STOCHASTIC INTEGRALS OF EMPIRICAL-TYPE PROCESSES WITH APPLICATIONS TO CENSORED REGRESSION

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Abstract

Motivated by the analysis of linear rank estimators and the Buckley-James nonparametric EM estimator in censored regression models, we study herein the asymptotic properties of stochastic integrals of certain two-parameter empirical processes. Applications of these results on empirical processes and their stochastic integrals to the asymptotic analysis of censored regression estimators are also given.

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Key Words and Phrases: Empirical processes, metric entropy, exponential inequalities, stochastic integrals, censored regression.
1. INTRODUCTION

Consider the linear regression model

\[ y_i = \alpha + \beta x_i + \varepsilon_i \quad (i = 1, 2, \ldots) \]  

(1.1)

where the \( \varepsilon_i \) are i.i.d. random variables with mean 0, and the \( x_i \) are either non-random or are independent random variables independent of \( \{\varepsilon_i\} \). Suppose that the responses \( y_i \) are not completely observable and that the observations are \( (x_i, z_i, \delta_i) \), where \( z_i = \min\{y_i, t_i\} \), \( \delta_i = I\{y_i \leq t_i\} \), and the \( t_i \) are independent random variables, independent of \( \{\varepsilon_i\} \). This is often called the "censored regression model" and the \( t_i \) are called the "censoring variables".

In 1979, Buckley and James [3] proposed the following method to estimate \( \alpha \) and \( \beta \). They started by replacing \( y_i \) by

\[ y_i^* = y_i \delta_i + E(y_i | y_i > t_i) (1 - \delta_i) \]  

(1.2)

and regressing the \( y_i^* \) (instead of the \( y_i \)) on the \( x_i \) to obtain

\[ \hat{\beta} = \frac{\sum_{i=1}^{n} y_i^* (x_i - \bar{x}_n)}{\sum_{i=1}^{n} (x_i - \bar{x}_n)^2} \]  

(1.3)

\[ \hat{\alpha} = \bar{y}_n^* - \hat{\beta} \bar{x}_n \]  

(1.4)

noting that \( E(y_i^*) = E(y_i) = \alpha + \beta x_i \), where \( \bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i \). Since \( E(y_i | y_i > t_i) \) in (1.2) is unknown, they replaced (1.3) by an iterative scheme in which \( E(y_i | y_i > t_i) \) is substituted by its successive estimates. Specifically, let \( e_{i}(b) = z_i - bx_i \) and order the uncensored \( e_{i}(b) \) as \( e_{(1)}(b) \leq \ldots \leq e_{(k)}(b) \), assuming that there are \( k \) uncensored observations. Let

\[ n_{i}(b) = \# \{ j : e_{j}(b) \geq e_{i}(b) \} \]  

(1.5)

where \#A denotes the number of elements of a set \( A \). Buckley and James first used the Kaplan-Meier estimator

\[ \hat{F}_{n, b}(u) = 1 - \prod_{i: e_{(i)}(b) \leq u} \frac{(n_{i}(b) - 1)/n_{i}(b)}{n_{i}(b) - 1}/n_{i}(b) \]  

(1.6)

to estimate the common distribution function \( F \) of \( e_{i} \overset{d}{=} \alpha + \varepsilon_i \). Assuming the \( x_i \) to be
nonrandom, they then replaced $E(y_i | y_i > t_i) = \beta x_i + E(e_i | e_i > t_i - \beta x_i)$ by

$$z_i(b) = bx_i + \int_{u > t_i - bx_i} ud\hat{F}_{n,b}(u) / (1 - \hat{F}_{n,b}(t_i - bx_i)).$$  \hfill (1.7)

Replacing (1.2) by $y_i^*(b) = y_i \delta_i + z_i(b)(1 - \delta_i)$, they proposed to estimate $\beta$ by iterative solution of the equation

$$b = \left( \frac{\sum_{i=1}^{n} (x_i - \bar{x})y_i^*(b)}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right),$$  \hfill (1.8)

in analogy with (1.3). Note that (1.8) is equivalent to the equation

$$W_n(b) = 0,$$

where

$$W_n(b) = \sum_{i=1}^{n} \delta_i (x_i - \bar{x})(y_i - bx_i) + \sum_{i=1}^{n} (1 - \delta_i)(x_i - \bar{x})(z_i(b) - bx_i).$$  \hfill (1.9)

Once a slope estimator $b^*$ is determined, an estimator of $\alpha$ can be obtained as the mean of $\hat{F}_{b^*}$.  

To analyze the asymptotic properties of the Buckley-James estimator, a crucial step is to study the random function $W_n(b)$ as $n \to \infty$. Of particular importance is the behavior of $W_n(b)$ for $b$ near $\beta$. Useful tools to study this kind of problems are provided by the concept of metric entropy of empirical-type processes and their stochastic integrals, which are discussed in Sections 2 and 3 below. Applications of these results to the random function $W_n(b)$, or more precisely, to a slight modification thereof, are discussed in Section 5. In this modification, we ignore the factors $1 - n_i^{-1}(b)$ in the Kaplan-Meier estimator (1.6) when $n_i(b)/n$ is too small, causing instability in the estimator. Specifically, we redefine $\hat{F}_{n,b}^*$ by

$$\hat{F}_{n,b}^*(u) = 1 - \prod_{i: e_i(b) < u} \left( 1 - p_n(n^{-1}n_i(b))/n_i(b) \right),$$  \hfill (1.10)

where $p_n$ is a smooth weight function on $[0,1]$ that will be specified in Section 5. In addition, we also use the weight function $p_n$ to modify the definition (1.7) of $z_i(b)$ in Section 5.
In Section 4, we apply the results of Sections 2 and 3 to another class of estimators of \( \beta \) in the censored regression model, introduced in [7] as extensions of the classical rank estimators with complete (uncensored) data. The rank estimators of \( \beta \) in [7] are defined by the equation

\[
S_n(b) = 0 ,
\]

(1.11)

where

\[
S_n(b) = \sum_{i=1}^{k} \psi \cdot p_n(\hat{F}_{n,b}(\epsilon_i(b))) \{x_i - \bar{x}(i,b)\} p_n(n^{-1}n_i(b)) ,
\]

(1.12)

\[
\bar{x}(i,b) = [\sum_{j=1}^{n} x_j I\{e_j(b) \geq \epsilon_i(b)\}] / n_i(b) ,
\]

(1.13)

\( \hat{F}_{n,b} \) is defined in (1.10), \( p_n \) is a smooth function on \([0,1]\) that will be specified in Section 4, and \( \psi \cdot p_n \) denotes the product of \( p_n \) and \( \psi \), which is a given "score function" (cf. [7]), i.e., \( \psi \cdot p_n(x) = \psi(x)p_n(x) \). Since the equation (1.11) may not have a solution, we define a rank estimator \( \tilde{\beta}_n \) of \( \beta \) as a zero-crossing of the step function \( S_n(b) \), i.e., the right and left hand limits \( S_n(\tilde{\beta}_n^+) \) and \( S_n(\tilde{\beta}_n^-) \) do not have the same sign. This zero-crossing notion of a solution of the equation \( W_n(b) = 0 \) was also used by James and Smith [5] to give a more precise definition of the Buckley-James estimator.

The functions \( W_n(b) \) and \( S_n(b) \), defined by (1.9) and (1.12) respectively, appear to be rather intractable analytically. An important step in our analysis of these functions is to express them using stochastic integrals of empirical-type processes. In particular, as shown in [7],

\[
S_n(b) = \int_{s=-\infty}^{\infty} \psi \cdot p_n(\hat{F}_{n,b}(s)) p_n(n^{-1}n_n(b,s)) \left[ dY_n(b,s) - \frac{X_n(b,s)}{\#_n(b,s)} dL_n(b,s) \right] ,
\]

(1.14)

where

\[
\#_n(b,s) = \sum_{j=1}^{n} I\{e_j(t_j-\beta x_j) \geq s+(b-\beta)x_j\} ,
\]

(1.15a)

\[
X_n(b,s) = \sum_{j=1}^{n} x_j I\{e_j(t_j-\beta x_j) \geq s+(b-\beta)x_j\} ,
\]

(1.15b)

\[
L_n(b,s) = \sum_{j=1}^{n} I\{e_j(t_j-\beta x_j) \leq s+(b-\beta)x_j\} ,
\]

(1.15c)

\[
Y_n(b,s) = \sum_{j=1}^{n} x_j I\{e_j(t_j-\beta x_j) \leq s+(b-\beta)x_j\} .
\]

(1.15d)
Here and in the sequel, $e_j = \alpha + \varepsilon_j$, $x \land y$ denotes $\min(x, y)$, and $x \lor y$ denotes $\max(x, y)$. We call the two-parameter processes $\#_n - \#_n$, $X_n - EX_n$, $L_n - EL_n$, $Y_n - EY_n$ empirical-type processes because they are similar to empirical processes and can be analyzed by techniques similar to those recently developed in empirical process theory; as will be shown in Section 2. In particular, these techniques enable us to obtain probability bounds, which are uniform in $b$ and $s$, in the approximation of the random function $\#_n(b, s) - \#_n(\beta, s)$ (or $L_n(b, s) - L_n(\beta, s)$, etc.) by its mean $E\#_n(b, s) - E\#_n(\beta, s)$. In Section 3, we apply these results to analyze stochastic integrals involving empirical-type processes. Making use of these stochastic integrals, we then study the asymptotic properties of $\hat{F}_{n, b}$, $S_n(b)$ and $W_n(b)$ in Sections 4 and 5.

2. METRIC ENTROPY AND CONVERGENCE PROPERTIES OF EMPIRICAL-TYPE PROCESSES

In this section we first review some recent results in empirical process theory due to Alexander [1] and then extend these results to the empirical-type processes (1.15). Let $\xi_1, \xi_2, \ldots$, be independent random variables taking values in a measurable space $(S, \mathcal{B})$ and let $P_i$ denote the probability distribution of $\xi_i$ (i.e., $P_i(B) = P(\xi_i \in B)$). Consider the empirical measure and process

$$\pi_n = n^{-1} \sum_{i=1}^{n} \delta_{\xi_i}, \quad \nu_n = n^{1/2}(\pi_n - \bar{P}_n),$$

where $\bar{P}_n = n^{-1} \sum_{i=1}^{n} P_i$ and $\delta_x$ denotes the unit point mass (delta function) at $x$. Let $\mathcal{G}$ be a class of real-valued measurable functions on $S$ such that $|f| \leq A$ for all $f \in \mathcal{G}$ and some $A > 0$. Let

$$\nu_n(f) = \int f d\nu_n = n^{-1/2} \sum_{i=1}^{n} (f(\xi_i) - Ef(\xi_i)) .$$

An important concept in Alexander's [1] analysis of $\sup_{f \in \mathcal{G}} \nu_n(f)$ is the "metric entropy" of $\mathcal{G}$ defined as follows. Given $\varepsilon > 0$, $p > 0$, and a probability measure $\mu$ on $(S, \mathcal{B})$, let
\[ N_p(\varepsilon, \mathcal{F}, u) = \min \{ k : \text{There exist } f_1, \ldots, f_k \in \mathcal{F} \text{ such that } \]
\[ \min_{i \leq k} \| f - f_i \|_p < \varepsilon \text{ for all } f \in \mathcal{F} \} , \]
\[ N^B_p(\varepsilon, \mathcal{F}, u) = \min \{ k : \text{There exist } f_1^U, f_1^L, \ldots, f_k^U, f_k^L \in \mathcal{F} \text{ such that } f_i^L \leq f_i \leq f_i^U \]
\[ \text{for some } i \text{ for every } f \in \mathcal{F}, \text{ and } \| f_i^U - f_i^L \|_p < \varepsilon \text{ for all } i \} . \]

The "metric entropy" and "metric entropy with bracketing" of \( \mathcal{F} \) in \( L^p(\mu) \) are \( \log N_p \) and \( \log N^B_p \), respectively.

Given a class \( \mathcal{F} \) with finite \( L^p(\mathcal{F}_n) \) entropy and \( \delta_0 > \delta_1 > \ldots > \delta_k > 0 \), there exist \( \mathcal{F}_j \subset \mathcal{F} \) \((j \leq m) \) such that \( |\mathcal{F}_j| = N_p(\delta_j, \mathcal{F}, \mathcal{F}_n) \) and for each \( f \in \mathcal{F} \) there exists \( f_j(f) \in \mathcal{F}_j \) with \( \| f - f_j(f) \|_p < \delta_j \). A basic idea in Alexander's probability bounds for \( \sup_{\mathcal{F}} |\nu_n(f)| \) is the following "chaining argument" (cf. also [4]). Writing
\[ \nu_n(f) = \nu_n(f_0(f)) + \sum_{j=0}^{K-1} \nu_n[f_{j+1}(f) - f_j(f)] + \nu_n[f - f_K(f)] , \]
we have
\[ P\{ \sup_{\mathcal{F}} |\nu_n(f)| > \lambda \} \leq |\mathcal{F}_0| \sup_{\mathcal{F}} P\{ |\nu_n(f)| > (1 - \varepsilon / 4)M \}
\[ + \sum_{j=0}^{K-1} |\mathcal{F}_j| |\mathcal{F}_{j+1}| \sup_{\mathcal{F}} P\{ |\nu_n[f_{j+1}(f) - f_j(f)]| > \eta_j \} \]
\[ + P\{ \sup_{\mathcal{F}} |\nu_n(f_K(f) - f)| > \varepsilon M / 8 + \eta_K \} \leq R_1 + R_2 + R_3 , \]
where the \( \eta_j > 0 \) are so chosen that \( \sum_{j=0}^{K} \eta_j < \varepsilon M / 8 \), and \( P^* \) denotes outer measure.

Bounds for the terms \( R_1 \) and \( R_2 \) in (2.2) are provided by Bennett's [2] inequality for sums of bounded independent random variables: If \( X_1, \ldots, X_n \) are independent random variables such that \( E X_i = 0 \) and \( |X_i| \leq A \), then for \( \alpha \geq n^{-1} \sum_1^n \text{Var}(X_i) \),
\[ P\{ |n^{-1} \sum_1^n X_i| > \lambda \} \leq 2 \exp\{-\frac{1}{2} M^2 v^{-1} g(AMn^{-1/2} A^{-1}) \} , \text{ where} \]
\[ g(\lambda) = 2 \lambda^{-2} ((1 + \lambda) \log(1 + \lambda) - \lambda) . \]

Making use of (2.2) and (2.3) together with an appropriate choice of the \( \delta_j \) and \( \eta_j \), Alexander [1] obtained sharp probability bounds for \( \sup_{\mathcal{F}} |\nu_n(f)| \) under a variety of metric entropy assumptions on \( \mathcal{F} \); the method to bound \( R_3 \) in (2.2) varies with these
assumptions on $\mathcal{F}$. In particular, he showed that for $\varepsilon > 0$, $0 < r < 2$ and $\theta > 0$, there exists $C = C(r, \theta, \varepsilon)$ such that if
\begin{equation}
\log N_\infty(\delta, \mathcal{F}, \mathbb{P}_n) \leq \theta \delta^{-r} \text{ for all } 0 < \delta \leq 1
\end{equation}
and if
\begin{equation}
M \geq C(\alpha^{(2-r)/4} \nu_n(r-2)/2(r+2))
\end{equation}
then analogous to (2.3),
\begin{equation}
P^*\{\sup_{\mathcal{F}} |\nu_n(f)| > M\} \leq 5 \exp\left\{-\frac{1}{2} (1-\varepsilon)M^2\alpha^{-1} g(AMn^{-1/2}n^{-1})\right\}
\end{equation}
where $\alpha \geq \sup_{\mathcal{F}} n^{-1} \sum_{i=1}^{n} \text{Var } f(\xi_i)$. The term $R_3$ in this case is handled by taking $\delta_K = \varepsilon M^{-1/16}$, so that
\begin{equation}
|\nu_n(f_K(f) - f)| \leq 2\nu_2 ||f_K(f) - f||_\infty \leq \varepsilon M/8.
\end{equation}

Let $\mathcal{D}$ be a class of measurable subsets of $S$ and let $\mathcal{F} = \{I_D: D \in \mathcal{D}\}$. Alexander [1] showed that if we replace (2.4) by
\begin{equation}
\log N_2^B(\delta, \mathcal{F}, \mathbb{P}_n) \leq \theta \delta^{-T} \text{ for all } 0 < \delta \leq 1
\end{equation}
then (2.6) still holds for $M$ satisfying both (2.5) and
\begin{equation}
M \leq \varepsilon n^{1/2}/16.
\end{equation}

Note that in this case with $f = I_D$, $\sup_{\mathcal{F}} |\nu_n(f)| = \sup_{\mathcal{F}} |\nu_n(D)|$ and $\alpha \geq \sup_{\mathcal{F}} n^{-1} \sum_{i=1}^{n} P_i(D)(1-P_i(D))$. The term $R_3$ in (2.2) is handled by taking $\delta_K = \varepsilon M^{-1/16}$ and using the bound
\begin{equation}
|\nu_n[f_k^U(f) - f] - |\nu_n[f_k^L(f) - f_k^L(f)]| + 2\nu_2 ||f_k^U(f) - f_k^L(f)||_1
\end{equation}
\begin{equation}
\leq |\nu_n[f_k^U(f) - f_k^L(f)]| + 2\nu_2 \delta_K^2,
\end{equation}
since $E||D^2 = ||E||_2^2$. Hence
\begin{equation}
R_3 \leq |\mathcal{F}_K| \sup_{\mathcal{F}} P\{|\nu_n[f_k^U(f) - f_k^L(f)]| > \delta_K\},
\end{equation}
which can then be bounded by using Bennett's inequality (2.3).

As a corollary of (2.6), we obtain the following result on empirical-type processes, which will be used in Section 3. Throughout the sequel, replacing $t_i - \beta x_i$ in
(1.15) by $t_i$, we shall assume without loss of generality that $\beta = 0$. We shall also restrict $b$ in (1.15) to a bounded interval $|b| \leq \rho$. For notational simplicity we shall write $\sup_{b,s} \sup_{b,s}$ to denote supremum over the region $|b| \leq \rho$ and $-\infty < s < \infty$.

**Lemma 1.** Let $(e_i, x_i, t_i)$, $i = 1, 2, \ldots$, be independent random vectors such that for some nonrandom constant $A$,

$$|x_i| \leq A \text{ for all } i.$$  

(2.10)

Let $Z_n(b,s)$ be any of the four empirical-type processes defined in (1.15) with $\beta = 0$. Let $u_n:[-\rho, \rho] \times (-\infty, \infty) \to (-\infty, \infty)$ be a nonrandom Borel function such that

$$|u_n(b,s)| \leq A, \quad |u_n(b,s) - u_n(b',s')| \leq A(|b-b'| + |s-s'|), \quad \text{for all } n, b, b', s, s'.$$  

(2.11)

Then for every $0 < \gamma < 1$ and $\varepsilon > 0$,

$$\sup_{|b-b'| \leq n^{-\gamma}} \left| \int_{s=-\infty}^{s=\infty} [u_n(b,s) - u_n(b',s)]d(Z_n(b,s) - EZ_n(b,s)) \right| = o(n^{(1-\gamma)/2+\varepsilon}) \text{ a.s.}$$  

(2.12)

**Proof.** We shall only consider the case $Z_n = Y_n$. First note that

$$\int_{s=-\infty}^{s=\infty} [u_n(b,s) - u_n(b',s)]dY_n(b,s) = \sum_{i=1}^{n} x_i [u_n(b, e_i - bx_i) - u_n(b', e_i - bx_i)] I_{\{e_i \leq t_i\}}.$$  

For fixed $n$, let $\psi_{b,b'}(e_i, x_i, t_i) = x_i [u_n(b, e_i - bx_i) - u_n(b', e_i - bx_i)] I_{\{e_i \leq t_i\}}$. Letting $\xi_i = (e_i, x_i, t_i)$, the class $\mathcal{F} = \{\psi_{b,b'}: |b| \leq \rho, |b'| \leq \rho\}$ clearly satisfies the entropy assumption (2.4) for every $r > 0$, in view of (2.10) and (2.11) (which in fact implies that $\log N_\infty(\delta, \mathcal{F}, \overline{p}_n) = O(\log \delta)$ as $\delta \to 0$). Moreover, by (2.11), there exists $A'$ such that $\text{Var} \psi_{b,b'}(e_i, x_i, t_i) \leq A' |b-b'|$ for all $i$. Hence the desired conclusion (2.12) follows from (2.6) with $M = n^{-\gamma/2+\varepsilon}$ and the Borel-Cantelli lemma. \[ \Box \]

We next modify Alexander's arguments sketched above to prove the following result, which will be used repeatedly in the subsequent sections.

**Theorem 1.** Let $e_1, e_2, \ldots$, be i.i.d. random variables whose common distribution function $F$ satisfies the Lipschitz condition $|F(x)-F(y)| \leq C|x-y|$ for all $x, y$ and some $C > 0$. Let $(x_i, t_i)$, $i = 1, 2, \ldots$, be independent random vectors that are independent of $\{e_n\}$. Assume that (2.10) holds and
\[ \sup_{n \to \infty} \frac{1}{h} \sum_{i=1}^{n} P\{s_{t_i} - bx_{t_i} \leq s + h \} = O(nh) \quad \text{as} \quad n \to \infty \quad \text{and} \quad h \to 0 \quad \text{with} \quad nh \to \infty, \quad (2.13) \]

\[ \sup_{t_i} E(|e_{t_i}^{\beta} - t_i|^r) < \infty \quad \text{for some} \quad r > 0. \quad (2.14) \]

Let \( Z_n(b,s) \) be any of the four empirical-type processes defined in (1.15) with \( \beta = 0 \).

For \( 0 < d \leq 1 \) let

\[ \alpha_{n,d} = \sup_{|b-b'| + |s-s'| \leq d} n^{-1} \text{Var}\{Z_n(b,s) - Z_n(b',s')\}. \quad (2.15) \]

Then for every \( 0 < \varepsilon < 1 \), as \( n \to \infty \) and \( M = o(n^{\frac{1}{2}}) \), but

\[ M/\{n^{(1-\varepsilon)/2} + n^{-(1-\varepsilon)/2}\} \to \infty, \]

\[ P\{ \sup_{|b-b'| + |s-s'| \leq d} \left| Z_n(b,s) - E Z_n(b,s) - Z_n(b',s') + E Z_n(b',s') \right| > M \} = o(\exp\{-\frac{1}{2} (1-\varepsilon) M^2 \alpha_{n,d}^{-1}\}). \quad (2.16) \]

Consequently, for every \( 0 \leq \gamma < 1 \) and \( \theta > 0 \),

\[ \sup_{|b-b'| + |s-s'| \leq n^{-(1-\gamma)}} \left| Z_n(b,s) - E Z_n(b,s) - Z_n(b',s') + E Z_n(b',s') \right| = O(n^{(1-\gamma)/2 + \theta}) \quad \text{a.s.} \quad (2.17) \]

**Proof.** We shall only consider the case \( Z_n = X_n \). To prove (2.16), note that the assumptions on \( M \) here satisfy Alexander's conditions (2.8) and (2.5) (with sufficiently small \( r \)). Let

\[ \Delta_n(b,s;b',s') = n^{-\frac{1}{2}} \{ Z_n(b,s) - E Z_n(b,s) - X_n(b',s') + E X_n(b',s') \}. \]

As in Alexander's argument outlined above, choose \( \delta_0 > \ldots > \delta_K \) with \( \delta_K \sim C \varepsilon n^{-\frac{1}{2}} \), where \( C \) is some positive constant depending on \( \varepsilon \). For fixed \( j = 0, 1, \ldots, K \), partition the interval \([-\rho, \rho]\) by points \( \beta_{j,v}^{(j)} < \beta_{j,v+1}^{(j)} \) such that \( \beta_{j,v+1}^{(j)} - \beta_{j,v}^{(j)} \leq \delta_j \)

\( (v = 1, 2, \ldots) \), with equality except possibly for the case \( v = 1 \) (\( \beta_1^{(j)} = -\rho \)). Thus, the number \( N_j \) of sub-intervals is the smallest integer \( \geq 2\rho/\delta_j \), so \( \log N_j \sim \log \delta_j \)

(in analogy with (2.4)). For \( j = 0, \ldots, K \) and \( b \in [-\rho, \rho] \), define \( \nu(b,j) \) by \( \beta_{\nu(b,j)}^{(j)} \leq b < \beta_{\nu(b,j)+1}^{(j)} \). In view of (2.14),

\[ \sup_i P\{|e_{i}^{\beta} - t_i|^r \geq \delta^{-1/r} \} = O(\delta) \quad \text{as} \quad \delta \to 0. \quad (2.18) \]

For \( j = 0, \ldots, K \), partition the interval \([-\delta^{-1/r}_j, \delta^{-1/r}_j]\) by points \( \sigma_{m+1}^{(j)} < \sigma_{m}^{(j)} \)

such that \( \sigma_{m+1}^{(j)} - \sigma_{m}^{(j)} \leq \delta_j \) (\( m = 1, 2, \ldots, M_j \)) with equality except possibly for the
case \( m = 1 (\sigma_1^{(j)} = -\delta_j^{-1/r}) \). Thus, the number \( M_j \) of such sub-intervals is the smallest integer \( \geq 2\delta_j^{-1/r-1} \), so \( \log M_j \sim \log \delta_j \). Let \( \sigma_0^{(j)} = -\infty, \sigma_{M_j + 2}^{(j)} = \infty \). For any given \( s \), define \( m(s,j) \) by \( \sigma_m^{(j)}(s,j) \leq s < \sigma_{m(s,j)+1}^{(j)} \). As in (2.1), note that

\[
\Delta_n^{(b,s;b',s')} = \Delta_n^{(b_0,0;\sigma_0^{(j)},\sigma_0^{(j)});b_0,0,\sigma_0^{(j)};b_0,0,\sigma_0^{(j)}}
+ \sum_{j=0}^{K-1} \left[ \Delta_n^{(b_0,0;\sigma_0^{(j)},\sigma_0^{(j)});b_0,0,\sigma_0^{(j)};b_0,0,\sigma_0^{(j)}} - \Delta_n^{(b_0,0;\sigma_0^{(j)},\sigma_0^{(j)});b_0,0,\sigma_0^{(j)};b_0,0,\sigma_0^{(j)}} \right]
+ \left[ \Delta_n^{(b_0,0;\sigma_0^{(j)},\sigma_0^{(j)});b_0,0,\sigma_0^{(j)};b_0,0,\sigma_0^{(j)}} - \Delta_n^{(b_0,0;\sigma_0^{(j)},\sigma_0^{(j)});b_0,0,\sigma_0^{(j)};b_0,0,\sigma_0^{(j)}} \right],
\]

and apply the chaining argument (2.2) with \( \nu_n \) replaced by \( \Delta_n \). Since

\[
|x_i I_{\{e_i \wedge t_i > s+bx_i\}}| \leq A \quad \text{and the } (e_i, x_i, t_i) \text{ are independent},
\]
we can apply Bennett's inequality (2.3) to obtain probability bounds as in Alexander's argument [1], noting that by the Lipschitz continuity of \( F \) and the assumption (2.13) on \( t_i \),

\[
\sup_{|b_1 - b_2|, |v|, |b_{1,s}|, |v|, |s_1 - s_2|, |v|, |s_1 - s_2|, |v|, |s_1 - s_2|} \text{Var}[\Delta_n^{(b_1,s;1;b_1,s_1)} - \Delta_n^{(b_2,s_2;1;b_2,s_2)}] = 0(n) \quad \text{as } n \to \infty \text{ and } h \to 0 \text{ such that } nh \to 0.
\]

The rest of the proof of (2.16) is similar to that in Alexander [1, proof of Theorem 2.3]. In particular, the last term in (2.19) can be handled by a "bracketing argument" as in (2.9), noting that \( n\delta_{K} \sim C_{\varepsilon}M^{-1/2} \) \( \to \infty \) and that \( X_n^{(b,s)} \) can be decomposed as monotone functions in \( b \) and \( s \):

\[
X_n^{(b,s)} = \sum_{j \leq n, x_j > 0} x_j I_{\{e_j \wedge t_j > s+bx_j\}} - \sum_{j \leq n, x_j < 0} |x_j| I_{\{e_j \wedge t_j > s-b|x_j|\}}.
\]

Setting \( M = n^{-\gamma/2+\theta} \) in (2.16) and noting that \( n^{-\gamma} = 0(n^{-\gamma}) \) as in (2.20), (2.17) follows from (2.16) and the Borel-Cantelli lemma.

In the preceding proof, the chain \( \delta_0, \ldots, \delta_K \) terminates with \( \delta_K \sim C_{\varepsilon}M^{-1/2} \), and therefore we can apply condition (2.13) with \( h = \delta_j \) (since \( \min_{j \leq K} \delta_j \to \infty \)). Since the chain \( \delta_0, \ldots, \delta_K \) in Alexander's proof of (2.16) under the assumption (2.4) also terminates with \( \delta_K \sim \varepsilon M^{-1/2} \), we can introduce the following relaxation of the
assumption (2.11) in Lemma 1, which we have shown to be a corollary of (2.6) by setting
\[ M = n^{-\gamma/2+\epsilon} \] (and therefore \( n(Mn^{-\frac{1}{2}}) \to \infty \)).

**LEMMA 2.** Suppose that in Lemma 1 we replace the assumption (2.11) by
\[ \sup_{b,s} |u_n(b,s)| = o(1) \quad \text{and} \quad \sup_{|b-b'|+|s-s'|<\epsilon} \frac{|u_n(b,s)-u_n(b',s')|}{|b-b'|+|s-s'|} = o(h) \quad \text{as} \quad n \to \infty \quad \text{and} \quad h \to 0 \quad \text{such that} \quad nh \to \infty \tag{2.21} \]

Then the conclusion (2.12) still holds for every \( 0 \leq \gamma < 1 \) and \( \epsilon > 0 \).

Under the assumptions of Theorem 1 we can further strengthen the conclusion (2.12) of Lemma 1 for our main result in Section 3. This is the content of

**LEMMA 3.** With the same notation and assumptions as in Theorem 1, let
\[ u_n: [-\rho, \rho] \times (-\infty, \infty) \to (-\infty, \infty) \] be nonrandom Borel functions satisfying (2.21). Then for every \( 0 \leq \gamma < 1 \) and \( \epsilon > 0 \),
\[ \sup_{|b-b'|<n^{-\gamma}, \epsilon < y < \infty} \left| \int_{s=-\infty}^{y} [u_n(b,s)-u_n(b',s)]d(Z_n(b,s)-EZ_n(b,s)) \right| = o(n^{(1-\gamma)/2+\epsilon}) \] a.s. \tag{2.22}

**Proof.** We shall only consider the case \( L_n(b, s) \). For fixed \( n \), denote \( L_n(b, s), \)
\( EL_n(b, s), u_n(b, s) - u_n(b', s) \) by \( L_b(s), \bar{L}_b(s), u_b, u_b' \) respectively, and let
\[ V(b', s) = \int_{-\infty}^{\infty} u(b', t) d(L_b(t) - \bar{L}_b(t)). \] As in the proof of Theorem 1, choose
\[ \delta_0 > \ldots > \delta_K \] and for \( j = 0, \ldots, K \), partition the real line by the points
\[ \sigma_{0} = -\infty < \sigma_{1} < \ldots < \sigma_{M_j+1} < \infty = \sigma_{M_{j+2}} \] and the interval \([-\rho, \rho]\) by the points
\[ \beta_{j} = -\rho < \ldots < \beta_{N_j} = \rho. \] Analogous to (2.19), we now have
\[ V(b, b', s) = V(b', s) - V(a, a', \sigma) = \left[ V(b, b', \sigma) - V(a, a', \sigma) \right] + \int_{\sigma}^{s} u(b, b', t) d(L_b(t) - \bar{L}_b(t)). \]

Note that for \( \sigma \leq s \),
\[ V(b, b', s) - V(a, a', \sigma) = \left[ V(b, b', \sigma) - V(a, a', \sigma) \right] + \int_{\sigma}^{s} u(b, b', t) d(L_b(t) - \bar{L}_b(t)). \]
Moreover, for $\sigma_m \leq s < \sigma_{m+1}$,

$$\int_{\sigma_m}^{\sigma_m+1} u_{b,b'} \, d(L_b - \tilde{L}_b) \leq \int_{\sigma_m}^{\sigma_m+1} u^+_{b,b'} \, d(L_b - \tilde{L}_b) + \int_{\sigma_m}^{\sigma_{m+1}} u^-_{b,b'} \, d(L_b - \tilde{L}_b) + \int_{\sigma_m}^{\sigma_{m+1}} |u_{b,b'}| \, d\tilde{L}_b.$$

The rest of the proof is similar to that of Theorem 1 and Lemma 1. □

An argument similar to the proof of Theorem 1 can also be used to prove the following result, which will be used in Sections 4 and 5.

**Lemma 4.** With the same notation and assumptions as in Theorem 1, for every $0 \leq \gamma < 1$ and $\theta > 0$,

$$\sup_{(b,s) : \text{Var} \ Z_n(b,s) \leq n^{-\gamma}} |Z_n(b,s) - EZ_n(b,s)| = O(n^{(1-\gamma)/2+\theta}) \quad \text{a.s.}$$

3. **Stochastic Integrals of Empirical-Type Processes**

In this section we apply the results of Section 2 to study stochastic integrals of the form

$$\int_{s=0}^{Y} U_n(b,s) \, dL_n(b,s) \quad \text{or} \quad \int_{s=0}^{Y} U_n(b,s) \, dY_n(b,s),$$

where $L_n$ and $Y_n$ are the empirical-type processes defined by (1.15c) and (1.15d), and $U_n(b,s)$ are random variables for which there exist nonrandom Borel functions $u_n(b,s)$ satisfying the following assumptions for some $\xi > 0$: For every $0 \leq \gamma < 1$ and $\varepsilon > 0$,

(A1) $\sup_{b-a \leq \infty, -\infty < s < \infty} |U_n(b,s) - u_n(b,s) - U_n(a,s) + u_n(a,s)| = O(n^{-\frac{\gamma}{2} - \gamma/2 + 2\varepsilon + \xi + \varepsilon}) \quad \text{a.s.}$

(A2) $\sup_{b,s} |U_n(b,s) - u_n(b,s)| = O(n^{-\frac{1}{2} + \xi + \varepsilon}) \quad \text{a.s.}$

(A3) For fixed $b \in [-\varrho, \varrho]$, $U_n(b,s)$ has bounded variation in $s$ and

$$\sup_{b \leq \varrho} \int_{s=-\infty}^{\infty} |dU_n(b,s)| = O(n^{\xi}) \quad \text{a.s.}$$

(A4) $n^{-\xi} u_n$ satisfies condition (2.21).
An example of such stochastic integrals is the linear rank statistic $S_n(b)$ defined in (1.12). In view of (1.14), we can express $S_n(b)$ in the form

$$S_n(b) = \int_{s=-\infty}^{\infty} U_n(b,s)dY_n(b,s) - \int_{s=-\infty}^{\infty} \tilde{U}_n(b,s)dL_n(b,s),$$

where $U_n(b,s) = \psi^* p_{n,b}(s) p_n^{-1}(b,s)$ and $\tilde{U}_n = U_nX_n/#$. Another example is given by (1.10), which can be expressed in the form

$$\log(1-F_{n,b}(y)) = -\int_{\infty<s<y} \log(1-p_n^{-1}(b,s)/#(b,s))dL_n(b,s).$$

Theorem 2 below, which will be applied to these two examples in Section 4, shows that under certain conditions we can approximate the stochastic integral $\int_{-\infty}^{\infty} U_n(b,s)dZ_n(b,s)$ by the nonrandom function $\int_{-\infty}^{\infty} u_n(b,s)dE_n(b,s)$ with $Z_n = L_n$ or $Y_n$, and also provides two kinds of error bounds for the approximation. The first kind of results, given in (3.3) below, shows that the difference between the stochastic integral and its nonrandom approximation is of the order $O(n^{1/3+\epsilon+\epsilon})$, where $\epsilon > 0$ can be arbitrarily small. Hence if $\xi < 1/2$, the approximation error is of the order $O(n)$. For example, in the case of the linear rank statistic $S_n(b)$ to be studied in Section 4, this implies that

$$\sup_{b \leq \rho} |S_n(b) - h_n(b)| = 0 \text{ a.s.},$$

where $h_n(b)$ is a nonrandom function defined in (4.3).

This result can be used to establish the consistency of the rank estimator $\tilde{\beta}_n$ (which is a zero-crossing of $S_n(b)$) under certain assumptions on $h_n(b)$. To prove that $n^{1/2}(\tilde{\beta}_n - \beta)$ has a limiting normal distribution, however, the order $O(n^{1/3+\epsilon+\epsilon})$ in the approximation of $S_n(b)$ by $h_n(b)$ is obviously too crude, and we need another kind of results, given by (3.2) in Theorem 2 below. Applying (3.2) to $S_n(b)$ yields that with probability 1,

$$S_n(b) = S_n(\beta) + [h_n(b) - h_n(\beta)] + O(n^{1/2}(\xi - \gamma/2+\epsilon)^+)$$

uniformly in $|b - \beta| \leq n^{-\gamma}$. Thus, if $\xi < \gamma/2$, we can approximate $S_n(b) - S_n(\beta)$ by $h_n(b) - h_n(\beta)$ with an error of the order $O(n^{1/2})$ for $|b - \beta| \leq n^{-\gamma}$. This result is important for establishing the asymptotic normality of $\tilde{\beta}_n$, as will be discussed further.
in Section 4. Hence, (3.2) enables us to dampen the factor \( n^{\xi} \) in the assumptions (A1) - (A4) on \( U_n \) by using the proximity of \( b \) to \( \beta \), and its usefulness will be illustrated by the applications in Sections 4 and 5.

**Theorem 2.** Let \( e_1, e_2, \ldots \), be i.i.d. random variables having a continuously differentiable density function \( f \) such that

\[
\int_{-\infty}^{\infty} (\sup_{s-t \leq s + d} |f'(t)|) ds < \infty \quad \text{for some} \quad d > 0 . \tag{3.1}
\]

Let \( (x_i, t_i), i = 1, 2, \ldots \), be independent random vectors that are independent of \( \{e_n\} \) and such that conditions (2.10), (2.13) and (2.14) are satisfied. Define \( L_n(b,s) \) and \( Y_n(b,s) \) by (1.15c) and (1.15d) with \( \beta = 0 \). Let \( U_n(b,s), u_n(b,s) \) be the same as above (satisfying (A1)-(A4) for some \( \xi > 0 \)). Then for every \( 0 \leq \gamma < 1 \) and \( \epsilon > 0 \),

\[
\begin{align*}
&\sup_{|b-a| \leq n^{-\gamma}, -\infty < y < \infty} \left| \int_{s=-\infty}^{y} U_n(b,s) dL_n(b,s) - \int_{s=-\infty}^{y} u_n(b,s) dEL_n(b,s) \right| = O(n^{(1-\gamma)/2+\xi+\epsilon}) \quad \text{a.s.} \tag{3.2}
\end{align*}
\]

\[
\begin{align*}
&\sup_{|b| \leq \rho, -\infty < y < \infty} \left| \int_{s=-\infty}^{y} U_n(b,s) dL_n(b,s) - \int_{s=-\infty}^{y} u_n(b,s) dEL_n(b,s) \right| = O(n^{(1+\xi+\epsilon)}) \quad \text{a.s.} \tag{3.3}
\end{align*}
\]

Moreover, (3.2) and (3.3) still hold if \( L_n \) is replaced by \( Y_n \).

**Proof.** For fixed \( n \), denote \( U_n(b,s), u_n(b,s), L_n(b,s), EL_n(b,s) \) by \( U_b(s) \), \( u_b(s), L_b(s) \) and \( \tilde{L}_b(s) \) respectively, to simplify the notation. Note that

\[
\begin{align*}
\int_{-\infty}^{\infty} U_b \ dL_b - \int_{-\infty}^{\infty} u_b \ d\tilde{L}_b - \int_{-\infty}^{\infty} U_a \ dL_a + \int_{-\infty}^{\infty} u_a \ d\tilde{L}_a \\
= \int_{-\infty}^{\infty} (U_b-u_b-U_a+u_a) \ dL_b + \int_{-\infty}^{\infty} U_a \ d(L_b-\tilde{L}_b-L_a) \\
+ \int_{-\infty}^{\infty} (u_b-u_a) \ d(L_b-\tilde{L}_b) + \int_{-\infty}^{\infty} (U_a-u_a) \ d(\tilde{L}_b-L_a) .
\end{align*}
\]

Since \( \sup_{n \geq 1, |b| \leq \rho} n^{-1} \int_{-\infty}^{\infty} dL_b \leq 1 \), it then follows from (A1) that

\[
\begin{align*}
&\sup_{|b-a| \leq n^{-\gamma}} \int_{-\infty}^{\infty} |U_b-u_b-U_a+u_a| \ dL_b = O(n^{(1-\gamma)/2+\xi+\epsilon}) \quad \text{a.s.}
\end{align*}
\]

[The rest of the text is not shown]
Likewise, by (A3) and Theorem 1,

$$\sup_{|b-a| \leq n^{-\gamma}} \int_{-\infty}^{\infty} |L_b - L_b - L_a + L_a| \, dU_a = O(n^{(1-\gamma)/2+\xi+\varepsilon}) \quad \text{a.s.}$$

By (A4) and Lemma 3,

$$\sup_{|b-a| \leq n^{-\gamma}} \int_{-\infty}^{\gamma} n^{-\xi}(u_b - u_a) \, d(L_b - L_b) = O(n^{(1-\gamma)/2+\varepsilon}) \quad \text{a.s.}$$

We shall show that

$$\sup_{|b-a| \leq n^{-\gamma}} \int_{-\infty}^{\gamma} (U_a - u_a) \, d(L_b - L_b) = O(n^{(1-\gamma)/2+\xi+\varepsilon}) \quad \text{a.s.} \quad (3.4)$$

Hence the desired conclusion (3.2) follows.

To prove (3.4), first note that

$$dL_b(s) - dL_a(s) = \sum_{j=1}^{n} E[f(s + bx_j)I_{t_j > s + bx_j} - f(s + ax_j)I_{t_j > s + ax_j}] \, ds \quad (3.5)$$

By (2.10) and (2.13),

$$\sup_{|b-a| \leq n^{-\gamma}} \left| \sum_{j=1}^{n} [f(s + bx_j) - f(s + ax_j)] I_{t_j > s + bx_j} + \sum_{j=1}^{n} f(s + ax_j) (I_{t_j > s + bx_j} - I_{t_j > s + ax_j}) \right|$$

$$\leq \sup_{s - A_0 < z < s + A_0} \left[ A n^{-\gamma} |f'(z)| + f(z) \sup_{|b| \leq 1} \sum_{j=1}^{n} p(s - A n^{-\gamma} t_j - bx_j < s + A n^{-\gamma}) \right].$$

Since \( \sup_{s - A_0 < z < s + A_0} f(z) \leq f(s + A_0) \sup_{s - A_0 < z < s + A_0} |f'(z)| \), (3.4) follows from (3.1), (3.5), and (A2).

To prove (3.3), apply (A2)-(A4) and Lemma 4 together with the bounds

$$\left| \int_{-\infty}^{\gamma} u_b \, dL_b - \int_{-\infty}^{\gamma} u_b \, dL_b \right| \leq \int_{-\infty}^{\gamma} |u_b - u_b| \, dL_b + \int_{-\infty}^{\gamma} u_b \, d(L_b - L_b)$$

$$\leq \int_{-\infty}^{\gamma} |u_b - u_b| \, dL_b + \int_{-\infty}^{\gamma} |L_b - L_b| \, dU_b + (|u_b(y) - u_b(y)| + |u_b(y)|) |L_b(y) - L_b(y)| \cdot \Box$$

4. APPLICATIONS TO CENSORED RANK ESTIMATORS

In this section we apply Theorems 1 and 2 to study the properties of the linear rank estimator \( \tilde{\beta}_n \) of the slope \( \beta \) in the censored regression model described in
Section 1. Since $\beta_n$ is defined as a zero crossing of the function $S_n(b)$ defined in (1.12), it is important to study the function $S_n(b)$ first. The function $S_n(b)$, however, is not a smooth function in $b$ and therefore one cannot apply standard techniques (based on Taylor's expansion of the random function defining the estimator in a neighborhood of the true parameter) that are commonly used to prove asymptotic normality of maximum likelihood estimators, $M$-estimators, etc. Moreover, $S_n(b)$ is not a monotone function in $b$, so one cannot make use of the monotonicity and contiguity arguments (cf. [6]) that have been applied to prove asymptotic normality of rank estimators of $\beta$ in the regression model (1.1) based on complete (uncensored) data $(x_i, y_i)$. Without loss of generality, we shall assume that $\beta = 0$. Theorems 1 and 2 enable us to approximate $S_n(b)$, in a neighborhood of $\beta(0)$, by $S_n(\beta) + \{h_n(b) - h_n(\beta)\}$, where $h_n$ is a nonrandom function which is much more tractable than $S_n(b)$. This is the content of

THEOREM 3. With the same notation and assumptions as in Theorem 2, define $\hat{F}_{n,b}$ by (1.10) and $S_n(b)$ by (1.14), where $\psi$ is a twice continuously differentiable function on $(0,1)$ such that for some $\theta > 0$ and $i = 0, 1, 2$,

$$|\psi^{(i)}(u)| = O(u^{-\theta-i} \cdot (1-u)^{-\theta-i}) \text{ as } u(1-u) \to 0,$$  \hspace{1cm} (4.1)

and the weight function $p_n$ is of the form

$$p_n(x) = p(n^\lambda (x - cn^{-\lambda})), \quad 0 \leq x \leq 1,$$  \hspace{1cm} (4.2a)

with $c > 0, 0 < \lambda < 1$, and $p$ being a twice continuously differentiable function on the real line such that

$$p(y) = 0 \text{ for } y \leq 0, \quad p(y) = 1 \text{ for } y \geq 1.$$  \hspace{1cm} (4.2b)

Define

$$A_{n,b}(y) = -\int_{-\infty < s < y} \left[ p_n(n^{-1}E_n(b,s))/E_n(b,s) \right] dE_n(b,s),$$  \hspace{1cm} (4.3)

$$h_n(b) = \int_{-\infty}^{\infty} \psi \cdot p_n(1-e^{-\lambda_n(b)s}) p_n(n^{-1}E_n(b,s)) \left[ dE_n(b,s) - \frac{EX_n(b,s)}{E_n(b,s)} dE_n(b,s) \right].$$
Then for every \(0 \leq \gamma < 1\) and \(\varepsilon > 0\),

\[
\sup_{|b-a| \leq n^{-\gamma}, -\infty < s < \infty} |\log(1 - \hat{F}_n(b, s)) - \Lambda_n(b, s) - \log(1 - \hat{F}_n(a, s)) + \Lambda_n(a, s)| = O(n^{-1/2 - \gamma/2 + 3\lambda + \varepsilon}) \text{ a.s.},
\]

(4.4)

\[
\sup_{b, s} |\log(1 - \hat{F}_n(b, s)) - \Lambda_n(b, s)| = O(n^{-1/2 + 3\lambda + \varepsilon}) \text{ a.s.},
\]

(4.5)

\[
\sup_{|b-a| \leq n^{-\gamma}} \left| S_n(b) - h_n(b) - S_n(a) + h_n(a) \right| = O(n^{(1-\gamma)/2 + (3+\Theta)\lambda + \varepsilon}) \text{ a.s.}
\]

(4.6)

**Proof.** To apply Theorem 2 we shall make use of the following inequality: For any twice continuously differentiable function \(g\) on \((0,1)\),

\[
|g(x_1) - g(x_2) - g(y_1) + g(y_2)| \leq \left( \sup_t |g'(t)| \right) |x_1 - x_2 - y_1 + y_2|
\]

\[
+ \left( \sup_t |g''(t)| \right) |y_1 - y_2| \left( |x_1 - x_2| + |y_1 - y_2| + |x_2 - y_2| \right).
\]

(4.7)

Since

\[
p_n(n^{-1}\#(b, s)) = 0 \text{ if } n(b, s) \leq cn^{1-\lambda},
\]

(4.8)

it follows from (1.10) that

\[
\log(1 - \hat{F}_n(b, u)) = -\int_{-\infty < s < u} \{p_n(n^{-1}\#(b, s))/n(b, s) + O(\#^2_n(b, s))\} dL_n(b, s).
\]

(4.9)

Let \(g_n(x) = n^{-3\lambda} p(n^{\lambda}(x-cn^{-\lambda}))/x\) for \(0 < x \leq 1\). Then \(\sup_{0 < x < 1} (|g_n'(x)| + |g_n''(x)|) = O(1)\). By (2.13) and the continuity of \(f\), as \(n \to \infty\) and \(h \to 0\) such that \(nh \to \infty\),

\[
\sup_{|b-b'| + |s-s'| \leq h} |n^{-1}E_n(b, s) - n^{-1}E_n(b', s')| = O(h).
\]

(4.10)

Hence it follows from Theorem 1, Lemma 4 and (4.7) that for every \(0 \leq \gamma < 1\) and \(\varepsilon > 0\),

\[
\sup_{|b-a| \leq n^{-\gamma}, -\infty < s < \infty} \left| g_n(n^{-1}\#(b, s)) - g_n(n^{-1}E_n(b, s)) - g_n(n^{-1}\#(a, s)) + g_n(n^{-1}E_n(a, s)) \right| = O(n^{-1/2 - \gamma/2 + \varepsilon}) \text{ a.s.},
\]

\[
\sup_{b, s} \left| g_n(n^{-1}\#(b, s)) - g_n(n^{-1}E_n(b, s)) \right| = O(n^{-1/2 + \varepsilon}) \text{ a.s.}
\]
Moreover, \( \int_{s=-\infty}^{\infty} |dg_n(n^{-1}\#_n(b,s))| \leq \sup_t |g_n(t)| \). Noting that

\[
\int_{-\infty<s<u} \left[ p_n(n^{-1}\#_n(b,s))/\#(b,s) \right] dL_n(b,s) = n^{\frac{3\lambda-1}{2}} \int_{-\infty<s<u} g_n(n^{-1}\#_n(b,s)) dL_n(b,s),
\]

the conclusions (4.4) and (4.5) follow from Theorem 2 (with \( \xi = 0 \)).

To prove (4.6), let \( \phi_n(x) = \psi \cdot p_n(1-e^{-x}) \) for \( x \geq 0 \), so that

\( \psi \cdot p_n(\hat{F}_{n,b}(s)) = \phi_n(-\log(1-\hat{F}_{n,b}(s))) \). Using (4.8), (4.9), and \( dL_n \leq |d\#_n| \), it can be shown that there exists \( K > 0 \) such that

\[
\sup_{b,s} |\log(1-\hat{F}_{n,b}(s))| \leq \log(Kn^\lambda) \quad \text{for all large } \ n. \tag{4.11}
\]

In view of (4.1) and (4.2), \( \sup_{2 \leq \varepsilon \leq K, n} n^{-\Theta_\omega} \lambda \left( |\phi_n(x)| + |\phi_n'(x)| + |\phi_n''(x)| \right) = O(1) \); moreover, \( \sup_{1 \leq \varepsilon \leq 1} n^{-2(2+\Theta)_\lambda} \left( |\phi_n(x)| + |\phi_n'(x)| + |\phi_n''(x)| \right) = O(1) \). Hence using a similar argument as before, we obtain the desired conclusion (4.6) for (1.14) by applying Theorem 2 to the cases \( U_n(b,s) = n^{-2(3+\Theta)_\lambda} \left( -\log(1-\hat{F}_{n,b}(s)) \right) \cdot p_n(n^{-1}\#_n(b,s)) \) and

\[
U_n(b,s) = n^{-2(3+\Theta)_\lambda} \phi_n(-\log(1-\hat{F}_{n,b}(s))) \cdot n^{-1} \chi_n(b,s) \cdot p_n(n^{-1}\#_n(b,s))/[n^{-1}\#_n(b,s)]
\]

respectively, making use of (4.4), (4.5) and Theorem 1 in this connection. \( \square \)

Suppose that \( \lambda \) in the weight function (4.2) is so chosen that \( 6(3+\Theta)_\lambda < 1 \).

Then by (4.6), with probability 1,

\[
S_n(b) - S_n(a) = h_n(b) - h_n(a) + o(n^{\frac{1}{3}}) \quad \text{uniformly in } a, b \in [-\rho, \rho] \quad \text{with } |b-a| \leq n^{-1/3}, \tag{4.12}
\]

\[
|S_n(b) - S_n(a) - h_n(b) - h_n(a)| = o(n^{2/3}) = o(n|b-a|) \quad \text{uniformly in } a, b \in [-\rho, \rho] \quad \text{with } |b-a| \leq n^{-1/3} \tag{4.13}
\]

Since \( n^{-1}|S_n(b) - h_n(b)| \to 0 \) a.s. for every fixed \( b \), it follows from (4.12) and (4.13) that

\[
\sup_{|b| \leq \rho} n^{-1}|S_n(b) - h_n(b)| \to 0 \quad \text{a.s.} \tag{4.14}
\]

Under certain assumptions on the nonrandom function \( h_n \), it can be shown by making use of (4.12) - (4.14) that the rank estimator \( \hat{\beta}_n \), which is a zero-crossing of \( S_n(b) \), is strongly consistent and asymptotically normal. The details are given in [7]. In particular, the following steps are used in [7] to prove the asymptotic normality of
\( \tilde{\beta}_n \) after establishing its consistency. First, by (4.12) and (4.13) with \( a = \beta \), we have with probability 1

\[
S_n(b) = S_n(\beta) + \{h_n(b) - h_n(\beta)\} + o(n^{-\frac{1}{2}}|b-\beta|) \quad \text{uniformly in } |b| \leq \rho. \tag{4.15}
\]

Secondly, an asymptotic analysis of the nonrandom function \( h_n(b) \) (defined in (4.3)) shows that under certain conditions,

\[
h_n(b) - h_n(\beta) \sim Cn|b-\beta| \quad \text{as } n \to \infty \text{ and } b \to \beta, \tag{4.16}
\]

for some nonrandom \( C \neq 0 \). Thirdly, a martingale central limit theorem can be used to show, under certain assumptions, that as \( n \to \infty \),

\[
n^{-\frac{1}{2}} S_n(\beta) \text{ has a limiting normal } N(0, \tau) \text{ distribution}, \tag{4.17}
\]

for some constant \( \tau \). After showing that \( \tilde{\beta}_n \) converges to \( \beta \) a.s. and recalling that \( \tilde{\beta}_n \) is a zero crossing of \( S_n(b) \), we then obtain from (4.15) - (4.17) that \( n^{\frac{1}{2}}(\tilde{\beta}_n - \beta) \) has a limiting \( N(0, \tau/C^2) \) distribution. In view of (4.14), a sufficient condition for the consistency of \( \tilde{\beta}_n \) is

\[
\lim_{n \to \infty} \inf_{|b-\beta| \geq \delta} n^{-1}|h_n(b)| > 0 \quad \text{for every } \delta > 0. \tag{4.18}
\]
5. APPLICATIONS TO THE BUCKLEY-JAMES ESTIMATOR

In this section we consider the Buckley-James estimator, which is a zero-crossing of the function $W_n(b)$ defined in (1.9). Instead of the Kaplan-Meier-type estimator (1.6) originally used by Buckley and James, we use here the modified version (1.10), involving a weight function $p_n$ as in Section 4, for the $\hat{F}_{n,b}$ in $z_i(b)$. In addition, we change the definition (1.7) of $z_i(b)$ as follows. Noting that

$$E(e_i|e_i > z) = \int_{s > z} sdF(s)/(1-F(z)) = z + \int_{s > z} (1-F(s))ds/(1-F(z)) ,$$

we replace (1.7) by

$$z_i(b) = t_i + \{ \int_{s > t_i - bx} \hat{F}_{n,b}(s)p_n(n^{-1} \#(b,s))ds\}/(1-\hat{F}_{n,b}(t_i - bx_i)) .$$

Using this definition of $z_i(b)$ in (1.9), we obtain that

$$W_n(b) - W_n(\beta) = (\beta - b) \sum_{i=1}^{n} (x_i - \bar{x}_n)^2 + \sum_{i=1}^{n} (1-\delta_i)(x_i - \bar{x}_n)[U_n(b,t_i - bx_i) - U_n(\beta,t_i - bx_i)] ,$$

where

$$U_n(b,z) = \{ \int_{s > z} (1-\hat{F}_{n,b}(s))p_n(n^{-1} \#(b,s))ds\}/(1-\hat{F}_{n,b}(z)) .$$

Our analysis of $W_n(b)$ depends on the following theorem on the approximation of $U_n(b,z)$ by the nonrandom function

$$u_n(b,z) = \int_{s > z} p_n(n^{-1}E_n(b,s)) \exp[\Lambda_n(b,s) - \Lambda_n,b(z)]ds ,$$

where $\Lambda_n,b$ is defined in (4.3). Without loss of generality we shall again assume that $\beta = 0$.

**THEOREM 4.** With the same notation and assumptions as in Theorem 2, define $\hat{F}_{n,b}$ by (1.10) and $U_n(b,z), u_n(b,z)$ by (5.2) and (5.3), where the weight function $p_n$ is of the form (4.2a) with $c > 0$, $0 < \lambda < \frac{1}{2}$ and $p$ being a twice continuously differentiable function satisfying (4.2b). Assume furthermore that
\[ M \triangleq \inf\{ a : P[\varepsilon_1 < a] = 1 \} < \infty, f(M) > 0, \quad \text{and} \quad \lim_{n \to \infty} \inf \frac{n}{\sum_{i=1}^{n} P[\varepsilon_i > M]} > 0. \quad (5.4) \]

Then for every \( 0 < \gamma < 1, \theta \geq 0 \) and \( \varepsilon > 0 \),

\[ \sup_{|b| \leq \gamma, z \geq -n^{\theta}} |U_n(b, z) - u_n(b, z)| = O(n^{-1/2 + \gamma} + \varepsilon) \quad \text{a.s.} \quad (5.5) \]

Moreover, if \( \gamma > \lambda \) and \( \theta < \gamma/2 \), then

\[ \sup_{|b| \leq \gamma, z \geq -n^{\theta}} |\int U_n(b, z) - u_n(b, z) - U_n(\tilde{b}, z+a) - u_n(\tilde{b}, z+a) - o(n^{-1/2}) \quad \text{a.s.} \quad (5.6) \]

**Proof.** From (4.8) and Lemma 4, it follows that

\[ p_n(n^{-1} \# n(b, s)) > 0 \Rightarrow \# n(b, s) > cn^{-\lambda} \quad \text{and} \quad \# n(b, s) \sim E\# n(b, s), \]

\[ p_n(n^{-1} E\# n(b, s)) > 0 \Rightarrow E\# n(b, s) > cn^{-\lambda} \quad \text{and} \quad \# n(b, s) \sim E\# n(b, s). \quad (5.7) \]

Since \( p_n(x) = 0 \) if \( x \leq cn^{-\lambda} \) or \( x \geq (c+1)n^{-\lambda} \) and since \( p_n(x) = O(n^{\lambda}) = O(x^{-1}) \)

for \( cn^{-\lambda} < x < (c+1)n^{-\lambda} \), it then follows that there exists \( K > 0 \) such that

\[ |p_n(x) - p_n(y)| \leq K|x-y|/x^2 \quad \text{if} \quad \frac{1}{2} \leq x/y \leq \frac{3}{2} \quad (x, y \in (0, 1)). \quad (5.8) \]

From (5.7) and (5.8) together with Lemma 4, we obtain that with probability 1,

\[ \left| \int_{z < y} \frac{p_n(n^{-1} \# n(b, s))}{n^{-1} \# n(b, s)} - \frac{p_n(n^{-1} E\# n(b, s))}{n^{-1} E\# n(b, s)} \right| d[n^{-1} L_n(b, s)] \]

\[ = O(n^{-1/2 + \varepsilon} \int_{z < y} (n^{-1} E\# n(b, s))^2 d[n^{-1} L_n(b, s)]) = O(n^{-1/2 + \varepsilon}/n^{-1} E\# n(b, y)), \quad (5.9) \]

uniformly in \( z < y \) with \( E\# n(b, y) > \frac{1}{2} cn^{1-\lambda} \). Here and in the sequel, \( \varepsilon \) is chosen to be an arbitrarily small positive number. Moreover, using integration by parts and

Lemma 4, it can be shown that with probability 1,

\[ \left| \int_{z < y} \frac{p_n(n^{-1} \# n(b, s))}{n^{-1} \# n(b, s)} d[n^{-1} L_n(b, s)] - n^{-1} E\# n(b, s)] \right| = O(n^{-1/2 + \varepsilon}/n^{-1} \# n(b, y)), \quad (5.10) \]

uniformly in \( z < y \) with \( \# n(b, y) > \frac{1}{2} cn^{1-\lambda} \), noting that by (5.8),

\[ |d[p_n(n^{-1} \# n(b, s))/n^{-1} \# n(b, s)]| = O((n^{-1} \# n(b, s))^{-2} d(n^{-1} \# n(b, s)) \). \]
We now apply (5.9) and (5.10) to prove (5.5). Let \( \hat{G}_{n,b} = 1 - \hat{G}_{n,b}, \ G_{n,b} = \exp(\Lambda_{n,b}) \).

It follows from (4.3) and (4.9) that

\[
\frac{\hat{G}_{n,b}(y)}{\hat{G}_{n,b}(z)} - \frac{G_{n,b}(y)}{G_{n,b}(z)} = \exp \left\{ \int_{z < s < y} \frac{p_n(n^{-1}n_{n}(b,s))}{\mathbb{E}_{n}(b,s)} \, dL_n(b,s) \right. \\
+ \left. \int_{z < s < y} \frac{p_n(n^{-1}E_{n}(b,s))}{\mathbb{E}_{n}(b,s)} \, dE_n(b,s) + O(n^{-1}) \right\} - 1. \tag{5.11}
\]

First consider the case \( \gamma = 0 \). From (5.7), (5.9) and (5.10), it follows that

\[
\sup_{|b| < \rho} \int_{z}^{M+\rho} |\hat{G}_{n,b}(s)p_n(n^{-1}n_{n}(b,s))/\hat{G}_{n,b}(z) - G_{n,b}(s)p_n(n^{-1}E_{n}(b,s))/G_{n,b}(z)| \, ds \\
= \int_{(M-1)vz}^{(M+\rho)vz} + \int_{z}^{(M-1)vz} = O(n^{-1+2+2\epsilon} + n^{-1+2+2\epsilon}|z|) \text{ a.s.}
\]

and therefore (5.5) follows. Here and in the sequel we use the convention \( \int_{v}^{u} = 0 \) if \( v \geq u \). Note in this connection that by (1.10), \( \hat{G}_{n,b}(z) \) remains constant for all \( z \geq \inf\{s : n_{n}(b,s) \leq n^{-1}\} \), and that \( G_{n,b}(z) \) remains constant for all \( z \geq \inf\{s : E_{n}(b,s) \leq n^{-1}\} \) by (4.3). Moreover, since \( |b_{x_{i}}| \leq A_{\rho} \) and \( e_{i} \leq M \) a.s., the range of integration in (5.2) or (5.3) can be restricted to be \( \leq M+\rho \).

We next consider the case \( \gamma > 0 \). Then by (5.4), with probability 1, as \( n \to \infty \) and \( s \to M \) such that \( M-s \geq n^{-Y+\epsilon} \),

\[
n^{-1}n_{n}(b,s) - n^{-1}E_{n}(b,s) - f(M)(M-s)n^{-1} \leq \sum_{i=1}^{n} p_{t_{i}>s+b_{x_{i}}} \text{ uniformly in } |b| \leq n^{-Y} \tag{5.12}
\]

since \( |b_{x_{i}}| \leq A_{n}^{-Y} = o(M-s) \). Moreover, by (4.3) and (5.4), as \( n \to \infty \) and \( \gamma \to M \) such that \( M-Y \geq n^{-Y+\epsilon} \),

\[
G_{n,b}(y) = \exp(\Lambda_{n,b}(y)) = (M-y)^{1+o(1)} \text{ uniformly in } |b| \leq n^{-Y}. \tag{5.13}
\]

To prove (5.5), it suffices to assume that \( \gamma \leq \lambda \). From (5.9)-(5.13), it then follows that with probability 1

\[
\int_{z}^{M-n^{-Y+\epsilon}} |\hat{G}_{n,b}(s)p_n(n^{-1}n_{n}(b,s))/\hat{G}_{n,b}(z) - G_{n,b}(s)p_n(n^{-1}E_{n}(b,s))/G_{n,b}(z)| \, ds \\
= \int_{(M-1)vz}^{(M-n^{-Y+\epsilon})} + \int_{z}^{(M-1)vz} = O(n^{-1+2+2\epsilon} + n^{-1+2+2\epsilon}|z|) \text{ uniformly in } z \text{ and in } |b| \leq n^{-Y}, \tag{5.14}
\]
noting in view of (5.12) and (5.4) that \( p_n(n^{-1}E_{n}(b,s)) = 1 \) for \( s \leq M^{-\gamma+\epsilon} \) and large \( n \), since \( \gamma \leq \lambda \). For \( M^{-\gamma+\epsilon} \leq s \leq M^{-\gamma} \), we use the bounds

\[ G_n(b,s)/G_n(b,z) \leq 1 \text{ if } s \geq z \], and

\[
\left| \hat{G}_n(b,s)p_n(n^{-1}E_{n}(b,s))/\hat{G}_n(b,z) - G_n(b,s)p_n(n^{-1}E_{n}(b,s))/G_n(b,z) \right|
\leq \left| \hat{G}_n(b,s)/\hat{G}_n(b,z) - G_n(b,s)/G_n(b,z) \right| p_n(n^{-1}E_{n}(b,s))
+ [G_n(b,s)/G_n(b,z)] |n^{-1}E_{n}(b,s) - n^{-1}E_{n}(b,s)| \sup_x |p'(x)| .
\]

From (5.9)-(5.11) and (5.15) together with Lemma 4, it follows that with probability 1,

\[
\int_{(M^{-\gamma+\epsilon})^{\lambda} z}^{M^{-\gamma} + \epsilon} \left| \hat{G}_n(b,s)p_n(n^{-1}E_{n}(b,s))/\hat{G}_n(b,z) - G_n(b,s)p_n(n^{-1}E_{n}(b,s))/G_n(b,z) \right| ds 
= O(n^{-1/2+\lambda+2\epsilon-\gamma}) \text{ uniformly in } z \text{ and in } |b| \leq n^{-\gamma} .
\]

From (5.14) and (5.16), we obtain (5.5) (with \( \epsilon \) replaced by \( \tilde{\epsilon} = 2\epsilon \), which can be arbitrarily small).

We now assume that \( \gamma > \lambda \) and \( \theta < \gamma/2 \) to prove (5.6). First note that for \( |b| \leq n^{-\gamma} \), \( \sup_i |bx_i| \leq An^{-\gamma} = o(n^{-\lambda}) \). Hence analogous to (5.12), we now have for \( |b| \leq n^{-\gamma} \),

\[
\#_{n}(b,s) \geq cn^{1-\lambda} \text{ and } s \rightarrow M \Rightarrow \#_{n}(b,s) - E_{n}(b,s) \sim f(M)(M-s) \sum_{i=1}^{n} P(t_i > s+bx_i) .
\]

Moreover, analogous to (5.13), we now have for \( |b| \leq n^{-\gamma} \)

\[
E_{n}(b,s) \geq cn^{1-\lambda} \text{ and } s \rightarrow M \Rightarrow G_{n,b}(s) = (M-s)^{1+\Theta(1)} .
\]

Since \( E_{n}(b,s) - f(M)(M-s) \sum_{i=1}^{n} P(t_i > s+bx_i) = O(n^{-1-\xi}) \) uniformly in \( |b| \leq n^{-\gamma} \) and \( s \geq M^{-n^{-\xi}} \), we obtain from Lemma 4 together with (5.7) and (5.8) the following refinement of (5.9) and (5.10): With probability 1,

\[
\left| \int_{zv(M^{-n^{-\xi}}) \leq s \leq \gamma} \left( \frac{p_n(n^{-1}E_{n}(b,s))}{\#_{n}(b,s)} dL_n(b,s) - \frac{p_n(n^{-1}E_{n}(b,s))}{E_{n}(b,s)} dE_{n}(b,s) \right) \right| = O(n^{-1/2-\xi/2+\epsilon/n^{-1}E_{n}(b,y)}) .
\]
From (5.11), (5.17), (5.18) and (5.19), it follows that with probability 1,

\[
\sup \int_{|b| \leq n^{-\xi}} \left| \frac{\hat{G}_{n,b}(s)p_n(n^{-1} \mathbb{E}_n(b,s))/\hat{G}_{n,b}(z) - G_{n,b}(s)p_n(n^{-1} \mathbb{E}_n(b,s))/G_{n,b}(z)}{\mathbb{V}(M-n^{-\xi})} \right| ds = O(n^{-\frac{1}{2}+2\varepsilon-\xi/2}), \text{ uniformly in } z, \tag{5.20}
\]

where \( \xi > 0 \) and \( \varepsilon > 0 \) are so chosen that

\[
\lambda > \xi > 4\varepsilon, \quad 3\xi + \theta_{1} + \varepsilon < \gamma/2, \quad 6\xi + 2\varepsilon + \theta < \frac{1}{2}. \tag{5.21}
\]

Since \( \xi < \lambda, \mathbb{P}(n^{-1} \mathbb{E}_n(b,s)) = 1 \) and \( n^{-1} \mathbb{E}_n(b,s) \geq \text{constant} \times n^{-\xi} \) for \( s \leq M-n^{-\xi} \) and large \( n \). Hence the same argument used to prove (4.4) and (4.5) of Theorem 3 can be used to show that

\[
\sup_{|b| \leq |b| \leq n^{-\xi}, s \leq M-n^{-\xi}} |\log \hat{G}_{n,b}(s) - \Lambda_{n,b}(s) - \log \hat{G}_{n,b}(s+a) + \Lambda_{n,b}(s+a)| = O(n^{-\frac{1}{2}-\gamma/2+3\xi+\varepsilon}) \text{ a.s.} \tag{5.22}
\]

\[
\sup_{|b| \leq |b| \leq n^{-\xi}} |\log \hat{G}_{n,b}(s) - \Lambda_{n,b}(s)| = O(n^{-\frac{1}{2}+3\xi+\varepsilon}) \text{ a.s.} \tag{5.23}
\]

From (5.22) and (5.23) together with the inequality (4.7) applied to \( g(x) = e^x \) with \( x \leq 1 \), it follows that

\[
\sup_{|b| \leq |b| \leq n^{-\xi}, n^{-\xi} \leq z \leq M-n^{-\xi}} \left| \int_{z}^{M-n^{-\xi}} \left( \frac{\hat{G}_{n,b}(s) - G_{n,b}(s)}{\hat{G}_{n,b}(z)} - \frac{\hat{G}_{n,b}(s)}{G_{n,b}(z+a)} \right) ds \right| = O(n^{-\frac{1}{2}-\gamma/2+3\xi+\varepsilon+\theta}) + O(n^{-1+6\xi+2\varepsilon+\theta}) \text{ a.s.} \tag{5.24}
\]

From (5.20), (5.21) and (5.24), (5.6) follows. \( \Box \)

Suppose that \( \lambda \) in the weight function \( p_n \) above is so chosen that \( \frac{1}{4} < \lambda < \frac{1}{2} \). Then making use of (5.1) and Theorem 4 and following the steps similar to those outlined at the end of Section 4 for the rank estimator \( \tilde{\beta}_n \), we can prove the consistency and asymptotic normality of the Buckley-James estimator under certain regularity conditions. The details are given in [8].
REFERENCES


