EXTENDED STOCHASTIC LYAPUNOV FUNCTIONS AND RECURSIVE ALGORITHMS IN LINEAR STOCHASTIC SYSTEMS

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Abstract

Herein we first review the method of stochastic Lyapunov functions in stability analysis of stochastic differential (or difference) equations and in convergence analysis of stochastic approximation schemes. A basic ingredient of this approach is to establish from the underlying system dynamics a supermartingale property, or an almost supermartingale property, of the stochastic Lyapunov function. Motivated by consistency problems of recursive estimators in time series models and linear stochastic systems, we then give an extension of this idea by working directly with the recursive inequalities induced by the system dynamics on these stochastic Lyapunov-type functions, instead of relying entirely on a supermartingale or almost supermartingale structure. Applications to recursive identification and adaptive control algorithms are also discussed.
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1. Introduction

In many scientific and engineering applications, the problem of parameter estimation not only involves the classical concept of efficiency of the estimator \( \hat{\theta}_n \) based on a sample of \( n \) observations \( Y_1, \ldots, Y_n \), but for on-line implementation the computational complexity of successively updating the estimator must also be considered. For example, a simple recursion of the form \( \hat{\theta}_n = g(n, \hat{\theta}_{n-1}, Y_n) \) requires much less storage and computational burden than an estimator of the form \( \hat{\theta}_n = g_n(Y_1, \ldots, Y_n) \) which needs the values of all the previous observations. In addition to the computational advantage for on-line identification, these recursive algorithms can also be easily tailored to track time varying parameters in dynamical systems (cf. [18]).

An important stochastic model in the time series and control systems literature is that governed by the linear difference equation

\[
A(q^{-1})y_n = B(q^{-1})u_{n-d} + C(q^{-1})\varepsilon_n,
\]

where \( \{y_n\} \), \( \{u_n\} \) and \( \{\varepsilon_n\} \) denote the output, input, and disturbance sequences, respectively, and

\[
A(q^{-1}) = 1 + a_1q^{-1} + \ldots + a_kq^{-k}, \quad B(q^{-1}) = b_0 + \ldots + b_hq^{-h},
\]

\[
C(q^{-1}) = 1 + c_1q^{-1} + \ldots + c_rq^{-r}
\]

are polynomials in the unit delay operator \( q^{-1} \). There is a large literature on recursive estimation of the parameter vector

\[
\theta = (a_1, \ldots, a_k, b_0, \ldots, b_h, c_1, \ldots, c_r)'
\]

of (1), for which many recursive estimators have been proposed. The recent monographs by Ljung and Söderström [18] and by Caines [1] provide excellent unified overviews of the subject.

A basic problem concerning a recursive identification algorithm is whether, or under what conditions, it converges. The seminal papers by Ljung [16], [17] represent a pioneering effort to study the consistency problem of various recursive identification algorithms in parameter estimation for the linear stochastic system (1). In these papers, Ljung introduced the method of studying the convergence of an identification algorithm, defined recursively by
\[ \hat{z}_n = \hat{z}_{n-1} + \hat{z}_n \, R^{-1} \, \phi_n \, e_n(\hat{z}_{n-1}), \]  

via the stability properties of an associated non-random ODE (ordinary differential equation). The \( \phi_n \) in (3) is a positive scalar \( \leq 1 \) such that \( \hat{z}_{n-1}^{\infty} = \infty \) and \( \hat{z}_{n-1}^{\infty} < \infty \). Typically, \( \phi_n \sim 1/n \). In (3), \( e_n(\theta_{n-1}) \) denotes the prediction error of the one-step-ahead predictor of \( y_n \), and \( \phi_n \) is a \((k+r+h+1)\)-dimensional vector satisfying the recursive relation

\[ \dot{\phi}_n = F(\theta_{n-1})\phi_{n-1} + G(\theta_{n-1})(y_{n-1}, \ldots, y_{n-k}, u_{n-d}, \ldots, u_{n-d-h}, \ldots, e_{n-1}^{(\theta_{n-2}), \ldots, e_{n-r}^{(\theta_{n-r-1})}}), \]

where \( F \) and \( G \) are matrix functions. The \( R_n \) in (3) is a positive definite matrix defined recursively by either of the following:

\[ R_n = R_{n-1} + \phi_n(\phi_n^T R_{n-1}), \]  

or

\[ R_n = \tau_n I, \quad \tau_n = \tau_{n-1} + \phi_n(\phi_n^T \tau_{n-1}). \]

The underlying heuristic idea in Ljung's approach is that if \( \theta_n \) should converge to a limit \( \theta^* \), then one may approximate \( e_i(\theta_{i-1}) \) and \( \phi_i \) in (3)-(5) by \( e_i(\theta^*) \) and \( \phi_i(\theta^*) \), which are stationary ergodic sequences under certain assumptions on the disturbances \( e_n \) and the inputs \( u_n \). This suggests that if \( R_n \) should also converge to a limit \( R^* \), then one may approximate (3) by

\[ \frac{\theta_n - \theta_{n-1}}{\phi_n e_n(\theta^*)} \approx R^{-1} \sum_{n} E[\phi_n(\theta^*) e_n(\theta^*)] \approx R^{-1} h(\theta^*), \]

where \( h(\theta^*) = E[\phi_n(\theta^*) e_n(\theta^*)] \) for all \( n \) by stationarity. Let \( t_n = \frac{\tau_n}{\tau_{n-1}} (\infty) \),

\[ \hat{\theta}(t) = \theta_n \text{ at } t = t_n, \]

and define \( \hat{\theta}(t) \) by linear interpolation when \( t_{n-1} < t < t_n \). Since \( t_n - \tau_{n-1} = \rho_n = 0 \), the left hand side of (6) can be approximated by

\[ \frac{d\theta}{dt} |_{t=t_n} \]

This suggests that the ODE

\[ \frac{dR}{dt} = R^{-1}(t) h(\theta(t)) \]

is a limiting version of the recursion (3) and that the constant solution (equilibrium) \( \theta(t) = \theta^* \) satisfies the equation \( h(\theta^*) = 0 \). Moreover, letting \( H(\theta) = E[\phi_n(\theta^*) \phi_n(\theta^*)] \), the same argument shows that (5a) and (5b) can be approximated by the ODEs

\[ \frac{dR}{dt} = H(\theta(t)) - R, \]

and

\[ \frac{dr}{dt} = tr H(\theta(t)) - r \text{ and } R = rI, \]

respectively. Hence the constant solution \( R(t) = R^* \) (or \( r(t) = tr R^* \)) of the ODE (8a) (or (8b)) satisfies the equation \( R^* = H(\theta^* \hat{\theta}(t)) \) (or \( tr R^* = tr H(\theta^*) \)).

The above argument shows that if the recursive scheme (3)-(5) should converge to a limit \( (\theta^*, R^*) \) a.s., then the limit \( (\theta^*, R^*) \) is an equilibrium of the ODE (7)-(8). Ljung showed further that under certain assumptions, a stability analysis of the ODE (7)-(8) can also characterize the set of limit points of the recursive
scheme. Different sets of assumptions have been proposed to ensure the validity of this approach, and we refer the reader to Ljung and Söderström [18, Chapter 4] for the details. Although the basic heuristic ideas of this approach are quite simple and the stability analysis of the ODE (7)-(8) can often be handled elegantly by using Lyapunov functions, the technical details of the argument and the verification of the underlying assumptions are difficult and tedious and require "persistent excitation" and other restrictive conditions on the inputs as well as stability assumptions on system dynamics.

Consider the ODE \( \frac{dx}{dt} = f(x) \), where \( x(t) = (x_1(t), \ldots, x_n(t))' \) and \( f = (f_1, \ldots, f_n)' \). In particular, for the ODE defined by (7) and (8b), \( x' = (\theta', r) \). A Lyapunov function \( V(x) \) associated with the ODE is a continuously differentiable nonnegative function such that

\[
<f, \text{grad } V> \leq 0. \tag{9}
\]

From (9) and the ODE \( \frac{dx}{dt} = f(x) \), it follows that

\[
\frac{d}{dt} V(x(t)) = <f(x(t)), \text{grad } V(x(t))> \leq 0. \tag{10}
\]

Hence \( V \) is non-increasing along the trajectories of the ODE and the set

\[
D = \{x : <f(x), \text{grad } V(x)> = 0\} \tag{11}
\]

is an invariant set of the ODE (i.e., any trajectory that starts in \( D \) remains there). Outside \( D \), \( V \) is strictly decreasing, but since \( V \) is bounded below, it cannot continue to decrease indefinitely. Hence \( x(t) \) must converge to \( D \) as \( t \to \infty \), showing that \( D \) is a globally asymptotically stable invariant set. The construction of such functions \( V \) is a basic ingredient in Ljung's stability analysis. Usually one cannot, however, define such a function \( V \) on the whole space, but can only define it with (9) holding on some bounded connected subset \( S \). In this case, Ljung's idea is to first choose \( S \) such that it is a domain of attraction of the invariant set \( D \) (i.e., any trajectory that starts in \( S \) converges to \( D \) as \( t \to \infty \)) and then to show that \( P(\theta_n \in R_n \in S \text{ infinitely often}) = 1 \). Carrying out this idea, however, often involves substantial technical difficulties.

Instead of working with a Lyapunov function associated with the limiting deterministic ODE (7)-(8), an obvious alternative is to develop an analogue for the original stochastic recursive algorithm (3)-(5). This is the idea behind the "stochastic Lyapunov function" approach introduced by Moore and Ledwich [19] and Solo [24] for convergence analysis of recursive identification algorithms. This approach, which will be discussed in Section 3, has a substantial history in the stability theory of stochastic dynamical systems and the almost sure (a.s.) convergence theory of stochastic approximation schemes. Section 2 reviews the method of stochastic Lyapunov functions, the supermartingale property of these functions and related martingale convergence theorems. Section 3 extends this method by replacing the supermartingale property with another generalization of the fundamental
property (10) of a Lyapunov function and by applying other martingale limit theorems (not restricted to convergence). Applications of this notion of "extended stochastic Lyapunov functions" to recursive identification, adaptive prediction and adaptive control will also be discussed in Sections 3 and 4.

2. Martingale convergence theorems and the method of stochastic Lyapunov functions in stochastic stability theory and stochastic approximation

Stochastic Lyapunov functions have been introduced as analogues of Lyapunov's auxiliary functions in the stability theory of deterministic ODEs to develop similar stability theorems for stochastic differential (or difference) equations and other Markov processes (cf. [5], [6]). Consider, for example, the stochastic differential equation (SDE)

\[ dY(t) = b(t, Y(t))dt + \sum_{i=1}^{m} \sigma_i(t, Y(t))dw_i(t), \]  

where \( Y(t), b(t, y), \sigma_i(t, y) \) are m-dimensional vectors and \( w_1(t), \ldots, w_m(t) \) are independent Wiener processes. Assume that \( b(t, y) \) and \( \sigma_i(t, y) \) are uniformly Lipschitz continuous in \( y \) and that \( b(t, 0) \equiv 0, \sigma_i(t, 0) \equiv 0 \), so that \( Y(t) \equiv 0 \) is a constant solution of (12). This constant solution is said to be "stable with probability 1" if it is stable with probability 1 and for all \( s \geq 0 \) and \( y \) in some neighborhood of 0,

\[ \lim_{y \to 0} \sup_{t \geq s} |Y_{s,Y}(t)| = 0, \]  

and is said to be "asymptotically stable with probability 1" if it is stable with probability 1 and for all \( s \geq 0 \) and \( y \) in some neighborhood of 0,

\[ P\{\lim_{t \to \infty} Y_{s,Y}(t) = 0\} = 1, \]

where \( Y_{s,Y}(t), t \geq s, \) denotes the solution of (12) with initial condition \( Y(s) = y \).

In the stability theory for a deterministic ODE \( dx/dt = f(x) \), the second (or direct) method of Lyapunov uses the existence of Lyapunov functions satisfying certain conditions to show that an equilibrium is stable or asymptotically stable. A basic property of these functions is (10), which can be written as

\[ D_0 V \leq 0, \quad \text{where} \quad (D_0 g)(u) = \lim_{\delta \to 0} g'(x(u(\delta))) - g(u), \]

and \( x^u(t) \) denotes the solution of the ODE with initial condition \( x^u(0) = u \). For the SDE (12) with time-varying coefficients, a Lyapunov-type function is a function of \( (t, y) \) and a natural analogue of the Lyapunov operator \( D_0 \) in (15) now takes the form

\[ (Lg)(t, y) = \lim_{\delta \to 0} \delta^{-1} E[g(t+\delta, Y(t+\delta)) - g(t, y)]. \]

Note that \( L \) is the infinitesimal generator of the space-time process \( (t, Y(t)) \).

A nonnegative function \( V \) defined on the domain \([0, \infty) \times B\), where \( B \) is some neighborhood of the equilibrium 0 of (12), is called a "stochastic Lyapunov function" associated with the SDE if

\[ L V \leq 0 \quad \text{on} \quad [0, \infty) \times B, \]
in analogy with (15). Like the stability theory of ODEs, the existence of such stochastic Lyapunov functions satisfying certain conditions implies that the equilibrium 0 of the SDE is stable or asymptotically stable with probability 1 (cf. [5]). A key tool in proving these results is that (17) implies that 
\( (V_t, V(t_B^\tau, y^B, \tau_B(t))), t \geq 0 \) is a supermartingale for all \( s \geq 0 \) and \( y \in B \), where \( \tau_B = \inf\{t \geq s : y^B(t) \notin B\} \). By Doob's martingale convergence theorem, the nonnegative supermartingale \( V_t \) converges a.s. as \( t \to \infty \). In addition to the martingale convergence theorem, one can also apply martingale inequalities to obtain bounds for probabilities of the type (13).

Roughly speaking, the stochastic Lyapunov function method is to transform a given stochastic dynamical system (possibly multidimensional, e.g., (12)) into a one-dimensional nonnegative process \( V_t \) so that the dynamics of the original system induce a supermartingale structure on \( V_t \). We next review the application of this method to prove the a.s. convergence of stochastic approximation schemes. Here an obvious candidate for a stochastic Lyapunov function does not yield a supermartingale but has an "almost supermartingale" property. To fix the ideas, consider the Robbins-Monro [21] scheme

\[
\begin{align*}
  u_{n+1} &= u_n - \rho_n y_n \\
  \hat{V}_n &= V(u_{n+1})
\end{align*}
\]

(18)
to find the root \( \theta \) of an unknown regression function in the regression model

\[
y_i = M(u_i) + \varepsilon_i \quad (i = 1, 2, \ldots),
\]

(19)
where \( \rho_n \) are positive constants such that \( \sum_1^\infty \rho_n^2 < \infty \) and \( \sum_1^\infty \rho_n = \infty \). The random disturbances \( \varepsilon_i \) in the regression model are assumed to form a martingale difference sequence with respect to an increasing sequence of \( \sigma \)-fields \( \mathcal{F}_i \) such that \( \sup_i E(\varepsilon_i^2 | \mathcal{F}_{i-1}) < \infty \) a.s., while the regression function \( M \) is assumed to satisfy

\[
M(\theta) = 0, \quad \inf_{\varepsilon \in [\theta - \delta, \theta + \delta]} I(\varepsilon - \theta) > 0 \quad \text{for all} \quad 0 < \delta < 1,
\]

\[
|\varepsilon| \leq c(|\theta - \varepsilon| + 1) \quad \text{for some} \quad c > 0 \quad \text{and all} \quad u.
\]

(20)

To prove that \( u_n \to \theta \) a.s., or equivalently that \( (u_n - \theta)^2 \to 0 \) a.s., a natural candidate for a stochastic Lyapunov function in the stochastic dynamical system (18) is the quadratic function \( \hat{V}(u) = (u - \theta)^2 \). The recursion (18) implies the following recursion for \( \hat{V}_n \):

\[
\begin{align*}
\hat{V}_n &= (u_{n+1} - \theta)^2 = V_{n-1} + \rho_n^2 M^2(u_n) - 2\rho_n M(u_n)(u_n - \theta) - 2\rho_n \varepsilon_n \chi_n - \rho_n \varepsilon_n^2.
\end{align*}
\]

(21)

Hence, by (20) and the fact that \( E(\varepsilon_n^2 | \mathcal{F}_{n-1}) = 0 \),

\[
E(\hat{V}_n | \mathcal{F}_{n-1}) \leq (1 + 2c^2 \rho_n^2) V_{n-1} + \rho_n^2 \{2c^2 + E(\varepsilon_n^2 | \mathcal{F}_{n-1})\} - 2\rho_n M(u_n)(u_n - \theta),
\]

(22)

and therefore

\[
E(\hat{V}_n | \mathcal{F}_{n-1}) \leq (1 + \alpha_{n-1}) V_{n-1} + \beta_{n-1} - \gamma_{n-1}, \quad \text{say},
\]

(23a)
where $\lambda_i, \beta_i, \gamma_i$ are nonnegative $\mathcal{F}_i$-measurable random variables such that
\[ \sum_{i=1}^{\infty} \lambda_i + \sum_{i=1}^{\infty} \beta_i < \infty \text{ a.s.} \] (23b)

Robbins and Siegmund [22] call a sequence $V_n \geq 0$ satisfying (23) a "nonnegative almost supermartingale", for which they prove the following convergence theorem:
\[ V_n \text{ converges a.s. and } \sum_{i=1}^{\infty} \gamma_i < \infty \text{ a.s.} \] (24)

The a.s. convergence of the Robbins-Monro scheme $u_n$ to $\theta$ follows easily from (20) and from the a.s. convergence of $|u_n - \theta| = V_{n-1}$ and that of
\[ \sum_{i=1}^{\infty} \mathbb{I}(u_i) (u_i - \theta) = \sum_{i=1}^{\infty} \gamma_i. \]

From (23a), it follows that $U_n = V_n / \sum_{i=1}^{n-1} (1+\alpha_i) - \sum_{i=1}^{n-1} \beta_i / \sum_{j=1}^{n} (1+\alpha_j)$ is a supermartingale. Although $U_n$ need not be nonnegative, it is bounded below by $-k$ on the event $\sum_{i=1}^{\infty} \beta_i < \sum_{i=1}^{\infty} \gamma_i$, and therefore $U_n$ converges a.s. on $\sum_{i=1}^{\infty} \beta_i < \sum_{i=1}^{\infty} \gamma_i$ for every $k = 1, 2, \ldots$. Since $\sum_{i=1}^{\infty} \beta_i < \infty$ a.s., $U_n$ converges a.s. Since $\sum_{i=1}^{\infty} \gamma_i < \infty$ a.s., it then follows that $V_n$ also converges a.s. This is the basic idea behind the Robbins-Siegmund theorem.

Starting with (20), we can in fact transform the $V_n$ into a nonnegative supermartingale if we further assume that $\sup_i E(\epsilon_i^2 | \mathcal{F}_{i-1}) \leq k$ (which has no loss of generality since we can apply the martingale convergence theorem locally to the events $\{ \sup_i E(\epsilon_i^2 | \mathcal{F}_{i-1}) \leq k \}$, $k = 1, 2, \ldots$, as in the preceding argument). Let
\[ W_n = (V_{n+1})_{i=n+1} \prod \{ 1 + (2c^2 + k) \epsilon_i^2 \}. \] (25)

From (20), it follows that $W_n$ is a nonnegative supermartingale, so the a.s. convergence of $W_n$, and hence of $V_n$ also, follows. This argument is due to Gladyshev [3].

The monograph by Nevel'son and Has'minskii [20] gives a systematic treatment of the a.s. convergence theory of other stochastic approximation schemes by using the method of stochastic Lyapunov functions.

3. Recursive identification and extended stochastic Lyapunov functions

As will be shown in the subsequent discussion, in applications to recursive identification and to adaptive prediction and control in the linear stochastic system (1), it provides much greater flexibility and convenience by not insisting that a stochastic Lyapunov-type function must always converge. Motivated by these applications, we generalize the Robbins-Siegmund theorem by dropping the condition $\sum_{i=1}^{\infty} \beta_i < \infty$ a.s. in the following.

Theorem 1. Let $\epsilon_n, n \geq 1$ be a martingale difference sequence such that
\[ \sup_n E(\epsilon_n^2 | \mathcal{F}_{n-1}) < \infty \text{ a.s.} \] Let $\alpha_n, \beta_n, \gamma_n, V_n$ be nonnegative $\mathcal{F}_n$-measurable random variables such that $\sum_{i=1}^{\infty} \alpha_i < \infty$ a.s., and let $w_n$ be $\mathcal{F}_n$-measurable. Suppose that for $n \geq 2$,
\[ V_n \leq (1 + \alpha_{n-1}) V_{n-1} + \epsilon_n \quad \text{a.s.} \tag{26} \]

(i) On the event \( \{ \epsilon_1^{\infty} E(\epsilon_i|\mathcal{F}_{i-1}^{2}) < \alpha \} \), \( V_n \) converges a.s. and
\[ \epsilon_1^{\infty} E(\epsilon_i|\mathcal{F}_{i-1}^{2}) < \infty \quad \text{a.s.} \]
(ii) For every \( \delta > 0 \),
\[ \max(V_n, \epsilon_i^{\infty} \epsilon_1^{\infty}) = O(\epsilon_i^{\infty} \epsilon_1^{\infty} \epsilon_i^{\infty}) \quad \text{a.s.} \]

Proof. (i) follows from Theorem 1 of Robbins and Siegmund [22]. To prove (ii), let \( Q_n = V_n / \epsilon_i^{\infty} \epsilon_1^{\infty} \). Then by (26),
\[ Q_n - Q_{n-1} + \epsilon_n \leq \epsilon_n \quad \text{a.s.} \]
where \( \epsilon_n \) follows from (27).

The desired conclusion then follows by a truncation argument and the strong law for martingales, as in the proof of Lemma 2(iii) of [10], noting that \( 1 - \epsilon_i^{\infty} \epsilon_1^{\infty} \epsilon_i^{\infty} < \infty \) a.s. since \( \epsilon_i^{\infty} \epsilon_1^{\infty} < \infty \) a.s.

A stochastic Lyapunov function of a stochastic dynamical system (possibly vector-valued) is a sequence of nonnegative random variables that have the supermartingale property, or more generally, an almost supermartingale property in the sense of (23a) and (23b). An "extended stochastic Lyapunov function" drops the assumption \( \epsilon_i^{\infty} \epsilon_1^{\infty} \epsilon_i^{\infty} < \infty \) in (23b) but reinforces (23a), which provides an inequality for \( E(V_n - V_{n-1}|\mathcal{F}_{n-1}^{2}) \), into an inequality (26) for \( V_n - V_{n-1} \) with an additional martingale difference term. To see the usefulness of this idea, we use Theorem 1 to derive the results of [10] and [12] on the strong consistency of the recursive least squares and extended least squares algorithms, and compare these results with those obtained by Moore and Ledwich [19] and Solo [24] using the a.s. convergence of a stochastic Lyapunov function.

As is well known, the linear stochastic system (1) can be written as a stochastic regression model
\[ y_n = \psi_n + \epsilon_n, \tag{27} \]
where \( \psi_n \) is defined in (2) and
\[ \psi_n = (\gamma_{n-1}, \ldots, \gamma_{n-k}, u_{n-d}, \ldots, u_{n-d-h}, \epsilon_{n-1}, \ldots, \epsilon_{n-r})' \tag{28} \]

It is commonly assumed that the \( \epsilon_n \) form a martingale difference sequence with respect to an increasing sequence of \( \sigma \)-fields \( \{ \mathcal{F}_n \} \) such that
\[
\sup_n E(|\epsilon_n|^\alpha |\mathcal{F}_{n-1}^{2}) < \infty \quad \text{a.s. for some } \alpha > 2.
\tag{29}
\]

The input \( u_n \) at stage \( n \) is assumed to be \( \mathcal{F}_n \)-measurable (i.e., involving only the current and past observations \( y_n, y_{n-1}, u_{n-1}, \ldots \), but no future observations).

In the "white-noise" case \( C(q^{-1}) = 1 \), the regressors \( \psi_n \) are observable random vectors such that \( \psi_n \) is \( \mathcal{F}_{n-1}^{2} \)-measurable, and a commonly used estimator
is that based on the least squares criterion. The least squares estimate
\[ \hat{\theta}_n = (\Sigma_1^{n-1} \psi_1^{n-1} i_1^{n-1} \psi_1^n) \text{ based on observations up to stage } n \]
and hence can be expressed in the recursive form
\[ \hat{\theta}_n = \hat{\theta}_{n-1} + p_{n-1} \bar{y}_n \]
where \[ p_{n-1} = \left( \Sigma_1^{n-1} \psi_1^{n-1} i_1^{n-1} \psi_1^n \right)^{-1} \] satisfies the recursion
\[ p_{n-1} = p_{n-2} + \bar{y}_n \]
Note that (30)-(31) is a special case of (3), (5a) with \[ \phi_n = \psi_n, \quad e_n = \bar{y}_n - \theta_{n-1} \]
\[ \phi_n = n^{-1} \bar{y}_n \psi_1^n = n^{-1} \bar{y}_n \]
In this special case, Ljung's ODE approach requires that
\[ n^{-1} \bar{y}_n \psi_1^n = n^{-1} \bar{y}_n \]
and a natural Lyapunov function in the stability analysis of the limiting ODE (7)-(8a) is
\[ V(\hat{\theta}_n, R(t)) = (\hat{\theta}_n - \theta)^\top R(t)(\hat{\theta}_n - \theta), \]
from [1, page 553], [16]. Instead of working with the Lyapunov function of the limiting ODE, an obvious alternative is to work directly with its analogous
\[ (\hat{\theta}_n - \theta)^\top R_n (\hat{\theta}_n - \theta) = n^{-1} V_n, \quad V_n = (\hat{\theta}_n - \theta)^\top p_{n-1} R_n (\hat{\theta}_n - \theta). \]
From (30) and (31), we obtain the following recursive relation for \( V_n \):
\[ V_n = V_{n-1} - (G_n - \theta)^\top \psi_n^n (1 - \psi_n^n \psi_n^n) + \psi_n^n \epsilon_1^n \psi_n^n \]
\[ + 2[(G_n - \theta)^\top \psi_n^n (1 - \psi_n^n \psi_n^n) \epsilon_1^n. \]
Hence \( V_n \) satisfies the inequality (26), which defines an extended stochastic
Lyapunov function, with \( \xi_n = 0 \),
\[ \xi_n = \bar{y}_n \psi_n^n \psi_n^n \quad n^{-1} = 2[(G_n - \theta)^\top \psi_n^n (1 - \psi_n^n \psi_n^n), \]
\[ \gamma_n = [(G_n - \theta)^\top \psi_n^n (1 - \psi_n^n \psi_n^n) \geq 0, \]
noting that \( \bar{y}_n \) is \( \mathcal{F}_n \)-measurable and that \( \psi_n^n \psi_n^n < 1 \). Since \( \bar{y}_n \psi_n^n \psi_n^n \leq 1 \), it then follows from Theorem 1(ii) that
\[ \max(V_n, \Sigma_1^{n-1} \psi_1^n) = O(\Sigma_1^{n-1} \psi_1^n \psi_1^n \epsilon_1^n) + O((\Sigma_1^{n-1} \psi_1^n \psi_1^n)^{2/3}) \quad \text{a.s.} \]
and therefore
\[ \sum_{i=1}^n \gamma_i = O(\Sigma_1^{n-1} \psi_1^n \psi_1^n \epsilon_1^n) \quad \text{a.s.}, \quad V_n = O(\Sigma_1^{n-1} \psi_1^n \psi_1^n \epsilon_1^n) \quad \text{a.s.} \]
From (29) and Lemma 2 of [10], it follows that
\[ \sum_{i=1}^n \psi_1^n \psi_1^n \epsilon_1^n = O(\Sigma_1^{n-1} \psi_1^n \psi_1^n) = O(\log \lambda_{\max}(\Sigma_1^{n-1} \psi_1^n) \quad \text{a.s.}, \]
where \( \lambda_{\max}(A) \) and \( \lambda_{\min}(A) \) denote the largest and smallest eigenvalues of a
symmetric matrix \( A \). Since
\[ V_n = \lambda_{\min} (\theta_n - \theta)^2 I_n - \theta_n \theta_n^2 \lambda_{\min} (P_n^{-1}) = \lambda_{\min} (\theta_n - \theta)^2 \lambda_{\min} (\Sigma_1 \psi_1 \psi_1^\top), \]
(36) and (37) give the following result of Lai and Wei [10] on the strong consistency
of $\theta_n$ in stochastic regression models. Moreover, from the definition (35) of $\gamma_n$, (36) and (37) also imply the result of [7] on adaptive predictors using the least squares identification method.

Theorem 2. Suppose that in the regression model (27), $\{e_n, \mathcal{F}_n, n \geq 1\}$ is a martingale difference sequence satisfying (29) and $\psi_n$ is $\mathcal{F}_{n-1}$-measurable.

(i) On the event $\{\lambda_{\min}(\Sigma_{1}^{n}\psi_i'\psi_i) \to \infty$ and $\log \lambda_{\max}(\Sigma_{1}^{n}\psi_i'\psi_i) = O(\lambda_{\min}(\Sigma_{1}^{n}\psi_i'\psi_i))\}$, $\theta_n \to \theta$ a.s.

(ii) Let $\hat{\theta}_n = \theta_n^{\psi_n}$ be the optimal predictor of $\gamma_n$ when $\theta$ is known, and let $\hat{\theta}_n = \theta_{n-1}^{\psi_n}$ be the adaptive predictor of $\gamma_n$ using the least squares estimate $\theta_{n-1}$ to replace $\theta$. Then for every $0 < \delta < 1$,

$$\frac{1}{\lambda_{\max}(\Sigma_{1}^{n}\psi_i'\psi_i)} \left(\sum_{i=1}^{n} \psi_i' \Sigma_{1}^{i-1} \psi_i \right) = O(\log \lambda_{\max}(\Sigma_{1}^{n}\psi_i'\psi_i)) \text{ a.s.}$$

For the linear system (1), the regressors $\psi_n$, however, are not completely observable in the "colored-noise" case with $r \geq 1$ moving average parameters $c_1$, ..., $c_r$, since the components $\varepsilon_{n-1}$, ..., $\varepsilon_{n-r}$ of $\psi_n$ are unobservable. Replacing the unobservable $\varepsilon_i$ by their estimates $\hat{\varepsilon}_i$ in the recursions (30) and (31) leads to the "extended least squares" algorithm of the form

$$\theta_n = \theta_{n-1} + P_n \phi_n (\psi_n - \theta_{n-1} \phi_n),$$

where

$$\phi_n = (\varepsilon_{n-1}, \varepsilon_{n-k}, \varepsilon_{n-d}, \varepsilon_{n-d-h}, \hat{\varepsilon}_{n-1}, ..., \hat{\varepsilon}_{n-r})'.$$

The estimates $\hat{\varepsilon}_i$ of $\varepsilon_i$ are given either by the residuals $\hat{\varepsilon}_i = y_i - \hat{y}_i \phi_i$, in which case (38) is called the AML method, or by the prediction errors $\hat{\varepsilon}_i = y_i - \hat{\theta}_i \phi_i$, in which case (38) is called the RML method (cf. [1], [18]). Ljung's stability analysis of the associated ODE shows that the following positive real assumption is needed for the extended least squares algorithm to be strongly consistent:

$$C(z) \hat{\phi}_1 + c_1 z + ... + c_r z^r \text{ has all zeros outside the unit circle},$$

and $\text{Re}(1/C(z)-1/2) > 0$ for all $|z| = 1$.

Under this assumption, Lai and Wei [12] showed that $V_n \hat{\phi}_n = (\theta_n - \theta) P_n^{-1}(\theta_n - \theta)$ again satisfies an inequality of the type (26) for the AML algorithm, and that for the RML algorithm, an inequality of this type holds on the event $\{\lim_{n \to \infty} \phi_n = 0\}$. In this way they obtained the following analogue of Theorem 2 for the extended least squares method under the positive real assumption (39).

Theorem 3. Assume the positive real condition (39).

(i) For the AML algorithm $\theta_n$ defined by (38) with $\hat{\varepsilon}_i = y_i - \theta_i \phi_i$,

$$\theta_n - \theta = O\left(\log \lambda_{\max}(\Sigma_{1}^{n}\phi_i'\phi_i) / \lambda_{\min}(\Sigma_{1}^{n}\phi_i'\phi_i)\right) \text{ a.s.}$$

Moreover, $\theta_n \to \theta$ a.s. if
\[ \lambda_{\min}(\Sigma_{i=1}^{n} \psi_i \psi_i') = \infty \quad \text{and} \quad \log \lambda_{\max}(\Sigma_{i=1}^{n} \psi_i \psi_i') = o(\lambda_{\min}(\Sigma_{i=1}^{n} \psi_i \psi_i')) \quad \text{a.s.} \]  

(41)

For every \( 0 < \varepsilon < 1 \),

\[ \frac{1}{n} \sum_{i=1}^{n} \left( \left( \psi_i - \phi_i \right) \left( \psi_i - \phi_i \right)^	op \right) = O(\log \lambda_{\max}(\Sigma_{i=1}^{n} \phi_i \phi_i')) \quad \text{a.s.} \]

On the event \( \lambda_{\max}(\Sigma_{i=1}^{n} \phi_i \phi_i') \to \infty \), \( \lambda_{\max}(\Sigma_{i=1}^{n} \phi_i \phi_i') \sim \lambda_{\max}(\Sigma_{i=1}^{n} \psi_i \psi_i') \quad \text{a.s.} \)  

(42)

(ii) For the RML algorithm \( \hat{\theta}_n \) defined by (38) with \( \hat{e}_i = y_i - \hat{\phi}_i - \hat{\phi}_i' \), (40) and (42) hold on the event \( \{ \lim_{n \to \infty} \hat{\phi}_n \hat{\phi}_n' = 0 \} \).

Earlier, Solo [24] and Moore and Ledwich [19] established the strong consistency of the AML algorithm under (39) and the persistent excitation assumption (52), which is much stronger than (41). Instead of working directly with the recursive inequalities for \( V_n \), they had to transform \( V_n \) into an almost supermartingale so that they could apply martingale convergence theorems. Thus, Solo introduced the stochastic Lyapunov function

\[ W_n = n^{-1}(V_n + 2f_n^2), \quad \text{where} \quad f_n = (\theta - \hat{\theta}_n)' \phi_n, \quad g_n = (1/C(q^{-1})-1/2) f_n. \]

Theorem 1 enables one to bypass this kind of transformations, and working directly with \( V_n \) as in [12] actually gives sharper results.

The excitation condition (41) for the strong consistency of the least squares estimator in Theorem 2 or the AML algorithm in Theorem 3 is in some sense the weakest possible, cf. [10]. As shown in [11], such excitation conditions on the stochastic regressors \( \psi_i \) can be translated into corresponding conditions involving the inputs alone. To ensure such excitation conditions in an adaptive control setting that will be discussed in the next section, one can either introduce occasional white-noise probing inputs when the data show inadequate information to estimate \( \theta \) by \( \hat{\theta}_n \), as in [12] and [13], or use the Caines-Lafortune [2] idea of "continually disturbed controls" of the form

\[ u_n = g_n(y_n, y_{n-1}, u_{n-1}, \ldots) + \eta_n, \]

where \( \{g_n\} \) represents a feedback control scheme and \( \{\eta_n\} \) represents an exogenous white-noise process satisfying certain assumptions.

4. Adaptive prediction, adaptive control and other applications of the method of extended stochastic Lyapunov functions

To begin with, consider the Robbins-Monro scheme (18) for finding the root of the regression function \( M \) satisfying (20). As shown in Section 3, application of the stochastic Lyapunov function \( V_n = \frac{1}{n} (u_{n+1} - \theta)^2 \) not only proves that \( u_n \to \theta \) a.s. but also shows that \( \frac{1}{n} \sum_{i=1}^{n} M(\theta)(u_i - \theta)^2 < \infty \) a.s. In the sequel we shall assume that \( M(\theta) = \lambda \) exists and is positive. It then follows that \( \frac{1}{n} \sum_{i=1}^{n} (u_i - \theta)^2 < \infty \) a.s. Assuming also that \( \rho_n \to 0 \), this in turn implies by Kronecker's lemma that

\[ \rho_n \frac{1}{n} (u_i - \theta)^2 \to 0 \quad \text{a.s.} \]

(43)
The consequence (43) of the method of stochastic Lyapunov functions has important implications on the adaptive control aspects of stochastic approximation. Motivated by the following kind of applications, Lai and Robbins [8], [9] studied the Robbins-Monro scheme (18) as a control algorithm in the regression model (19). Suppose that in (19), \( u_i \) is the dosage level of a drug given to the \( i \)th patient who turns up for treatment and that \( y_i \) is the response of the patient. Suppose also that the optimal response is \( y^* \), which we shall assume to be 0 without loss of generality (replacing \( y_i \) by \( y_i - y^* \) if necessary). To achieve this mean response \( y^* = 0 \), if the unique (by (20)) root \( \theta \) of the equation \( M(\theta) = 0 \) were known, then the dosage levels should all be set at \( \theta \). Since \( \theta \) is usually unknown, how can the dosage levels \( u_i \) be chosen so that they approach \( \theta \) rapidly? More precisely, the control problem is choose the design levels \( u_1, \ldots, u_n \) so as to minimize in some sense, at least in the long run as \( n \to \infty \), the loss (or regret, due to ignorance of \( \theta \))

\[
L_n = \frac{\lambda^2}{\sqrt{1}} \sum_{i=1}^{n} (u_i - \bar{\theta})^2 \left( -n^{-1}[M(u_i) - M(\bar{\theta})]^2 \right) .
\] (44)

Note that by choosing \( \sigma_n \sim (cn)^{-1} (c > 0) \) in the Robbins-Monro scheme (18), (43) implies that

\[
n^{-1}L_n \to 0 \quad \text{a.s.} \quad (45)
\]

Suppose that the \( \epsilon_i \) are i.i.d. with mean 0 and variance \( \sigma^2 > 0 \). The Robbins-Monro scheme (18) with \( \sigma_n \sim (cn)^{-1} \) is asymptotically normal if \( 0 < c < 2. \lambda \). In fact, \( n^{-1/2}(u_n - \bar{\theta}) \) converges in distribution to \( N(0, \sigma^2/(c(2\lambda - c))) \), and an asymptotically optimal choice of \( \sigma_n \) is \( \sigma_n \sim (\lambda n)^{-1} \) (cf. [8]). For this asymptotically optimal choice, not only do we have

\[
n^{-1/2}(u_n - \bar{\theta}) \overset{D}{\to} N(0, \sigma^2/\lambda^2) ,
\] (46)

but the regret \( L_n \) defined in (44) also has the logarithmic order

\[
L_n \sim (\sigma^2/\lambda^2) \log n \quad \text{a.s.} \quad (47)
\]
as shown in [8]. The constant \( \sigma^2/\lambda^2 \) in (46) and (47) is in some sense minimal when the \( \epsilon_i \) are normal and \( M \) is linear, in which case the stochastic approximation scheme \( u_n \) (with \( \sigma_n = (\lambda n)^{-1} \)) is equal to the maximum likelihood estimator of \( \theta \) at stage \( n \) (cf. [8], [9]).

Since \( \lambda = M(\theta) \) is usually unknown in practice, Lai and Robbins [8], [9] considered an adaptive stochastic approximation scheme (18) with \( \sigma_n = (n\lambda_n)^{-1} \), where \( \lambda_n \) is a consistent estimator of \( \lambda \) at stage \( n \). It is shown that for such adaptive stochastic approximation schemes, (46) and (47) still hold. Wei [25] subsequently extended these results to multivariate adaptive stochastic approximation schemes for multidimensional regression functions.

There are analogous results for adaptive control of the linear stochastic system (1). The problem is to choose the inputs \( u_i \) so that \( \sum_{i=1}^{n} (y_i - y^*_i)^2 \) is minimized in
some sense, at least in the long run as \( n \to \infty \), for a given sequence of target values \( \{ y^*_i \} \) for the outputs. The special case \( y^*_i \equiv 0 \) corresponds to the "regulation problem". When the system parameters are known and \( b_0 \neq 0 \), the optimal controller chooses the inputs \( u_i \) at stage \( i \) so that \( E(y_{i+d} | \mathcal{F}_i^c) = y^*_{i+d} \). To focus on the main ideas, we shall only consider the case of unit delay \( d = 1 \). The outputs of this optimal input sequence are given by \( y_i = y^*_i + \epsilon_i \). In view of this, we define the "regret" (due to ignorance of the system parameters) of an input sequence \( \{ u_i \} \) to be

\[
L_n = \frac{1}{n} \sum_{i=1}^{n} (y_i - y^*_i - \epsilon_i)^2 = \frac{1}{n} \sum_{i=1}^{n} (E(y_i | \mathcal{F}_{i-1}) - y^*_i)^2 ,
\]

in analogy with (44). An input sequence is called "globally convergent" if (45) holds, cf. [4].

An important breakthrough in the long-standing problem of developing, for the stochastic system (1), on-line recursive control algorithms that are globally convergent is due to Goodwin, Ramadge and Caines [4]. Their idea is to use the "stochastic gradient method" for recursive estimation of the parameters:

\[
\begin{align*}
\theta_n &= \hat{\theta}_{n-1} + a \phi_n (y_n - \theta_n \phi_n) / r_n , \\
r_n &= r_{n-1} + \phi_n \phi_n ,
\end{align*}
\]

where \( a > 0 \), \( \phi_n \) is the same as in (38) with \( \hat{\epsilon}_{i} = y_i - \theta_{i-1} \phi_i \), and \( \theta_0 \), \( \phi_0 \), \( r_0 > 0 \) are arbitrary initial conditions. The inputs \( u_i \) are then determined by the equation

\[
\hat{\epsilon}_{n-1} \phi_n = y^*_n .
\]

Under certain assumptions including the positive real condition

\[
C(z) \triangleq 1 + c_1 z + \ldots + c_r z^r \text{ has all zeros outside the unit circle,}
\]

and \( \text{Re}(C(z)-a/2) > 0 \) for all \( |z| = 1 \),

a stochastic Lyapunov function argument similar to that for the Robbins-Monro scheme presented above can be used in conjunction with an analysis of the system dynamics to show that (45) holds for the regret \( L_n \), defined by (48), of this input sequence. Actually Goodwin et al. [4] reparametrized the model (1) and used the stochastic gradient algorithm to estimate the transformed parameters. However, as shown in [15], a modification of their argument can be used to handle the original model without reparametrization.

In analogy with the result (47) for adaptive stochastic approximation, it is natural to ask whether one can improve the global convergence result of Goodwin et al. [4] and have a logarithmic order for the regret \( L_n \). To answer this question, Lai and Wei [12] used the AML algorithm (38) (in which \( \hat{\epsilon}_i = y_i - \theta_i \phi_i \)) for recursive estimation of the system parameters to construct recursive control algorithms whose regrets satisfy
\[ L_n = O(\log n) \text{ a.s.}, \quad (52) \]

under certain assumptions including stability of the polynomials \( A(z) \) and \( B(z) \) and the positive real condition (39). A basic tool in their analysis is Theorem 3(i) on the strong consistency of the AML method and on the associated adaptive predictors (42). In particular, the desired order of \( \log n \) in (52) follows from the order of \( \log \lambda_{\max}(\Sigma_1^n, \Phi_1^n) \) in (42). As pointed out in Section 3, Theorem 3 can be obtained by applying Theorem 1 to the extended stochastic Lyapunov function \( V_n \overset{\Delta}{=} (\theta - \theta)^T \Sigma_n^{-1}(\theta - \theta) \). Thus, while the method of stochastic Lyapunov functions has led to the global convergence result (45) of Goodwin et al. [4], the method of extended stochastic Lyapunov functions is also instrumental in the theory of adaptive controllers satisfying the much stronger property (52).

In the white-noise case \( C(z) = 1 \), the AML algorithm reduces to the least squares estimator (30). For the regulation problem (i.e., \( Y_i^* = 0 \)) in this case, Lai and Wei [13] constructed adaptive controllers using recursive least squares for system identification such that

\[ \limsup_{n \to \infty} \frac{L_n}{\log n} \leq (k+h) \limsup_{n \to \infty} E(\varepsilon_n^2 | \mathcal{F}_{n-1}^2), \quad (53) \]

which is a stronger result than (40). The constant on the right hand side of (53) is in some sense best possible (cf. [7], [13]). In particular, the factor \( k+h \) corresponds to the number of parameters to be estimated if \( b_0 \) were known, noting the linear constraint (50) induced by the inputs on the estimates of the \((k+h+1)\)-dimensional parameter vector \( \theta \).

For the case of colored noise, although the extended least squares algorithm (38) has been called "approximate maximum likelihood" (AML), it does not arise from maximization of the log-likelihood function, as in the off-line (non-recursive) maximum likelihood estimator, when the \( \varepsilon_i \) are assumed to be normally distributed with mean \( 0 \) and variance \( \sigma^2 \). Using a linear approximation to the log-likelihood function, Söderström [23] derived a recursive estimator, commonly called RML, for the system parameters of (1). This method is recently refined in [14], where the underlying idea is to make use of an auxiliary recursive estimator based on instrumental variables and to linearize the log-likelihood function only in a neighborhood of the auxiliary estimator. Under certain conditions, the recursive estimator is shown in [14] to be asymptotically normal; moreover, the covariance matrix of the limiting normal distribution is minimal when the \( \varepsilon_i \) are i.i.d. \( N(0, \sigma^2) \) random variables.

The method of extended stochastic Lyapunov functions can also be used to provide an analogue of Theorem 2 for this estimator, as shown in [14].

Making use of this new recursive estimator, an adaptive control algorithm can be constructed for the linear stochastic system (1) whose regret satisfies

\[ \limsup_{n \to \infty} \frac{L_n}{\log n} \leq (k+h+r+1) \limsup_{n \to \infty} E(\varepsilon_n^2 | \mathcal{F}_{n-1}^2), \quad (54) \]
Moreover, for the regulation problem \( (\phi_j^* \equiv 0) \), (54) can be further strengthened into
\[
\limsup_{n \to \infty} \frac{1}{\log n} \max_k \{ (k+r)+h \} \limsup_{n \to \infty} E(\epsilon_n^2 | \mathcal{F}_{n-1}) \tag{55}
\]
which is a generalization of (53) that corresponds to the case \( r = 0 \). The details are given in [15].

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