLE STOCKING POINTS, DELIVERY EACH PERIOD

This section deals with the problem of controlling the inventory level for a single item at each of a group of \( n \) stocking points. It is assumed that a random quantity of resupply is available each period, and is offered in the fixed priority order 1, 2, \( \ldots \), \( n \) to the stocking points. That is, the available delivery quantity each period is first offered to stocking point 1. Then any balance remaining is offered to stocking point 2. Continuing in this manner, the balance remaining is offered to the \( n \)th stocking points in order, and the \( n \)th stocking point has the opportunity to accept delivery only if the requirements of all other stocking points have been met. The total time to run through this sequence of \( n \) offers of the available delivery quantity is assumed negligible in comparison with the length of the basic time period. It is further assumed that the distribution of delivery quantity and the distributions of demands on the individual stocking points are all mutually independent.

Systems with characteristics resembling these occur frequently in practice. For example, the \( n \) stocking points might represent \( n \) grocery stores visited in rotation each morning by a truck carrying dairy products.

It is assumed that each stocking point is following one of the three stocking policies defined in sections 2, 3, and 4 above, i.e. policy I, II, or III. However, all need not be following the same policy. A necessary condition for the existence of a stationary distribution of
stock level at each of the stocking points is formulated, and a method for computing the stationary distributions is shown.

The stationary distributions of stock level are then used to compute stationary-one-period expected costs. A basis for selecting an optimum stocking policy for the system of \( n \) stocking points is formulated.

Stationary Distribution of Stock Level After Delivery

In what follows, the superscript \( i \) will be used to denote variables and distributions associated with the \( i^\text{th} \) stocking point.

The sequence \( \{\xi_j^i\} \) of demands on the \( i^\text{th} \) stocking point is assumed to be a sequence of independent, identically distributed random variables with density function \( \phi^i(\xi) \). Furthermore these sequences are mutually independent for the \( n \) stocking points, and are also independent of \( \{R_j\} \), the sequence of random variables representing the delivery quantities available to the system.

The basic stability condition which is necessary to prevent the mean stock level at any of the stocking points from tending to minus infinity is that the average quantity available for delivery exceed the average demand. In other words,

\[
(1) \quad \sum_{i=1}^{n} \int_{0}^{\infty} \xi \phi^i(\xi) \, d\xi < E_H(R),
\]
where \( H(r) \) is the distribution of \( R \). Since, for the reorder policies considered here, delivery is not accepted by the stocking point unless the quantity on hand is below a specified finite level, the mean stock level cannot drift off to plus infinity at any of the stocking points.

If the first stocking point is following one of the stocking policies I, II, or III, and if condition (1) is satisfied, it is clear that the stationary distribution \( F^1(y) \) of the stock level \( Y^1 \) after delivery is obtained as in section 5. The fact that other stocking points rely for resupply on the delivery amounts not accepted by the first stocking point clearly has no effect on \( F^1(y) \). However, as discussed later, the optimum values of \( s^1 \) and \( S^1 \) will in general depend on the demand distributions and stocking policies at the other stocking points.

Considering next the second stocking point, it is immediately evident that the distribution of quantity of goods available to the second stocking point is not the same in each period. Corresponding to a given initial distribution \( F^1_0(t) \), there will be a sequence \( \{H^2_j(r)\} \) of distributions of delivery quantity available to the second stocking point. If the first stocking point is following stocking policy III with critical levels \( s^1 \) and \( S^1 \), then

\[
(2) \quad H^2_{j+1}(r) = \begin{cases} 
\int_{-\infty}^{\infty} \int_{-\infty}^{s^1+\xi} \int_{0}^{r+S^1+\xi-t} \phi(\xi) \, dH^1_{j}(\rho) \, dF^1_{j}(t) \, d\xi \\
\quad + H^1_j(r) \int_{0}^{\infty} \int_{s^1+\xi}^{\infty} \phi(\xi) \, dF^1_{j}(t) \, d\xi & \text{for } r > 0 \\
0 & \text{for } r \leq 0,
\end{cases}
\]
where \( F_j^1(t) = (T_j^1)^{t} F_0^1(t) \) and \( H_j^1(r) = H(r) \) for all \( j \). The first of these equations is obtained by considering the possible ways in which the event \( R_{j+1}^2 < r \) can occur. If \( X_j^1 + \xi_{j+1}^1 < s^1 \), then the first stocking point will accept delivery of the quantity \( s^1 + \xi_{j+1}^1 - X_j^1 \), if available, and thus \( R_{j+1}^2 < r \) if \( R_{j+1}^1 < r + s^1 + \xi_{j+1}^1 - X_j^1 \). If \( X_j^1 + \xi_{j+1}^1 > s^1 \), then no delivery to the first stocking point takes place, and \( R_{j+1}^2 < r \) if \( R_{j+1}^1 < r \). The second of the equations (2) follows from the condition \( R_{j+1}^1 \geq 0 \). If the first stocking point is following stocking policy I or II, relations similar to those of (2) may be obtained for \( H_{j+1}^2(r) \).

If the second stocking point is following stocking policy III, and if the initial distribution of stock at the second stocking point is \( F_0^2(t) \), then the distribution of the stock level \( X_{j+1}^2 \) at time \( j+1 \) is given by

\[
(3) F_{j+1}^2(y) = T_{j+1}^2 F_j^2(y) = (T_j^2)^{j+1} F_0^2(y)
\]

\[
= \begin{cases} 
\int_0^\infty \int_{-\infty}^{y+\xi} \int_0^{y+\xi-t} \varphi^2(\xi) \, dH_j^2_{j+1}(r) \, dF_j^2(t) \, d\xi & \text{for } y \leq s^2 \\
\int_0^\infty \int_{s^2+\xi}^{y+\xi} \varphi^2(\xi) \, dF_j^2(t) \, d\xi \\
\quad + \int_0^\infty \int_{-\infty}^{s^2+\xi} \int_0^{y+\xi-t} \varphi^2(\xi) \, dH_j^2_{j+1}(r) \, dF_j^2(t) \, d\xi & \text{for } s^2 < y \leq S^2.
\end{cases}
\]
Similar equations hold if the second stocking point is following policy I or II.

Referring back to equations (2), one notes that if $F^1_j(t)$ is assumed to converge in the limit to a unique stationary distribution $F^1(t)$, independent of $F^1_0(t)$, then $H^2_{j+1}$ also converges in the limit to a unique stationary distribution $H^2(r)$, independent of $F^1_0(t)$. Then it is assumed that the sequence $\{T^2_j\}$ of operators converges in the limit to an operator $T^2$, and that a unique stationary distribution $F^2(t)$ of stock level after delivery at the second stocking point exists. $F^2(t)$ is then a fixed point of $T^2$. This stationary distribution in turn determines a stationary distribution $H^3(r)$ of stock available to the third stocking point.

In general, suppose the $k^{th}$ stocking point is following stocking policy III, and that the stationary distribution $F^k(t)$ of stock level after delivery has been determined. Then the stationary distribution of the delivery quantity remaining for use by the $(k+1)^{st}$ stocking point is

\[
H^{k+1}(r) = \begin{cases} 
\int_0^\infty \int_{-\infty}^{r+S^k_+} \int_{s^k_+}^{r+S^k_+} \phi^k(\xi)dH^k(\rho)dF^k(t)d\xi \\
+ H^k(r) \int_0^\infty \int_{s^k_+}^{\infty} \phi^k(\xi)dF^k(t)d\xi 
\end{cases} \quad \text{for } r > 0
\]

(4) $H^{k+1}(r) = \begin{cases} 
0 
\end{cases} \quad \text{for } r \leq 0$.\]
This result follows from equations (2) and the analysis above. Since \( \phi^k(\xi) \) is assumed continuous, \( dH^{k+1}(r) \) may be represented as

\[
(5) \quad dH^{k+1}(r) = \begin{cases} 
  dr \int_r^\infty \int_{r+S^k-t}^\infty \phi^k(\rho-r-t) \, dF^k(t) \, dH^k(\rho) \\
  + dH^k(r) \int_0^\infty \int_{s+\xi}^\infty \phi^k(\xi) \, dF^k(t) \, d\xi 
  & \text{for } r > 0 \\
  \int_0^\infty \int_{s+\xi}^\infty \int_{s+\xi-t}^\infty \phi^k(\xi) \, dH^k(\rho) \, dF^k(t) \, d\xi 
  & \text{for } r = 0.
\end{cases}
\]

If the \( k \)th stocking point is following stocking policy I, then instead of equations (4) one obtains

\[
(6) \quad H^{k+1}(r) = \begin{cases} 
  \int_{-\infty}^\infty \int_0^\infty \int_{r+S^k-t}^\infty \phi^k(\xi) \, dH^k(\rho) \, d\xi \, dF^k(t) 
  & \text{for } r > 0 \\
  0 
  & \text{for } r \leq 0.
\end{cases}
\]

This result can be obtained through an analysis similar to that above, or by letting \( S^k = S^k \) in equation (4). If the \( k \)th stocking point is following stocking policy II, then instead of equation (4) one obtains

\[
(7) \quad H^{k+1}(r) = \begin{cases} 
  \int_0^\infty \int_{-\infty}^{s+\xi} \phi^k(\xi) \, dF^k(t) \, d\xi + H^k(r) \int_0^\infty \int_{s+\xi}^\infty \phi^k(\xi) \, dF^k(t) \, d\xi 
  & \text{for } r > 0 \\
  0 
  & \text{for } r \leq 0.
\end{cases}
\]
This result can be obtained through an analysis paralleling that used to derive (4). However, it may be observed that (7) is obtained as a limiting case of equations (4) by letting $S^k$ approach infinity.

Having obtained $H^{k+1}(r)$, one can then compute the stationary distribution $F^{k+1}(t)$ of stock level after delivery at the $(k+1)^{st}$ stocking point. Proceeding in this manner, it is clear that one can compute in turn the stationary distributions for each of the $n$ stocking points.

Selection of Optimum Stocking Policy

In computing stationary one-period expected costs for the $k^{th}$ stocking point, it is assumed that if stock remains on hand at the close of a period, a linear holding cost $c_k^1$ is incurred, and that if demand has exceeded supply during a period, a linear shortage cost $c_k^2$ is incurred. Further, it is assumed that if delivery of additional supply is accepted, a fixed cost $K_k$ is incurred. If the expected stationary one-period cost for the $k^{th}$ stocking point is denoted by

$L^k(s^1, s^2, s^2, \ldots, s^k, S^k)$, indicating the dependence of $F^k(t)$ on $(s^1, s^2, s^2, \ldots, s^k, S^k)$, then

(8) $L^k(s^1, s^2, \ldots, s^k, S^k) = c_k^1 \int_0^\infty \int_0^{\xi} (t-\xi) \phi^k(\xi) \, dF^k(t) \, d\xi$

$- c_k^2 \int_0^\infty \int_{-\infty}^{\xi} (t-\xi) \phi^k(\xi) \, dF^k(t) \, d\xi$

$+ K_k [1 - dH^k(0)] \int_0^\infty \int_{-\infty}^{S^k+\xi} \phi^k(\xi) \, dF^k(t) \, d\xi$. 
The final term of this expression is the product of $k^k$, of the probability that a non-zero quantity of goods is available for delivery, and of the probability that the stock level at the end of the period is below $s_k$.

Forming the total cost function $L$ for the system of $n$ stocking points,

$$L(s^1, S^1, \ldots, s^n, S^n) = \sum_{k=1}^{n} L(s^1, S^1, \ldots, s^k, S^k).$$

The optimum stocking policy is defined as that set of numbers $(s^1, S^1, \ldots, s^n, S^n)$ which maximizes $L(s^1, S^1, \ldots, s^n, S^n)$, subject to the restrictions

$$0 \leq s^k \leq S^k < \infty \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Another problem of interest which could be investigated through use of the above results is the determination of the optimum delivery priority sequence for the $n$ stocking points. For example, if a delivery truck makes a daily round of the $n$ stocking points, there are $n!$ possible delivery sequences to choose among. Using the above results, it would be possible to determine an optimum stocking policy, and the corresponding value of stationary one-period expected costs, for each of these possible delivery sequences. Then to each of these delivery sequences, one could assign an additional cost factor which is a function of the distance travelled, number of hours required to carry out the sequence of deliveries, or both. By considering the sum of these two cost factors for each of the $n!$ possible delivery sequences, one could select that delivery sequence which resulted in the smallest total cost.
7. MULTIPLE STOCKING POINTS, UNCERTAIN DELIVERY

In this section, the multiple stocking point model in the previous section is extended to the case in which deliveries are not available each period. Instead, the resupply process is assumed to be as in the model of section 5. In other words, the times between availabilities of deliveries are a sequence of identically distributed random variables, and the delivery size distribution is dependent on the number of periods since the last delivery.

When a delivery becomes available, it is offered in the fixed priority sequence 1, 2, ... n to the stocking points, as in the model of section 6. It is assumed that each stocking point is following one of the stocking policies defined in sections 2,3, and 4, i.e. policy I, II, or III. However, the stocking points need not all follow the same policy. It should be noted that under these policies each stocking point makes its decision on acceptance of delivery without regard to the stock level at any of the other stocking points. However, the stocking policies are interdependent through the process of determining the set of stocking policies which minimizes the limiting one-period expected costs for the entire group of n stocking points.

Below, the problem of two stocking points is first considered. Then the method of extending the analysis to an arbitrary number of stocking points is outlined. For the case of two stocking points, a
stability condition necessary for the existence of a stationary distribution of stock level after delivery is formulated, and functional equations satisfied by this distribution are derived. Then relations for the limiting distribution of stock at the beginning of an arbitrary period are derived, and the method of determining an optimum set of stocking policies for the system indicated.

Stationary Distribution of Stock Level After Delivery

Here, functional equations for the joint stationary distribution of stock level after delivery are derived for the two stocking point model described above.

It is assumed that deliveries of a random size \( R_j \) become available to the system of two stocking points at a random sequence \( \{ \eta_j \} \) of times; furthermore, it is assumed that \( \Pr(\eta_1 = 0) = 1 \), and that the sequence \( \{ D_j \} = \{ \eta_{j+1} - \eta_j \} \) of times between delivery availabilities is a sequence of independent, identically distributed random variables. Letting the random variable \( D \) represent the time between any two successive delivery availabilities, the probability distribution of \( D \) is given by

\[
(1) \quad \Pr(D = k) = d_k \quad \text{for} \quad k = 1, 2, \ldots ,
\]

where
\[ \sum_{k=1}^{\infty} d_k = 1 \]

It is assumed that the conditional distribution \( H_i(r) \) of available delivery quantity, given that \( i \) periods have elapsed since the last delivery, is otherwise independent of the time period and is defined by

\[ H_i(r) = \Pr[R_{n_{j+1}} < r | D_j = i] \quad \text{for } i = 1, 2, \ldots, \]

and for all \( j \).

The sequences \( \{\xi_1\} \) and \( \{\zeta_1\} \) of demands at the first and second stocking points are assumed to each be sequences of independent, identically distributed random variables, with density functions \( \phi(\xi) \) and \( \psi(\zeta) \) respectively. Furthermore, these two distributions, the delivery time distribution, and the delivery quantity distributions are assumed to be mutually independent, except for the dependence noted above of the distribution of the quantity available for delivery on the time between deliveries.

The random variable \( X_j \) will represent the stock level at the first stocking point at time \( j \), after taking into account any delivery received at time \( j \), and \( Y_j \) will represent the comparable stock level for the second stocking point. The joint distribution of \( X_j \) and \( Y_j \) will be denoted \( F_j(x,y) \). The stocking policies considered will again be those introduced in sections 2, 3, and 4. Although the sequence \( \{X_j, Y_j\} \) of pairs does not in general form a Markov chain, the subsequence
\{X_{\eta_j}, Y_{\eta_j}\} of stock levels after delivery does form a Markov chain. In other words, given the stock levels at time \(\eta_k\), the conditional probability of an event at time \(\eta_l\), where \(l > k\), is the same as the conditional probability of the same event given the stock levels at times \(\eta_1, \eta_2, \ldots, \eta_k\). This Markov process will be investigated.

If both stocking points are following stocking policy III, i.e. if the first stocking point accepts delivery up to its maximum storage capacity \(S^1\) if the stock level before delivery is below \(s^1\), and otherwise rejects delivery, and if the second stocking point follows the same procedure with critical levels \(s^2\) and \(S^2\), then the resulting one-step transition relations are

\[
X_{\eta_j} = \begin{cases} 
X_{\eta_j} - \xi_{\eta_j} & \text{if } X_{\eta_j} - \xi_{\eta_j} > s^1 \\
X_{\eta_j} - \xi_{\eta_j} + R_{\eta_j} & \text{if } X_{\eta_j} - \xi_{\eta_j} < s^1 \text{ and } X_{\eta_j} - \xi_{\eta_j} + R_{\eta_j} < S^1 \\
S^1 & \text{if } X_{\eta_j} - \xi_{\eta_j} < s^1 \text{ and } X_{\eta_j} - \xi_{\eta_j} + R_{\eta_j} \geq S^1,
\end{cases}
\]

and
where all the above summations are carried out over the range $\eta_{j+1} \leq i < \eta_{j+1}$.

These relations determine a stationary transition operator $T$, which transforms the distribution $F_{\eta_j}$ into $F_{\eta_{j+1}}$, i.e.

\[
(5) \quad F_{\eta_{j+1}} = T F_{\eta_j}
\]

Following the methods which have been used repeatedly above, this transformation can be written out in detail as
\[
\sum_{i=1}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{x+\xi} \int_{0}^{y+\xi} \int_{-\infty}^{x+\xi-t} \phi(i)(\xi)\psi(i)(\xi) \, dH_{1}(r) \, dF_{\eta_{j}}(t,u) \, d\xi d\xi \\
\text{if } x \leq s^{1}, \text{ any } y
\]

\[
\sum_{i=1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \{ \int_{s^{1}+\xi}^{x+\xi} \int_{y+\xi}^{y+\xi-t} \int_{\infty}^{\infty} \phi(i)(\xi)\psi(i)(\xi) \, dH_{1}(r) \, dF_{\eta_{j}}(t,u) \} \, d\xi d\xi \\
\text{if } s^{1} < x \leq s^{1}, y \leq s^{2}
\]

\[
\sum_{i=1}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \{ \int_{s^{1}+\xi}^{x+\xi} \int_{s^{2}+\xi}^{\infty} \int_{0}^{\infty} \int_{s^{1}+\xi}^{x+\xi-t} \phi(i)(\xi)\psi(i)(\xi) \, dH_{1}(r) \, dF_{\eta_{j}}(t,u) \} \, d\xi d\xi \\
\text{if } s^{1} < x \leq s^{1}, s^{2} < y \leq s^{2}
\]

(6) \quad F_{\eta_{j+1}}(x,y) = \sum_{i=1}^{\infty} d_{i} \left\{ \int_{0}^{\infty} \int_{s^{1}+\xi}^{x+\xi} \phi(i)(\xi) \, dF_{\eta_{j}}(t,\infty) \, d\xi \\
+ \int_{0}^{\infty} \int_{s^{1}+\xi}^{x+\xi-t} \phi(i)(\xi) \, dH_{1}(r) \, dF_{\eta_{j}}(t,\infty) \, d\xi \right\} \\
\text{if } s^{1} < x \leq s^{1}, y > s^{2}
\]

\[
\sum_{i=1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \{ \int_{s^{1}+\xi}^{x+\xi} \int_{-\infty}^{y+\xi} \int_{\infty}^{\infty} \phi(i)(\xi)\psi(i)(\xi) \, dH_{1}(r) \, dF_{\eta_{j}}(t,u) \} \, d\xi d\xi \\
\text{if } x > s^{1}, y \leq s^{2}
\]
\[
\sum_{i=1}^{\infty} d_i \left\{ \int_{0}^{\infty} \int_{s^2 + \xi}^{y + \xi} \psi^{(1)}(\xi) dF_{\eta_j}(\infty, u) d\xi + \int_{0}^{\infty} \int_{s^1 + \xi}^{\infty} \int_{-\infty}^{s^2 + \xi} \int_{0}^{x + \xi} y + \xi - u \right. \\
+ \int_{-\infty}^{s^1 + \xi} \int_{s^2 + \xi}^{\infty} \int_{s^1 + \xi - t + y + \xi - u}^{\infty} \phi^{(1)}(\xi) \psi^{(1)}(\xi) dH_1 (r) dF_{\eta_j}(t, u) d\xi d\xi d\xi \\
\left. \quad \text{if } x > s^1, s^2 < y \leq s^2 \right. \\
\quad \text{if } x > s^1, y > s^2,
\]

where \( F_{\eta_j}(\infty, u) \) and \( F_{\eta_j}(\infty, y) \) designate the marginal distributions of \( X_{\eta_j} \) and \( Y_{\eta_j} \), respectively. Similar transition relationships can be formulated for stocking policies I and II, or for any combination of the three basic stocking policies.

A necessary stability condition to ensure that the mean stock levels at both stocking points remain finite under any combination of the three basic stocking policies is that the average quantity available for delivery exceed the average demand. In other words,

\[
E(D) \left\{ \int_{0}^{\infty} \xi \phi(\xi) d\xi + \int_{0}^{\infty} \xi \psi(\xi) d\xi \right\} < \sum_{i=1}^{\infty} d_i E_{H_1}(R).
\]

It is assumed that if this condition is satisfied, a unique joint stationary distribution \( F(x,y) \) of stock level at the two stocking points exists, subject to possible additional restrictions on \( H(r) \). If both stocking points are following policy III, then \( F(x,y) \) is found by solving the equations
\[
\sum_{i=1}^{\infty} \int_0^\infty \int_{-\infty}^{x+\xi} \int_{-\infty}^{y+\xi-t} \phi(i)(\xi) \psi(i)(\xi) dH_i(r) dF(t,u) d\xi d\xi
\]
if \( x \leq s^1 \), any \( y \)

\[
\sum_{i=1}^{\infty} \int_0^s \int_{s+\xi}^s \int_{s+\xi}^{y+\xi-t-u} \int_{-\infty}^{x+\xi} \int_{-\infty}^{y+\xi-t-u} \phi(i)(\xi) \psi(i)(\xi) dH_i(r) dF(t,u) d\xi d\xi
\]
if \( 1 < x \leq s^1, y \leq s^2 \)

\[
\sum_{i=1}^{\infty} \int_0^s \int_s^{x+\xi} \int_{s+\xi}^{y+\xi-t} \int_{-\infty}^{y+\xi-t-u} \int_{-\infty}^{x+\xi} \int_{-\infty}^{y+\xi-t-u} \phi(i)(\xi) \psi(i)(\xi) dH_i(r) dF(t,u) d\xi d\xi
\]
if \( 1 < x \leq s^1, s^2 < y \leq s^2 \)

(8) \( F(x,y) = \sum_{i=1}^{\infty} \int_0^s \int_{s+\xi}^{x+\xi} \phi(i)(\xi) dF(t,u) d\xi \)

+ \( \int_0^s \int_{-\infty}^{s+\xi-t} \phi(i)(\xi) dH_i(r) dF(t,u) d\xi \)
if \( 1 < x \leq s^1, y > s^2 \)

\[
\sum_{i=1}^{\infty} \int_0^\infty \int_{-\infty}^{x+\xi} \int_{-\infty}^{y+\xi-t} \phi(i)(\xi) \psi(i)(\xi) dH_i(r) dF(t,u) d\xi d\xi
\]
if \( x > s^1, y \leq s^2 \)
If both stocking points are following stocking policy I, i.e., accepting delivery up to a fixed storage capacity whenever available, an analysis similar to the above gives the equations

\[
F(x,y) = \begin{cases} 
\sum_{i=1}^{\infty} \int_0^\infty \int_0^\infty \int_{-\infty}^{+\xi} \int_{-\infty}^{T+\xi-t} \phi(i) \psi(i) dH_i(r) dF(t,u) d\xi d\xi 
& \text{if } x \leq S_1, \text{ any } y \\
\sum_{i=1}^{\infty} \int_0^\infty \int_0^\infty \int_{-\infty}^{+\xi} \int_{-\infty}^{T+\xi-t+y+\xi-u} \phi(i) \psi(i) dH_i(r) dF(t,u) d\xi d\xi 
& \text{if } x > S_1, y \leq S_2 \\
1 & \text{if } x > S_1, y > S_2
\end{cases}
\]

satisfied by the joint stationary distribution of stock level at the two stocking points. This result also follows from letting \( s_1 = S_1 \) and \( s_2 = S_2 \) in equations (5) above. Similarly, if both stocking points are following stocking policy II, i.e., accepting the full available delivery if the stock level is below a specified level, and otherwise rejecting delivery, then one obtains the equations
which are satisfied by the joint stationary distribution of stock level after delivery at the two stocking points. These equations also result from letting $S^1 \to \infty$ and $S^2 \to \infty$ in equations (8) above. Equations similar to (8), (9) and (10) above for the joint distribution of stock level after delivery can be obtained for any other pairing of policies for the two stocking points (for example, stocking policy III for the first stocking point and II for the second).

The above analysis can be extended to more than two stocking points. For example, if three stocking points are considered, all following stocking policy III, the one-step transitions for the stock levels $X_{\eta_{j+1}}$, $Y_{\eta_{j+1}}$, and $Z_{\eta_{j+1}}$ are given by equations (3) and (4) above, and by
\[
\begin{align*}
Z_{\eta_{j+1}} &= \begin{cases} 
Z_{\eta_j} - \Sigma y_{i+1} & \text{if } Z_{\eta_j} - \Sigma y_{i+1} \geq s^3 \text{ or } Z_{\eta_j} - \Sigma y_{i+1} < s^3 \text{ and } Y_{\eta_j} < S^2 \\
Z_{\eta_j} - \Sigma y_{i+1} + Y_{\eta_j} - \Sigma x_{i+1} - x_{i+1} - R_{\eta_j} & S^1 - S^2 \\
& \text{if } Z_{\eta_j} - \Sigma y_{i+1} < s^3, Y_{\eta_j} = S^2, \text{ and } Y_{\eta_{j+1}} < S^3 \\
S^3 & \text{if } Z_{\eta_j} - \Sigma y_{i+1} < s^3, Y_{\eta_j} = S^2, \text{ and } Y_{\eta_{j+1}} \geq S^3 
\end{cases}
\end{align*}
\]

where the summations extend over \( \eta_{j+1} \leq i \leq \eta_{j+1} \) as before. From these relations, the transition operator can be constructed as above, and thus the functional equations which must be satisfied by the joint stationary distribution of stock level obtained.

Stationary Distribution of Stock Level at Beginning of Arbitrary Period

Above, the method of determining the joint stationary distribution \( F(x, y) \) of stock level after delivery for two stocking points was investigated. Here it will be assumed that this distribution exists and has been obtained, and the problem of finding the joint stationary distribution \( G(x, y) \) of stock level at the beginning of an arbitrary period will be considered.
As formulated above in section 5, the limiting distribution of the number of periods since the last delivery is given in terms of the distribution of times between deliveries by

\[ \delta_k = \frac{1}{E(D)} \sum_{i=k+1}^{\infty} d_i \quad \text{for} \quad k = 0, 1, 2, \ldots. \]

Then given that the last delivery took place \( k \) periods earlier, the stock level at the first stocking point will be below \( x \) and at the second stocking point below \( y \) if and only if the stock levels after the last delivery, less the demands during the intervening \( k \) periods, are less than \( x \) and \( y \) respectively. Multiplying the conditional probability of this event by \( \delta_k \), the probability that the last delivery took place \( k \) periods earlier, and summing over all \( k \), one obtains the result

\[ G(x, y) = \sum_{k=0}^{\infty} \delta_k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{x+\xi} \int_{-\infty}^{y+\xi} \phi^{(k)}(\xi) \psi^{(k)}(\xi) dF(t, u) \, d\xi d\xi \]

for all \( x, y \),

where

\[ \int_{-\infty}^{x} \phi^{(0)}(\xi) \, d\xi = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \]

and

\[ \int_{-\infty}^{y} \psi^{(0)}(\xi) \, d\xi = \begin{cases} 0 & \text{if } y \leq 0 \\ 1 & \text{if } y > 0 \end{cases} \]
This result holds for any pairing of the three stocking policies described above; the selected stocking policies enter into the result only through the computation of $F(x, y)$.

If one wishes to find the joint stationary distribution of stock level at the end of an arbitrary period, rather than the beginning, the above computation must be modified to include the demands during one additional period. Thus the joint stationary distribution $G^*(x, y)$ of stock level at the end of an arbitrary period is given by

$$(14) \quad G^*(x, y) = \sum_{k=0}^{\infty} \delta_k \int_0^{\infty} \int_0^{x+5} \int_{-\infty}^{y+5} \phi^{(k+1)}(\xi) \psi^{(k+1)}(\xi) dF(t, u) d\xi d\xi$$

for all $x, y$.

Results similar to the above hold for the case of three or more stocking points. The above joint stationary distributions can be used to compute stationary one-period expected costs for the system of two stocking points. Then assuming that the criterion for optimization of the system is the minimization of these expected costs, the optimum policy can be determined by finding that set $(s^1, s^1, s^2, s^2)$ of values, subject to the restrictions $s^1 \leq s^1 \leq \infty$ and $s^2 \leq s^2 \leq \infty$ which minimize these costs.
8. POLICY IV--AUTOMATIC RESUPPLY, PROVISION FOR EMERGENCY DELIVERY

This section deals again with the problem of controlling the inventory level of a single item at a single stocking point. The stocking policy considered is a combination of policy II (as defined in section 3) applied to a sequence of random delivery quantities for normal resupply, plus an \((s,S)\) policy to provide for emergency resupply.

Under this policy, the inventory level at the stocking point is examined at the beginning of each period. If the stock level is below a fixed level \(S\) (corresponding to \(s\) in policy II), the stocking point accepts delivery of a random quantity \(R\) of goods. If the stock level is above \(S\), this routine delivery is omitted. (It should be noted that this model includes the special case in which the routine delivery quantity is fixed rather than random.) It is assumed that if the decision to accept the routine delivery is made, this delivery takes place prior to the beginning of the following period.

In the first version of this model, it is assumed that the routine delivery takes place immediately. Then considering the stock level including this delivery, a second ordering decision is made. If the stock level is below an emergency level \(s\), where \(s < S\), an order is placed for a quantity of goods sufficient to bring the stock level up to \(S\). It is assumed that this delivery takes place prior to the beginning of the following period.
In the second version of this model, it is assumed that the second ordering decision is made without knowing the quantity to be delivered in the routine delivery. This would be the case, for example, if the decision to take the routine delivery represents a decision to initiate production of a standard production lot of random size. Then if the stock level at the beginning of the period is below the emergency level $s$, say at $x$, an order is placed for an additional quantity $S - x$. Again it is assumed that this delivery takes place prior to the beginning of the following period.

In both versions, it will be noted that if the routine delivery is reduced to zero, i.e. if

$$\Pr(R = 0) = 1,$$

then the stocking policy reduces to the $(s,S)$ model discussed in [1] and [5]. On the other hand, if $s = -\infty$, both versions reduce to models of the policy II type, as discussed in section 3 above.

The study of this model was motivated by an interest in investigating the effect of introducing into an inventory system following an $(s,S)$ policy an automatic delivery each period of a fixed or random quantity of resupply. Stocking policies of this type are known to be in use by industrial firms. Also, the Army is considering systems with essentially the stocking policies studied here for use in the supply of combat units. In both applications, the routine deliveries are expected to supply the major portion of the demand, with corrective action in the form of either rejecting
the routine delivery or ordering additional resupply taken only infrequently. Thus one would ordinarily expect the mean delivery rate for routine deliveries to be approximately equal to the mean demand rate.

Below, functional equations satisfied by the stationary distribution of stock level at the beginning of a period are derived for both versions of the above model. Cost structures are then assumed, and examples carried out to illustrate the selection of optimum decision parameters.

Stationary Distribution of Stock Level: First Version

Here, the first version of the model outlined above is considered. The random variable $X_j$ will represent the stock level at the beginning of the $(j+1)^{st}$ period, i.e. at time $j$, and the random variable $\xi_j$ will represent the demand on the stocking point during the $j^{th}$ period. It is assumed that the sequence $\{\xi_j\}$ of demands is a sequence of independent random variables, each distributed with known continuous density function $\varphi(\xi)$.

If the stock level $X_j$ at time $j$ is less than the fixed level $S$, then a random quantity $R_{j+1}$, with known distribution $H(r)$ independent of demand, is delivered to the stocking point during the $(j+1)^{st}$ period. However, if the stock level $X_j$ is not less than $S$,
the routine delivery during the \((j+1)\)st period is omitted. Then assuming that the quantity \(R_{j+1}\) is known, as would be the case if this delivery were immediate, a second ordering decision is made at time \(j\). If the new stock level is below an emergency level \(s\), an additional order of a quantity sufficient to bring the stock level up to \(S\) is made, for delivery during the \((j+1)\)st period. Under these conditions, the sequence \(\{X_j\}\) of stock levels forms a Markov chain, with one-step transitions

\[
(1) \quad X_{j+1} = \begin{cases} 
X_j - \xi_{j+1} & \text{if } X_j \geq S \\
X_j + R_{j+1} - \xi_{j+1} & \text{if } X_j < S, \ X_j + R_{j+1} \geq s \\
S - \xi_{j+1} & \text{if } X_j < S, \ X_j + R_{j+1} < s.
\end{cases}
\]

If \(F_j(x)\) is the distribution function for \(X_j\), one then obtains as the relation, in differential form, between the distributions of \(X_j\) and \(X_{j+1}\)

\[
(2) \quad dF_{j+1}(x) = \begin{cases} 
dx \left[ \int_x^\infty \phi(t-x) dF_j(t) + \int_0^\infty \int_{x-r}^S \phi(t+r-x) dF_j(t) dH(r) \right] \\
\quad + \int_0^\infty \int_{-\infty}^{S-r} \phi(S-x) dF_j(t) dH(r) \right] \text{ for } x < s \\
\int_x^\infty \phi(t-x) dF_j(t) + \int_0^\infty \int_{x-r}^S \phi(t+r-x) dF_j(t) dH(r) \\
\quad + \int_0^\infty \int_{-\infty}^{S-r} \phi(S-x) dF_j(t) dH(r) \right] \text{ for } s \leq x < S \\
\int_x^\infty \phi(t-x) dF_j(t) + \int_0^\infty \int_{x-S}^{x-r} \phi(t+r-x) dF_j(t) dH(r) \\
\quad \text{ for } x \geq S.
\end{cases}
\]
These equations are obtained by considering the possible ways in which the event \( X_{j+1} = x \) can occur. Since the stocking policy calls for discontinuing routine deliveries if the stock level rises above \( S \) and for placing additional orders if the stock level drops below \( s \), the stock level will return with probability one from any stock level to the interval \([s, S]\) in a finite number of periods. It is assumed that there exists a unique stationary distribution \( F(x) \) for this process, independent of the initial distribution \( F_0(x) \) of stock level. No stability condition relating the demand and routine resupply distributions is necessary if \( s \) and \( S \) are finite.

Since the demand distribution has a continuous density function \( \phi(x) \), the stationary distribution \( F(x) \) has a density function \( f(x) \) which satisfies the relations

\[
\int_{s}^{\infty} \phi(t-x)f(t)\,dt + \int_{0}^{\infty} \int_{s-r}^{S} \phi(t+r-x)f(t)\,dt\,dH(r) + \int_{0}^{\infty} \int_{s-r}^{S-r} \phi(S-x)f(t)\,dt\,dH(r) \text{ for } x < s
\]

\[
\int_{s}^{\infty} \phi(t-x)f(t)\,dt + \int_{0}^{\infty} \int_{X-r}^{S} \phi(t+r-x)f(t)\,dt\,dH(r)
\]

\[
+ \int_{0}^{\infty} \int_{S-r}^{r} \phi(S-x)f(t)\,dt\,dH(r) \text{ for } s \leq x < S
\]

\[
\int_{s}^{\infty} \phi(t-x)f(t)\,dt + \int_{0}^{\infty} \int_{x-S}^{S} \phi(t+r-x)f(t)\,dt\,dH(r) \text{ for } x \geq S.
\]

It should be noted that \( f(x) \) is continuous, except for a simple discontinuity at \( S \).
Having obtained a stationary distribution \( f(x) \), it is then possible to compute stationary one-period expected costs. First an appropriate cost structure must be defined. It is assumed that the unit cost of goods supplied by the routine delivery is \( a_1 \); if the routine delivery is rejected, a fixed cost of \( K_1 \) is incurred. If an emergency order is placed, it is assumed that a fixed cost of \( K_2 \) plus a unit cost of \( a_2 \) are incurred. At the beginning of each period, prior to deliveries, a linear holding cost \( c_1 x \) or a linear shortage cost \( -c_2 x \) is assessed, depending on whether the stock level \( x \) is positive or negative. Then if both routine and emergency deliveries are assumed to take place immediately after the beginning of the period, the stationary one-period expected costs are given by

\[
(4) \quad L(s,S) = K_1[1-F(s)] + a_1 \int_0^\infty H(r) \, dF(s) \\
+ \int_{-\infty}^S \int_0^{s-x} \left(K_2 + a_2(S-x-r)\right) \, dH(r) \, dF(x) \\
+ c_1 \int_0^\infty x \, dF(x) - c_2 \int_{-\infty}^0 x \, dF(x).
\]

The optimum stocking policy is determined by finding the pair \( (s,S) \) of numbers which minimizes \( L(s,S) \).

**Example 1**

Suppose demand has an exponential distribution

\[
\varphi(\xi) = \lambda e^{-\lambda \xi} \quad \text{for} \quad \xi \geq 0.
\]
Then equations (3) take the form

\[
\begin{align*}
\int_{S}^{\infty} \phi(t)f(t) dt + \int_{-\infty}^{S} \phi(t+r)f(t) dt dH(r) \\
+ \int_{-\infty}^{\infty} \int_{S-r}^{S} \phi(s)f(t) dt dH(r) & \quad \text{for } x < s \\
\int_{-\infty}^{\infty} \int_{-\infty}^{S-r} \phi(s)f(t) dt dH(r) & \quad \text{for } s \leq x < S \\
\int_{x}^{\infty} \phi(t)f(t) dt + \int_{x-S}^{x-r} \phi(t+r)f(t) dt dH(r) & \quad \text{for } x \geq S.
\end{align*}
\]

Through differentiation with respect to \( x \), one obtains

\[
(5) \quad f'(x) - \lambda f(x) = \begin{cases} 
0 & \text{for } x < s \\
-\lambda \int_{-\infty}^{x-r} f(x-r) dH(r) & \text{for } s \leq x < S \\
-\lambda f(x) - \lambda \int_{x-S}^{\infty} f(x-r) dH(r) & \text{for } x \geq S.
\end{cases}
\]

The first of these equations is satisfied by the solution

\[
(6) \quad f(x) = C e^{\lambda(x-s)} \quad \text{for } x < s.
\]

Using this solution, the second equation becomes

\[
f'(x) - \lambda f(x) = -\lambda \int_{-\infty}^{x-S} f(x-r) dH(r) - \lambda \int_{x-S}^{\infty} C e^{\lambda(x-r-s)} dH(r).
\]
Then letting $y = x - s$ and taking the Laplace transform, one obtains

$$
(7) \quad (\omega - \lambda) U(\omega) - f(s+) = -\lambda U(\omega) V(\omega) - \frac{C\lambda}{\lambda - \omega} [V(\omega) - V(\lambda)],
$$

where

$$
(8) \quad U(\omega) = \int_{0}^{\infty} e^{-\omega y} f(y + s) \, dy
$$

and

$$
(9) \quad V(\omega) = \int_{0}^{\infty} e^{-\omega r} \, dH(r).
$$

From the continuity at $s$ of equations (3),

$$
\begin{align*}
\frac{d}{ds} f(s) &= C, \\
\end{align*}
$$

Then after rearrangement, equation (7) becomes

$$
(10) \quad U(\omega) = \frac{C}{\omega - \lambda} \left. \frac{\omega - \lambda + \lambda(V(\omega) - V(\lambda))}{\omega - \lambda + \lambda V(\omega)} \right|_{\omega - \lambda + \lambda V(\omega)}.
$$

A general inversion of this result does not exist. However, suppose the delivery size distribution is exponential with mean $r_0$. Then

$$
V(\omega) = \frac{1}{1 + \omega r_0}
$$

and

$$
(11) \quad U(\omega) = \frac{C}{1 + \lambda r_0} \left\{ \frac{\omega r_0 (1 + \lambda r_0) + 1}{\omega (\omega r_0 - \lambda r_0 + 1)} \right\}.
$$
Considering first the case \( 1/r_o \neq \lambda \), \( U(\omega) \) can be rewritten as

\[
U(\omega) = \frac{c}{1 - (\lambda r_o)^2} \left\{ \frac{1}{\omega} - \frac{(\lambda r_o)^2}{\omega - \lambda + 1/r_o} \right\}.
\]

After inverting this transform and making the substitution \( x = y + s \), one obtains the density function

\[(12) \quad f(x) = \frac{c}{(1 - \lambda r_o)^2} (1 - (\lambda r_o)^2 \exp[(\lambda - 1/r_o)(x-s)]) \]

for \( s \leq x < S \).

For the region \( x \geq s \), substitution of (6) and (12) into the third line of (5) gives

\[
f'(x) = -\frac{c\lambda}{1 - (\lambda r_o)^2} \int_{x-S}^{\infty} \left\{ \frac{1}{r_o} e^{-r/r_o} - \lambda r_o \exp[(\lambda - 1/r_o)(x-s)] e^{-\lambda r_o^2} \right\} dr
\]

\[-\frac{c\lambda}{r_o} \int_{x-S}^{\infty} \exp[\lambda(x-s)] \exp[-(\lambda + 1/r_o)x] \, dr \]

\[= -\frac{c\lambda}{1 - (\lambda r_o)^2} (1 - \lambda r_o \exp[(\lambda - 1/r_o)(S-s)]) \exp[-1/r_o(x-S)] \]

for \( x > S \).

Then integrating from \( x \) to \( \infty \), and making use of the fact that \( f(x) \) must vanish at \( \infty \), one obtains

\[(13) \quad f(x) = \frac{c\lambda r_o}{1 - (\lambda r_o)^2} (1 - \lambda r_o \exp[(\lambda - 1/r_o)(S-s)]) \exp[-1/r_o(x-S)] \]

for \( x \geq S \).
From the condition that the density function integrate to one, the constant $C$ is evaluated as

\[
C = \left\{ \frac{1}{\lambda} + \frac{\Delta}{\sqrt{1-(\lambda r_o)^2}} + \frac{\lambda r_o^2}{\sqrt{[1-(\lambda r_o)^2](1-\lambda r_o)}} \left[ (\lambda r_o)^2 e^{(\lambda-1/r_o)\Delta} + 1 - 2\lambda r_o \right] \right\}^{-1},
\]

where $\Delta = S-s$. Thus with this constant, the density function for the stationary distribution of stock level at the beginning of a period for the case $1/r_o \neq \lambda$ is given by

\[
f(x) = \begin{cases} 
C e^\lambda(x-s) & \text{for } x < s \\
\frac{C}{1-(\lambda r_o)^2} \left[ 1 - (\lambda r_o)^2 e^{(\lambda-1/r_o)(x-s)} \right] & \text{for } s \leq x < S \\
\frac{C\lambda r_o}{1-(\lambda r_o)^2} \left[ 1 - \lambda r_o e^{(\lambda-1/r_o)\Delta} \right] e^{-1/r_o(x-s)} & \text{for } x \geq S.
\end{cases}
\]

In the case $1/r_o = \lambda$, $U(\omega)$ becomes

\[
U(\omega) = \frac{C(2\omega + \lambda)}{2\omega^2} = C \left\{ \frac{1}{\omega} + \frac{\lambda}{2\omega^2} \right\}.
\]

After inverting this transform and making the substitution $x = y + s$, one obtains the density function

\[
f(x) = C[1 + \frac{\lambda}{2}(x-s)] \quad \text{for } s \leq x < S;
\]
this function is the limit of (12) as $\lambda \to 1/r_0$. Then substituting (6) and (16) into the third line of (5) gives

\[
f'(x) = -C \lambda^2 \left[ \int_{x-S}^{x-S} \left[ 1 + \frac{\lambda}{2} (x-r-s) \right] e^{-\lambda r} dr + \int_{x-S}^{\infty} e^{\lambda (x-r)} e^{-2\lambda r} dr \right]
\]

\[= -\frac{C \lambda}{2} (1 + \lambda \Delta) e^{-\lambda (x-S)} \quad \text{for } x > S.
\]

Then integrating from $x$ to $\infty$ and making use of the fact that $f(x)$ must vanish at $\infty$, one obtains

\[
f(x) = \frac{C}{2} (1 + \lambda \Delta) e^{-\lambda (x-S)} \quad \text{for } x \geq S;
\]

this function is the limit of (13) as $\lambda \to 1/r_0$. The constant $C$ is evaluated as above, and is

\[
C = \left[ \frac{3}{2\lambda} + \frac{3\Delta}{2} + \frac{\lambda^2}{4} \right]^{-1}.
\]

Thus with this constant, the stationary distribution of stock level at the beginning of a period for the case $1/r_0 = \lambda$ is given by

\[
f(x) = \begin{cases} C e^{\lambda (x-S)} & \text{for } x < s \\ C(1 + \frac{\lambda}{2} (x-s)) & \text{for } s \leq x < S \\ \frac{C}{2} (1 + \lambda \Delta) e^{-\lambda (x-S)} & \text{for } x \geq S. \end{cases}
\]
Stationary one-period expected costs may now be calculated for this example. Assuming that consideration is restricted to stocking policies such that \( s \geq 0 \), equation (4) becomes

\[
L(s, \Delta) = \frac{C_1 r_o^2}{1 - (\lambda r_o)^2} \left[ K_1 - a_1 r_o \right] \left( 1 - \lambda r_o e^{\frac{(\lambda - 1/r_o)\Delta}{r_o}} \right) + a_1 r_o 
+ \frac{C_1 \lambda r_o}{1 - (\lambda r_o)^2} \int_0^\infty \left[ \frac{1}{r_o} e^{-r/r_o} e^{\lambda(x-s)} \right] \, dx 
+ \frac{c_1 C_1 \lambda r_o}{1 - (\lambda r_o)^2} \int_0^\infty \left[ \frac{1}{r_o} e^{-(\lambda r_o)^2} e^{-1/r_o(x-s)} \right] \, dx 
+ \frac{c_1 C}{1 - (\lambda r_o)^2} \int_0^s x(1 - (\lambda r_o)^2) e^{-1/r_o(x-s)} \, dx 
+ c_1 C \int_0^s x e^{\lambda(x-s)} \, dx - c_2 C \int_0^s x e^{\lambda(x-s)} \, dx 
\]

\[
= \frac{C_1 r_o^2}{1 - (\lambda r_o)^2} \left[ K_1 - a_1 r_o \right] \left( 1 - \lambda r_o e^{\frac{(\lambda - 1/r_o)\Delta}{r_o}} \right) + a_1 r_o 
+ \frac{c}{\lambda(1 + \lambda r_o)} \left( K_2 + a_2(\Delta + 1/\lambda) \right) + \frac{C(c_1 + c_2)}{\lambda^2} e^{-\lambda s} 
+ \frac{C c_1}{1 - (\lambda r_o)^2} \left\{ (\lambda r_o)^3 e^{\frac{(\lambda - 1/r_o)\Delta}{r_o}} \left[ \frac{r_o(s + \Delta)}{1 - \lambda r_o} + \frac{r_o^2(2 - \lambda r_o)}{(1 - \lambda r_o)^2} \right] \right\} 
+ \lambda r_o^2 \Delta + \frac{\Delta(\Delta + 2s)}{2} + \left[ 1 - \frac{(\lambda r_o)^3}{1 - \lambda r_o} \right] \frac{s}{\lambda} 
+ \lambda r_o^3 - \frac{\lambda^2 r_o^4}{(1 - \lambda r_o)^2} - \frac{1 - (\lambda r_o)^2}{\lambda^2} \right\},
\]
where $\Delta = S - s$. For a given value of $r_o$, the pair of values $(s, \Delta)$ which minimize this expression, subject to the restrictions $0 \leq s < \infty$ and $0 \leq \Delta < \infty$, can be determined by the usual analytical or numerical methods for finding the minimum of a function of two variables.

It should be noted that if $r_o \to 0$, then equation (20) reduces to

\begin{equation}
L(s, \Delta) = \frac{1}{1 + \lambda \Delta} \left\{ K_2 + \frac{c_1 + c_2}{\lambda} e^{-\lambda s} + c_1 \left[ s - \frac{1}{\lambda} + \lambda \Delta s + \frac{\lambda^2}{2} \right] + c_2 (\Delta + \frac{1}{\lambda}) \right\},
\end{equation}

the stationary one-period expected costs for the $(s, S)$ model studied by Karlin in [3]. As he shows, this function takes on its minimum for

\begin{equation}
\Delta = \sqrt{\frac{2K_2}{\lambda c_1}}
\end{equation}

and

\begin{equation}
s = \max \left\{ 0, \frac{1}{\lambda} \log \frac{c_1 + c_2}{\sqrt{2\lambda K_2 c_1} + c_1} \right\}.
\end{equation}

In comparing the policy discussed here with the regular $(s, S)$ policy, it is of interest to know when the present policy leads to a lower value of stationary one-period expected costs. In the present example, this is equivalent to asking when $L(s, \Delta)$ of equation (20), considered as a function of $r_o$, has its minimum for some value of $r_o > 0$. Evaluating the derivative of $L(s, \Delta)$ with respect to $r_o$ at $r_o = 0$, one obtains after a lengthy calculation the simple result
\(\frac{\partial L(s, \Delta)}{\partial \omega} \bigg|_{\omega = 0} = a_1 - \frac{K_2 \lambda}{1 + \Delta} - a_2.\)

Thus the present policy is better than the regular \((s, S)\) policy if

\(a_1 < \frac{K_2 \lambda}{1 + \Delta} + a_2.\)

In particular, if the unit cost of routine deliveries is less than the unit cost of emergency orders, i.e. \(a_1 < a_2\), the present policy is always better under the assumptions of this example. Or if \(\Delta\) is fixed at its optimum value (22) for the regular \((s, S)\) policy, the \((s, S)\) policy can clearly be improved upon if

\(a_1 < \frac{K_2 \lambda}{1 + \sqrt{\frac{2K_2}{c_1}}} + a_2.\)

Stationary Distribution of Stock Level: Second Version

Here, functional equations satisfied by the stationary distribution of stock level at the time of ordering are derived for the second version of the inventory model outlined above.

In this version, the random variable \(X_j\) again represents the stock level at the beginning of the \((j+1)\)th period, i.e. at time \(j\), and the random variable \(\xi_j\), the demand on the stocking point during the \(j\)th period. It is assumed that the sequence \(\{\xi_j\}\) of demands is
a sequence of independent random variables, each distributed with the known continuous density function \( \phi(t) \).

If the stock level \( X_j \) at time \( j \) is less than the reorder level \( S \), then a random quantity \( R_{j+1} \), with known distribution \( H(r) \) independent of demand, is supplied to the stocking point at the close of the \((j+1)\)st period. This will be regarded as the routine resupply. Furthermore, if \( X_j \) is less than the emergency reorder level \( s \), where \( s \leq S \), an additional quantity \( S - X_j \) is ordered for immediate delivery. However if the stock level \( X_j \geq S \), then the regular delivery at the close of the \((j+1)\)st period is omitted.

Under the above conditions, the sequence \( \{X_j\} \) of stock levels forms a Markov chain, with one-step transitions

\[
(27) \quad X_{j+1} = \begin{cases} 
X_j - \xi_{j+1} & \text{if } X_j \geq S \\
X_j - \xi_{j+1} + R_{j+1} & \text{if } s \leq X_j < S \\
S - \xi_{j+1} + R_{j+1} & \text{if } X_j < s .
\end{cases}
\]

Then if \( F_j(x) \) is the distribution function for \( X_j \), one obtains as the relation, in differential form, between the distributions of \( X_j \) and \( X_{j+1} \)
\[
(28) \quad dF_{j+1}(x) = \begin{cases} 
\int_{-\infty}^{\infty} \phi(t-x)dF_j(t) + \int_{s}^{\infty} \phi(t+r-x)dH(r)dF_j(t) \\
\quad + \int_{-\infty}^{s} \int_{0}^{\infty} \phi(S+r-x)dH(r)dF_j(t) \\
\quad \quad \quad \text{for } x < s \\
\int_{s}^{x} \int_{t}^{\infty} \phi(t+r-x)dH(r)dF_j(t) \\
\quad + \int_{-\infty}^{x-t} \phi(t+r-x)dH(r)dF_j(t) \\
\int_{-\infty}^{s} \int_{0}^{\infty} \phi(S+r-x)dH(r)dF_j(t) \\
\quad \quad \quad \text{for } s \leq x < S \\
\int_{x}^{\infty} \phi(t-x)dF_j(t) + \int_{s}^{\infty} \phi(t+r-x)dH(r)dF_j(t) \\
\quad + \int_{x-t}^{s} \int_{x}^{\infty} \phi(s+r-x)dH(r)dF_j(t) \\
\quad \quad \quad \text{for } x \geq S.
\end{cases}
\]

Since the stochastic policy calls for strong corrective action if the stock level falls outside the interval \( s \leq x < S \), it appears reasonable to assume that there exists a unique limiting stationary distribution \( F(x) \) for this process, independent of the initial distribution \( F_0(x) \). No stability condition relating the demand and routine resupply distributions is necessary if \( s \) and \( S \) are finite.

Since the demand distribution has a continuous density function \( \phi(t) \), the stationary distribution \( F(x) \) has a continuous density function \( f(x) \) which satisfies the relations
\begin{align*}
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&\int_{S}^{\infty} \phi(t-x)f(t)dt + \int_{S}^{\infty} \int_{0}^{\infty} \phi(t+r-x)f(t)dH(r)dt \\
&+ \int_{-\infty}^{S} \int_{0}^{\infty} \phi(S+r-x)f(t)dH(r)dt \quad \text{for } x < s
\end{align*}

(29) \quad f(x) = \begin{cases} 
\int_{S}^{\infty} \phi(t-x)f(t)dt + \int_{X}^{S} \int_{0}^{\infty} \phi(t+r-x)f(t)dH(r)dt \\
+ \int_{S}^{X} \int_{X-t}^{\infty} \phi(t+r-x)f(t)dH(r)dt \\
+ \int_{-\infty}^{S} \int_{0}^{\infty} \phi(S+r-x)f(t)dH(r)dt \quad \text{for } s \leq x < S
\end{cases}
\begin{align*}
&\int_{x}^{\infty} \phi(t-x)f(t)dt + \int_{S}^{\infty} \int_{X-t}^{\infty} \phi(t+r-x)f(t)dH(r)dt \\
&+ \int_{-\infty}^{S} \int_{X-S}^{\infty} \phi(S+r-x)f(t)dH(r)dt \quad \text{for } x \geq S.
\end{align*}

A question of interest is the selection of the optimum levels \(s\) and \(S\) for this inventory model. For a given cost structure, the optimum values will be defined to be those which minimize the stationary one-period expected costs. It will be assumed here that the cost of goods supplied by routine delivery is a unit cost \(a_1\) times the quantity delivered; if the routine delivery is cancelled in any period, a fixed cost \(K_1\) is incurred. The cost of an emergency order is assumed to be a fixed cost \(K_2\), plus a unit cost \(a_2\) times the quantity ordered. It is assumed that at the end of the period, just before the routine delivery arrives, a holding cost \(c_1y\) is assessed if the stock level \(y\) is greater than zero, and a shortage cost \(-c_2y\) is assessed if the stock
level is below zero. Under these assumptions, the stationary expected one-period costs are given by

\begin{equation}
L(s, S) = K_1 (1 - F(S)) + a_1 E_R(R) F(S) + \int_{-\infty}^{S} [K_2 + a_2(S-x)] f(x) dx \\
+ c_1 \int_{0}^{\infty} y g(y) dy - c_2 \int_{-\infty}^{0} y g(y) dy ,
\end{equation}

where \( g(y) \) is the density function for the stock level at the end of the period, before the routine delivery is received, and is given by

\begin{equation}
g(y) = \begin{cases} 
\int_{0}^{\infty} \phi(t-y)f(t) dt + \phi(S-y) \int_{-\infty}^{S} f(t) dt & \text{if } y < s \\
\int_{y}^{\infty} \phi(t-y)f(t) dt + \phi(S-y) \int_{-\infty}^{S} f(t) dt & \text{if } s \leq y < S \\
\int_{y}^{\infty} \phi(t-y)f(t) dt & \text{if } y \geq S .
\end{cases}
\end{equation}

Example 2

Suppose demand has an exponential distribution

\[ \phi(\xi) = \lambda e^{-\lambda \xi} \quad \text{for} \quad \xi \geq 0, \]

and that the routine resupply distribution \( H(r) \) has a density function \( h(r) \). Then substituting into equations (29), one obtains
\begin{align*}
\int_S^\infty \phi(t)f(t)dt + \int_S^x \int_0^{\infty} \phi(t+r)f(t)dH(r)dt \\
+ \int_S^x \int_{x-t}^{\infty} \phi(t+r)dH(r)dt \\
+ \int_{-\infty}^S \int_0^{\infty} \phi(S+r)f(t)dH(r)dt \quad \text{for } x < s \\
\int_x^\infty \phi(t)f(t)dt + \int_s^x \int_{x-t}^{\infty} \phi(t+r)dH(r)dt \\
+ \int_{-\infty}^S \int_{x-S}^{\infty} \phi(S+r)f(t)dH(r)dt \quad \text{for } x \geq S.
\end{align*}

Differentiating with respect to \( x \), one obtains

\begin{align*}
(33) \quad f'(x) - \lambda f(x) &= \begin{cases} 
0 & \text{for } x < s \\
-\lambda \int_0^{x-S} f(x-r) dH(r) & \text{for } s < x < S \\
-\lambda f(x) - \lambda \int_{x-S}^{x-S} f(x-r)dH(r) - \lambda h(x-S)F(s) & \text{for } x > S.
\end{cases}
\end{align*}
In the region $x < s$, the solution

$$(34) \quad f(x) = C e^{\lambda(x-s)}$$

satisfies (33).

Considering next the region $s < x < S$, one notes that the right side of equation (33) for this region is a convolution of the delivery quantity distribution with the unknown stock level distribution $f(x)$. This suggests the possibility of obtaining a solution for the region $x \geq s$ by Laplace transform methods, then truncating the solution to the region $s \leq x < S$. Letting $y = x - s$, and taking the Laplace transform of both sides of the second line of equation (33), one obtains

$$\int_0^\infty e^{-\omega y} \left[ f'(y+s) - \lambda f(y+s) \right] dy = -\lambda \int_0^\infty \int_0^y e^{-\omega y} f(y+s-r) \, dH(r) \, dy$$

or

$$(\omega - \lambda) U(\omega) - f_s = -\lambda U(\omega) V(\omega),$$

where

$$U(\omega) = \int_0^\infty e^{-\omega y} f(y+s) \, dy$$

and

$$V(\omega) = \int_0^\infty e^{-\omega r} \, dH(r).$$
Solving for the transform $U(\omega)$ of the stock level distribution, and noting that by continuity of $f(x)$

$$f(s^+) = C,$$

one obtains

$$U(\omega) = \frac{C}{\omega - \lambda(1 - V(\omega))} \cdot$$  \hspace{1cm} (35)

A general inversion of this result does not exist. However, suppose the delivery size distribution is exponential with mean $r_o$. Then

$$V(\omega) = (1 + \omega r_o)^{-1}.$$  \hspace{1cm} (36)

Considering first the case for $1/r_o \neq \lambda$,

$$U(\omega) = \frac{C(\omega r_o + 1)}{\omega^2 r_o + \omega(1-\lambda r_o)} = \frac{C}{1 - \lambda r_o} \left\{ \frac{1}{\omega} - \frac{\lambda r_o}{\omega - \lambda + 1/r_o} \right\}.$$  \hspace{1cm} (37)

Inverting this transform, and making the substitution $x = y + s$, one obtains the result

$$f(x) = \frac{C}{1 - \lambda r_o} \left[ 1 - \lambda r_o \exp[(\lambda - 1/r_o)(x-s)] \right] \text{ for } s \leq x \leq S.$$  \hspace{1cm} (38)

Then for the region $x \geq S$, substitution of this result into the third line of equation (33) gives
\[ f'(x) = -\frac{\lambda C}{1-\lambda r_o} \int_{x-S}^{x-S} \left\{ \frac{1}{r_o} \exp(-r/r_o) - \lambda \exp[(\lambda - 1/r_o)(x-s)] \exp(-\lambda r) \right\} \, dr \]

\[ -\frac{C}{r_o} \exp[-1/r_o(x-S)] \]

\[ = -\frac{C}{r_o(1-\lambda r_o)} \left[ 1 - \lambda r_o \exp[(\lambda - 1/r_o)(S-s)] \exp[-1/r_o(x-S)] \right]. \]

Integrating from \( x \) to \( +\infty \), and using the fact that \( f(x) \) must vanish at \( +\infty \), one obtains

\[ (39) \quad f(x) = \frac{C}{1-\lambda r_o} \left[ 1 - \lambda r_o \exp[(\lambda - 1/r_o)(S-s)] \exp[-1/r_o(x-S)] \right] \]

for \( x \geq S \).

It is easily seen that this function and the function (12) above coincide at \( x = S \), thus satisfying the continuity condition at this point. Using the condition that

\[ \int_{-\infty}^{\infty} f(x) \, dx = 1, \]

one evaluates the constant \( C \) as

\[ (40) \quad C = \left\{ \frac{1}{\lambda} + \frac{\Delta}{1-\lambda r_o} + \frac{r_o}{(1-\lambda r_o)^2} \left[ (\lambda r_o)^2 \exp[(\lambda - 1/r_o)\Delta] + 1 - 2\lambda r_o \right] \right\}^{-1} \]

where \( \Delta = S-s \). Thus the stationary distribution of stock level for \( 1/r_o \neq \lambda \) is given by
(41) \( f(x) = \begin{cases} 
C e^{\lambda(x-s)} & \text{for } x \leq s \\
\frac{C}{1-\lambda r_o} \left[ 1-\lambda r_o \exp[(\lambda-1/r_o)(x-s)] \right] & \text{for } s \leq x \leq S \\
\frac{C}{1-\lambda r_o} \left[ 1-\lambda r_o \exp[(\lambda-1/r_o)\Delta] \right] \exp[-1/r_o(x-S)] & \text{for } x \geq S,
\end{cases} \)

where \( C \) is given by equation (40).

If \( 1/r_o = \lambda \), then by a similar process one obtains the result

(42) \( f(x) = \begin{cases} 
C e^{\lambda(x-s)} & \text{for } x \leq s \\
C[1 + \lambda(x-s)] & \text{for } s \leq x \leq S \\
C(1 + \lambda\Delta) e^{-\lambda(x-S)} & \text{for } x \geq S,
\end{cases} \)

where

\[
C = \frac{2\lambda}{(2 + \lambda\Delta)^2};
\]

these results also follow from taking limits in the results above for the case \( 1/r_o \neq \lambda \).

Next, substituting the solution (41) into equations (31), one obtains, after simplification, the density function

(43) \( g(y) = \begin{cases} 
C \left[ (1+\lambda r_o) - B r_o e^{-\lambda\Delta} \right] e^{\lambda(y-s)} & \text{for } y < s \\
C \left\{ \frac{1}{1-\lambda r_o} - \frac{(\lambda r_o)^2}{1-\lambda r_o} \exp[(\lambda-1/r_o)(y-s)] - B r_o e^{-\lambda\Delta} e^{\lambda(y-s)} \right\}^2 & \text{for } s \leq y < S \\
C B e^{-1/r_o(y-S)} & \text{for } y \geq S
\end{cases} \)
for the stationary distribution of stock level at the end of a period, where

\[ B = \frac{\lambda r_o}{1 - (\lambda r_o)^2} \left[ 1 - \lambda r_o \exp(\lambda - 1/r_o) \Delta \right] \]

Then if consideration is restricted to stocking policies such that \( s \geq 0 \), the stationary one-period expected costs, under the assumptions made in formulating equation (30), are found to be

\[ L(s, \Delta) = \frac{Cr_o}{1 - \lambda r_o} \left[ K_1 - a_1 r_o \right] \left[ 1 - \lambda r_o \exp(\lambda - 1/r_o) \Delta \right] + a_1 r_o 
+ \frac{c}{\lambda} \left( K_2 + a_2 (\Delta + \frac{1}{\lambda}) + C \left[ 1 + \lambda r_o \right] - B \lambda r_o e^{-\lambda \Delta} \right) \frac{c_1 + c_2}{\lambda^2} 
+ c_1 C \left\{ \frac{r_o (\lambda r_o)^2}{1 - \lambda r_o} \left[ \frac{r_o}{(1 - \lambda r_o)^2} - \frac{1}{\lambda} \right] \exp(\lambda - 1/r_o) \Delta \right\} 
+ \frac{1}{1 - \lambda r_o} \frac{\Delta (\Delta + 2s)}{2} + \frac{r_o^2}{1 - \lambda r_o} \frac{\lambda r_o^2}{(1 - \lambda r_o)^3} - \frac{1}{\lambda} (1 + \lambda r_o) 
+ \frac{s}{\lambda} \left( 1 + \lambda r_o \right) - \frac{(\lambda r_o)^2 r_o}{(1 - \lambda r_o)^2} s \right\} \]

It should be noted that if \( r_o = 0 \), this expression reduces to equation (21), the stationary one-period expected cost function for the \( (s, S) \) model.
Considering $L(s,\Delta)$ as a function of the mean size $r_o$ of the routine delivery, and differentiating with respect to this variable, one obtains

$$\frac{\partial L(s,\Delta)}{\partial r_o} \bigg|_{r_o=0} = \frac{\lambda}{1 + \lambda \Delta} (K_1 - K_2) + a_1 - a_2.$$

The condition that the present policy be an improvement over the regular $(s,S)$ policy is that the minimum of $L(s,\Delta)$ with respect to $r_o$, where $r_o \geq 0$, occur for a positive value of $r_o$. If the above derivative is negative, this condition is fulfilled. In particular, if $K_1 = K_2$, the present policy will be an improvement over the $(s,S)$ policy if $a_1 < a_2$, i.e., if the unit cost of routine deliveries is less than the unit cost of emergency deliveries.
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