AN ANALYTIC APPROACH TO FINITE FLUCTUATION PROBLEMS IN PROBABILITY

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GLEN E. BAXTER

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Introduction and Definitions

During the past 15 years a great deal of research has been done on finite fluctuation problems in probability theory. Basically, the problem is to calculate the distribution of random variables defined in terms of partial sums $S_n$ of a sequence $\{X_k\}$ of independent and identically distributed random variables. One feature of this work is that emphasis is placed on finding the exact distribution of variables defined by a finite number of the $S_n$ as opposed to finding limit theorems involving distributions of $S_n$ for large $n$. Our purpose here is to show that one relatively simple method will give all the interesting results on finite fluctuation problems which have thus far been discovered. This method also gives a number of interesting new results concerning questions not previously considered.

It is amusing to note that the real interest in finite fluctuation problems was inspired by a limit theorem of Erdős and Kac [11]. In 1946 they proved the following result: Let $X_1, X_2, \ldots$ be independent random variables each having mean 0 and variance 1 and such that the central limit theorem is applicable. Let $S_k = X_1 + \cdots + X_k$ and let $N_n$ denote the number of $S_k$'s, $1 \leq k \leq n$, which are positive. Then
\[ \lim_{n \to \infty} P \left\{ \frac{N_n}{n} < \alpha \right\} = \frac{2}{\pi} \arcsin \alpha^{1/2}, \quad (0 \leq \alpha \leq 1). \]

A short time after the appearance of the result (0.1), E. Sparre Andersen [1] showed that the same result holds if \( X_1, X_2, \ldots \) are independent and have a common symmetric and continuous distribution. In proving (0.1) Andersen presented the first significant result concerning finite fluctuation problems. He proved that, if \( X_1, X_2, \ldots \) are independent and have a common continuous and symmetric distribution,

\[ P \{ N_n = m \} = (-1)^n \left( \begin{array}{c} -\frac{1}{2} \\ m \end{array} \right) \left( \begin{array}{c} -\frac{1}{2} \\ n-m \end{array} \right). \]

Still later, Andersen [2,3] extended his own result (0.2) to the case in which the \( X_k \)'s are neither symmetric nor have continuous distributions. He showed that \( P \{ N_n = m \} \) depends only on \( P[S_k > 0]\) \( (k = 1, 2, \ldots, n) \) according to the identity

\[ \sum_{n=0}^{\infty} \lambda^n \sum_{m=0}^{n} u^m P[N_n = m] = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (u^k P[S_k > 0] + P[S_k \leq 0]) \right]. \]

The importance of the form of this identity was further demonstrated by Andersen [4,5] in other examples.

The next significant contribution to the theory of finite fluctuation problems was made independently by Pollaczek [16] and by Spitzer [18]. Spitzer showed that the distribution of \( M_n = \max(S_0, S_1, \ldots, S_n) \), where \( S_0 = 0 \), depends only on the distribution of \( S_k \) \( (k = 1, 2, \ldots, n) \) according to the identity
\[ (0.4) \quad \sum_{n=0}^{\infty} \lambda^n \int e^{i t M} \, dP = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} \left( \int_{\{S_k > 0\}} e^{i t S_k} \, dP + P(S_k \leq 0) \right) \right]. \]

Pollaczek had found earlier essentially the same identity (0.4). The discovery by Spitzer of the identity (0.4) inspired a great deal of work on finite fluctuation problems by a number of people. We will mention here as a final result of interest one further identity presented by Wendel [22] in 1960. Let \( S_0, S_1, \ldots, S_n \) be ordered according to decreasing value, and denote these ordered sums by \( R_{n0} \geq R_{n1} \geq \cdots \geq R_{nn} \). Wendel's result states that

\[ (0.5) \quad \sum_{n=0}^{\infty} \lambda^n \sum_{m=0}^{n} U^m \int e^{i t R_{nm}} \, dP \]

\[ = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} \left( \int_{\{S_k > 0\}} e^{i t S_k} \, dP + U^k P(S_k > 0) \right) \right. \]

\[ + \left. U^k \int_{\{S_k < 0\}} e^{i t S_k} \, dP + P(S_k \leq 0) \right] \]

We are now in a position to make two observations. First, the results stated in (0.3), (0.4), and (0.5) are all examples of one basic functional identity. In Part I of this report we establish a method by which the above identities (and others) may be derived. The virtue of the method lies in its simplicity and its applicability. Our approach is basically analytical in contrast with the combinatorial approach of Andersen and Spitzer and Feller [13], and the algebraic treatment of Wendel. Next, we observe that none of the results stated above deals with joint distributions. In Part II of this
report we consider the problem of joint distributions using methods similar to those developed in Part I. Certain mathematical difficulties require us to restrict our attention in Part II to variables \( X_k \) which take on only integral values. For this restricted case we find results which are very similar to those listed in (0.3), (0.4), and (0.5).

Before stating a typical result of Part II, let us note a curious fact. In each identity (0.3), (0.4), and (0.5), the right hand side is a product of two types of functions. If we set

\[
(0.6) \quad f_+(\lambda, t) = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} \int_{\{S_k > 0\}} e^{itS_k} \, dP \right]
\]

and

\[
(0.7) \quad f_-(\lambda, t) = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} \int_{\{S_k < 0\}} e^{itS_k} \, dP \right],
\]

then Spitzer's identity (0.4), for example, can be written

\[
(0.4) \quad \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int e^{-\lambda t} \, dP = f_+(\lambda, t) f_-(\lambda, 0).
\]

Similar expressions hold for (0.3) and (0.5).

The results of Part II will show that product formulas like (0.4) hold also for joint distributions involving the "range" \( R_n = \max (S_0, S_1, \ldots, S_n) - \min (S_0, S_1, \ldots, S_n) \). Specifically, let \( X_1, X_2, \ldots \) be independent and
identically distributed lattice variables \( (X_k = 0, \pm 1, \pm 2, \ldots) \) and let 
\[ A_k = \delta_{k0} - \lambda P(X_1 = k). \]
Let \( D_n = \det(A_{j-1}) \), \( i, j = 0, 1, \ldots, n \), let

\[
\begin{vmatrix}
A_0 & A_1 & \cdots & A_n \\
A_{-1} & A_0 & \cdots & A_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{-n+1} & A_{-n+2} & \cdots & A_1 \\
z^n & z^{n-1} & \cdots & 1
\end{vmatrix},
\]

\[ (z = e^{it}) \]

and let

\[
\begin{vmatrix}
A_0 & A_1 & \cdots & A_{n-1} & z^{-n} \\
A_{-1} & A_0 & A_{n-2} & z^{-n+1} \\
\vdots & \vdots & \vdots & \vdots \\
A_{-n} & A_{-n+1} & \cdots & A_1 & 1
\end{vmatrix},
\]

\[ (z = e^{it}) \]

Our analogue of (0.4) is then

\[
(0.10) \quad \sum_{k=0}^{\infty} \lambda^k \int_{\{ R_k < n \}} e^{itM} dP = f_n^+(\lambda, t) f_n^-(\lambda, 0).
\]

Of course, the joint distribution of \( \max(S_0, S_1, \ldots, S_n) \) and 
\( \min(S_0, S_1, \ldots, S_n) \) can be calculated from the result (0.10).

The polynomials \( f_n^+(\lambda, t) \) and \( f_n^-(\lambda, t) \) defined in (0.8) and (0.9) have an extremely important biorthogonality relation which is crucial in our methods.
If we let \( \phi(t) \) denote the characteristic function of \( X_1 \) (lattice variable), then it will be shown that

\[
(0.11) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} [e^{i\lambda t} f_n^-(\lambda, t)] [e^{-i\lambda t} f_m^+(\lambda, t)] [1 - \lambda \phi(t)] \, dt = \delta_{nm}.
\]

Perhaps even more important is the observation that (0.11) implies that \( f_n^+(\lambda, t) [1 - \lambda \phi(t)] \) is a Fourier series with zero coefficients of \( e^{ikt} \) for \( k = 1, 2, \ldots, n \). As we will see later, this last fact connects very closely the functions \( f_+^+(\lambda, t) \) and \( f_n^+(\lambda, t) \).

In section 4 of Part I we apply our method to the problem of change of sign. Although limiting distributions for changes of sign are known [9], and although distributions for the number of changes in sign among \( S_0, S_1, \ldots, S_n \) have been given for special variables \( X_k \) [12], this problem is generally conspicuous in its absence from papers on finite fluctuation problems. There appear to be no "invariant" results of the type (0.2) for the number of changes of sign. It is hoped that the discussion of section 4 Part I will shed some light on the essentially more difficult character of this problem. The results presented in section 4 are in large part joint work of the author with E. Sparre Andersen.

We now list for future reference the definitions of variables which will be used in the sequel. In all cases \( S_0, S_1, \ldots, S_n \) will denote the partial sums of a sequence of independent and identically distributed random variables \( \{X_k\} \).
$N_n$: the number of positive partial sums among $S_0, S_1, \ldots, S_n$

$L_n$: the first index $k$ such that $S_k = \max(S_0, S_1, \ldots, S_n)$

$L_n$: the first index $k$ such that $S_k = \min(S_0, S_1, \ldots, S_n)$

$M_n = \max(S_0, S_1, \ldots, S_n)$

$\bar{M}_n = \min(S_0, S_1, \ldots, S_n)$

$R_n = \max(S_0, S_1, \ldots, S_n) - \min(S_0, S_1, \ldots, S_n)$

$(0.12)$ $C_n$: the number of indices $k$ such that either $S_{k-1} > 0$ and $S_k < 0$, or $S_{k-1} < 0$ and $S_k > 0$

$R_{n,k}$: the partial sums $S_0, S_1, \ldots, S_n$ are ordered according to decreasing value and the ordered sums are denoted by

$$R_{n,0} \geq R_{n,1} \geq \cdots \geq R_{n,n}$$

$L_{n,k}$: the index $m$ such that $S_m = R_{n,k}$.

In order that the index variables $L_{n,k}$ be well defined we need to make a convention. If several partial sums among $S_0, S_1, \ldots, S_n$ have the same value, we will order these partial sums according to increasing value of the subscripts. This will always give a unique ordering of the partial sums. In the sequel it will be convenient at times to analyze a simple picture called a "path". A path is just a plot (with connecting lines) of possible values of $S_0, S_1, \ldots, S_n$. Such a path is drawn below to illustrate the variables defined above.
Fig. 1.
1. A factorization problem:

In this section we will find the distribution of the variables $L_n$ and $M_n$ by a procedure which brings clearly into focus the central mathematical problem of the method of Part I. The procedure used in this section is closely related to the Wiener - Hopf factorization technique of classical analysis. We point out, however, that it is more general than the Wiener - Hopf technique because of special properties of the functions with which we are dealing.

In all we will derive four identities similar to those listed in (0.3), (0.4), and (0.5). In the notation of (0.6) and (0.7) these identities are as follows:

Identities:

i) (Andersen [4])

\[
\sum_{n=0}^{\infty} \lambda^n \sum_{m=0}^{n} U^m P(L_n = m) = f_+ (\lambda U, 0) f_- (\lambda, 0),
\]

ii) \[
\sum_{n=0}^{\infty} \lambda^n \int_{\{L_n = n\}} e^{\text{itS}} n \, dP = f_+ (\lambda, t),
\]

iii) \[
\sum_{n=0}^{\infty} \lambda^n \int_{\{L_n = 0\}} e^{\text{itS}} n \, dP = f_- (\lambda, t),
\]

iv) (Spitzer [15])

\[
\sum_{n=0}^{\infty} \lambda^n \int e^{\text{itM}} n \, dP = f_+ (\lambda, t) f_- (\lambda, 0).
\]
It will be most convenient to prove ii) and iii) first.

Proof: Step I: For notational convenience let

\[ (1.2) \quad p_n(t) = \int_{\{L_n=n\}} e^{itS} dP \quad \text{and} \quad q_n(t) = \int_{\{L_n=0\}} e^{itS} dP. \]

If \( \varphi(t) \) denotes the characteristic function of \( X_1 \), then

\[ (1.3) \quad \varphi^n(t) = \int e^{itS} dP = \sum_{k=0}^{n} \int_{\{L_n=k\}} e^{itS} dP. \]

A typical "path" satisfying the condition \( L_n = k \) is illustrated below for the case \( k = 2 \) and \( n = 5 \). It is easy to see that

![Diagram](image-url)
such a path can be described by separate conditions on the sets of variables $X_1, \ldots, X_k$ and $X_{k+1}, \ldots, X_n$. If we let $\sim_{n-k}$ denote the variable "L" applied to $Y_1 = X_{k+1}, \ldots, Y_{n-k} = X_n$, then $\{L_n = k\} = \{L_k = k\} \cap \sim_{n-k} = 0\}$. Thus, using the independence and identical distribution property of the $X_k$'s we may write

$$\int e^{itS_n} dP = \int e^{itS_k} e^{it(S_n - S_k)} dP_{\sim_{n-k}} = \pi_k(t) q_{n-k}(t).$$

From (1.3) and (1.4) we get that for all $n \geq 0$

$$\varphi^n(t) = \sum_{k=0}^{n} p_k(t) q_{n-k}(t).$$

In terms of the generating functions $P(\lambda, t)$ and $Q(\lambda, t)$ of $p_k(t)$ and $q_k(t)$ relation (1.5) gives one equation in the two unknown functions $P(\lambda, t)$ and $Q(\lambda, t)$:

$$\frac{1}{1 - \lambda \varphi} = P(\lambda, t) Q(\lambda, t).$$

**Step II**: We prove next a lemma related to Wiener - Hopf factorization which shows that (1.5) is an "algebraic" condition which uniquely determines both $p_k(t)$ and $q_k(t)$. D. Ray [17] seems to have been the first person to use a Wiener - Hopf factorization procedure in fluctuation problems.
Lemma 1.1: Let \( \varphi(t) \) be a known Fourier-Stieltjes transform of a function of bounded variation and let \( \{ P_k(t) \} \) and \( \{ q_k(t) \} \) be sequences of functions which satisfy

\[
\begin{align*}
(1) & \quad P_0(t) = q_0(t) = 1 \\
(2) & \quad P_k(t) = \int_{0^+}^\infty e^{itx} \, dG(x) \quad (G(x) \text{ of B.V.}) \\
(3) & \quad q_k(t) = \int_{-\infty}^{0^+} e^{itx} \, dG(x) \quad (G(x) \text{ of B.V.}) \\
(4) & \quad \varphi^n(t) = \sum_{k=0}^{n} P_k(t) q_{n-k}(t).
\end{align*}
\]

Then, \( P_k(t) \) and \( q_k(t) \) are uniquely determined for all \( k \).

**Proof:** Assume that \( q_m(t) \) and \( p_m(t) \) have been determined for all \( m < k \).

From (4) with \( n = k \)

\[
\varphi^n(t) = P_k(t) + q_k(t) + \sum_{m=1}^{k-1} p_m(t) q_{k-m}(t).
\]

Thus, by the induction hypothesis

\[
P_k(t) + q_k(t) = \int_{-\infty}^{\infty} e^{itx} \, dK(x),
\]

where \( K(x) \) is known. Properties (2) and (3) together with the uniqueness theorem for Fourier-Stieltjes transforms implies
\[ p_k(t) = \int_{0^+}^{\infty} e^{itx} \, dK(x) \quad \text{and} \quad q_k(t) = \int_{-\infty}^{0^+} e^{itx} \, dK(x). \]

We note in passing that the lemma would remain true if the ranges of integration in (2) and (3) were replaced by \( A \) and \( A' \) for any linear Borel set, \( A \).

**Step III:** We have left only to demonstrate that \( f_+(\lambda, t) \) and \( f_-(\lambda, t) \) are generating functions of quantities which satisfy the conditions of Lemma 1.1 and identities ii) and iii) will follow. By (0.6) and (0.7) we have \( 1/(1 - \lambda \Phi(t)) = f_+(\lambda, t) f_-(\lambda, t) \), showing condition (4) is satisfied. To see that (1), (2), and (3) are also satisfied it suffices to observe that sums and products of terms,

\[ \int_{0^+}^{\infty} e^{itx} \, dG(x) \quad \text{or} \quad \int_{-\infty}^{0^+} e^{itx} \, dG(x) \]

are again terms of the same type. Since \( \log f_+(\lambda, t) \) (or \( \log f_-(\lambda, t) \)) is of type (1.9), we are finished with the proof.

To prove identities i) and iv) we merely repeat the first step of the proof just given.

**Proof of iv):** Consider

\[ \int e^{itM_n} \, dP = \sum_{k=0}^{n} \int_{[L_n = k]} e^{itM_n} \, dP = \sum_{k=0}^{n} \int_{[L_k = k] \cap [L_{n-k} = 0]} e^{itS_k} \, dP \]

\[ = \sum_{k=0}^{n} p_k(t) q_{n-k}(0). \]
Identity iv) follows from (1.10) by taking generating functions of both sides.

**Proof of i):** Consider

\[
\sum_{k=0}^{n} U^k \int_{L_n = k} 1 \cdot dP = \sum_{k=0}^{n} U^k \int_{L_k \cap (L_{n-k} = 0)} 1 \cdot dP
\]

\[
= \sum_{k=0}^{n} U^k p_k(0) q_{n-k}(0).
\]

Clearly, (1.11) is equivalent to identity i).

The most important part of the proof just given from the viewpoint of future considerations is the demonstration of an explicit pair of functions \( f_+(\lambda, t) \) and \( f_-(\lambda, t) \) (with obvious properties) satisfying

\[
f_+(\lambda, t) f_-(\lambda, t) = 1/[1 - \lambda \varphi(t)].
\]

The construction of these explicit functions is one of the central mathematical problems of our method of the next section. As we will see, it is not always possible to find "factors" explicitly. The crux of our future methods is a straightforward generalization of Lemma 1.1 in which condition (1.5b) is replaced by

\[
P(\lambda, t) [1 - \lambda \varphi(t)] = Q(\lambda, t).
\]
2. An operator:

The distribution of each variable listed in (0.12) is theoretically determined as soon as the characteristic function $\varphi(t)$ of $X_1$ is specified. It is therefore of some interest to see if the results of the previous section can be expressed explicitly in terms of $\varphi(t)$. To accomplish this end we introduce a notation.

**Notation:** For any function

$$\psi = \int_{-\infty}^{\infty} e^{itx} dH(x), \quad (H(x) \text{ is of B.V.}),$$

let

$$\psi^+ = \int_{0}^{\infty} e^{itx} dH(x) \quad \text{and} \quad \psi^- = \int_{-\infty}^{0} e^{itx} dH(x).$$

Suppressing dependence on $t$, we can now write the expressions in (0.6) and (0.7) in the form

$$f^+_\lambda = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^+ \right],$$

$$f^-_\lambda = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^- \right].$$

The identities can be now easily written in terms of $\varphi$, with, of course, the notations $+$ and $-$. 

Actually, much more than a notation has been introduced. Relation (2.2) defines operators $+$ and $-$ which are fundamental to the theory of finite fluctuation problems. We will show how these operators lead to a simple and yet powerful method for analyzing any particular problem. Our method is an expansion of one presented by the author in a previous paper [6]. The
algebraic treatment of fluctuation problems used by Wendel [21] seems to have many features in common with our method.

There are three important properties of the operator \( + \) defined by (2.2).

Similar properties hold for the operator \( - \).

Properties:

- **P1. Linear:** \((a\psi_1^+ + b\psi_2^-)^+ = a\psi_1^+ + b\psi_2^+\), where \(a\) and \(b\) are complex numbers.

- **P2. Idempotent:** \((\psi^+)^+ = \psi^+\)

- **P3. Closure:** The class \(A^+\) of all functions \(\psi^+\), where \(\psi\) is of form (2.1) is closed under addition, multiplication, and multiplication by a complex constant.

We note in particular that \((\psi^-)^+ = 0 = (\psi^+)^-\) and that \(\psi = \psi_1^+ + \psi_2^-\) implies \(\psi_1^+ = \psi^+\) and \(\psi_2^- = \psi^-\). The best way to explain the method of \( + \) operators is to illustrate its use in a few simple examples. In all examples \(\varphi\) will denote the characteristic function of \(X_1\).

**Example 2.1:** All positive sums. In our first example we consider in some detail the evaluation of \(P(N_n = n)\). Let

\[
(2.4) \quad \varphi_n = \int_{\{N_n=n\}} e^{itS_n} dP.
\]

By independence of the \(X_k\'s\)

\[
(2.5) \quad \varphi_n = \int_{\{N_n=n\}} e^{itS_n} e^{itX_{n+1}} dP = \int_{\{N_n=n\}} e^{itS_{n+1}} dP.
\]
We have interpreted the product $\varphi \varphi_n$ as "adding a step to the end of a path which satisfies the condition $N_n = n."$ The last integral in (2.5) is of the form (2.1) and we can apply the $+$ operator to it. The operator has the interpretation of eliminating all paths from the range of integration which have $S_{n+1} \leq 0$. Thus,

$$\int_{[N_{n+1} \geq n+1]} \text{e}^{-itS_{n+1}} \, dP = \varphi_{n+1} ,$$

$(\varphi_0 = 1)$.

Letting

$$\psi = \sum_{n=0}^{\infty} \lambda^n \varphi_n ,$$

Fig. 3.
we get from (2.7)

(2.8) \[ \psi = 1 + \lambda (\varphi \psi)^+ = 1 + \lambda \sum_{n=0}^{\infty} \lambda^n (\varphi \varphi_n)^+. \]

The interchange of the summation and + operator in (2.8) can easily be justified. We note that there is a unique power series solution \( \psi \) to (2.8), since by equating coefficients of like powers of \( \lambda \) on both sides of (2.8) we are led back to the recurrence relation (2.6).

To find the solution of (2.8) we first note that (2.8) and P2. imply \( (\psi - 1)^+ = \psi - 1. \) Since \( 1^+ = 0 \), we have also that \( \psi - 1 = (\psi - 1)^+ = \psi^+. \)

Thus, we can rewrite (2.8) in the form

(2.9) \[ [\psi(1 - \lambda \varphi)]^+ = 0 \text{ and } \psi^- = 1. \]

But

(2.10) \[ \psi = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^+ \right] \]

is a function with \( \psi^- = 1 \) such that

\[ \psi(1 - \lambda \varphi) = \exp \left[ -\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^- \right] \in A^- \]

(See P3.).

Thus, \([\psi(1 - \lambda \varphi)]^+ = 0; \) and the solution is given in (2.10). To find the generating function for \( P(N_n = n) \) from \( \psi \) we merely set \( t = 0. \)
Example 2.2: First non-positive sum among $S_1, S_2, \ldots, S_n$. Let

$$\psi_n = \int \left\{ \begin{array}{l} n-l=n-1 \\ S_n<n \end{array} \right\} e^{-itS_n} dP,$$

(n $\geq 1$),

and let $\varphi_n$ be given by (2.4). From (2.5) and an obvious interpretation of the operator, we get

$$\varphi_n e = \int \left\{ \begin{array}{l} n=n \\ S_n<n+1 \end{array} \right\} e^{-itS_n+1} dP = \psi_n+1 $$

Equivalently,

$$\psi_n+1 = \varphi_n - (\varphi_n)^+ = \varphi_n - \varphi_n+1$$

In terms of generating functions, (2.13) states

$$\sum_{n=1}^{\infty} \lambda^n \psi_n = 1 - (1 - \lambda \varphi) \psi = 1 - \exp \left[ - \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^+ \right]$$

The last equality in (2.14) follows from a use of (2.10).

Example 2.3: Maximum at endpoint. In this example we exhibit a slight but very useful variation of the argument of example 1. Let

$$\varphi_n = \int_{\left[ L_n=n \right]} e^{-itS_n} dP$$
Furthermore, let \( \mathcal{L}_n \) be defined for \( Y_1 = X_2, \ldots, Y_n = X_{n+1} \) exactly as \( L_n \) is defined for \( X_1, \ldots, X_n \). Then

\[
\varphi_n = \int_{\{L_n = n\}} e^{itS_{n+1}} dP
\]

(2.16)

We have interpreted \( \varphi_n \) as "adding a step at the beginning of a path satisfying \( L_n = n \)." The "new" path will satisfy the condition \( L_{n+1} = n+1 \) if, and only if, \( S_{n+1} > 0 \).

\[
(2.17) \quad (\varphi_n)^+ = \int_{\{L_{n+1} = n+1\}} e^{itS_{n+1}} dP = \varphi_{n+1} \quad (\varphi_0 = 1).
\]

Fig. 4.

Thus,

Relations (2.17) and (2.6) are identical, showing that \( \varphi_n \) in (2.15) and \( \varphi_n \)
in (2.4) are the same. Once again (2.10) gives the generating function for the expressions in (2.15).

Earlier we mentioned the dual properties P1. - P3. of the + and - operators. Assuming that there is some justice in mathematics, we should expect a dual for every result thus far found. We expect a dual of example 1 concerning all non-negative sums of the form

\[ (2.18) \quad \sum_{n=0}^{\infty} \lambda^n \int_{[L_n=0]} e^{- \lambda t S_n} dP = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi_k)^- \right]. \]

Corresponding to example 2, we expect that

\[ (2.19) \quad \sum_{n=0}^{\infty} \lambda^n \int_{\{L_{n-1}=0 \}} e^{- \lambda t S_n} dP = 1 - \exp \left[ - \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi_k)^+ \right] \]

Fortunately, the results (2.18) and (2.19) are quite easy to prove using the methods described in the examples above, showing that there is some justice in mathematics after all.

3. **Several major results:**

In this section we will indicate the power of the method of + operators by briefly describing how identities (0.3), (0.4), and (0.5) can be derived from this method. We also give two results which seem to have appeared previously only in special cases.
Example 3.1: Positive partial sums (Andersen [4]). Let

\begin{equation}
\varphi_n = \sum_{m=0}^{n} u^m \int_{\{N_n=m\}} e^{itS_n} \, dP, \quad (\varphi_0 = 1).
\end{equation}

then,

\begin{equation}
\varphi \varphi_n = \sum_{m=0}^{n} u^m \int_{\{N_n=m\}} e^{itS_{n+1}} \, dP.
\end{equation}

Applying the operators we find

\begin{equation}
U(\varphi_n) = \sum_{m=0}^{n} u^{m+1} \int_{\left\{ \begin{array}{l}
N_{n+1}=m+1 \\
S_{n+1}>0
\end{array} \right\}} e^{itS_{n+1}} \, dP
= \sum_{m=1}^{n+1} u^m \int_{\left\{ \begin{array}{l}
N_{n+1}=m \\
S_{n+1}>0
\end{array} \right\}} e^{itS_{n+1}} \, dP,
\end{equation}

and

\begin{equation}
(\varphi_n)\varphi = \sum_{m=0}^{n} u^m \int_{\left\{ \begin{array}{l}
N_{n+1}=m \\
S_{n+1}<0
\end{array} \right\}} e^{itS_{n+1}} \, dP.
\end{equation}

It is easily seen that equality still holds in (3.2) and (3.3) if the summation on the right is extended to \( m = 0, 1, \ldots, n + 1 \). By adding,
(3.4) \[ U(\varphi \varphi_n)^+ + (\varphi \varphi_n)^- = \varphi_{n+1} \]

In terms of the generating function \( \psi \) of \( \varphi_n \) relation (3.4) states that

(3.5) \[ \psi = 1 + \lambda U(\varphi \varphi)^+ + \lambda (\varphi \varphi)^- \]

or equivalently,

(3.6) \[ [\psi(1 - \lambda U\varphi)]^+ + [\psi(1 - \lambda \varphi) - 1]^- = 0 \]

Each term in (3.6) must necessarily vanish. If we set \( P = \psi(1 - \lambda \varphi) \) and \( Q = \psi(1 - \lambda U\varphi) \), we have \( P - 1 \in A^+ \), \( Q \in A^- \), and

(3.7) \[ \frac{P}{Q} = \frac{1 - \lambda \varphi}{1 - \lambda U\varphi} \]

Thus, we are led to a "factorization" problem of the type discussed in section 1. Using the unique factors

\[ P = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k}(U^k(\varphi^k)^+ - (\varphi^k)^-) \right] \]

\[ Q = \exp \left[ - \sum_{k=1}^{\infty} \frac{\lambda^k}{k}(U^k(\varphi^k)^- - (\varphi^k)^+) \right] \]

of (3.7), we are led to the result

(3.8) \[ \psi = P/[1 - \lambda \varphi] = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k}(U^k(\varphi^k)^+ + (\varphi^k)^-) \right] \]
The identity of (0.3) follows from (3.8) by setting $t = 0$.

**Example 3.2:** Distribution of $\max(S_0, S_1, \ldots, S_n)$ (Spitzer [18]).

Let

$$
(3.9) \quad \varphi_n = \int e^{-itM_n} dP
$$

and let $\tilde{M}_n$ denote the maximum of the partial sums of $X_2, \ldots, X_{n+1}$ (including $S_0 = 0$). Then,

$$
(3.10) \quad \varphi_n^* = \int e^{-it(\tilde{M}_n + X_1)} dP.
$$

But $\tilde{M}_n + X_1 = M_{n+1}$, if $M_{n+1} > 0$. Thus,

$$
(3.11) \quad (\varphi_n^*)^+ = \int_{[M_{n+1} > 0]} e^{-itM_{n+1}} dP = \varphi_{n+1} - P(M_{n+1} = 0)
$$

$$
= \varphi_{n+1} - P(N_{n+1} = 0)
$$

The generating function, say $C$, of $P(N_n = 0)$ is found by equating coefficients of $U^0$ (with $t = 0$) in (3.8); i.e.

$$
(3.12) \quad C = \sum_{n=0}^{\infty} \lambda^n P(N_n = 0) = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} P(S_k \leq 0) \right]
$$

Thus, if $\psi$ is the generating function of $\varphi_n$ we get from (3.11)

$$
(3.13) \quad \frac{\psi}{C} = 1 + \lambda(\varphi_n^*)^+.
$$
whose solution according to example 2.1 is simply

\[ \psi C = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k) \right] \]

**Example 3.3: Order Statistics (Wendel [22]).** Let

\[ \varphi_n = \sum_{k=0}^{n} u^k \int e^{itR_{nk}} dP \]

and let \( \tilde{R}_{nk} \) be the order statistics for the partial sums of \( X_2, \ldots, X_{n+1} \). Then,

\[ \varphi \varphi_n = \sum_{k=0}^{n} u^k \int e^{it(\tilde{R}_{nk} + X_1)} dP, \]

where a step has been added at the beginning of a path (see Fig. 5.). The relative positions of the points on the "old" path are unchanged by the addition of a new point at the origin. Thus, \( \tilde{R}_{nk} + X_1 = R_{n+1,k} \) if \( R_{n+1,k} > 0 \),

\[ R_{42} = \tilde{R}_{32} + X_1 \]
\[ R_{44} = \tilde{R}_{33} + X_1 \]

Fig. 5
and \( \tilde{R}_{nk} + X_n = R_{n+1,k+1} \) if \( R_{n+1,k+1} \leq 0 \). In no case is \( \tilde{R}_{nk} + X_n \equiv S_0 \).

In particular, we note

\[
U(\varphi \varphi_n)_- = \sum_{k=0}^{n} U^{k+1} \int_{[S_0 \neq R_{n+1,k+1} \leq 0]} e^{itR_{n+1,k+1}} dP
\]

\[
= \sum_{k=0}^{n+1} U^{k} \int_{[R_{n+1,k} \leq 0]} e^{itR_{n+1,k}} dP - \sum_{k=0}^{n+1} U^{k} P(R_{n+1,k} = S_0).
\]

But, \( R_{n+1,k} = S_0 \) if, and only if, \( N_{n+1} = k \). Thus,

\[
(\varphi \varphi_n)_+ + U(\varphi \varphi_n)_- = \varphi_{n+1} - \sum_{k=0}^{n+1} U^{k} P(N_{n+1} = k).
\]

If \( K \) is the generating function given in (0.3) and \( \psi \) is the generating function of \( \varphi_n \), then (3.17) gives

\[
\frac{\psi}{K} = 1 + \lambda(\varphi \varphi_n)^+ + \lambda U(\varphi \varphi_n)^-.
\]

This last equation is the "dual" of (3.5), and we can therefore immediately write from (3.8)

\[
\psi = K \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} ((\varphi^k)^+ + U^k(\varphi^k)^-) \right].
\]

Finally, we state without proof two generalizations of examples previously considered. These identities appear to be new, although special cases have appeared previously.
Example 3.4: Order indices. Let

\begin{equation}
\varphi_n = \sum_{k,m=0}^{n} v^k U^m \int_{\{L_{nk}=m\}} e^{itR_{nk}} dP.
\end{equation}

Then, the generating function \( \psi \) of \( \varphi_n \) in terms of the notation (0.6) and (0.7) is

\begin{equation}
\psi = f_+^1(\lambda U, t) f_+^1(\lambda v, 0) f_-^1(\lambda UV, t) f_-^1(\lambda, 0).
\end{equation}

Setting \( t = 0 \) in (3.20) gives a result of Darling [10] on order indices.

Example 3.5: Distribution of kth positive sum. Let

\begin{equation}
\varphi_n = \sum_{k=1}^{n} U^k \int_{\left\{ \begin{array}{c}
N_n = k \\
S_n > 0
\end{array} \right\}} e^{itS} dP, \quad (n \geq 1).
\end{equation}

Then, the generating function of \( \varphi_n \) is

\begin{equation}
\psi = \frac{U}{1 - U} [1 - \exp \left( \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (U^k - 1) (\varphi^k)^+ \right)]
\end{equation}

Setting \( t = 0 \) in (3.22) gives results of Andersen [5].

4. Change of sign:

We now apply our method to a much more difficult problem than those considered up to now. It is hoped that the discussion of this section will provide
a deeper insight into the real character of the previous identities as well as explain why the change of sign is basically a more difficult problem. A number of the ideas presented here were conceived in joint work with E. Sparre Andersen.

To analyze change of sign we consider two functions:

\[ (4.1) \quad \varphi_n = \sum_{k=0}^{n} U^k \int_{\mathcal{C}_n = k \atop S_n > 0} e^{itS_n} dP, \quad (\Phi = \sum_{0}^{\infty} \lambda^n \varphi_n), \]

and

\[ (4.2) \quad \psi_n = \sum_{k=0}^{n} U^k \int_{\mathcal{C}_n = k \atop S_n < 0} e^{itS_n} dP, \quad (\Psi = \sum_{0}^{\infty} \lambda^n \psi_n). \]

It follows that

\[ (4.3) \quad (\varphi_n^\dagger)^+ = \sum_{k=0}^{n} U^k \int_{\mathcal{C}_{n+1} = k \atop S_n > 0, S_{n+1} > 0} e^{itS_{n+1}} dP \]

and

\[ (4.4) \quad U(\psi_n^\dagger)^+ = \sum_{k=1}^{n+1} U^k \int_{\mathcal{C}_{n+1} = k \atop S_{n+1} > 0, S_n < 0} e^{itS_{n+1}} dP. \]
From (4.3) and (4.4) and a similar argument with the - operator, we find

\[
\begin{align*}
(4.5) \quad \begin{cases} 
(\phi_n^+) + U(\psi_n^+) = \phi_{n+1} \\
U(\phi_n^-) + (\psi_n^-) = \psi_{n+1}
\end{cases} \quad (\phi_0 = 0) \\
(\psi_0 = 1).
\end{align*}
\]

The recurrence relation (4.5) implies an obvious pair of equations for the generating functions \( \Phi \) and \( \Psi \). It will be convenient to write these equations in matrix form:

\[
(4.6) \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 - \lambda \phi & -U\lambda \phi \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \phi \\ \Psi \end{pmatrix}^+ + \begin{pmatrix} 1 & 0 \\ -U\lambda \phi & 1 - \lambda \phi \end{pmatrix} \begin{pmatrix} \phi \\ \Psi \end{pmatrix}^-
\]

Relation (4.6) is extremely similar in appearance to (3.6). However, in order to use the method described following (3.6), we have to expand (4.6) somewhat by introducing two auxiliary functions \( \widetilde{\phi} \) and \( \widetilde{\Psi} \). Letting

\[
\mathcal{M} = \begin{pmatrix} \widetilde{\phi} & \phi \\ \widetilde{\Psi} & \Psi \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

we write in place of (4.6)

\[
(4.7) \quad I = \begin{bmatrix} 1 - \lambda \phi & -U\lambda \phi \\ 0 & 1 \end{bmatrix} \mathcal{M}^+ + \begin{bmatrix} 1 & 0 \\ -U\lambda \phi & 1 - \lambda \phi \end{bmatrix} \mathcal{M}^-.
\]

Proceeding as before we let

\[
P = \begin{pmatrix} 1 & 0 \\ -U\lambda \phi & 1 - \lambda \phi \end{pmatrix} \mathcal{M}
\]

\[
(4.8) \quad Q = \begin{pmatrix} 1 - \lambda \phi & -U\lambda \phi \\ 0 & 1 \end{pmatrix} \mathcal{M}.
\]
Then, \( P - I \) has elements from \( A^+ \), \( Q \) has elements from \( A^- \), and

\[
(4.9) \quad PQ^{-1} = \frac{1}{1 - \lambda \varphi} \begin{bmatrix} 1 & \lambda \varphi \\ -\lambda \varphi & (1 - \lambda \varphi)^2 - \lambda \varphi \end{bmatrix}.
\]

Thus, we are led to a matrix "factorization". Lack of commutativity in this case rules out the previous method of constructing explicit factors using exponentials. This does not necessarily mean that explicit formulas similar to \( (2.3) \) do not exist for \( P \) and \( Q \). We will show below, however, that there do not exist formulas for \( P \) and \( Q \) with the same basic simplicity of their counterparts in \( (2.3) \). For certain special distributions of \( X_1 \), the factors \( P \) and \( Q \) can be determined. For such examples in the Bernoulli case and for the characteristic function \( \varphi = 1/(1 + t^2) \), the reader is referred to [8].

We return for a moment to the result of example 2.1. Using the recurrence relation \( (2.6) \) we can write the \( \varphi_n \) of \( (2.4) \) as an iteration of the \( + \) operator as follows:

\[
(4.10) \quad \begin{align*}
\varphi_0 &= 1 \\
\varphi_1 &= \varphi^+ \\
\varphi_2 &= (\varphi \varphi^+)^+ \\
\varphi_3 &= (\varphi (\varphi \varphi^+)^+)^+ \\
\ldots.
\end{align*}
\]

However, formula \( (2.10) \) gives a second way of writing \( \varphi_1, \varphi_2, \ldots \).
That is,

\[
\varphi_2 = \frac{1}{2} (\varphi^2)^+ + \frac{1}{2} \varphi^2
\]

\[(4.11)\]

\[
\varphi_3 = \frac{1}{3} (\varphi^3)^+ + \frac{1}{2} (\varphi^2)^+ \varphi^+ + \frac{1}{6} \varphi^3
\]

\[\ldots\]

Thus, we see that \( \varphi_n \) does not actually involve iterations of the \( + \) operator but is in fact a polynomial in \( \varphi^+, (\varphi^2)^+, \ldots, (\varphi^n)^+ \). Moreover, these polynomials have a generating function with a closed form (2.10). The reduction of \( \varphi_n \) to a polynomial in \( (\varphi^k)^+ \) is quite clearly connected with the invariant results of type \( (0.2) \) discovered by Andersen. Let us now return to the change of sign problem.

There appear to be two possible reasons why we cannot write explicit factors \( P \) and \( Q \) of (4.9). It may be that the elements of \( P \) and \( Q \) are expressible as polynomials in \( (\varphi^k)^+ \), but these polynomials do not have a nice closed-form generating function. The lack of invariant results of the type \( (0.2) \) for change of sign makes this possibility unlikely. On the other hand it is possible that the elements of \( P \) and \( Q \) are not polynomials in \( (\varphi^k)^+ \) at all. As we will now show, this latter possibility is actually the fact.

From (4.11) we see that the coefficient of \( U^n \) in \( \varphi_n \) is

\[(4.12)\]

\[
\varphi_{nn} = \begin{cases} \frac{\text{itS}}{n!} e^{-n} \text{dP} \\ \text{if} \left\{ \begin{array}{l} C_n = n \\ S_n > 0 \end{array} \right. \end{cases}
\]
But, $C_n = n$ and $S_n > 0$ are possible only when $n$ is odd. Now, if a simplification of $P$, or equivalently, of $\varphi_n$ to a polynomial in $(\varphi^k)^+$ is possible, then in particular the terms $\varphi_{33} = (\varphi(\varphi^+)^-)^+\varphi_{55} = (\varphi(\varphi(\varphi^+)^-)^+)\varphi_{55} = (\varphi(\varphi(\varphi(\varphi^+)^-)^+)\varphi_{55}$, etc. will also simplify. We look first at $\varphi_{33} = (\varphi^2\varphi^+)^+ - (\varphi(\varphi^+)^+)^+$. By (4.11), $(\varphi(\varphi^+)^+)^+$ is already a polynomial in $(\varphi^k)^+$. Let us postulate the existence of constants $A$, $B$, and $C$ so that as an identity in $\varphi$

(4.13) $\varphi^2\varphi^+ = A(\varphi^3)^+ + B(\varphi^2)^+ \varphi^+ + C\varphi^3$. 
Clearly, we need include only terms homogeneous of degree 3 on the right in (4.13). Letting \( \varphi = e^{i \lambda} + 1 \) in (4.13) and equating coefficients of \( e^{i \lambda t} \) on both sides, we deduce that

\[
\begin{align*}
1 &= A + B + C \\
2 &= 3A + 2B \\
1 &= 3A
\end{align*}
\]

The only solution is \( A = 1/3, B = 1/2, \) and \( C = 1/6. \) Thus, (4.13) becomes, with the use of (4.11),

\[
(4.14) \quad (\varphi^2 \varphi^+)^+ = \frac{1}{3} (\varphi^3)^+ + \frac{1}{2} (\varphi^2)^+ \varphi^+ + \frac{1}{6} \varphi^+^3 = (\varphi(\varphi \varphi^+)\varphi^+)\varphi^+
\]

In other words, \( \varphi_{33} = (\varphi^2 \varphi^+)^+ - (\varphi(\varphi \varphi^+)\varphi^+)\varphi^+ = (\varphi(\varphi \varphi^+)\varphi^+)\varphi^+ \) is identically zero if (4.13) holds. This is clearly impossible.

In a sense we have been too restrictive in our demands for a formula for \( P \) and \( Q. \) As Andersen pointed out, the distribution of \( N_n \) depends only on the probability that \( S_k \) is positive \((k = 1, 2, \ldots, n). \) We should expect the distribution of \( C_n \) to depend at least on the probability of a change of sign at the \( k \)th step. In operator notation, this means allowing terms of the form \( (\varphi(\varphi^k)^+)\varphi^+ \) and \( (\varphi(\varphi^k)^-)\varphi^+ \) in the formula. More generally, a formula for \( P \) and \( Q \) might include products of terms of the form

\[
(\varphi^1(\varphi^2)^+ (\varphi^3)^+ \cdots (\varphi^m)^+)^+,
\]

which involve one iteration of the operator. Unfortunately, it can be shown by an argument similar to that used above that \( \varphi_{n n} \) cannot be expressed as a sum of terms involving only one iteration of the operator. In general, it appears (although a proof has not been given) that \( \varphi_{nn} \) with \( n = 2^k - 1 \) cannot be reduced to an expression of terms with at most \( k - 2 \) iterations of the operator.
5. A difference system

Thus far, no consideration has been given to joint distributions of the variables listed in the introduction. In this and subsequent sections we will develop a slight modification of the method of + operators which is useful for finding joint distributions. We consider here only the case in which $X_k$ takes on only integral values, with one minor exception to this rule in Lemma 5.2. The characteristic function $\varphi$ is now a Fourier series and we will use the notation

$$1 - \lambda \varphi = \sum_{k=-\infty}^{\infty} A_k e^{ikt}.$$  

Once again we start with a procedure which is designed to bring the mathematical problem into sharp focus. We examine the expressions

$$\varphi_{n,k} = \int \frac{itS_k}{M_k \leq n, L_k = 0} dP, \quad (\varphi_n = \sum_{k=0}^{\infty} \lambda^k \varphi_{n,k}),$$  

and

$$\psi_{n,k} = \int \frac{itS_k}{M_k > n, L_k = 0} dP, \quad (\psi_n = \sum_{k=0}^{\infty} \lambda^k \psi_{n,k}).$$
It is clear that $\varphi_{n,k}$ and $\psi_{n,k}$ are polynomials of at most degree $n$ in $e^{it}$ and $e^{-it}$, respectively. A fundamental property of these functions can now be stated.

**Lemma 5.1:** For every $n \geq 0$,

$$
\varphi_n(1 - \lambda \varphi) = \sum_{k=-\infty}^{0} C_k e^{ikt} + \sum_{k=n+1}^{\infty} C_k e^{ikt}, \quad (C_0 = 1),
$$

(5.4)

$$
\psi_n(1 - \lambda \varphi) = \sum_{k=-\infty}^{-(n+1)} C'_k e^{ikt} + \sum_{k=0}^{\infty} C'_k e^{ikt}, \quad (C'_0 = 1).
$$

**Proof:** Step I: First, we note two limit relations which were essentially proved in Part I:

$$
\lim_{n \to \infty} \varphi_n = \Phi = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda}{k} \int_{0}^{\infty} e^{itx} \, dP[S_k < x] \right].
$$

(5.5)

and

$$
\lim_{n \to \infty} \psi_n = \Psi = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda}{k} \int_{0}^{\infty} e^{itx} \, dP[S_k < x] \right].
$$

(5.6)

These limits have the important product property

$$
\Phi \Psi = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda}{k} P[S_k = 0] \right]/[1 - \lambda \varphi].
$$

(5.7)
Step II: Next, we focus our attention on

\[ \Phi_{n,k} - \Phi_{n-1,k} = \int_{\mathcal{E}_{n,k}} \frac{itS_k}{e^{S_k}} dP \]

\[
= \sum_{m=0}^{k} \int_{\mathcal{E}_{m,k}} \frac{itS_m}{e^{S_m}} \cdot \frac{it(S_m - S_k)}{e^{S_k - S_m}} dP.
\]

(5.8)

A typical "path" satisfying the conditions of integration in the sum of (5.8) is illustrated below for \( k = 7 \) and \( m = 3 \).

Fig. 6

Any such path can be described by separate conditions on the sets \( X_1, \ldots, X_m \) and \( X_{m+1}, \ldots, X_k \). Letting \( \tilde{L}_{k-m} \) and \( \tilde{M}_{k-m} \) denote the usual variables defined for \( Y_1 = X_{m+1}, \ldots, Y_{k-m} = X_k \).
\[
\begin{aligned}
\begin{cases}
L_k = 0 \\
M_k = n \\
L_k' = n
\end{cases}
= 
\begin{cases}
L_m = m \\
\bar{L}_m = 0 \\
S_m = n
\end{cases}
\cap 
\begin{cases}
\bar{L}_{k-m} = 0 \\
\bar{M}_{k-m} \geq n
\end{cases}
\end{aligned}
\]

Thus, if
\[
\alpha_{nm} = P(L_m = 0, L_m = m, S_m = n), \quad (\alpha_n = \sum_{m=0}^{\infty} \lambda^m \alpha_{nm}),
\]
we get
\[
\phi_{n,k} - \phi_{n-1,k} = e^{i \int} \sum_{m=0}^{k} \alpha_{nm} \psi_{n,k-m'}
\]
or equivalently,
\[
\phi_n - \phi_{n-1} = \alpha_n e^{i \int} \psi_n.
\]

By a similar argument, we find
\[
\psi_n - \psi_{n-1} = \beta_n e^{-i \int} \phi_n,
\]
where
\[
\beta_{nm} = P(L_m = 0, \bar{L}_m = m, S_m = -n), \quad (\beta_n = \sum_{m=0}^{\infty} \lambda^m \beta_{nm}).
\]

**Step III:** We now observe that the known limits \(\phi\) and \(\psi\) together with the difference equations (5.10) and (5.11) uniquely determine \(\phi_n\) and \(\psi_n\).
To see this, write

\[
\begin{pmatrix}
\varphi_{n-1} \\
\psi_{n-1}
\end{pmatrix} =
\begin{pmatrix}
1 & -\alpha_n e^{i\lambda t} \\
-\beta_n e^{-i\lambda t} & 1
\end{pmatrix}
\begin{pmatrix}
\varphi_n \\
\psi_n
\end{pmatrix} = \mathcal{M}_n
\begin{pmatrix}
\varphi_n \\
\psi_n
\end{pmatrix}.
\]

Without regard to justifying limits for the moment, let us iterate (5.13)

to obtain

\[
\begin{pmatrix}
\varphi_{n-1} \\
\psi_{n-1}
\end{pmatrix} = \mathcal{M}_n \mathcal{M}_{n+1} \cdots \mathcal{M}_1 \begin{pmatrix}
\varphi \\
\psi
\end{pmatrix} = \mathcal{M} \begin{pmatrix}
\varphi \\
\psi
\end{pmatrix}.
\]

It can be shown using \( \sum |\alpha_n| < \infty \) and \( \sum |\beta_n| < \infty \) that the matrix product in (5.14) actually exists and, moreover, the elements \( m_{ij} \) of \( \mathcal{M} \) have the form

\[
\begin{align*}
m_{11} &= \sum_{k=-\infty}^{0} A_{k} e^{ik\lambda t}, \\
m_{12} &= e^{i\lambda t} \sum_{k=0}^{\infty} A_{k} e^{ik\lambda t}, \\
m_{21} &= e^{-i\lambda t} \sum_{k=\infty}^{0} A_{k} e^{ik\lambda t}, \\
m_{22} &= \sum_{k=0}^{\infty} A_{k} e^{ik\lambda t}.
\end{align*}
\]

(5.15)

where the series converge absolutely. Thus, from (5.14) and (5.15), using the

fact that \( 1/\varphi \) and \( 1/\psi \) are "one-sided" Fourier series.
\[ (5.16) \quad \frac{\psi_{n-1}}{\psi} = \frac{m_1}{\psi} + \frac{m_2}{\phi} = \sum_{k=-\infty}^{0} B_k e^{ik\lambda} + \sum_{k=n}^{\infty} B_k e^{ik\lambda} \]

A comparison of the constant terms on both sides of (5.16) together with (5.7) yields (5.4). This ends the proof of Lemma 5.1.

The basic mathematical problem of the method we will soon develop is to construct functions \( \varphi_n \) which have a property like (5.4) with respect to a given function. The proof above shows that the solutions \( \varphi_n \) and \( \psi_n \) of a difference system (5.10) and (5.11) have this rather unusual property with respect to the inverse product of their limits. A similar result is important in the continuous case (See Lemma 5.2). However, in the lattice case we can simplify the construction of \( \varphi_n \) and \( \psi_n \) by use of the coefficient formula for Fourier series. From (5.4)

\[ (5.17) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_n (1 - \lambda \varphi) e^{i\lambda t} \, dt = \delta_{n0} \, (m = 0, 1, \ldots, n). \]

Substituting from (0.8) and (0.9) we see that in terms of the notation of (5.1) we have the explicit formulas \( \psi_n = f_n^{-1} (\lambda, t) \) and \( \varphi_n = \left(D_{n-1}/D_n\right) f_n^+ (\lambda, t). \)

Szegö [14] has studied polynomials \( g_n(z) \) and \( h_n(z) \) of degree \( n \) in \( z \) and \( 1/z \), respectively, which are biorthogonal with respect to an integrable weight function \( f(t) \) on \(-\pi \leq t \leq \pi\). That is,

\[ (5.18) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(z) h_m(z) f(t) \, dt = \delta_{nm}, \quad (z = e^{it}). \]
By direct substitution one can see that $e^{i\text{nt}_f}$ and $e^{-i\text{nt}_f}$ of (0.8) and (0.9) have the biorthogonality property (5.18) with respect to $1 - \lambda \varphi$.

For further discussion of this relation to Szegő polynomials see [7,15,20].

As a final consideration of this section we show how the procedure of the proof above is useful in obtaining a special but very elegant invariant result for joint distributions.

**Lemma 5.2:** Let $X_1, X_2, \ldots$ be independent and have a common absolutely continuous distribution function. Then,

$$P(N_n = n, L_n = n) = \frac{1}{2n}. \quad (5.19)$$

**Proof:** Set

$$\varphi_k(x, t) = \begin{cases} e^{itS_k} dF, & 0 < k < x \\ 0, & k \geq 0 \end{cases} \quad (\varphi(x, t) = \sum_{k=0}^{\infty} \lambda^k \varphi_k(x, t)). \quad \qquad (5.20)$$

Then, using that $P(S_k = 0) = 0$ ($k \geq 1$),

$$\lim_{x \to \infty} \varphi(x, 0) = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)_0 \right] = \exp \sum_{k=1}^{\infty} \frac{\lambda^k}{2k} \quad \quad \quad (5.21)$$

Moreover, by an argument similar to (5.8) through (5.11) we have

$$\frac{d\varphi(x, t)}{dx} = \alpha(x) e^{itx} \varphi(x, t). \quad \quad \quad (5.22)$$
where bar indicates conjugate and where

\[(5.23) \quad \int_{0}^{x} \alpha(\xi) \, d\xi = \sum_{k=1}^{\infty} \lambda^k r(L_k = k, N_k = k, S_k < x)\]

Letting \( t = 0 \) in (5.22), we get simply

\[(5.24) \quad \frac{d}{dx} \varphi = \alpha(x) \varphi, \quad (\varphi(0,0) = 1).\]

Solving (5.24) we get

\[(5.25) \quad \varphi(\infty, 0) = \exp \left[ \int_{0}^{\infty} \alpha(\xi) \, d\xi \right] = \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{2k} \right]\]

and the result follows from (5.23).

6. A projection operator.

The result of Lemma 5.1 can be stated in a more convenient manner using the following notation:

Notation: For any function

\[(6.1) \quad \psi = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \quad \text{with} \quad \sum |c_k| < \infty,\]

let \(m < n\)

\[(6.2) \quad [\psi]^n_m = \sum_{k=m}^{n} c_k e^{ikt}.\]

In this new notation the results (5.4) become

\[(6.3) \quad \left[ \varphi_n(1 - \lambda \varphi) \right]^0_n = \left[ \psi_n(1 - \lambda \varphi) \right]^0_{-n} = 1.\]
It is apparent that the bracket notation is actually a projection operator on the space of Fourier series (6.1). In this respect it is similar to the + operator of (2.2). Unfortunately, the closure property P3. of the + operator does not hold for the bracket. Nonetheless, the notation together with the next lemma will form the basis of a simple yet powerful method for working certain problems of joint distributions.

Lemma 6.1: Let \( \psi \) be given as in (6.1) and let \( D_n(\psi) = \det (C_{j-l}) \), \( (i, j = 0, 1, \ldots, n) \). If \( D_n(\psi) \neq 0 \), then

\[
\varphi_n = \frac{1}{D_n(\psi)} \left| \begin{array}{ccc}
C_0 & C_1 & \cdots & C_n \\
C_{-1} & C_0 & \cdots & C_{n-1} \\
& \vdots & \ddots & \vdots \\
& & & \vdots \\
C_{-m+1} & C_{-n+2} & \cdots & C_1 \\
z^n & z^{n+1} & \cdots & 1
\end{array} \right| (z = e^{it})
\]

is a polynomial of at most degree \( n \) in \( e^{it} \) which satisfies

\[
(6.5) \quad [\varphi_n \psi]_0^n = 1.
\]

In the arguments below, it will be obvious that the polynomials we seek exist and are the unique solutions of equations like (6.5). Thus (6.4) will give us the explicit form of our answer. One special case of (6.4) which will be of interest later is that in which \( \psi \) is a power series, i.e. \( C_k = 0, (k<0) \).
In this special case there exists a fixed sequence of constants \( \{\alpha_k\} \) such that for every \( n \)

\[
\varphi_n = \sum_{k=0}^{n} \alpha_k e^{ikt}.
\]

We now turn to a few examples to illustrate the use of (6.2) and (6.4) in fluctuation problems.

**Example 6.1: Maximum and all non-negative sums.** Let

\[
\varphi_{n,k} = \int_{\{M_k \leq n \}} e^{its_k} dP, \quad (\varphi_n = \sum_{k=0}^{n} \lambda^k \varphi_{n,k}).
\]

Using \( \varphi \) to denote as usual the characteristic function of \( X_1 \),

\[
\varphi_{n,k} = \int_{\{M_k \leq n \}} e^{its_{k+1}} dP.
\]

It follows that

\[
[\varphi_{n,k}]_0^n = \int_{\{M_{k+1} \leq n \}} e^{its_{k+1}} dP = \varphi_{n,k+1}.
\]

or equivalently,

\[
\lambda[\varphi_{n,k}]_0^n = \varphi_n - 1.
\]

Now \( \varphi_n \) is a polynomial of at most degree \( n \) in \( e^{it} \), and therefore

\[
[\varphi_n (1 - \lambda \varphi)]_0^n = 1.
\]

By (6.4) and (0.3) the solution is

\[
\varphi_n = \frac{D^{n-1} (1 - \lambda \varphi)}{D^{n-1} (1 - \lambda \varphi)} f_n^*(\lambda, t).
\]
Example 6.2: Maximum at end-point and range. Consider this time

\[(6.11) \quad \varphi_{n,k} = \int_{\{L_k = k\}} e^{i t S_k} dP, \quad \varphi_n = \sum_{k=0}^{\infty} \varphi_{n,k}.\]

Let \(L_k\) and \(R_k\) be the usual variables defined for \(X_2, \ldots, X_{n+1}\). Then

\[\varphi_{n,k} = \int_{\{L_k = k\}} e^{i t S_{k+1}} dP,\]

where a step has been added at the beginning of a path. Now, \(L_{k+1} = k+1\) and \(R_{k+1} \leq n\) are both satisfied for the new path if and only if, \(1 \leq S_{k+1} \leq n\) (See Fig. 7, below). Thus, for \(n \geq 1\) we have

\[\varphi_{n,k} = \int_{\{L_k = k\}} e^{i t S_{k+1}} dP,\]
(6.12) \[ [\varphi_{n,k}]_{1}^{n} = \varphi_{n,k+1} \]

Introducing generating functions and noting that \( \varphi_{n} \) is a polynomial in \( e^{it} \) of at most degree \( n \) with constant term 1, it follows that

(6.13) \[ [\varphi_{n}(1 - \lambda \varphi)]_{1}^{n} = 0 \text{ and } [\varphi_{n}]_{0}^{0} = 1 \]

The solution of (6.13) is exactly \( \varphi_{n} = f_{n}^{+} \), where \( f_{n}^{+} \) is given in (0.8).

Example 6.3: Distribution of maximum \( M_{n} \). It is interesting to see if by the present methods one can derive for lattice variables the distributions of \( N_{n} \), \( M_{n} \), and \( R_{nk} \) found in Part I. We will illustrate how this can be done for the variable \( M_{n} \). In the process we will unfold an important technique of our method. Let

(6.14) \[ \varphi_{n,k} = \int_{\{M_{k} \leq n\}} e^{itS_{k}} dP, \quad (\varphi_{n} = \sum_{k=0}^{n} \lambda^{k} \varphi_{n,k}), \]

so that

\[ [\varphi_{n,k}]_{-\infty}^{n} = \left[ \int_{\{M_{k} \leq n\}} e^{itS_{k+1}} dP \right]_{-\infty}^{n} = \varphi_{n,k+1} \]

thus, noting that \( [\varphi_{n}]_{-\infty}^{n} = \varphi_{n} \), we have

(6.15) \[ [\varphi(1 - \lambda \varphi)]_{-\infty}^{n} = 1 \]
We try a solution of (6.15) of the form

\begin{equation}
\varphi_n = \hat{\varphi}_n \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda}{k} (\varphi^k)^{-1} \right] = \hat{\varphi}_n f_-(\lambda, t),
\end{equation}

where \( \hat{\varphi}_n \) is a polynomial in \( e^{it} \) of degree at most \( n \). Now,

\[ [\varphi_n (1 - \lambda \varphi)]^{-1} \bigg|_{-\infty}^{+} = \left[ \frac{\hat{\varphi}_n}{f_+} \right]^{-1} = 0, \]

so that we are looking for a polynomial \( \hat{\varphi}_n \) satisfying

\begin{equation}
\left[ \frac{\hat{\varphi}_n}{f_+} \right]^n_0 = 1.
\end{equation}

Since \( 1/f_+ \) is a power series in \( e^{it} \), it follows from (6.6) that there is a sequence \( \{\alpha_k\} \) of constants for which

\[ \hat{\varphi}_n = \sum_{k=0}^{n} \alpha_k e^{ikt}. \]

Moreover, since \( \varphi_n \to 1/(1 - \lambda \varphi) \) as \( n \) becomes infinite, by (6.16)

\[ \sum_{k=0}^{\infty} \alpha_k e^{ikt} = f_+(\lambda, t). \]

Finally, the generating function for the joint distributions of the maximum \( M_k \) and \( S_k \) is
$$\phi = \sum_{n=0}^{\infty} (\varphi_n - \varphi_{n-1}) e^{int}$$

$$= f_-(\lambda, t) \sum_{n=0}^{\infty} (\hat{\varphi}_n - \hat{\varphi}_{n-1}) e^{int}$$

$$= f_-(\lambda, t) \sum_{k=0}^{\infty} \alpha_k e^{ik(t+\tau)}$$

$$= f_+(\lambda, t + \tau) f_-(\lambda, t).$$

Spitzer's identity (6.4) is obviously included in (6.18).

As a final remark, let us note that the bracket operator (6.2) and the method of this section has an obvious analogue for the non-lattice case. The method and arguments of the examples is exactly the same for this more general case except that no analogue of the explicit formula (6.4) seems to exist.

7. Applications of the method:

To demonstrate the power of the "projection" method of the last section, we now take up some analogues for joint distributions of the identities of section 3.

**Example 7.1:** Maximum and positive partial sums. We let

$$\varphi_{n,k} = \sum_{m=0}^{k} \sum_{m=0}^{\infty} \mathbb{P}(N_k = m, M_k < n) e^{itS_k}$$

Then, it can be shown by the usual argument that

$$U[\varphi_{n,k}]_n + [\varphi_{n,k}]_{-\infty} = \varphi_{n,k+1}.$$
In terms of generating functions

\begin{equation}
(7.2) \quad [\varphi_n (1 - \lambda U \varphi)]^n_1 + [\varphi_n (1 - \lambda \varphi)]^0_{-\infty} = 1.
\end{equation}

We try a solution of the form

\begin{equation}
(7.3) \quad \varphi_n = \hat{\varphi}_n \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda}{k} (\varphi^k)^{-} \right],
\end{equation}

where \( \hat{\varphi}_n \) is a polynomial of degree \( n \) in \( e^{it} \) with constant term \( 1 \).

For any such polynomial \( [\varphi_n (1 - \lambda \varphi)]^0_{-\infty} = 1 \), so that the problem is reduced to constructing \( \hat{\varphi}_n \) satisfying

\begin{equation}
[\hat{\varphi}_n \psi]^n_1 = 0 \quad \text{and} \quad [\hat{\varphi}_n \psi \psi]^n_1 = 1,
\end{equation}

where

\begin{equation}
(7.4) \quad \psi = (1 - \lambda U \varphi) \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda}{k} (\varphi^k)^{-} \right] = \sum_{k=-\infty}^{\infty} B_k e^{ikt}.
\end{equation}

Thus,

\begin{equation}
(7.5) \quad \hat{\varphi}_n = \frac{1}{D_{n-1} (\psi)} \begin{vmatrix}
B_0 & B_1 & \cdots & B_n \\
\vdots & \ddots & \ddots & \vdots \\
B_{-n+1} & B_{-n+2} & \cdots & B_1 \\
\zeta & \zeta^{-1} & \cdots & 1
\end{vmatrix}, \quad (z = e^{it}).
\end{equation}

In summary, the generating function of the quantities in (7.1) is

\begin{equation}
(7.6) \quad \varphi_n = \hat{\varphi}_n \exp \left[ \sum_{k=1}^{\infty} \frac{\lambda}{k} (\varphi^k)^{-} \right],
\end{equation}

where \( \hat{\varphi}_n \) is given in (7.5) and (7.4).
Example 7.2. Maximum and range. To obtain information on $M_n$ and $\tilde{M}_n$, it is easiest to find the joint distribution of $M_n$ and $R_n$. Clearly, the joint distribution of $M_n$ and $\tilde{M}_n$ will follow. In this case we let

\[(7.7)\]

$$
\varphi_{n,k} = \sum_{m=0}^{k} U^m \int_{\begin{array}{c}
I_k = m \\
R_k < n
\end{array}} e^{itM_k} dP,
$$

$$
\varphi_n = \sum_{k=0}^{\infty} \lambda^k \varphi_{n,k}.
$$

By the bracket method, adding a step to the beginning of a path, it can be shown that

\[(7.8)\]

$$
U[\varphi_{n,k}]_n = \sum_{m=1}^{k+1} U^m \int_{\begin{array}{c}
I_{k+1} = m \\
R_{k+1} < n
\end{array}} e^{itM_{k+1}} dP
$$

$$
= \varphi_{n,k+1} - P[I_{k+1} = 0, R_{k+1} \leq n].
$$

The generating function $\psi_n(0)$ of the second term on the right in (7.8) was evaluated in section 5. We found $\psi_n(0) = f_n^r(\lambda, 0)$, where $f_n^r(\lambda, t)$ is given in (6.9). Thus, from (7.8)

$$
\varphi_n = \psi_n(0) + \lambda U[\varphi_{n,k}]_n,
$$

or equivalently,

\[(7.9)\]

$$
[\varphi_n(1 - \lambda U \varphi)]_n = 0 \quad \text{and} \quad [\varphi_n]_0^0 = \psi_n(0).
$$
The solution of (7.9) is \( \varphi_n = \psi_n(0) f_n^+(\lambda U, t) \), where \( f_n^+(\lambda, t) \) is given by (0.8). In summary, the generating function of the quantities defined by (7.7) is

\[
(7.10) \quad \varphi_n = f_n^+(\lambda U, t) f_n^-(\lambda, 0)
\]

**Example 7.3: First positive sum with restricted minimum.** For our final example we consider the evaluation of

\[
(7.11) \quad \psi_{n,k} = \begin{cases} \sum_{k=0}^{\infty} \lambda^k \psi_{nk} & \psi_n = \sum_{k=1}^{\infty} \lambda^k \psi_{nk} \end{cases}
\]

But in section 5, we showed that

\[
\varphi_{n,k} = \begin{cases} \sum_{k=0}^{\infty} \lambda^k \psi_{nk} 
\end{cases}
\]

has generating function \( \varphi_n = f_n^-(\lambda, t) \). Thus

\[
(7.12) \quad [\varphi_{n,k}]_{-n} = \varphi_{n,k+1} + \psi_{n,k+1}
\]

or equivalently,

\[
(7.13) \quad \psi_n = 1 - [\varphi_n(1 - \lambda \phi)]_{-n}
\]
Now \([\psi_n]_1^\infty = \psi_n\), so that actually

\[(7.14) \quad [\varphi_n(1 - \lambda \varphi)]\cdot_{-n}^0 = 1\]

and

\[(7.15) \quad \psi_n = 1 - \left[(1 - \lambda \varphi) \varphi_n\right]_0^\infty.\]

Relation (7.14) is the one which determines that \(\varphi_n = f_n^{-}(\lambda, t)\). In summary, the generating function of the quantities defined in (7.11) is

\[(7.16) \quad \psi_n = 1 - \left[(1 - \lambda \varphi) f_n^{-}(\lambda, t)\right]_0^\infty.\]

8. Examples.

To illustrate the results previously derived we consider here a series of three specific examples. The computations involved are straightforward applications of the method described in [14, § 5.3] and will be omitted.

We will use the notation

\[(8.1) \quad \tilde{f}_n^+ (z) = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1} & A_n \\ A_{-1} & A_0 & \cdots & A_{n-2} & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{-n+1} & A_{-n+2} & \cdots & A_0 & A_1 \\ z^n & z^{n-1} & \cdots & z & 1 \end{bmatrix}\]

(with a similar formula for \(\tilde{f}_n^- (z)\)) to denote the numerator of the expression \(f_n^+ (z)\) given in (0.8). Note that \(D_n(1 - \lambda \varphi) = \tilde{f}_n^+(0)\) and
that \( \hat{f}_n^+(z) = \hat{f}_n^+(z) / \hat{f}_n^+(0) \).

**Example 1: Bernoulli variables.** Let \( X_k \) have the distribution \( P(X_k = 1) = p \) and \( P(X_k = -1) = q \). The determinant in (8.1) becomes essentially bordered. We find that

\[
\hat{r}_n^+(z) = \sum_{k=0}^{n} \lambda^k \frac{p^k}{\lambda} z^k \left( r_1^{n+1-k} - r_2^{n+1-k} \right)
\]

\[
(8.2)
\begin{align*}
&= \frac{r_1^{n+1} - (\lambda p z)^{n+1}}{1 - \lambda p z} - \frac{r_2^{n+1} - (\lambda p z)^{n+1}}{1 - \lambda p z},
\end{align*}
\]

where

\[
r_1 = \frac{1 + \sqrt{1 - 4 \lambda^2 pq}}{2}, \quad r_2 = \frac{1 - \sqrt{1 - 4 \lambda^2 pq}}{2}.
\]

The formula for \( \hat{r}_n^-(z) \) is found from (8.2) by interchanging \( p \) and \( q \) and replacing \( z \) by \( 1/z \).

Let us examine the results of section 5 in terms of the formula (8.2).

From (5.2) and (8.2) we see that \( (z = e^{it}) \)

\[
\hat{\varphi}_n = \sum_{k=0}^{\infty} \lambda^k \left\{ \begin{array}{c}
M_k \leq n \\
\tilde{I}_k = 0
\end{array} \right\} e^{itS_k} dP
\]

\[
(8.3)
\begin{align*}
&= \left[ \frac{r_1^{n+1} - (\lambda p z)^{n+1}}{1 - \lambda p z} - \frac{r_2^{n+1} - (\lambda p z)^{n+1}}{1 - \lambda p z} \right]/\left( r_1^{n+2} - r_2^{n+2} \right).
\end{align*}
\]
Moreover, system (5.10 - 5.11) will be satisfied by \( \psi_n \) and the corresponding \( \psi_n \) where \( \alpha_n \) and \( \beta_n \) are given by

\[
\alpha_n = \frac{\lambda^n p^n (r_1 - r_2)}{r_1^{n+1} - r_2^{n+1}}, \quad \beta_n = \frac{\lambda^n q^n (r_1 - r_2)}{r_1^{n+1} - r_2^{n+1}}.
\]

Finally, let us observe that \( f_+ (\lambda, t) \) and \( f_- (\lambda, t) \) of (0.6 - 0.7) can be found from (8.2) by letting \( n \) become infinite. We have

\[
f_+ = \lim_{n \to \infty} \frac{\tilde{f}_n(z)}{\tilde{f}_n(0)} = \frac{1}{1 - \frac{\lambda p}{r_1} e^{it}} \quad \text{and} \quad f_- = \lim_{n \to \infty} \frac{\tilde{f}_n(z)}{\tilde{f}_{n+1}(0)} = \frac{1}{r_1 - \lambda q e^{-it}}.
\]

It follows that \( f_+ f_- = 1/(1 - \lambda p e^{it} - \lambda q e^{-it}) \) and, of course, the conditions of Lemma 1.1 are satisfied. Inversions of the transforms seem particularly easy in this case.

**Example 2:** Two-sided geometric. For our next example we take the distribution

\[
\begin{align*}
P(X_k = \pm m) &= \frac{1}{2} pq^m \quad (m \geq 1) \\
P(X_k = 0) &= p
\end{align*}
\]

It turns out that \( \tilde{r}_n(z) \) in this case has the form

\[
\tilde{r}_n(z) = (zq)^{n+1} \left( 1 - \frac{\lambda p}{r_2} \right)^n + (1 - zq) \frac{r_1 - q^2}{r_1 - r_2} \sum_{k=0}^{n} \left[ (zq(1 - \frac{\lambda p}{r_2})^k \frac{r_1}{r_2} \right]^n \left[ zq(1 - \frac{\lambda p}{r_2})^k \frac{r_1}{r_2} \right]^{n-k},
\]

\[
= (1 - zq) \frac{r_2 - q^2}{r_1 - r_2} \sum_{k=0}^{n} \left[ zq(1 - \frac{\lambda p}{r_2})^k \frac{r_1}{r_2} \right]^n \left[ zq(1 - \frac{\lambda p}{r_2})^k \frac{r_1}{r_2} \right]^{n-k},
\]

\[
\quad - (1 - zq) \frac{r_2 - q^2}{r_1 - r_2} \sum_{k=0}^{n} \left[ zq(1 - \frac{\lambda p}{r_2})^k \frac{r_1}{r_2} \right]^n \left[ zq(1 - \frac{\lambda p}{r_2})^k \frac{r_1}{r_2} \right]^{n-k}.
\]
where

\[
    r_1 = \frac{(1 - \lambda p + q^2) + \sqrt{(1 - \lambda p + q^2)^2 - 4q^2(1 - \lambda p)^2}}{2}
\]

\[
    r_2 = \frac{(1 - \lambda p + q^2) - \sqrt{(1 - \lambda p + q^2)^2 - 4q^2(1 - \lambda p)^2}}{2}
\].

By appropriate division, the various quantities \( f_n^+(z) \), \( f_n^-(z) \), etc. can be constructed from (8.7). We find \( f_n^- (z) \) from (8.7) simply by replacing \( z \) by \( \frac{1}{z} \). As \( n \) becomes infinite

\[
    f_n^+(z) \to f_+ = \frac{1 - zq}{1 - zq(1 - \lambda p^2)} = \frac{1}{r_1}, (z = e^{it}),
\]

(8.8)

\[
    f_n^-(z) \to f_- = \frac{1 - \bar{z}q}{r_1 - \bar{z}q(1 - \lambda p^2)} = \frac{1}{r_1}, (z = e^{it}).
\]

**Example 3:** Two-sided exponential. Finally, we consider a sequence \( \{X_k\} \) whose distributions are continuous, each having density \( f(x) = \frac{1}{2} \cdot e^{-|x|} (\ - \infty < x < \infty) \). The computations in this case can be deduced from Example 2 by an appropriate limiting process. We take \( q = 1 - p = e^{-\Delta x} \), \( z = e^{i\xi \Delta x} = e^{it} \), \( n = x / \Delta x \), and \( k = y / \Delta x \), and then let \( \Delta x \to 0 \). In this way we pass from Fourier series in \( e^{it} \) in Example 2 to Fourier transforms in \( e^{i\xi x} \) in this example. The counterpart of (8.1) exists under this limit and we find that
\[ F(x, \xi) = \frac{\sigma^2}{4 \sqrt{1 - \lambda}} e^{-\left(\frac{\lambda}{2} + \mu\right)x} - \frac{\mu^2}{4 \sqrt{1 - \lambda}} e^{-\left(\frac{\lambda}{2} + \sigma\right)x} + \frac{\sigma \lambda}{4 \sqrt{1 - \lambda}} e^{-\left(\frac{\lambda}{2} + \mu\right)x} \cdot \frac{e^{(i\xi - \sqrt{1 - \lambda})x} - 1}{i\xi - \sqrt{1 - \lambda}} e^{-\left(\frac{\lambda}{2} + \sigma\right)x} \cdot \frac{e^{(i\xi + \sqrt{1 - \lambda})x} - 1}{i\xi + \sqrt{1 - \lambda}}, \]

where

\[ \sigma = 1 + \sqrt{1 - \lambda}, \quad \mu = 1 - \sqrt{1 - \lambda}. \]

The counterpart of the determinant \( D_n(1 - \lambda \varphi) = \tilde{F}_n^+(0) \) is found by letting \( 1 \xi \to -\infty \). We thus find

\[ D(x, 1 - \lambda \varphi) = \frac{\sigma^2}{4 \sqrt{1 - \lambda}} e^{-\left(\frac{\lambda}{2} + \mu\right)x} - \frac{\mu^2}{4 \sqrt{1 - \lambda}} e^{-\left(\frac{\lambda}{2} + \sigma\right)x}, \]

Dividing (8.9) by (8.10) gives a number of quantities with very interesting probability interpretation. By formal analogy with the result \( \tilde{F}_n^+(z) = \tilde{F}_n^+(z)/\tilde{F}_n^+(0) \) and its use in Examples 6.1, 7.2 and (5.2), we deduce that for the variables \( X_k \) of this example
\[ \varphi(x, \xi) = \sum_{k=0}^{\infty} \lambda^k \int \frac{e^{i\xi M_k}}{D(x, 1 - \lambda \varphi)} \, dP \]
\[ = \frac{\mathcal{Z}(x, \xi)}{D(x, 1 - \lambda \varphi)} \]
\[ = \sum_{k=0}^{\infty} \lambda^k \int \frac{e^{i\xi M_k}}{D(x, 1 - \lambda \varphi)} \, dP = \frac{\psi(x, \xi)}{D(x, 1 - \lambda \varphi)}. \]

We expect that the \( \varphi(x, \xi) \) and \( \psi(x, \xi) \) of (8.11) satisfy a system analogous to (5.10 - 5.11). A tedious calculation will show that for

\[ \alpha(x) = \frac{\lambda e^{-(\frac{\lambda}{2} + 1)x}}{D(x, 1 - \lambda \varphi)}, \]

the function \( \varphi(x, \xi) \) satisfies

\[ \frac{d \varphi(x, \xi)}{dx} = \alpha(x) \varphi(x, \xi) \]

Taking limits as \( x \) becomes infinite, we find, moreover,

\[ f_+^*(\xi) = \lim_{x \to \infty} \varphi(x, \xi) = 1 - \frac{\lambda}{1 + \sqrt{1 - \lambda}} \frac{1}{1 - i\xi - \sqrt{1 - \lambda}} \]
\[ = \frac{1}{1 - \frac{1 + \sqrt{1 - \lambda}}{1 - i\xi}} \]

Clearly, \( |f_+^*(\xi)|^2 = [1 - \lambda/(1 + \xi^2)]^{-1} \), as was expected.
REFERENCES


