OPTIMAL ISSUING POLICIES IN INVENTORY MANAGEMENT

BY

WILLIAM P. PIERSKALLA

TECHNICAL REPORT NO. 7
August 28, 1965

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT GP-3739

PROGRAM IN OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
OPTIMAL ISSUING POLICIES IN INVENTORY MANAGEMENT

by

William P. Pierskalla

TECHNICAL REPORT NO. 7

August 28, 1965

PREPARED UNDER THE AUSPICES

OF

NATIONAL SCIENCE FOUNDATION GRANT GP-3739

PROGRAM IN OPERATIONS RESEARCH

STANFORD UNIVERSITY

STANFORD, CALIFORNIA
# Table of Contents

<table>
<thead>
<tr>
<th>Chapter 1</th>
<th>Introduction</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Introduction and Characterization of the Model</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>The Truncation Point $S_0$</td>
<td>9</td>
</tr>
<tr>
<td>Chapter 2</td>
<td>Multiple Demands on the Stockpile</td>
<td>11</td>
</tr>
<tr>
<td>2.1</td>
<td>Modification of Assumption (6)</td>
<td>11</td>
</tr>
<tr>
<td>2.2</td>
<td>General Relationships</td>
<td>13</td>
</tr>
<tr>
<td>2.3</td>
<td>Bounds on the Optimal Policy</td>
<td>33</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>Addition of Penalty Costs</td>
<td>61</td>
</tr>
<tr>
<td>3.1</td>
<td>The Case for FIFO</td>
<td>62</td>
</tr>
<tr>
<td>3.2</td>
<td>The Case for FIFO When $L(S)$ is Linear</td>
<td>81</td>
</tr>
<tr>
<td>3.3</td>
<td>The Case for LIFO</td>
<td>92</td>
</tr>
<tr>
<td>Chapter 4</td>
<td>Field Life Functions Which Are Not Convex or Concave</td>
<td>101</td>
</tr>
<tr>
<td>4.1</td>
<td>Lemmas</td>
<td>103</td>
</tr>
<tr>
<td>4.2</td>
<td>Optimal Policies</td>
<td>113</td>
</tr>
<tr>
<td>Chapter 5</td>
<td>The Dynamic Inventory-Depletion Model</td>
<td>140</td>
</tr>
<tr>
<td>5.1</td>
<td>$L(S)$ Concave with $0 \leq L'(S) \leq -1$</td>
<td>142</td>
</tr>
<tr>
<td>5.2</td>
<td>$L(S)$ Concave or Convex with Slope $&lt; -1$</td>
<td>152</td>
</tr>
<tr>
<td>5.3</td>
<td>The Problem of Stockouts</td>
<td>162</td>
</tr>
<tr>
<td>Chapter 6</td>
<td>A Stochastic Field Life Function</td>
<td>165</td>
</tr>
<tr>
<td>Chapter 7</td>
<td>Batches of Items of the Same Age in the Stockpile</td>
<td>178</td>
</tr>
<tr>
<td>Chapter 8</td>
<td>Summary and Conclusions</td>
<td>187</td>
</tr>
<tr>
<td>Bibliography</td>
<td></td>
<td>191</td>
</tr>
</tbody>
</table>
Chapter I
Introduction

1.1 Introduction and Characterization of the Model

The general inventory depletion problem can be described as the problem of finding an issue policy which maximizes or minimizes a prescribed function when the inventory itself is changing in quality over time. The change in quality may be either an appreciation or a deterioration of the useful life, the field life, of each item in the inventory as long as the item remains in the stockpile. An issue policy is a selected order of issue of the items in the stockpile when demands for the items are made from the field.

In 1958 Derman and Klein [2] and Lieberman [9] presented some analytic results concerning a more specific formulation of the general model. They obtained optimal policies of the form LIFO, last in first out, or FIFO, first in first out. The advantage of LIFO or FIFO policies is twofold. In practice, they are the most easily understood and most easily implemented policies. Second, they require only a knowledge of the relative ages of the items in the inventory and not their exact ages. However their model contains several restrictive assumptions which limit the application of the results to most real situations.

It is the primary purpose of this work to eliminate and/or modify some of the assumptions of the model and still, whenever possible, obtain LIFO or FIFO as optimal policies.
In order to be more specific as to which assumptions will be changed or removed, it will be advantageous to characterize the model explicitly.

The Model

A. Assumptions

(1) At the beginning of the process, a stockpile has \( n \) indivisible identical items of varying ages \( S_1 < S_2 < \ldots < S_n \) where \( S_1 > 0 \). The ages \( S_i \) are called the initial ages of the items.

(2) Each item has a field life \( L(S) \) which is a known non-negative function of the age \( S \) of the item upon being issued.

(3) Items are issued successively until either the entire stockpile is depleted or the remaining items in the stockpile have no further useful life, i.e. \( L(S) = 0 \) for the remaining items.

(4) No penalty or installation costs are associated with the issuance of an item from the stockpile.

(5) New items are never added to the stockpile after the process starts.

(6) An item is issued from the stockpile only when the entire life of the preceding item issued is ended.

(7) At the beginning of the process each item has positive field life, i.e., \( L(S_i) > 0 \) for all \( i = 1, 2, \ldots, n \).
B. Objective

The objective is to find the issue policy which maximizes the total field life of the stockpile. An issue policy which achieves this maximum is called an optimal policy.

Since assumption (2) requires that $L(S)$ be known, this model is called deterministic. Moreover for all of the results in the following chapters, it is assumed that $L(S)$ is a deteriorating field life function. The appreciating field life function has not been studied since the mathematical techniques used in the case of deteriorating $L(S)$ generally carry over to the appreciating case with minor modifications. In addition, appreciating functions do not have the range of applications in the inventory context as do depreciating functions.

Many of the earlier results based on the above model are quite interesting and several of them are used in the subsequent chapters of this work. Some of these earlier results will now be presented.

Derman and Klein [2] stated the first major theorem giving sufficient conditions for LIFO optimality. However as Zehna [11] pointed out the statement was not quite correct. Zehna gave the corrected statement: "If $L(S)$ is a convex monotone function and LIFO is optimal for $n = 2$, then LIFO is optimal for all $n > 2$." Shortly after the Derman and Klein paper, Lieberman [9] presented three theorems which have been of benefit in further development of inventory depletion theory. In particular, his Theorem 3 is used many times in later chapters. The three theorems are:
Theorem 1: (i) If \( \frac{dL}{dS} = L'(S) \geq -1 \) and (ii) LIFO is an optimal policy when \( n = 2 \), then LIFO is an optimal policy when \( n = 3, 4, \ldots \).

Theorem 2: (i) If \( L(S) \) is a convex monotone function or if \( L'(S) \geq -1 \) and (ii) FIFO is an optimal policy when \( n = 2 \), then FIFO is an optimal policy for \( n = 3, 4, \ldots \).

Theorem 3: (i) If \( L'(S) \geq -1 \) and (ii) \( L(S) \) is a nonincreasing or nondecreasing concave function, then FIFO is an optimal policy.

Zehna proved that Theorem 3 can be generalized to "Suppose \( L(S) \) is a concave differentiable function. Then FIFO is an optimal policy for \( n \geq 2 \) if and only if \( L'(S) \geq -1 \)."

With the Derman-Klein-Zehna theorem above, and the first two theorems of Lieberman, the search for an optimal policy reduced to a search for sufficient conditions when \( n = 2 \). Bomberger [1], Eilon [7], and Zehna presented many results establishing such sufficient conditions. In addition Zehna presented two theorems for \( L'(S) < -1 \) which are often quoted in later chapters. His Theorems 2.4 and 2.6 are combined in "If \( L(S) \) is a convex or a concave differentiable function and \( L'(S) < -1 \) for all \( S > 0 \), then LIFO is optimal for \( n \geq 2 \)."

But with the exception of Zehna's Chapters 4 and 5 and Eilon [4], none of the papers have considered removing any of the restrictive assumptions of the model. Looking at the assumptions of the model, it
is apparent that there is a broad area of inventory depletion problems which are not covered by the model.

Assumption (6) implicitly assumes that there is only one demand source withdrawing items from the stockpile. Zehna and Eilon [4] independently approached this problem and both proved the result that if $L(S) = aS + b$ for $b > 0 > a > -1$, then for a stockpile of $n$ items FIFO is optimal for one or more demand sources. Zehna also proved (Theorem 4.2) if $L(S)$ is either a convex or a concave differentiable function with $L'(S) < -1$, then LIFO is optimal for two demand sources and (Theorem 4.3) if FIFO (LIFO) is optimal for one and for two demand sources, then FIFO (LIFO) is optimal for more than two demand sources. Moreover Zehna demonstrated that for general $L(S)$ concave and differentiable with slope $\geq -1$, FIFO is not always an optimal policy. In Chapter 2, however, we are able to show that when $L(S)$ is any concave nonincreasing function with slope $\geq -1$, the FIFO issuing policy and the optimal issuing policy have the same upper and lower bounds for more than one demand source. Since not all policies are included in these bounds, then FIFO can be called a suboptimal policy in the sense that it differs from the optimal policy by not more than their common upper and lower bounds. Furthermore, it is shown that if the number of demand sources is greater than or equal to one half of the number of items in the stockpile, then FIFO is optimal.

All of the previous papers have assumed that there are no penalty costs, such as installation or work stoppage costs, when an item is issued from the stockpile. However, in most real situations, penalty
costs are incurred, and if they are significant relative to the useful lives of some of the items in the stockpile, the optimal policy may change when these costs are considered. For example, if the policy which is optimal without penalty costs says to issue many items which have very little useful field life, then the penalty cost could more than outweigh the usefulness of the items. In Chapter 3 assumption (4) is removed and replaced by the assumption that there is a constant penalty cost, \( p \), incurred every time an item is issued from the inventory. Theorems are then proved which reduce the search for the optimal policy from a search of \( n! \) policies to the search of \( n \) policies. When \( L(S) \) is linear an optimal policy can be specified exactly.

The most common assumption about \( L(S) \) in the literature and in Chapters 2 and 3 is that \( L(S) \) is a convex or concave function. It may be the case, however, that in a real problem \( L(S) \) may be a concave function for some period of time and then a convex function after that time. Curves of this type will be called S-shaped curves and in Chapter 4 a special case of the S-shaped curve is examined and optimal policies obtained. The special case is \( L(S) \) is concave nonincreasing and strictly positive for all \( S \in [0, t] \), \( L(S) = L(t) = c \) for all \( S \in [t, \infty) \), and the left hand derivative of \( L(S) \), \( L^-(S) \), has \( L^-(S) \geq -1 \) for all \( S \in (0, t) \). We then show that for some \( i = 1, \ldots, n \) the optimal policy is to issue the first \( i \) items by FIFO and the remaining \( n - i \) items by LIFO. When the concave part of \( L(S) = aS + b \) then sufficient conditions are given which locate the specific \( i = 1, \ldots, n \) which yields the optimal policy.
Probably the most restrictive assumption of the model is assumption (5). Assumption (5) states that new items are never added to the stockpile after the process starts. This assumption makes the model completely static and not the dynamic representation of a "going concern" which is what most real inventory problems are. In Chapter 5 this assumption is removed and the dynamic inventory depletion model is presented. Here again we consider \( L(S) \) concave nonincreasing and \( L^-(S) \geq -1 \), then if FIFO is optimal in the static model, FIFO is optimal when \( N \) items are added to the inventory at different times in the future. When \( L(S) \) is concave or convex and \( L^-(S) < -1 \), then a policy called generalized-modified-LIFO (GML) is optimal. GML is the policy where LIFO is used until a new item arrives then the new item is immediately issued to the demand source which has the least life remaining on its item currently in consumption.

For Chapters 3, 4, and 5 the results are given for one or more demand sources. Also, in Chapters 2, 3, 4, and 5, we have been concerned with the deterministic model, i.e., \( L(S) \) is a known function. Zehna considered the case where the field life of an item is a nonnegative random variable, \( X(S) \), dependent on the age, \( S \), of the item upon being issued. In this case the objective function now becomes: maximize the total expected return (utility) of the stockpile. Zehna obtained two theorems for this stochastic model. The first theorem (5.1) holds only when the distribution of \( X(S) \) has an increasing mean value function as \( S \) increases. The second theorem (5.2) holds when the mean value function is decreasing and since this work is concerned with
deteriorating inventories we state his Theorem 5.2: "Suppose for each \( S \geq 0 \), \( X(S) \) has density

\[
\frac{1}{[L(S)]^{\alpha+1} \Gamma(\alpha + 1)} x^\alpha e^{-\frac{x}{L(S)}} \quad \text{for} \quad x \geq 0
\]

where \( L(S) = e^{-kS} \), \( k > 0 \) and integer \( \alpha > -1 \). Then LIFO is optimal when \( n = 2 \)." The stochastic model presented in Chapter 6 differs somewhat from Zezma's model. In Chapter 6 \( X(S) \) can take on any one of a countable number of values, \( L_1(S) \) with probability \( p_1 \) where \( i = 1, \ldots, M, \sum_{i=1}^{M} p_i = 1 \), and \( p_1 \geq 0 \) (\( M \) may be replaced by \( +\infty \)). If \( L_i(S) = a_i S + b_i \) where \( b_i > 0 > a_i > -1 \), \( L_i(S) < L_{i+1}(S) \) for all \( i \) and any \( S < S_o \) (where \( S_o \) is defined in the next section), and \( L_i(S_o) = L_{i+1}(S_o) = 0 \) for all \( i \), then FIFO is optimal for \( n \) items in the stockpile and one demand source. This result says that if we know that each item deteriorates according to some linear field life function then even though the specific function is unknown for any item, FIFO is a policy which maximizes the total expected return. If we change \( L_i(S) = a_i S + b_i \) above to \( L_i(S) \) is concave and differentiable with \( 0 \geq L_i(S) > L_{i+1}(S) > -1 \) for all \( i \) and \( S < S_o \), then FIFO is optimal for \( n = 2 \).

In Chapter 7 we consider the case when there are batches of items of the same age in the stockpile. Eilin [5] considered this problem in regard to the obsolescence of commodities which are subject to deterioration in the stockpile. However he did not consider the batch assumption's effect on the optimality of LIFO or FIFO. In Chapter 7
this latter consideration is made. A general result is proved: If \( L(S) \) is continuous and if FIFO (LIFO) is optimal in the case of no batches, then FIFO (LIFO) is optimal when batches are permitted.

1.2 The Truncation Point \( S_o \)

Assumption (2) of the model requires that \( L(S) \) be a nonnegative function of \( S \). In the case where we assume that \( L(S) \) is a concave decreasing function then there is a point, say \( S_o \), such that \( L(S) > 0 \) for all \( S \in [0, S_o) \) and \( L(S) \leq 0 \) for all \( S \geq S_o \). Thus for all \( S \geq S_o \), \( L(S) \) must be redefined to be identically zero and \( S_o \) is a finite truncation point. In another case, e.g., \( L(S) = \frac{1}{S} \) for all \( S > 0 \) then as \( S \to +\infty \), \( L(S) \to 0 \) and \( S_o = +\infty \) is called the truncation point.

In general, if \( L(S) \) is a decreasing function of \( S \) and \( L(0) > 0 \), then \( S_o = +\infty \) is a truncation point for \( L(S) \) if and only if

\[
S_o = \inf \{ S \in [0, \infty) | L(S) \leq 0 \}
\]

and then \( L(S) \) is redefined to be

\[
L(S) = \begin{cases} 
L(S) > 0 & \text{for all } S \in [0, S_o) \\
0 & \text{for all } S \geq S_o 
\end{cases}
\]

(Ref. Zehna [11].)

From a practical point of view it makes little sense to permit \( L(S) \) to be arbitrarily large for some \( S \). Hence we will assume that there is some number \( k < \infty \) such that \( L(S) < k \) for all \( S \) of interest. If \( L(S) = \frac{1}{S} \), as shown in the example above, we will assume this \( L(S) \)
applies only to those $S > 0$ such that $L(S) = \frac{1}{3} < k$. Then if a finite number, $n$, of items are issued by any policy $A$, the total field life, $Q_A$, is bounded by $0 < Q_A < nk = K$ for all policies $A$ and any $n$ items $0 \leq S_1 < S_2 < \cdots < S_n$. 
Chapter 2

Multiple Demands on the Stockpile

2.1 Modification of Assumption (6)

The model, as previously defined, contains the implicit hypothesis that there is only one demand source withdrawing items from the stockpile. Except for Zehna [11], Chapter 4, all of the previous work done on the deterministic inventory depletion model necessarily requires this single demand source assumption. Zehna, however, proved that when \( L(S) = aS + b \ (b > 0 > a > -1) \) then FIFO is optimal for one or more demand sources. In addition, he showed that if \( L(S) \) is either a convex or a concave differentiable function with \( L'(S) < -1 \), then LIFO is optimal for one or more demand sources.

One of the objectives of the present work, is to remove the assumption of a single demand source. We will denote the number of demand sources requesting items from the stockpile by the letter "\( \nu \)." \( \nu \) is an integer and is bounded by \( 1 \leq \nu \leq n \) where \( n \) is the number of items initially in the stockpile. We do not consider \( \nu > n \) since the policy of issuing the \( n \) items to the \( n \) demand sources cannot be improved upon in terms of maximizing the total field life of the stockpile. The demand sources will be denoted by \( M_1, M_2, \ldots, M_\nu \).
Since assumption (6) contains the implicit assumption of a single demand source we will modify assumption (6) as follows:

(6)' An item is issued from the stockpile whenever any demand source has consumed the entire useful field life of the item previously issued to it. If two or more demand sources request a new item at the same time, the new items will be issued to them in the same order as they received their last previously issued items.

A policy is said to be feasible if a demand on the stockpile is always satisfied, provided (i) the stockpile is not empty and (ii) the remaining items in the stockpile have positive field life. In seeking the optimal policy we will only be concerned with the optimal policy which belongs to the class of feasible policies.

Before proceeding further, it will be useful to define the notation which is used to describe a policy. An issuing policy for \( v \) demand sources:

1. List the items assigned to a particular demand source in their order of use from the first item used until the last item used, and

2. separate the items for different demand sources by a semicolon.

For example, a policy \( A \) can be described as follows:

\[
A = [S_{11}, S_{12}, \ldots, S_{11}; S_{21}, \ldots, S_{21}; \ldots; S_{v1}, \ldots, S_{v1}]
\]
Thus A is the issuing policy which assigns
items \( S_{11}, S_{12}, \ldots, S_{1v} \) to demand source \( M_1 \) in that order,
items \( S_{21}, S_{22}, \ldots, S_{2v} \) to demand source \( M_2 \) in that order, ...
items \( S_{v1}, S_{v2}, \ldots, S_{vv} \) to demand source \( M_v \) in that order.

Note that \( \sum_{j=1}^{v} i_j = n \) if all items are assigned. It is obvious that
the choice of \( M_1, M_2, \ldots, M_v \) for the particular assignment of items
above was arbitrary. Hence the \( v! \) policies obtained by permuting the
\( M_i \)'s are equivalent policies in the sense that the total field life
obtained from the \( n \) items is unchanged regardless of how the demand
sources are labelled.

It is assumed that the process begins by issuing \( v \) items, one to
each \( M_1, M_2, \ldots, M_v \).

2.2 General Relationships

Among the items which have a deteriorating field life function
there are several interesting relationships which will be useful at
various times throughout the subsequent chapters. For this reason these
relationships have been gathered together and stated as lemmas in this
section.

**Lemma 2.1:** Let \( L(S) \) be a continuous nonincreasing function with
\( L'(S) \geq -1 \) for \( 0 < S \leq S_0 \). Let \( v = 1 \). If the items in the stock-
pile are issued according to FIFO, the field life of any item at the
time of issue is strictly positive.
Proof of Lemma 2.1: If $S_o = +\infty$, the lemma is trivially true. Hence assume $S_o < +\infty$. By FIFO $S_n$ is the first item issued and by assumption (7) of the model $L(S_n) > 0$. Now assume the lemma is true for the first $k$ items issued and it will be proved true for the first $k + 1$ items issued.

Since items are withdrawn from the stockpile in decreasing order of their index numbers (under FIFO), let the $k^{th}$ item issued be denoted by $S_j$ and the $(k + 1)^{st}$ item issued by $S_{j-1}$. Let $x$ denote the total field life of the first $k - 1$ items (under FIFO). Then the inductive hypothesis is:

$$L(S_j + x) > 0 \quad (2.2.1)$$

which implies that $S_j + x < S_o$. Now it must be proved that

$$L(S_{j-1} + x + L(S_j + x)) > 0 \quad (2.2.2)$$

or in other words that

$$S_{j-1} + x + L(S_j + x) < S_o.$$ 

Now by hypothesis $L(S) \geq -1$ for all $S$ with $0 < S \leq S_o$ and since $L(\cdot)$ is continuous and since $S_j + x < S_o$ by (2.2.1) we can form

$$\frac{L(S_j + x) - L(S_o)}{S_j + x - S_o} \geq -1 \quad (2.2.3)$$
hence
\[ L(S_j + x) - L(S_o) \leq S_o - S_j - x \]
and since \( L(S_o) = 0 \) we obtain
\[ L(S_j + x) + x + S_j \leq S_o \]  \hspace{1cm} (2.2.4)

But \( S_{j-1} + x < S_j + x \), hence in (2.2.4)
\[ S_{j-1} + x + L(S_j + x) < S_j + x + L(S_j + x) \leq S_o \]

which proves (2.2.2). Therefore by induction the lemma is proved.

q.e.d.

The next lemma is concerned with the effect on total field life when \( M \) items of arbitrary ages are combined with the inventory of \( n = N \) items and the process then starts. We assume a FIFO issuing policy is used.

**Lemma 2.2:** Let \( L(S) \) be a continuous nonincreasing function with \( L'(S) \geq -1 \) for all \( S \) with \( 0 < S \leq S_o \). Let \( v = 1 \). Denote by \( Q_{F,N} \), the total field life obtained by issuing the \( n = N \) items according to FIFO. Let \( M \geq 1 \) additional items of initial ages \( S^* \leq S^*_2 \leq \cdots \leq S^*_M \leq S_o \) be combined with the original \( N \) items. Let the \( N + M \) items be issued by FIFO and denote the total field life by \( Q_{F,N+M} \). Let \( S^*_i \neq S^*_j \) for all \( i, j \).

Then \( Q_{F,N+M} \geq Q_{F,N} \) for any finite \( N, M \).
Proof of Lemma 2.2: The proof will be by induction. Let \( M = 1 \) and \( N \geq 1 \). Three cases are possible.

**Case 1** \( S^*_1 < S_1 \)

\[
Q_{F_N} \leq Q_{F_N} + L(S^*_1 + Q_{F_N}) = Q_{F_{N+1}}
\]

**Case 2** \( S_N < S^*_1 \)

For this case we will use induction on \( N \) to show \( Q_{F_{N+1}} \geq Q_{F_N} \).

Let \( N = 1 \). We have since \( L(S) \) is nonincreasing \( L(S_1) \geq L(S^*_1) \) and since \( L(\cdot) \) is continuous and \( L(\cdot) \geq -1 \) for \( 0 < S < S_0 \) then

\[
\frac{L(S_1) - L(S_1 + L(S^*_1))}{-L(S^*_1)} \geq -1
\]

\[
L(S_1) - L(S_1 + L(S^*_1)) \leq L(S^*_1)
\]

and

\[
Q_{F_1} = L(S_1) \leq L(S^*_1) + L(S_1 + L(S^*_1)) = Q_{F_2}
\]

Now assume true for \( N = J \) and prove true for \( N = J + 1 \). Let \( x \) be the total field life from issuing items \([S^*_1, S_{j+1}, S_j, \ldots, S_2]\) by FIFO and let \( y \) be the total field life from issuing \([S_{j+1}, S_j, \ldots, S_2]\) by FIFO. Then the inductive assumption states

\[
x \geq y.
\]

(2.2.5)

16
We must show

\[
Q_{F_{N+1}} = x + L(S_1 + x) \geq y + L(S_1 + y) = Q_{F_N} \tag{2.2.6}
\]

(i) if \( x = y \), then (2.2.6) holds with equality

(ii) if \( x > y \), then by lemma 2.1, \( S_1 + x < S_0 \) and \( S_1 + y < S_0 \)

and since \( L(\cdot) \) is continuous and \( L^-(S) \geq -1 \) for \( S \leq S_0 \)

we have

\[
\frac{L(S_1 + x) - L(S_1 + y)}{x - y} \geq -1
\]

\[
L(S_1 + x) - L(S_1 + y) \geq -x + y
\]

and

\[
L(S_1 + x) + x \geq L(S_1 + y) + y
\]

which proves (2.2.6); hence by induction the lemma is true for this case.

\[\text{Case 3} \quad S_1 < \cdots < S_i < S_i^* < S_{i+1} < \cdots < S_N\]

Let \( x \) denote the total field life of the \( N^{th}, \ N-1^{st}, \ldots, i+1^{st} \)

items issued by FIFO (i.e., of items \( S_N, S_{N-1}, \ldots, S_{i+1} \)). Then

\( S_i^* + x > S_i + x > S_i \) and since \( L(\cdot) \) is nonincreasing

\[
L(S_i^* + x) \leq L(S_i + x). \tag{2.2.7}
\]

Now let

\[
S_i^* + x = T_1^*
\]
\[ S_j + x = T_j \quad \text{for all } j = 1, \ldots, i. \]

Then (2.2.7) becomes \( L(T_1) \leq L(T_i) \) and \( 0 < S_1 + x < S_2 + x < \cdots < S_1 + x < S_1^* + x \) is rewritten as

\[ 0 < T_1 < T_2 < \cdots < T_i < T_1^* \] (2.2.8)

but (2.2.8) shows that we now have case 2 above with \( N = i \) hence

\[ x + Q_{F_{i+1}} \geq x + Q_{F_1} \quad \text{by case 2.} \]

But

\[ Q_{F_{N+1}} = x + Q_{F_{i+1}} \geq x + Q_{F_1} = Q_{F_N}. \]

Therefore \( Q_{F_{N+1}} \geq Q_{F_N} \) in all three cases and since the three cases exhaust all possibilities the lemma is proved for \( M = 1 \) and \( N \geq 1 \).

Let \( M > 1 \) and \( N \geq 1 \).

Assume the lemma is true for \( M > 1 \) and consider adding \( M + 1 \) items of initial ages \( \{S_{i}^{*}\}_{i=1}^{M+1}, (S_{i}^{*} < S_{i+1}^{*}) \). Ignoring \( S_{i}^{*} \) temporarily, the total field life of the remaining items \( Q_{F_{N+M}} \) satisfies

\[ Q_{F_{N+M}} \geq Q_{F_N} \]

by the inductive assumption. Then adding \( S_{M+1}^{*} \) can only increase the total field life by the case \( M = 1 \) i.e., \( Q_{F_{N+M+1}} \geq Q_{F_{N+M}} \geq Q_{F_N} \) and by induction the lemma is proved.

q.e.d.

It is very important to know the ordering of a FIFO assignment to \( v > 1 \) demand sources. Lemma 2.3 below gives such an assignment under a deteriorating field life function with slope \( \geq -1 \).
**Lemma 2.3:** Let \( L(S) \) be a continuous nonincreasing function with \( L^-(S) \geq -1 \) for all \( S \) such that \( 0 < S \leq S_o \). Let \( \nu \geq 1 \). Then starting from the oldest item \( S_n \), FIFO assigns every \( \nu^{th} \) item to the same demand source, i.e., without loss of generality we can arbitrarily let \( M_1 \) receive \( S_n \), \( M_2 \) receive \( S_{n-1} \) etc. to start, then

- demand source \( M_1 \) receives items indexed by \( n - k \nu \)
- demand source \( M_2 \) receives items indexed by \( n - k \nu - 1 \)
- ...
- demand source \( M_j \) receives items indexed by \( n - k \nu - j + 1 \)
- ...
- demand source \( M_\nu \) receives items indexed by \( n - k \nu - \nu + 1 \)

for \( k = 0, 1, 2, \cdots \) until all items have been assigned. Conversely if the assignment of items is as given above, then the assignment is FIFO.

**Proof of Lemma 2.3:** The proof proceeds by the use of induction in several parts.

Let \( k = 1 \).

Now we have arbitrarily assigned
\[ S_n \rightarrow M_1 \]
\[ S_{n-1} \rightarrow M_2 \]
\[ \ldots \]
\[ S_{n-v+1} \rightarrow M_v \]

and since \( L(\cdot) \) is nonincreasing

\[ L(S_n) \leq L(S_{n-1}) \leq \ldots \leq L(S_{n-v+1}) \] (2.2.9)

hence the next assignment is \( S_{n-v} \) to \( M_1 \) since \( L(S_n) \) is the smallest field life. Now assume the lemma is true for assigning \( S_{n-v-i+1} \) to \( M_i \) for \( i \in \{1, 2, \ldots, v-1\} \) we must prove

\[ S_{n-v} \text{ is assigned to } M_{i+1}. \] (2.2.10)

To prove (2.2.10) it is useful to show for all \( j \) such that

\[ 2 \leq j \leq i \] that

\[ L(S_{n-v-j+2} + L(S_{n-j+2})) \leq L(S_{n-v-j+1} + L(S_{n-j+1})) + L(S_{n-j+1}). \] (2.2.11)

Inequality (2.2.11) states that the field life obtained from the two items assigned to \( M_{j-1} \) is less than the field life obtained from the two items assigned to \( M_j \). For simplicity let \( x = L(S_{n-j+2}) \) and \( y = L(S_{n-j+1}) \) and since \( L(\cdot) \) is nonincreasing \( x \leq y \).

(i) If \( x = y \) then (2.2.11) is obviously true.

(ii) If \( x < y \), then since \( L(\cdot) \) is continuous, since \( L^{-1}(S) \geq -1 \) for \( S \leq S_o \) and since by lemma 2.1 we have \( S_{n-v-j+2} + y < S_o \) then
\[
\frac{L(S_{n-v-j+2} + x) - L(S_{n-v-j+2} + y)}{x - y} \geq -1
\]

hence

\[
L(S_{n-v-j+2} + x) + x \leq L(S_{n-v-j+2} + y) + y
\]

\[
\leq L(S_{n-v-j+1} + y) + y
\]

since

\[
L(S_{n-v-j+2} + y) \leq L(S_{n-v-j+1} + y)
\]

and (2.2.11) is satisfied, for all \( j = 2, \ldots, i \).

Now we can telescope (2.2.11) into the inequality

\[
L(S_{n-v} + L(S_n)) + L(S_n) \leq L(S_{n-v-j+1} + L(S_{n-j+1})) + L(S_{n-j+1})
\]

for all \( j = 1, \ldots, i \)

and we will show

\[
L(S_{n-i}) \leq L(S_{n-v} + L(S_n)) + L(S_n).
\]

(2.2.12)

Then, if (2.2.12) holds we have by (2.2.12) and (2.2.9) that demand source \( M_{i+1} \) (who received \( S_{n-i} \)) is the next source to demand an item from the stockpile which by FIFO and the inductive assumption is item \( S_{n-v-i} \). We now prove (2.2.12). By lemma 2.1, \( S_{n-i} + L(S_n) < S_0 \) and since \( L(\cdot) \) is continuous and \( L'(S) \geq -1 \) for \( S \leq S_0 \) we have
\[
\frac{L(S_{n-i} + L(S_n)) - L(S_{n-i})}{L(S_n)} \geq -1
\]

\[
L(S_{n-i}) \leq L(S_n) + L(S_{n-i} + L(S_n))
\]

\[
\leq L(S_n) + L(S_{n-v} + L(S_n))
\]

since \[
L(S_{n-i} + L(S_n)) \leq L(S_{n-v} + L(S_n)) .
\]

Hence (2.2.12) holds and \( S_{n-v-i} \) is assigned to \( M_{i+1} \). Therefore by induction the lemma is true for \( k = 1 \).

Assume the lemma is true for \( k = t \) and it will be proved true for \( k = t + 1 \). For all \( j = 1, \ldots, v \) let \( x_j \) be the total field life of all the items assigned to \( M_j \) up through cycle \( t - 1 \), i.e., of items \( S_{n-j+1}, S_{n-v-j+1}, \ldots, S_{n-(t-1)v-j+1} \). Then the inductive assumption states

\[
x_1 \leq x_2 \leq \cdots \leq x_j \leq \cdots \leq x_v . \tag{2.2.13}
\]

In order to assert that \( M_1 \) receives item \( S_{n-(t+1)v} \), it is necessary to prove

\[
L(S_{n-tv} + x_1) + x_1 \leq L(S_{n-tv-j} + x_{j+1}) + x_{j+1} \tag{2.2.14}
\]

for all \( j = 1, \ldots, v - 1 \).

If \( x_{j+1} = x_1 \), then (2.2.14) obviously holds since \( L(\cdot) \) is nonincreasing.

If \( x_{j+1} > x_1 \), then since \( L(\cdot) \) is continuous and nonincreasing and by lemma 2.1, \( S_{n-tv} + x_{j+1} < S_0 \), thus
\[ \frac{L(S_{n-t\nu} + x_{j+1}) - L(S_{n-t\nu} + x_1)}{x_{j+1} - x_1} \geq -1 \]

implies

\[ L(S_{n-t\nu} + x_1) + x_1 \leq L(S_{n-t\nu} + x_{j+1}) + x_{j+1} \]
\[ \leq L(S_{n-t\nu} - j + x_{j+1}) + x_{j+1}. \]

And (2.2.14) holds for all \( j = 1, \ldots, \nu - 1 \) since \( j \) was arbitrary.

Therefore item \( S_{n-(t+1)\nu} \) is issued to \( M_1 \) and the lemma has been proved for the first assignment in the \( t + 1 \)st cycle.

Now assume the lemma is true for the \( j \)th assignment in the \( t + 1 \)st cycle and it will be proved true for the \( j + 1 \)st assignment in the \( t + 1 \)st cycle \( (j + 1 \leq \nu) \).

Let

- \( y_1 \) be the total field life of items issued to \( M_1 \) up through cycle \( t \).
- \( y_2 \) be the total field life of items issued to \( M_2 \) up through cycle \( t \).
- \( \ldots \).
- \( y_j \) be the total field life of items issued to \( M_j \) up through cycle \( t \).

Then by the inductive assumption on \( t \) and on \( j \)

\[ x_{j+1} \leq x_{j+2} \leq \cdots \leq x_{\nu} \leq y_2 \leq \cdots \leq y_j \quad (2.2.15) \]

where \( y_1 = x_1 + L(S_{n-t\nu-i+1} + x_i) \) for \( i = 1, \ldots, j \).
It must be shown that

\[ x_{j+1} + L(S_{n-tv-j} + x_{j+1}) \leq x_k + L(S_{n-tv-k+1} + x_k) \]

\[ \leq y_i + L(S_{n-(t+1)v-i+1} + y_i) \] \hspace{1cm} (2.2.16)

for all \( k = j + 2, \ldots, v \) and \( v = 1, \ldots, j \).

But (2.2.16) follows immediately by the same reasoning as used above.

If \( x_{j+1} = x_k \), then the first inequality in (2.2.16) holds since \( L(\cdot) \) is nonincreasing. If \( x_{j+1} < x_k \) then since \( S_{n-tv-j} + x_k < S_0 \) by lemma 2.1 then

\[ \frac{L(S_{n-tv-j} + x_k) - L(S_{n-tv-j} + x_{j+1})}{x_k - x_{j+1}} \geq -1 \]

implies

\[ L(S_{n-tv-j} + x_{j+1}) + x_{j+1} \leq L(S_{n-tv-j} + x_k) + x_k \]

\[ \leq L(S_{n-tv-k+1} + x_k) + x_k , \]

for \( k = j + 2, \ldots, v \) since \( k \) was arbitrary. Similarly if \( x_k = y_i \) then the second inequality in (2.2.16) is obviously satisfied, and if \( x_k < y_i \) then since \( S_{n-tv-k+1} + y_i < S_0 \) by lemma 2.1, then

\[ L(S_{n-tv-k+1} + x_k) + x_k \leq L(S_{n-tv-k+1} + y_i) + y_i \]

\[ \leq L(S_{n-(t+1)v-i+1} + y_i) + y_i \]

for all \( i = 1, \ldots, j \) since \( i \) was arbitrary. Thus both inequalities in (2.2.16) hold.
But (2.2.16) implies that $M_{j+1}$ is in need of an item before $M_{j+2}, \ldots, M_v, M_1, \ldots, M_j$ in that order. Therefore the next item to be assigned must be assigned to $M_{j+1}$. However the last item assigned (by the inductive assumption) was item $S_{n-(t+1)v-j+1}$ and since we are following FIFO then $S_{n-(t+1)v-j}$ is the next item and is assigned to $M_{j+1}$. But this last assignment is precisely what this lemma states it should be. Hence by induction on $k = t$ and on $j$ the lemma is proved. The converse is obviously true since we assign the oldest item each time an assignment is made.

q.e.d.

There is an interesting corollary to this lemma which states exactly how many items each demand source receives under FIFO issuance when the field life function is as given in lemma 2.3.

**Corollary 2.3.1:** Let $L(S)$ be a continuous nonincreasing function with $L^{-}(S) \geq -1$ for all $S$ such that $0 < S \leq S_o$. If FIFO is used to assign the $n$ items to $v \geq 1$ demand sources, then demand source $M_j$ receives exactly $\left\lfloor \frac{n-j}{v} \right\rfloor + 1$ items ($j = 1, \ldots, v$) where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

**Proof of Corollary 2.3.1:** By lemma 2.3 any demand source $M_j$ receives indexed by $n - kv - j + 1$ for $k = 0, 1, \ldots, t$ where $t$ is the integer such that

$$n - tv - j + 1 \geq 1.$$
Thus \( n - tv - j \geq 0 \) and \( t \leq \frac{n - j}{v} \). But \( t \) is the largest integer satisfying this condition, hence \( t = \left[ \frac{n - j}{v} \right] \). Now since \( k = 0, 1, \ldots, t \) items then \( M_j \) receives exactly
\[
\sum_{k=0}^{\left[ \frac{n-1}{v} \right]} 1 = 1 + \left[ \frac{n - j}{v} \right] \text{ items.}
\]
q.e.d.

For example, let \( n = 11, v = 3 \) then

\[ M_1 \] receives \( 1 + \left[ \frac{11 - 1}{3} \right] = 1 + \left[ \frac{10}{3} \right] = 4 \) items

\[ M_2 \] receives \( 1 + \left[ \frac{11 - 2}{3} \right] = 1 + \left[ \frac{9}{3} \right] = 4 \) items

\[ M_3 \] receives \( 1 + \left[ \frac{11 - 3}{3} \right] = 1 + \left[ \frac{8}{3} \right] = 3 \) items

(total 11 items)

The next two lemmas are very useful in the proofs of theorems in subsequent chapters. In the case \( v = 1 \) demand source, lemma 2.4 states that the FIFO issuance of a set of \( n \) items has a greater total field life than another set of \( n \) items also issued by FIFO whenever the initial ages of each member of the second set of items is at least as great as its corresponding member in the first set. This result holds under fairly general \( L(S) \). Lemma 2.5 generalizes lemma 2.4 to the case of more than one demand source. We have stated the \( v = 1 \) and \( v \geq 1 \) cases separately since the proof of lemma 2.5 becomes quite simple once lemma 2.4 has been proved.
Lemma 2.4: Let $L(S)$ be a continuous nonincreasing function with $L^-(S) \geq -1$ for all $S$ such that $0 < S \leq S_0$. Let $\nu = 1$. Consider two sets of $n$ items which the following characteristics:

$$I = \{S_1, \ldots, S_n | S_i < S_{i+1} < S_0 \text{ for all } i = 1, \ldots, n - 1\}$$

$$II = \{\hat{S}_1, \ldots, \hat{S}_n | \hat{S}_i < \hat{S}_{i+1} < S_0 \text{ for all } i = 1, \ldots, n - 1\}$$

and $S_i \leq \hat{S}_i$ for all $i = 1, \ldots, n$. Denote by $Q_F$ and $\hat{Q}_F$ the total field life by FIFO issuance of the items in sets I and II respectively. Then $Q_F \geq \hat{Q}_F$.

Proof of Lemma 2.4: The proof is by induction.

Let $n = 1$. Since $L(\cdot)$ is nonincreasing then

$$Q_F = L(S_1) \geq L(\hat{S}_1) = \hat{Q}_F.$$

Now assume the lemma is true for $n = k$ items and it will be proved true for $n = k + 1$ items. Let $x$ and $\hat{x}$ denote the total field life obtained by the FIFO issuance of the first $k$ items issued (i.e., the $k$ oldest items) of sets I and II respectively. Thus by the inductive assumption $x \geq \hat{x}$. If $x = \hat{x}$, then

$$Q_F = L(S_1 + x) + x \geq L(\hat{S}_1 + \hat{x}) + \hat{x} = \hat{Q}_F$$

since $L(\cdot)$ is nonincreasing. If $x > \hat{x}$, then since $L(\cdot)$ is continuous nonincreasing and $L^-(S) \geq -1$ for $S \leq S_0$ and by lemma 2.1, $S_1 + x < S_0$ and $\hat{S}_1 + \hat{x} < S_0$ then

$$\frac{L(S_1 + x) - L(S_1 + \hat{x})}{x - \hat{x}} \geq -1.$$
implies

\[ Q_F = L(S_1 + x) + x \geq L(S_1 + \hat{x}) + \hat{x} \geq L(S_1 + \hat{x}) + \hat{x} = \hat{Q}_F. \]

And by induction the lemma is proved. q.e.d.

**Lemma 2.5**: Let \( L(S) \) and sets I and II have the same properties as in lemma 2.4. Let \( v \geq 1 \). Denote by \( Q_{F, n, v} \) and \( \hat{Q}_{F, n, v} \) the total field life by FIFO issuance of the \( n \) items to the \( v \) demand sources with the items from sets I and II respectively. Then

\[ Q_{F, n, v} \geq \hat{Q}_{F, n, v}. \]

**Proof of Lemma 2.5**: Using lemma 2.3 the following table for the assignment of the \( n \) items to the \( v \) demand sources can be constructed

<table>
<thead>
<tr>
<th>Demand Source</th>
<th>Set I</th>
<th>Set II</th>
<th>Set I</th>
<th>Set II</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 )</td>
<td>([S_n, \ldots, S_{n-kv}])</td>
<td>([\hat{S}<em>n, \ldots, \hat{S}</em>{n-kv}])</td>
<td>( x_1 )</td>
<td>( \hat{x}_1 )</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>([S_{n-1}, \ldots,)</td>
<td>([\hat{S}_{n-1}, \ldots,)</td>
<td>( x_2 )</td>
<td>( \hat{x}_2 )</td>
</tr>
<tr>
<td></td>
<td>( S_{n-kv-1} ])</td>
<td>( \hat{S}_{n-kv-1} ])</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( M_v )</td>
<td>([S_{n-v+1}, \ldots,)</td>
<td>([\hat{S}_{n-v+1}, \ldots,)</td>
<td>( x_v )</td>
<td>( \hat{x}_v )</td>
</tr>
<tr>
<td></td>
<td>( S_{n-(k+1)v+1} ])</td>
<td>( \hat{S}_{n-(k+1)v+1} ])</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where the subscripts on the \( S \)'s are such that \( n - kv - i \geq 1 \) for all \( k = 0, 1, \ldots \) and \( i = 0, 1, \ldots, v - 1 \), i.e., the inventory is...
exhausted. Note that lemma 2.3 tells us that the subscripts on the items for a particular demand source are the same for both sets I and II. Hence the items assigned to $M_j$ from sets I and II obey the conditions

(i) $S_{n-kv-j+1} \leq \hat{S}_{n-kv-j+1}$ for all $k = 0, 1, 2, \ldots$

(ii) there are the same number of items assigned to $M_j$ from set I as there is from set II.

But these conditions hold for all $M_j$, $j = 1, \ldots, v$. Hence by lemma 2.4 $x_j \geq \hat{x}_j$ for all $j = 1, \ldots, v$ and

$$Q_{F_{n,v}} = \sum_{j=1}^{v} x_j \geq \sum_{j=1}^{v} \hat{x}_j = \hat{Q}_{F_{n,v}}.$$ 

q.e.d.

Zehna [11] stated that for $L(S)$ concave and differentiable with $0 \geq L'(S) \geq -1$ for $0 \leq S \leq S_o$ when there are $v = 2$ demand sources and either $n = 3$ or $n = 4$ items in the stockpile, then FIFO is the optimal issuing policy. He did not present the proof of this statement and since these results are essential to the proof of Theorem 2.6, the proof is presented here.

**Lemma 2.6:** Let $L(S)$ be a concave function with $L''(S) \geq -1$ for $0 < S \leq S_o$. Let $v = 2$ and $n = 3$ or $n = 4$. Then FIFO is the optimal issue policy.
Proof of Lemma 2.6: The proof will proceed by the elimination of all non-FIFO policies. By Lieberman [9] Theorem 3, it is only necessary to consider allocations to each demand source, \( M_1 \) and \( M_2 \), which are FIFO within the allocation. Hence the possible policies are:

\[
\begin{align*}
\text{n = 4} & \\
1 &= [S_4 ; S_3, S_2, S_1] \\
2 &= [S_4, S_3 ; S_2, S_1] \\
3 &= [S_4, S_2 ; S_3, S_1] \\
4 &= [S_4, S_1 ; S_3, S_2] \\
5 &= [S_3 ; S_4, S_2, S_1] \\
6 &= [S_2 ; S_4, S_3, S_1] \\
7 &= [S_1 ; S_4, S_3, S_2] \\
\text{n = 3} & \\
A &= [S_3 ; S_2, S_1] \\
B &= [S_2 ; S_3, S_1] \\
C &= [S_1 ; S_3, S_2]
\end{align*}
\]

Policies 1, 5, and A may be eliminated immediately since they are not feasible (they contradict assumption (6)'). A general result is now proved which eliminates policies 4, 7, and C. Consider the two policies \( I = [A, S_2 ; B, S_1] \) and \( II = [A, S_1 ; B, S_2] \) where A and B are any items in FIFO order. Let \( y \) and \( x \) denote the total field life of the items represented by A and B respectively. It will be shown that for \( x \geq y \geq 0 \)

\[
Q_I = L(S_2 + y) + y + L(S_1 + x) + x \geq L(S_1 + y) + y + L(S_2 + x) + x = Q_{II} .
\]

(2.2.17)
If $x = y$ then (2.2.17) holds with equality. If $x > y$ then since $S_2 + y < S_0$ by lemma 2.1 and since $L(\cdot)$ is concave for $S \leq S_0$ then

$$\frac{L(S_2 + y) - L(S_1 + y)}{S_2 - S_1} \geq \frac{L(S_2 + x) - L(S_1 + x)}{S_2 - S_1}.$$ 

With $S_2 > S_1$ we then have $L(S_2 + y) + L(S_1 + x) \geq L(S_2 + x) + L(S_1 + y)$ and (2.2.17) holds for all $x \geq y$.

Now in policy 4 $A = S_4$, $B = S_3$ and $L(S_4) \leq L(S_3)$

in policy 7 $A = \emptyset$, $B = S_4$, $S_3$ and $0 < L(S_4) + L(S_3 + L(S_4))$

in policy C $A = \emptyset$, $B = S_3$ and $0 < L(S_3)$.

We apply the above result for $q_1 \geq q_{11}$ and see that policy 4 is dominated by policy 3, policy 7 is dominated by policy 6, and policy C is dominated by policy B.

Thus for $n = 3$, A and C have been eliminated, hence B is optimal but by lemma 2.3, policy B is FIFO.

For $n = 4$ by lemma 2.3, policy 3 is FIFO. It is necessary to show that policy 3 dominates policies 2 and 6. That is, show

$$L(S_4) + L(S_2 + L(S_4)) + L(S_3) + L(S_1 + L(S_3))$$

$$\geq L(S_4) + L(S_3 + L(S_4)) + L(S_2) + L(S_1 + L(S_2))$$

(2.2.18)

and

$$L(S_4) + L(S_2 + L(S_4)) + L(S_3) + L(S_1 + L(S_3))$$

$$\geq L(S_4) + L(S_3 + L(S_4)) + L(S_1 + L(S_4) + L(S_3 + L(S_4))) + L(S_2).$$

(2.2.19)
For (2.2.18) since \( S_3 + L(S_4) < S_o \) and \( S_1 + L(S_2) < S_o \) by lemma 2.1 and since \( L(\cdot) \) is concave for \( S \leq S_o \) then

\[
\frac{L(S_3 + L(S_4)) - L(S_2 + L(S_4))}{S_3 - S_2} \leq \frac{L(S_3) - L(S_2)}{S_3 - S_2}
\]

implies

\[
L(S_3 + L(S_4)) + L(S_2) \leq L(S_2 + L(S_4)) + L(S_3) .
\] (2.2.20)

Furthermore since \( L(\cdot) \) is nonincreasing

\[
L(S_4) + L(S_1 + L(S_2)) \leq L(S_4) + L(S_1 + L(S_3)) .
\] (2.2.21)

Combining (2.2.20) and (2.2.21) we obtain (2.2.18). For (2.2.19) since \( L(\cdot) \) is nonincreasing and by lemma 2.2

\[
L(S_3) \leq L(S_4) + L(S_3 + L(S_4))
\]

implies

\[
L(S_1 + L(S_3)) \geq L(S_1 + L(S_4) + L(S_3 + L(S_4))).
\] (2.2.22)

Combining (2.2.22) and (2.2.20) we obtain (2.2.19). Hence policy 3, which is FIFO, dominates policies 2 and 6. FIFO is optimal for \( n = 4 \).

q.e.d.

In the next section we will use some of the foregoing lemmas and corollaries to prove some interesting results on optimal inventory depletion policies when \( v > 1 \).
2.3 Bounds on the Optimal Policy

As Zehna [11] points out, the extension of the results for \( \nu = 1 \) to the case \( \nu \geq 1 \) when \( L(S) \) is concave nonincreasing is not a simple matter. He gives a counterexample to show that such an extension is not possible in general. However, for the particular case \( L(S) = aS + b \), \((b > 0 > a > -1)\), for \( 0 \leq S \leq S_o \), the results for \( \nu = 1 \) and \( \nu \geq 1 \) coincide, \textit{viz.} FIFO is optimal in both cases.

Presented below are a set of theorems which provide upper and lower bounds on the optimal policy when \( \nu > 1 \) and \( L(S) \) is concave nonincreasing for \( S \leq S_o \). These bounds for the optimal policy coincide with the bounds for the FIFO policy for the same \( n \) items and \( \nu > 1 \). And since not all policies are included in these bounds, the optimal policy and the FIFO policy are "close" in the sense that the difference between the optimal policy and the FIFO policy cannot exceed the difference between their common upper and lower bounds.

Since \( L(S) \) takes the same form for all of the theorems and lemmas of this section we will say:

\( L(S) \) has property \( \Omega \) if \( L(S) \) is a concave nonincreasing function for all \( S \) such that \( 0 \leq S \leq S_o \) and \( L^-(S) \geq -1 \) for \( 0 < S \leq S_o \).

Theorem 2.1: Let \( L(S) \) have property \( \Omega \). Let \( \nu > 1 \). Denote by \( Q^*_{n,\nu} \), the total field life obtained from the \( n \) items in the stockpile when the number of demand sources is \( \nu \) and when an optimal issuing policy is followed. Then

\[
Q^*_{n,\nu} \leq Q^*_{n,\nu+1} \text{ for any } \nu = 1, \ldots, n-1.
\]
Proof of Theorem 2.1: Let the optimal policy which achieves $Q_{n,v}^*$ be denoted by $A = [S_{i_{11}}, S_{i_{12}}, \ldots, S_{i_{1j_1}}, S_{i_{21}}, \ldots, S_{i_{2j_2}}; \ldots; S_{i_{v1}}, \ldots, S_{i_{vj_v}}]$. Now since $n > v$ then at least one of the subscripts $j_1, j_2, \ldots, j_v$ is an integer strictly greater than 1 (i.e., at least one demand source must have two or more items assigned to it). Let us say $j_k > 1$. Then the total field life contributed by demand source $M_k$ to the total field life $Q_{n,v}^*$ is given by

$$Q_{M_k} = L(S_{i_{kl}}) + L(S_{i_{k2}} + L(S_{i_{k1}})) + \cdots + L(S_{i_{kj_k}} + L(S_{i_{kl}}) + \cdots).$$

(2.3.1)

Now consider the following issuing policy $B_{v+1}$ for the case of $v + 1$ demand sources:

Issue the same items in the same order to all demand sources $M_i$ for all $i \neq k$ as are issued to them when policy $A$ is followed.

Issue item $S_{i_{kj_k}}$ to demand source $M_{v+1}$ and issue the remaining $j_k - 1$ items to demand source $M_k$ in the same order as under policy $A$.

Let $Q_{B_{n,v+1}}$ denote the total field life obtained from policy $B_{v+1}$. We will show $Q_{B_{n,v+1}} \geq Q_{n,v}^*$.

Now the total field life contributed by demand sources $M_i$ for all $i \neq k$ is the same for both policy $A$ and policy $B_{v+1}$. Hence we only need to examine the field life contributed by $M_k$ and $M_{v+1}$. Let

$$x = L(S_{i_{kl}}) + L(S_{i_{k2}} + L(S_{i_{k1}})) + \cdots + L(S_{i_{kj_k-1}} + L(S_{i_{kl}}) + \cdots).$$

(2.3.2)
then

\[ Q_{M_k} = x + L(S_{k,j_k} + x) \]  \hspace{1cm} (2.3.3)

by using (2.3.1). We must show

\[ x + L(S_{k,j_k}) \geq Q_{M_k} \]  \hspace{1cm} (2.3.4)

but \( L(*) \) is nonincreasing hence \( L(S_{k,j_k}) \geq L(S_{k,j_k} + x) \) since \( x > 0 \). Therefore (2.3.4) holds. But \( x \) is the field life contributed by \( M_k \) and \( L(S_{k,j_k}) \) is the field life contributed by \( M_{\nu+1} \) under policy \( B_{\nu+1} \).

Therefore

\[ Q_{B_{n,\nu+1}} = \sum_{i=1}^{\nu} Q_{M_i} + x + L(S_{i,j_k}) \]

\[ \geq \sum_{i=1}^{\nu} Q_{M_i} \]

\[ = Q^*_{n,\nu}. \]

Now \( Q^*_{n,\nu+1} \) is the optimal policy for \( \nu + 1 \) demand sources, hence

\[ Q^*_{n,\nu+1} \geq Q_{B_{n,\nu+1}} \geq Q^*_{n,\nu}. \]

q.e.d.
Theorem 2.2: Let \( L(S) \) have property \( \Omega \). Let \( v \geq 1 \). Then when
the FIFO issuing policy is followed

\[
Q_{n,v}^F \leq Q_{n,v+1}^F \quad \text{for any } v = 1, \ldots, n - 1.
\]

Proof of Theorem 2.2: By lemma 2.1 (applied to each demand source
separately) the FIFO issuance of the \( n \) items in the stockpile results
in each item having positive field life on issuance under either \( F_{n,v} \)
or \( F_{n,v+1} \). Furthermore in any FIFO ordering of the \( n \) items for any
\( v \leq n \) there are then exactly \( n \) terms \( L(S_1 + \cdots) \) for
\( i = 1, \ldots, n \). Hence there is a one - one correspondence between the
terms in \( Q_{n,v}^F \) and \( Q_{n,v+1}^F \) where this correspondence is established
on the basis of the index letter \( i \) for \( L(S_i + \cdots) \) and
\( i = 1, \ldots, n \). Now using lemma 2.3

\[
Q_{n,v}^F = L(S_n) + L(S_{n-v} + L(S_n)) + \cdots \quad M_1
\]

\[
+ L(S_{n-1}) + L(S_{n-v-1} + L(S_n)) + \cdots \quad M_2
\]

\[
+ \quad \vdots \quad \vdots \quad \vdots
\]

\[
+ L(S_{n-v+1}) + L(S_{n-2v+1} + L(S_{n-v+1})) + \cdots \quad M_v
\]

(2.3.5)
\[ Q_{n,v+1} = L(S_n) + L(S_{n-v-1} + L(S_n)) + \cdots + L(S_{n-1}) + L(S_{n-v-2} + L(S_n)) + \cdots + \]
\[ + L(S_{n-v}) + L(S_{n-2v} + L(S_{n-v})) + \cdots + L(S_{n-v+1}) + L(S_{n-2v} + L(S_{n-v})) + \cdots + L(S_{n}) + L(S_{n-2v-1} + L(S_{n-v})) + \cdots + L(S_{n-v+1}) \]
\[ (2.3.6) \]

Now choose any \( L(S_i + x_i) \) for \( i = 1, \ldots, n \) belonging to \( Q_{n,v} \) and the corresponding \( L(S_i + y_i) \) for \( i = 1, \ldots, n \) belonging to \( Q_{n,v+1} \). We will show

\[ L(S_i + y_i) \geq L(S_i + x_i) \quad \text{for all } i = 1, \ldots, n ; \]

but since \( L(\cdot) \) is nonincreasing, it is only necessary to show \( x_i \geq y_i \) for all \( i = 1, \ldots, n \).

**Case 1:** \( i \in \{ n - v, n - v + 1, \ldots, n \} \)

Then \( y_i = 0 \) and since \( x_i \geq 0 \) we have \( x_i \geq y_i \) \[ (2.3.7) \]

**Case 2:** \( 1 \leq i \leq n - v - 1 \)

Then

\[ x_i = L(S_{i+t_v}) + L(S_{i+(t-1)v} + L(S_{i+t_v})) + \cdots + L(S_{i+v} + \cdots) \]
\[ y_i = L(S_i + s(v+1)) + L(S_i + (s-1)(v+1)) + L(S_i + s(v+1)) + \cdots + L(S_i + v + 1) \]

where these equations follow from lemma 2.3. Now \( s \leq t \) since by lemma 2.3 every \( \nu \)th item is assigned to the \( j \)th demand source (say \( M_j \) receives \( S_{i_j} \)) under \( Q_{F, n} \) and every \( (v + 1) \)th item is assigned under \( Q_{F, n} \). Hence when the \( F_{n, v} \) policy is followed, the demand source which receives \( S_{i_j} \) will have already received more (or equal) items than the demand source which receives \( S_{i_j} \) under \( F_{n, v} \). Hence \( x_i \) and \( y_i \) have the following policies

\[ F_{x_i} = [S_{i+t}, S_{i+(t-1)v}, \ldots, S_{i+v}] \]

\[ F_{y_i} = [S_{i+s(v+1)}, S_{i+(s-1)(v+1)}, \ldots, S_{i+v+1}] \]

But

\[ i + v < i + v + 1 \Rightarrow S_{i+v} < S_{i+v+1} \]

\[ i + 2v < i + 2(v + 1) \Rightarrow S_{i+2v} < S_{i+2(v+1)} \]

\[ \vdots \]

\[ i + sv < i + s(v + 1) \Rightarrow S_{i+sv} < S_{i+s(v+1)} \quad (2.3.8) \]

Now consider the FIFO policy of issuing the \( s \) items \( S_{i+v}, \ldots, S_{i+sv} \) and denote this policy by \( A \) i.e.,

\[ A = [S_{i+sv}, S_{i+(s-1)v}, \ldots, S_{i+v}] \]
Now by (2.3.8) and lemma 2.4

\[ Q_A \geq y_1 \]

where \( Q_A \) is the field life from policy A. Furthermore, since \( s \leq t \) then by lemma 2.2

\[ Q_A \leq x_1 \]

Thus \( x_1 \geq y_1 \). And since the choice of \( L(S_i + x_i) \) was arbitrary for \( 1 \leq i \leq n - v - 1 \)

\[ x_i \geq y_i \quad \text{for all } i \text{ with } 1 \leq i \leq n - v - 1. \]

\( (2.3.9) \)

Combining (2.3.7) and (2.3.9) we have

\[ x_i \geq y_1 \quad \text{for all } i = 1, \ldots, n, \]

therefore

\[ L(S_i + x_i) \leq L(S_i + y_i) \quad \text{for all } i = 1, \ldots, n \]

and

\[ Q_{n,v}^n = \sum_{i=1}^{n} L(S_i + x_i) \leq \sum_{i=1}^{n} L(S_i + y_i) = Q_{n,v+1}^n. \]

q.e.d.
Theorem 2.3: Let $\mathbf{L}(S)$ have property $\mathcal{P}$. Let $\nu \geq 1$. If one item is added to the initial stockpile of $n$ items prior to the issuance of any of the items, then

$$Q_{F,n,\nu} \leq Q_{F,n+1,\nu}$$

when the FIFO issuing policy is followed.

Proof of Theorem 2.3: Before beginning the proof it should be noted that for $\nu = 1$, this theorem reduces to lemma 2.2.

Let $S_{n+1}$ denote the initial age of the new item. We consider three cases:

Case 1. $S_{n+1} < S_1$ and say $S_1$ is assigned to $M_j$ for some $j \in \{1, \ldots, \nu\}$. Then by lemma 2.3

$$Q_{F,n,\nu} = \mathbf{L}(S_n) + \mathbf{L}(S_{n-\nu} + \mathbf{L}(S_n)) + \cdots$$

$$+ \mathbf{L}(S_{n-1}) + \mathbf{L}(S_{n-\nu-1} + \mathbf{L}(S_{n-1})) + \cdots$$

$$+ \mathbf{L}(S_{n-2}) + \mathbf{L}(S_{n-\nu-2} + \mathbf{L}(S_{n-2})) + \cdots$$

$$+ \cdots + \mathbf{L}(S_{n-j+1}) + \cdots + \mathbf{L}(S_1 + \mathbf{L}(S_{n-j+1}) + \cdots)$$

$$+ \cdots + \mathbf{L}(S_{n-\nu+1}) + \cdots$$  \hspace{1cm} (2.3.12)
\[ Q_{F_{n+1, \nu}} = L(S_n) + \cdots \]
\[ + L(S_{n-1}) + \cdots \]
\[ + \cdots + L(S_{n-j+1}) + \cdots \]
\[ + L(S_{n-j}) + \cdots \]
\[ + \cdots + L(S_{n+1} + L(S_{n-j+1}) + \cdots ) \]
\[ + L(S_{n-j}) + \cdots \]
\[ + \cdots + L(S_{n-\nu+1}) + \cdots . \]  
(2.3.13)

and

\[ Q_{F_{n+1, \nu}} - Q_{F_{n, \nu}} = L(S_{n+1} + L(S_{n-j}) + \cdots ) > 0 \]

by lemma 2.1. Therefore \( Q_{F_{n+1, \nu}} > Q_{F_{n, \nu}} \) for this case.

**Case 2**

\( S_n < S_{n+1} < \sigma_0 \)

\( Q_{F_{n, \nu}} \) is still given by (2.3.12); however \( Q_{F_{n+1, \nu}} \) now becomes

\[ Q_{F_{n+1, \nu}} = L(S_{n+1}) + L(S_{n-\nu+1} + L(S_{n+1})) + \cdots \]
\[ + L(S_n) + L(S_{n-\nu} + L(S_n)) + \cdots \]
\[ + L(S_{n-1}) + L(S_{n-\nu-1} + L(S_{n-1})) + \cdots \]
\[ + \cdots + L(S_{n-j+1}) + L(S_{n-\nu-j+1} + L(S_{n-j+1})) + \cdots \]
\[ + \cdots + L(S_{n-\nu+2}) + L(S_{n-2\nu+2} + L(S_{n-\nu+2})) + \cdots . \]
And
\[ Q_{F_{n+1, \nu}} - Q_{F_{n, \nu}} = L(S_{n+1}) + L(S_{n-\nu+1} + L(S_{n+1})) + \cdots \\
- [L(S_{n-\nu+1}) + L(S_{n-2\nu+1} + L(S_{n-\nu+1})) + \cdots] \geq 0 \]

by lemma 2.2 since \( Q_{F_{n+1, \nu}} - Q_{F_{n, \nu}} \) represents the difference of the two policies

\[ A = [S_{n+1}, S_{n-\nu+1}, S_{n-2\nu+1}, \ldots, S_{n-k\nu+1}, \ldots] \]

\[ B = [S_{n-\nu+1}, S_{n-2\nu+1}, \ldots, S_{n-k\nu+1}, \ldots] \]

where \( B \) has the same items as \( A \) after \( A \) has issued its first item \( S_{n+1} \).

Therefore

\[ Q_{F_{n+1, \nu}} \geq Q_{F_{n, \nu}} \]

**Case 3** \( S_i < S_{n+1} < S_{i+1} \) for any \( i = 1, \ldots, n-1 \)

Then for items \( S_n \) down through \( S_{i+1} \) the two total field life functions are identical. Let \( x_j \) denote the total field life contributed by \( M_j \) \((j = 1, \ldots, \nu)\) down through item \( S_{i+1} \). Without loss of generality let \( S_{n+1} \) be assigned to \( M_j \). Now we can rewrite (2.3.12) in the following manner:
\[ Q_{n,v} = x_1 + L(S_{n-tv} + x_1) + \cdots \]
\[ + x_2 + L(S_{n-tv-1} + x_2) + \cdots \]
\[ + \cdots \]
\[ + x_j + L(S_{n-(t-1)v-j+1} + x_j) + \cdots \]
\[ + x_{j+1} + L(S_{n-(t-1)v-j} + x_{j+1}) + \cdots \]
\[ + \cdots \]
\[ + x_v + L(S_{n-tv+v+1} + x_v) + \cdots \]

where \( n - tv < i + 1 \), and \( S_{n-(t-1)v-j+1} = S_1 \) when numbering from above.

And

\[ Q_{n+1,v} = x_1 + L(S_{n-tv+1} + x_1) + \cdots \]
\[ + x_2 + L(S_{n-tv} + x_2) + \cdots \]
\[ + \cdots \]
\[ + x_j + L(S_{n+1} + x_j) + \cdots \]
\[ + x_{j+1} + L(S_{n-(t-1)v-j+1} + x_{j+1}) + \cdots \]
\[ + x_{j+2} + L(S_{n-(t-1)v-j} + x_{j+2}) + \cdots \]
\[ + \cdots \]
\[ + x_v + L(S_{n-(t-1)v-v+2} + x_v) + \cdots \]

By induction we will prove \( Q_{n+1,v} \geq Q_{n,v} \) for this case. Let \( S_1 = S_1 \)

then \( S_{n-(t-1)v-j+1} = S_1 \) and

43
\[ Q_{n+1,v} - Q_{n,v} = L(S_{n+1} + x_j) + L(S_1 + x_{j+1}) - L(S_1 + x_j) . \quad (2.3.14) \]

Now by the definition of the \( x_i \)'s and since FIFO is being followed then by lemma 2.3

\[ x_j \leq x_{j+1} \leq x_{j+2} \leq \ldots \leq x_v \leq x_1 \leq \ldots \leq x_{j-1} . \quad (2.3.15) \]

If \( x_j = x_{j+1} \) then (2.3.14) has

\[ Q_{n+1,v} - Q_{n,v} = L(S_{n+1} + x_j) > 0 \quad \text{and} \quad Q_{n+1,v} > Q_{n,v} . \]

If \( x_j < x_{j+1} \), since \( S_{n+1} + x_{j+1} < S_o \) by lemma 2.1 and since \( L(\cdot) \) is concave for \( S \leq S_o \) then

\[
\frac{L(S_{n+1} + x_{j+1}) - L(S_{n+1} + x_j)}{x_{j+1} - x_j} \leq \frac{L(S_1 + x_{j+1}) - L(S_1 + x_j)}{x_{j+1} - x_j}
\]

implies

\[
L(S_{n+1} + x_j) + L(S_1 + x_{j+1}) \geq L(S_{n+1} + x_{j+1}) + L(S_1 + x_j)
\]

\[
> L(S_1 + x_j) .
\]

Therefore in (2.3.14) we have

\[ Q_{n+1,v} > Q_{n,v} . \]
Now assume $Q_{F_{n+1},v} - Q_{F_{n},v} \geq 0$ for $S_1 = S_k$ (i.e., $S_k < S_{n+1} < S_{k+1}$) and it will be proved true for $S_1 = S_{k+1}$ (i.e., $S_{k+1} < S_{n+1} < S_{k+2}$) for $k = 1, 2, \ldots, n - 2$.

Now for $S_1 = S_{k+1}$ we have

$$Q_{F_{n+1},v} - Q_{F_{n},v} = L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_{m+1} + \cdots)$$

$$- \left[ L(S_{k+1} + x_j) + L(S_k + x_{j+1}) + \cdots + L(S_{m} + \cdots) \right]$$

(2.3.16)

where we assume that $S_1$ is assigned to $M_{m+1}$ under $F_{n+1},v$.

Now using (2.3.15) and since $S_{n+1} + x_{j+1} < S_o$ by lemma 2.1 and $L(\cdot)$ is concave for $S \leq S_o$ then if $x_{j+1} > x_j$, then

$$\frac{L(S_{n+1} + x_{j+1}) - L(S_{n+1} + x_j)}{x_{j+1} - x_j} \leq \frac{L(S_{k+1} + x_{j+1}) - L(S_{k+1} + x_j)}{x_{j+1} - x_j}.$$

This implies

$$L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) \geq L(S_{k+1} + x_j) + L(S_{n+1} + x_{j+1}).$$

(2.3.17)

If $x_{j+1} = x_j$, then (2.3.17) holds with equality. Now adding the same quantities to both sides of (2.3.17) we obtain
\[ L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots) \]
\[ \geq L(S_{k+1} + x_j) + L(S_{n+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots). \]

(2.3.18)

But by the inductive assumption

\[ L(S_{n+1} + x_{j+1}) + L(S_k + x_{j+2}) + L(S_{k-1} + x_{j+3}) + \cdots + L(S_1 + x_{m+1} + \cdots) \]
\[ \geq L(S_k + x_{j+1}) + L(S_{k-1} + x_{j+2}) + \cdots + L(S_1 + x_m + \cdots) \]

(2.3.19)

where the left side of (2.3.19) is just the right side of (2.3.18) after omitting \( L(S_{k+1} + x_j) \). Hence we can write the new inequality using (2.3.18) and (2.3.19)

\[ L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots) \]
\[ \geq L(S_{k+1} + x_j) + L(S_{k} + x_{j+1}) + L(S_{k-1} + x_{j+2}) + \cdots + L(S_1 + x_m + \cdots). \]

(2.3.20)

However, (2.3.20) is precisely what we want to show for (2.3.16). Therefore \( Q_{n+1, \nu}^{*} - Q_{n, \nu}^{*} \geq 0 \). And by induction case 3 has been proved.

Combining cases 1, 2, and 3 the theorem is proved.

\textit{q.e.d.}

\textbf{Theorem 2.4:} Let \( L(S) \) have property \( \Omega \). Let \( \nu \geq 1 \). If one item is added to the initial stockpile of \( n \) items prior to the issuance of any of the items, then

\[ Q_{n, \nu}^{*} \leq Q_{n+1, \nu}^{*} \]

when an optimal issuing policy is followed.
Proof of Theorem 2.4: Let $\psi_n$ denote the optimal policy which yields total field life $Q^*_{n,\nu}$. Let $A$ be the policy where the original $n$ items are issued according to $\psi_n$ and the new item, $S_{n+1}$, is issued last, to the demand source which first finishes its consumption of its items under $\psi_n$. Then if the field life of item $S_{n+1}$ at the time of issuance is denoted by $x$ we have $x \geq 0$, and

$$Q^*_{n,\nu} \leq Q^*_{A,n+1,\nu} \leq Q^*_{n+1,\nu}.$$

q.e.d.

Before presenting the next theorem, we should point out an interesting extension of Theorems 2.3 and 2.4.

**Corollary 2.4.1:** Let $L(S)$ have property $\Omega$. Let $\nu \geq 1$. If $M \geq 1$ items are added to the initial stockpile of $n$ items prior to the issuance of any of the items, then

(i) $Q^*_{F,n+M,\nu} \geq Q^*_{F,n,\nu}$ if the FIFO issuing policy is followed

(ii) $Q^*_{n+M,\nu} \geq Q^*_{n,\nu}$ if an optimal issuing policy is followed.

**Proof of Corollary 2.4.1:** We will just prove (i) since the proof for (ii) follows mutatis mutandis.

In Theorem 2.3 we have already proved the corollary true for $M = 1$. Assume the corollary is true for $M > 1$ and consider adding $M + 1$ items to the stockpile. Ignoring item $S_{M+1}$ temporarily, the total field life of the remaining items satisfies $Q^*_{F,n+M,\nu} \geq Q^*_{F,n,\nu}$ by the
inductive hypothesis. Then adding \( S_{M+1} \) can only increase the total field life by \( M = 1 \) hence by Theorem 2.3

\[
Q_{F_{n+M+1}, v} \geq Q_{F_{n+M}, v} \geq Q_{F_n, v}.
\]

q.e.d.

Theorem 2.5: Let \( L(S) \) have property \( \emptyset \). Let \( v \geq 1 \). If \([\frac{1}{2}(n + 1)] \leq v \leq n\), then any feasible policy which assigns more than two items to any demand source has a lower total field life than some policy which assigns at most two items to each demand source.

Before beginning the proof of this theorem two things should be pointed out. First, although the theorem doesn't explicitly state the improved policy, the proof does state it. Second, if we call the set "G" the set of all policies which issue at most two items to each demand source, the theorem states that for any feasible policy for issuing the \( n \) items, there is a member of \( G \) which dominates it. The theorem does not state that this member of \( G \) is a feasible policy.

Indeed, this may not be the case at all. At this point though, it should be noted that FIFO \( \in G \) and by lemma 2.3 FIFO is feasible. Theorem 2.6 will show that of all the policies in \( G \), FIFO maximizes the total field life for the \( n \) items; hence FIFO is the optimal policy.

Proof of Theorem 2.5: Since \( v \) is an integer which is greater than or equal to \( \frac{1}{2} \) the number of items in the stockpile then if \( i \) demand sources have \( k_i > 2 \) items assigned to them, there are at least

\[
\sum_i (k_i - 2)
\] demand sources which have only one item assigned to them (since all demand sources must have at least one item by the initial assignment).
We only need to consider one demand source with \( k_1 > 2 \) items and \( k_1 - 2 \) demand sources with only one item each since the same procedure (following) applies to all other demand sources with \( k_j > 2 \) items assigned to them.

Let \( 1 > 2 \) items be assigned to \( M_k \). In particular let these items be denoted by \( S_{k_1} < S_{k_2} < \cdots < S_{k_1} \). Let \( M_j \) be a demand source with only one item assigned to it.

Let \( \psi = [S_{t_1}^{i}, \ldots, S_{t_1}^{r_2}, S_{t_1}^{r_1}; S_{j_1}^{r_1}] \) be the part of any arbitrary feasible policy which assigns \( S_{t_1}^{i}, \ldots, S_{t_1}^{r_2}, S_{t_1}^{r_1} \) to \( M_k \) and \( S_{j_1}^{r_1} \) to \( M_j \) where \( S_{t_1}^{i}, \ldots, S_{t_1}^{r_2}, S_{t_1}^{r_1} \) is any permutation of the items \( S_{k_1}, \ldots, S_{k_1} \).

We will now show \( S_{j_1}^{r_1} < S_{k_1}^{r_1} \). Assume to the contrary that \( S_{j_1}^{r_1} > S_{k_1}^{r_1} \). Then since \( L(\cdot) \) is nonincreasing \( L(S_{j_1}^{r_1}) \leq L(S_{k_1}^{r_1}) \). We have two cases:

**Case (i):** \( S_{k_1}^{r_2} \neq S_{t_1}^{r_1} \) then \( S_{k_1}^{r_2} \) is issued before item \( S_{t_1}^{r_1} \.

Let \( x \) be the total field life up to but not including the issuance of item \( S_{k_2}^{r_2} \). If \( x = 0 \) then \( L(S_{j_1}^{r_1}) \leq L(S_{k_1}^{r_2}) \) above. If \( x > 0 \) then

\[
\frac{L(S_{k_2}^{r_2} + x) - L(S_{k_2}^{r_2})}{x} \geq -1
\]

implies

\[
L(S_{j_1}^{r_1}) \leq L(S_{k_2}^{r_2}) \leq L(S_{k_2}^{r_2} + x) + x . \tag{2.3.21}
\]
But (2.3.21) says that policy \( \psi \) is infeasible since item \( S_{j_1} \) is consumed prior to \( S_{k_2} \) hence some \( S_{t_j} \) should be assigned to \( M_{j_1} \) rather than \( M_{k_1} \). This result contradicts the hypothesis that \( \psi \) is feasible. Therefore in this case \( S_{j_1} < S_{k_2} \).

**Case (ii):** \( S_{k_2} = S_{t_1} \) then \( S_{k_1} \neq S_{t_1} \) and item \( S_{k_1} \) is issued before item \( S_{t_1} \). Since we are assuming \( S_{j_1} > S_{k_2} \) then \( S_{j_1} > S_{k_1} \) and \( L(S_{j_1}) \leq L(S_{k_1}) \). By the same argument as in case (i) above we obtain

\[
L(S_{j_1}) \leq L(S_{k_1}) \leq L(S_{k_1} + x) + x
\]

and we obtain the contradiction that some \( S_{t_j} \) should be issued to \( M_{j_1} \) rather than \( M_{k_1} \). Hence in this case also \( S_{j_1} > S_{k_1} > S_{k_2} \). Thus

\[
S_{j_1} < S_{k_2} \quad \text{(2.3.22)}
\]

Now from Lieberman [9] Theorem 3 we have that the FIFO issuance of \( S_{k_1}, \ldots, S_{k_l} \) yields a greater total field life than any other permutation such as given by \( S_{t_1}, \ldots, S_{t_1} \) in policy \( \psi \). Therefore if we let policy \( A \) be

\[
A = [S_{k_1}, \ldots, S_{k_2}, S_{k_1}; S_{j_1}]
\]

then

\[
Q_A > Q_\psi
\]
Now policy A may not be a feasible policy; however since we only wish to show that there exists a policy which belongs to $G$ which is better than $\psi$, in the sense of greater total field life, we do not need feasibility for $A$. It will be shown that the policy which belongs to $G$ has field life $Q$ and $Q \geq Q_A$. Thus $Q \geq Q_\psi$.

Now since $S_{j_1} < S_{k_2}$ and $S_{k_1} < S_{k_2}$, we consider two cases.

Case 1: $S_{j_1} < S_{k_1}$

Then policy $B = [S_{k_1}, S_{k_{1-1}}, \ldots, S_{k_2}, S_{j_1}, S_{k_1}]$ results in a greater total field life than policy $A$. The proof of this statement follows:

Let $Q_A$ and $Q_B$ be the total field life from policy $A$ and policy $B$ respectively.

Let $x = L(S_{k_1}) + \ldots + L(S_{k_2}) + L(S_{k_1}) + \ldots$ then

\[
Q_A = x + L(S_{k_1}) + L(S_{j_1})
\]

\[
Q_B = x + L(S_{j_1}) + L(S_{k_1})
\]

we must show $Q_B \geq Q_A$. Now $x > 0$ by lemma 2.1 and $S_{k_1} - S_{j_1} > 0$.

Furthermore by lemma 2.1 $x + S_{k_1} < S_0$ and $L(\cdot)$ is concave for $S \leq S_0$ by hypothesis, thus

\[
\frac{L(S_{k_1} + x) - L(S_{j_1} + x)}{S_{k_1} - S_{j_1}} \leq \frac{L(S_{k_1}) - L(S_{j_1})}{S_{k_1} - S_{j_1}}
\]

51
which implies

\[ L(S_{k_1} + x) + L(S_{j_1}) \leq L(S_{j_1} + x) + L(S_{k_1}) \]

hence \( Q_B \geq Q_A \).

**Case 2** \( S_{k_1} < S_{j_1} \).

Then policy \( C = [S_{k_1}, S_{k_1+1}, \ldots, S_{k_2}, S_{j_1}; S_{k_2}, S_{k_1}] \) results in a greater total field life than policy \( A \).

Let \( y = L(S_{k_1}) + \ldots + L(S_{j_2}) \) then

\[ Q_A = y + L(S_{k_2} + y) + L(S_{k_1} + y + L(S_{k_2} + y)) + L(S_{j_1}) \]

\[ Q_C = y + L(S_{j_1} + y) + L(S_{k_2}) + L(S_{k_1} + L(S_{k_2})) \]

We must show \( Q_C \geq Q_A \).

By lemma 2.2 since \( y > 0 \), \( y + L(S_{k_2} + y) \geq L(S_{k_2}) \). Now since \( L(\cdot) \) is nonincreasing

\[ L(S_{k_1} + y + L(S_{k_2} + y)) \leq L(S_{k_1} + L(S_{k_2})) \quad (2.3.23) \]

By lemma 2.1, \( S_{k_2} + y < S_o \) and since \( L(\cdot) \) is concave for \( S \leq S_o \) and since \( S_{k_2} - S_{j_1} > 0 \) then
\[ \frac{L(S_{k_2} + y) - L(S_{j_1} + y)}{S_{k_2} - S_{j_1}} \leq \frac{L(S_{k_2}) - L(S_{j_1})}{S_{k_2} - S_{j_1}} \]

which implies

\[ L(S_{k_2} + y) + L(S_{j_1}) \leq L(S_{j_1} + y) + L(S_{k_2}) \quad (2.3.24) \]

Upon combining (2.3.23) and (2.3.24) we have proved

\[ Q_C \geq Q_A \]

Note that policy C has reduced the problem by assigning only 1 - 1 items to \( M_k \) and 2 items to \( M_j \). We will now show that there exists a policy D which is better than policy B where D assigns 1 - 1 items to \( M_k \) and 2 items to \( M_j \).

Let \( D = [S_{k_1}, S_{k_1 - 1}, \ldots, S_{k_2}, S_{k_1}, S_j, S_{j_1}] \) where \( S_j < S_{j_1} < S_{k_1} \).

But the same argument as in case 2 above applies since policy B presents the same situation relative to policy D that policy A presents to policy C. Thus \( Q_D \geq Q_B \). Hence in either case \( S_j < S_{j_1} \) or \( S_{k_1} < S_j \), we have a policy \( Q_D \geq Q_A \) or \( Q_C \geq Q_A \) which is better than policy A and which assigns 1 - 1 items to \( M_k \) and 2 items to \( M_j \).

This reduction process continues. If \( i - 1 > 2 \), we consider demand source \( M_{j_2} \) with only one item, \( S_{j_2} \) then

\[ A^* = [S_{k_1}, S_{k_1 - 1}, \ldots, S_{k_2}, S_{k_1}, S_{j_2}] \]

53
where $S_{k_1}^*$ is either $S_{k_1}$ or $S_{j_1}$ of the first iteration. In the same manner that it was shown that $S_{j_1}^* < S_{k_2}$, it is also the case that $S_{j_2} < S_{k_1}^*$ (actually $S_{j_2} < S_{k_2}$ also but this is not necessary at this point). Then policy $A^*$ is dominated by

$$C^* = [S_{k_1}, S_{k_1 - 1}, \ldots, S_{k_4}, S_{j_1}^*; S_{j_2}^*] \text{ if } S_{j_2} > S_{k_1}$$

or

$$D^* = [S_{k_1}, S_{k_1 - 1}, \ldots, S_{k_4}, S_{j_1}^*; S_{j_2}^*; S_{j_2}] \text{ if } S_{j_2} < S_{k_1}$$

which reduces the problem to $1 - 2$ items assigned to $M_x$ and 2 items assigned to $M_y$.

We must now show that it is better to go from an $i = 3$ problem to an $i = 2$ problem and then by reduction the theorem has been proved.

Let $S_{k_1} < S_{k_2} < S_{k_3}$ be the $i = 3$ items assigned to $M_x$ and let $S_{j_1}$ be the single item assigned to $M_y$. Now $S_{j_2} < S_{k_2}$ by the same reasoning as given before.

**Case 1a:** $S_{j_1} > S_{k_1}$ and let $A = [S_{k_3}, S_{k_2}, S_{k_1}; S_{j_1}]$;
$B = [S_{k_3}, S_{j_1}; S_{k_2}, S_{k_1}]$

$$Q_A = L(S_{k_3}) + L(S_{k_2} + L(S_{k_3})) + L(S_{k_1} + L(S_{k_2} + L(S_{k_3}))) + L(S_{j_1})$$

$$Q_B = L(S_{k_3}) + L(S_{j_1} + L(S_{k_3})) + L(S_{k_2}) + L(S_{k_1} + L(S_{k_2})).$$
We must show $Q_B \geq Q_A$. Now by lemma 2.2

$$L(S_{k_3}) + L(S_{k_2} + L(S_{k_3})) \geq L(S_{k_2})$$

therefore since $L(\cdot)$ is nonincreasing

$$L(S_{k_1} + L(S_{k_2} + L(S_{k_3}))) \leq L(S_{k_1} + L(S_{k_2})). \quad (2.3.25)$$

By lemma 2.1 $S_{k_2} + L(S_{k_3}) < S_o$ and since $L(\cdot)$ is concave for $S \leq S_o$

$$\frac{L(S_{k_2} + L(S_{k_3})) - L(S_{j_1} + L(S_{k_3}))}{S_{k_2} - S_{j_1}} \leq \frac{L(S_{k_2}) - L(S_{j_1})}{S_{k_2} - S_{j_1}}.$$

Thus

$$L(S_{k_2} + L(S_{k_3})) + L(S_{j_1}) \leq L(S_{j_1} + L(S_{k_3})) + L(S_{k_2}) \quad (2.3.26)$$

and combining (2.3.25) and (2.3.26) we have $Q_B \geq Q_A$ as desired.

**Case 2a**

$S_{j_1} < S_{k_1}$

Let $C = [S_{k_3}, S_{k_2}, S_{j_1}, S_{k_1}]$ then

$$Q_C = L(S_{k_3}) + L(S_{k_2} + L(S_{k_3})) + L(S_{j_1} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3}))) + L(S_{k_1}).$$
We must show \( Q_C > Q_A \). By lemma 2.1 \( S_{k_1} + L(s_{k_1}) + L(s_{k_2} + L(s_{k_3})) < S_0 \) and since \( L(\cdot) \) is concave \( S \leq S_0 \)

\[
\frac{L(s_{k_1} + L(s_{k_3} + L(s_{k_2} + L(s_{k_3})))) - L(s_{j_1} + L(s_{k_3} + L(s_{k_2} + L(s_{k_3}))))}{s_{k_1} - s_{j_1}} \leq \frac{L(s_{k_1}) - L(s_{j_1})}{s_{k_1} - s_{j_1}}
\]

and

\[
L(s_{k_1} + L(s_{k_3} + L(s_{k_2} + L(s_{k_3})))) + L(s_{j_1}) \leq L(s_{j_1} + L(s_{k_3} + L(s_{k_2} + L(s_{k_3})))) + L(s_{k_1})
\]

(2.3.27)

Hence by (2.3.27), \( Q_C > Q_A \). Now let

\[
D = [s_{k_2}, s_{k_1}, s_{k_2}, s_{j_1}]
\]

the proof that \( Q_D > Q_C \) is the same as given in case la only with the proper subscripts interchanged. Hence \( Q_D > Q_A \), and we have shown a better policy exists where 2 items are assigned to \( M_k \) and 2 items are assigned to \( M_j \).

By reduction, the theorem is proved.

q.e.d.
Theorem 2.6: Let L(S) have property Ω. If \( \left\lfloor \frac{1}{2}(n + 1) \right\rfloor \leq v \leq n \), then FIFO is the optimal issuing policy.

Proof of Theorem 2.6: Note that not only does FIFO \( \in G \) and FIFO is feasible but also that FIFO issues all \( n \) of the items, i.e., none of the items deteriorate to zero in the stockpile.

We will now show that an optimal policy for the conditions given in this theorem also must issue all of the items. This last statement is proved by contradiction. Assume that the optimal policy allows at least one item, say \( S_j \), to expire in the stockpile. Then since \( \left\lfloor \frac{1}{2}(n + 1) \right\rfloor \leq v \leq n \) there is at least one demand source which receives only one item, say \( S_i \). In addition \( S_i < S_j \) or else by lemma 2.1, \( S_j \) would have positive field life upon the consumption of \( S_i \), i.e., \( S_j + L(S_i) < S_0 \), and \( S_j \) would then be issued. Thus assume \( S_i < S_j \). Now by Lieberman [9] Theorem 3 we have

\[
L(S_j) + L(S_i + L(S_j)) \geq L(S_i)
\]

where equality holds only if \( L'(S) = -1 \) over the range of \( S_i \) and \( S_j \), and strict inequality holds at all other times. Therefore letting \( S_j \) deteriorate to zero in the stockpile can't be optimal. And we obtain a contradiction to the assumption of optimality. But \( S_j \) was a general item which deteriorated in the stockpile, thus the contradiction obtained applies to all \( S_j \), and the optimal policy must issue all \( n \) items.

Thus the optimal policy as well as the FIFO policy issues all items in the stockpile. Now in looking at all policies in \( G \) we can restrict
our attention to looking at only those policies which issue all \( n \) items. Let \( A \in G \) be one of these policies and consider any two demand sources \( M_i \) and \( M_j \) under policy \( A \).

**Case 1** \( M_i \) receives \( S_{i_1}, S_{i_2} \) with \( S_{i_1} < S_{i_2} \)

\( M_j \) receives \( S_{j_1}, S_{j_2} \) with \( S_{j_1} < S_{j_2} \)

Then if the four items \( S_{i_1}, S_{i_2}, S_{j_1}, S_{j_2} \) are not assigned to \( M_i \) and \( M_j \) according to FIFO, then the total field life can be increased by a FIFO assignment since by lemma 2.6 FIFO is optimal for \( n = 4, \nu = 2 \).

**Case 2** \( M_i \) receives \( S_{i_1} \)

\( M_j \) receives \( S_{j_1}, S_{j_2} \) with \( S_{j_1} < S_{j_2} \)

Again, if the three items \( S_{i_1}, S_{j_1}, S_{j_2} \) are not assigned to \( M_i \) and \( M_j \) according to FIFO then the total field life can be increased by a FIFO assignment. By lemma 2.6 FIFO is optimal for \( n = 3, \nu = 2 \).

**Case 3** \( M_i \) receives \( S_{i_1} \); \( M_j \) receives \( S_{j_1} \). Then FIFO is obviously optimal (there is only one policy).

Thus the total field life from all demand sources can be improved until every demand source has a FIFO ordering of its items relative to every other demand source. We will call such an ordering a pairwise-FIFO ordering. It should be noted that any other ordering results in a lower total field life hence pairwise-FIFO is optimal.
We must now show that pairwise-FIFO is the same as FIFO for the total assignment of the \( n \) items to the \( v \) demand sources. Assume the items are in pairwise-FIFO order. Now relabel the demand sources such that

\[
\begin{align*}
M_v & \text{ has item } S_n \text{ assigned to it } \\
M_{v-1} & \text{ has item } S_{n-1} \text{ assigned to it } \\
\vdots & \vdots \\
M_p & \text{ has item } S_{n-i+1} \text{ assigned to it } \\
M_1 & \text{ has item } S_{n-v+1} \text{ assigned to it }.
\end{align*}
\]

(2.3.28)

This relabelling is possible since no two of the items \( S_n, \ldots, S_{n-v+1} \) can be assigned to the same demand source under pairwise-FIFO. Now consider the demand source \( M_p \) which has the two items \( S_{n-i+1} \) and \( S_i \) assigned to it, for any \( p = 1, \ldots, v \). We must show that \( S_i = S_{n-v-i+1} \). Then by lemma 2.3 we have a FIFO ordering for the total assignment (since \( p \) was arbitrary). The proof of \( S_i = S_{n-v-i+1} \) is by contradiction. Assume \( S_i \neq S_{n-v-i+1} \).

Case 1 \[ S_i > S_{n-v-i+1} \]

Now from above we know \( S_i < S_{n-v+1} \) and since \( S_i > S_{n-v-i+1} \) there are at most \( (n - v) - (n - v - i + 1) - 1 = i - 2 \) items, with initial life greater than \( S_i \), which are available for assignment to demand.
sources $M_v, \ldots, M_{p+1}, M_{v'}, \ldots, M_{p+1}$ are the first $1 - 1$ demand sources to consume their initial items. Hence some item $S_{j_1} \prec S_{i_1}$ must be assigned to one of these $1 - 1$ demand sources, say demand source $M_{p+j}$ ($j \geq 1$). Then the pairwise ordering for $M_p$ and $M_{p+j}$ is $[S_{n-i+1}, S_{i_1}; S_{n-i+1+j}, S_{j_1}]$; but $S_{n-i+1} < S_{n-i+1+j}$ and $S_{i_1} > S_{j_1}$ is not a FIFO ordering, hence we obtain a contradiction to the assumption of pairwise-FIFO. Therefore $S_{i_1} \not\prec S_{n-v-i+1}$.

Case 2

$S_{i_1} \prec S_{n-v-i+1}$

As shown above (2.3.28) there are $1 - 1$ demand sources with items whose initial ages are greater than $S_{n-i+1}$. And since $S_{i_1} \prec S_{n-v-i+1}$ there are at least $(n - v + 1) - (n - v - i + 1) = i$ items such that $S_{n-v-i+1+j} > S_{i_1}$ for $j = 0, 1, \ldots, i - 1$ and these items are available for issuance to the first $1 - 1$ demand sources requesting items viz. $M_v, \ldots, M_{p+1}$.

Hence there is at least one $S_{n-v-i+1+j}$ which must be issued to one of the $M_t$ where $t = 1, 2, \ldots, i - 1$. But then the issue policy for $M_p$ and $M_t$ is $[S_{n-i+1}, S_{i_1}; S_{n-i+1-k}, S_{n-v-i+1+j}]$ where $k = p - t$. But $S_{n-i+1} > S_{n-i+1-k}$ and $S_{i_1} < S_{n-v-i+1+j}$ is not a FIFO ordering; hence we obtain a contradiction to pairwise-FIFO. Therefore $S_{i_1} \not\prec S_{n-v-i+1}$.

Combining cases 1 and 2 we have $S_{i_1} = S_{n-v-i+1}$ and by lemma 2.3 since $p$ was arbitrary, FIFO is optimal.

q.e.d.
Chapter 3
Addition of Penalty Costs

The next problem of interest will be to examine the implications of adding penalty costs to the model. These penalty costs can be considered as issuing costs or installation costs.

The removal of assumption (4) which states that there are no penalty costs is important not only because it is often the case in practical situations that there is an installation or work-stoppage cost but also because the optimal policy in the model without penalty costs may no longer be optimal when penalty costs are added. This latter point can be seen, for example, in the case where the optimal policy in the case of no penalty costs issues a large number of items whereas some suboptimal policy may issue only a few items. Then if the penalty cost is sufficiently large the policy which was optimal in the no penalty case could easily become the worst policy after subtracting the penalty costs.

Henceforth in this chapter it is assumed that there is a constant penalty cost, $p$, associated with the issuance of each item from the stockpile. Furthermore it is assumed that $p$ is defined in the same units of measure as $L(S)$. 
In previous sections \( Q_{A_1} \) was defined as the total field life obtained from the issuance of \( i \) items in accordance with policy \( A \). This notation will be retained and in addition a return function \( R_{A_1} \) will be used where \( R_{A_1} = Q_{A_1} - ip \). That is, \( R_{A_1} \) is the total return obtained from the issuance of \( i \) items in accordance with policy \( A \); it equals the total field life less the total penalty cost. The objective will be to find a policy which will maximize \( R \) over all possible policies.

It is conceivable that in issuing an item which has positive field life, the net increase (if any) in the total field life may be more than offset by the penalty cost incurred. Because of this event, we will also remove the assumption that an item must be issued if it has positive field life. In its place we will merely assume that any item with zero field life will not be issued. Furthermore we will assume that there is no cost associated with the disposal of items which are not issued.

It should be noted that to start the process, it may no longer be optimal to issue \( \nu \) items to \( M_1, \ldots, M_\nu \). If the optimal policy calls for the issuance of only \( i < \nu \) items, then the \( i \) items would be issued immediately and the process would terminate.

### 3.1 The Case for FIFO

It will be useful to define: for \( j \leq n \)

1. \( A_{j,\nu} \) is any policy of issuing \( j \) items to \( \nu \) demand sources
(ii) $F_{j,v}$ is the policy of issuing the same $j$ items as are issued in (i) to $v$ demand sources by FIFO.

(iii) $F_{(j,v)^*}^*$ is the policy of issuing the youngest $j$ items to $v$ demand sources by FIFO.

In other words, if the FIFO issuance of any $j$ items is

$$F_{j,v} = [S_{1_1}, S_{1_2}, \ldots , S_{1_i} \ldots; S_{v_1}, \ldots, S_{v_{i_v}}]$$

where $S_{k_j} > S_{k_{j+1}}$ for all $k = 1, \ldots, v$ and $j = 1, \ldots, i_k$

then

$A_{j,v}$ is any permutation of the above $S_{k_j}$'s.

Now $F_{(j,v)^*}^*$ does not necessarily include any of the $j$ items of policy $F_{j,v}$.

$$F_{(j,v)^*}^* = [S_{j}, S_{j-v}, \ldots; S_{j-1}, S_{j-v-1}, \ldots; \ldots; S_{j-v+1}, S_{j-2v+1}, \ldots].$$

$R_{A_{j,v}}^*, Q_{A_{j,v}}^*, R_{F_{j,v}}^*, Q_{F_{j,v}}^*, R_{F_{(j,v)^*}}^*$ and $Q_{F_{(j,v)^*}}^*$ are defined with respect to the above policies.

**Lemma 3.1:** Let $L(S)$ be a concave function with $L^-(S) \geq -1$ for $0 < S \leq S_0$ and $L^+(0) \leq 0$. Let $v \geq 1$. If FIFO is the issuing policy which maximizes the total field life for any $j$ items in inventory, then
\[ R_{F, j, \nu} \geq R_{A, j, \nu} \quad \text{for any } j = 1, \ldots, n. \]

Proof of Lemma 3.1: If \( j \leq \nu \) then \( R_{F, j, \nu} = R_{A, j, \nu} \) since exactly \( j \) demand sources receive an item.

If \( \nu < j \leq n \) then since FIFO maximizes the total field life for any \( j \) items (by hypothesis) we have

\[ Q_{F, j, \nu} > Q_{A, j, \nu}. \]

Hence

\[ R_{F, j, \nu} - R_{A, j, \nu} = Q_{F, j, \nu} - j \cdot p - [Q_{A, j, \nu} - j \cdot p] = Q_{F, j, \nu} - Q_{A, j, \nu} \geq 0. \]

And

\[ R_{F, j, \nu} \geq R_{A, j, \nu} \quad \text{for all } j = 1, \ldots, n \]

since \( j \) was arbitrary.

q.e.d.

Lemma 3.2: Let \( L(S) \) be a concave function with \( L^{-}(S) \geq -1 \) for \( 0 < S \leq S_0 \) and \( L^{+}(0) \leq 0 \). Let \( \nu \geq 1. \) Then

\[ R_{F, (j, \nu)}^{*} \geq R_{F, j, \nu} \quad \text{for any } j = 1, \ldots, n. \]
Proof of Lemma 3.2: If $F(j, \nu)^* \simeq F_j, \nu$ (where $\simeq$ means "is the same as") then $R_{F(j, \nu)^*} = R_{F_j, \nu}^*$.

If $F(j, \nu)^* \nleq F_j, \nu$ then $F_j, \nu$ must contain at least one item which has initial age greater than $S_j$ (note that $S_j$ is the oldest item in $F(j, \nu)^*$).

Now by lemma 2.5, $Q_{F(j, \nu)^*} \geq Q_{F_j, \nu}$ hence

$$R_{F(j, \nu)^*} - R_{F_j, \nu}^* = Q_{F(j, \nu)^*} - j \cdot p - [Q_{F_j, \nu}^* - j \cdot p]$$

$$= Q_{F(j, \nu)^*} - Q_{F_j, \nu}^* \geq 0.$$  

And since $j$ was arbitrary $R_{F(j, \nu)^*} \geq R_{F_j, \nu}^*$ for any $j = 1, \ldots, n$. q.e.d.

Using these two lemmas, we obtain the following interesting theorem.

Theorem 3.1: Let $L(S)$ be a concave function with $L^-(S) \geq -1$ for $0 < S \leq S_o$ and $L^+(0) \leq 0$. Let $\nu \geq 1$. If FIFO is the issuing policy which maximizes the total field life for any $j$ items in inventory, then the optimal issuing policy must be one of the $n$ policies

$F(1, \nu)^*, F(2, \nu)^*, \ldots, F(n, \nu)^*.$

Proof of Theorem 3.1: The proof is immediate from lemmas 3.1 and 3.2. q.e.d.
Corollary 3.1: If \( L(S) = aS + b \) where \( b > 0 > a > -1 \), then the
optimal issue policy must be one of the \( n \) policies

\[
F(1,v)*, F(2,v)*, \ldots, F(n,v)*
\]

Proof of Corollary 3.1: By Zehna [11] Theorems 4.1 and 4.3 and
Lieberman [9] Theorem 3, FIFO is optimal for all \( n \) and \( v \), thus by
Theorem 3.1 the above result follows.

q.e.d.

Corollary 3.2: Let \( L(S) \) be concave with \( L^-(S) \geq -1 \) for \( 0 < S < S_0 \)
and \( L^+(0) \leq 0 \). Let \( v = 1 \). Then the optimal policy is one of the
\( n \) policies

\[
F(1,1)*, F(2,1)*, \ldots, F(n,1)*
\]

Proof of Corollary 3.2: Again by Lieberman [9] Theorem 3, FIFO is
optimal for all \( n \); hence by Theorem 3.1 the result follows.

q.e.d.

Thus Theorem 3.1 and Corollaries 3.1 and 3.2 state that is is only
necessary to search \( n \) policies until the optimal \( F(j,v)* \) is found;
then issue the \( j \) newest items by FIFO and discard the remaining
\( n - j \) items without issuing them even if they have positive field life.
Their positive field life is offset by the penalty cost of installation.

In certain cases it is possible to select analytically the optimal
policy from the \( n \) policies \( F(1,v)*, \ldots, F(n,v)* \). These cases
involve placing special restrictions on the derivative of \( L(S) \) and
relating these restrictions to the penalty cost.
Theorem 3.2: Let \( L(S) \) be a concave function with \( L^-(S) \geq -1 + \frac{1}{K} \) for \( 0 < S \leq S_o \), where \( K > 1 \) is any finite real number and with \( L^+(0) \leq 0 \). Let \( \nu \geq 1 \). If

\[
0 < p \leq \left( \frac{1}{K} \right)^\nu L(S_n)
\]

and if FIFO maximizes the total field life \( Q^*_F(i, \nu) \) for all \( i \) and \( \nu \), then \( F(n, \nu)^* \) is the optimal policy. Furthermore

\[
R^*_F(n, \nu)^* \geq R^*_F(n-1, \nu)^* \geq \cdots \geq R^*_F(1, \nu)^* .
\]

The proof of Theorem 3.2 will be aided by the following lemma.

Lemma 3.3: Let \( L(S) \) be a concave function with \( L^-(S) \geq -(1 - \frac{1}{K}) \) for \( 0 \leq S \leq S_o \), where \( K > 1 \) is some real number and \( L^+(0) \leq 0 \). Let \( S_{j_1}, \ldots, S_{j_i} \) be any \( i \) items with \( S_{j_k} < S_{j_{k+1}} \) for all \( k = 1, \ldots, i-1 \) and \( S_{j_i} < S_o \). Then if

\[
F_1 = \begin{bmatrix} S_{j_1} & S_{j_{i-1}} & \cdots & S_{j_i} \end{bmatrix}
\]

and

\[
F_{i-1} = \begin{bmatrix} S_{j_{i-1}} & \cdots & S_{j_i} \end{bmatrix}
\]

are two FIFO policies for issuing the \( i \) and \( i-1 \) youngest items respectively then

\[
Q^*_F(i) - Q^*_F(i-1) - \left( \frac{1}{K} \right)^{i-1} L(S_{j_i}) \geq 0 .
\]
Proof of Lemma 3.3: The proof will be by induction. Consider $i = 2$, it must be shown that

$$Q_{F_2} - Q_{F_1} - \left(\frac{1}{K}\right)L(S_{j_2}) \geq 0$$

where

$$Q_{F_2} = L(S_{j_2}) + L(S_{j_1} + L(S_{j_2}))$$

and

$$Q_{F_1} = L(S_{j_1}).$$

Now by lemma 2.1 $S_{j_1} + L(S_{j_2}) < S_o$ hence

$$\frac{L(S_{j_1} + L(S_{j_2})) - L(S_{j_1})}{L(S_{j_2})} \geq -1 + \frac{1}{K}$$

$$\Rightarrow L(S_{j_1} + L(S_{j_2})) + L(S_{j_2}) - L(S_{j_1}) - \frac{1}{K} L(S_{j_2}) \geq 0$$

as required.

Now assume the lemma is true for $i - 1$ and it will be shown to be true for $i$. Let $x$ be the total field life from the issuance of the $i - 1$ oldest items by FIFO i.e., $A_x = [S_{j_{i-1}}, S_{j_{i-2}}, \ldots, S_{j_2}]$ and let $y$ be the total field life from the FIFO issuance of $A_y = [S_{j_{i-1}}, S_{j_2}]$. Then by lemma 2.1 $S_{j_1} + x < S_o$ and $S_{j_1} + y < S_o$ and by lemma 2.2, $x \geq y$ and actually $x > y$ since $L^-(S) > -1$,
\[
\frac{L(S_{j_1} + x) - L(S_{j_1} + y)}{x - y} \geq -1 + \frac{1}{K}
\]

\[
\Rightarrow L(S_{j_1} + x) + x - [L(S_{j_1} + y) + y] \geq \frac{1}{K} (x - y)
\]

but

\[
L(S_{j_1} + x) + x = Q_{F_1}
\]

and

\[
L(S_{j_1} + y) + y = Q_{F_{i-1}}
\]

thus

\[
Q_{F_1} - Q_{F_{i-1}} \geq \frac{1}{K} (x - y)
\]

now subtract \((\frac{1}{K})^{i-1} L(S_{j_1})\) from both sides to obtain

\[
Q_{F_1} - Q_{F_{i-1}} - \left(\frac{1}{K}\right)^{i-1} L(S_{j_1}) \geq \frac{1}{K} \left[ x - y - \left(\frac{1}{K}\right)^{i-2} L(S_{j_1}) \right]
\]

but by the inductive assumption on \(i - 1\)

\[
x - y - \left(\frac{1}{K}\right)^{i-2} L(S_{j_1}) \geq 0
\]

and since \(\frac{1}{K} > 0\) then

\[
Q_{F_1} - Q_{F_{i-1}} - \left(\frac{1}{K}\right)^{i-1} L(S_{j_1}) \geq 0.
\]

q.e.d.
Proof of Theorem 3.2: To show \( F(n, v)^* \) is the optimal policy, it is sufficient to show

\[
R_E(n, v)^* \geq R_E(n-1, v)^* \geq \ldots \geq R_E(1, v)^*
\]

since by Theorem 3.1 it is only necessary to consider \( F(n, v)^*, \ldots, F(1, v)^* \).

It will be shown that \( R_E(j+1, v)^* \geq l_{j+1} R_E(j, v)^* \) for any \( j = 1, \ldots, n - 1 \).

If \( j \leq v - 1 \) then

\[
R_E(j+1, v)^* - R_E(j, v)^* = L(S_{j+1}) - p
\]

\[
\geq L(S_{j+1}) - \left( \frac{1}{K} \right) L(S_n)
\]

but \( 1 > \frac{1}{K} > 0 \)

\[
\geq L(S_{j+1}) - L(S_n) \geq 0
\]

since \( L(\cdot) \) is nonincreasing and \( n \geq v \geq j + 1 \).

Hence

\[
R_E(j+1, v)^* \geq R_E(j, v)^* \quad \text{for all } j = 1, \ldots, v - 1.
\]

(3.1.1)

If \( n - 1 \geq j > v - 1 \) then consider policies \( F(j+1, v)^* \) and \( F(j, v)^* \)

which by lemma 2.3 can be written as
\[
\begin{align*}
M_1 & \quad [\bar{S}_{j+1}, \bar{S}_{j-v+1}, \bar{S}_{j-2v+1}, \ldots ] \quad [\bar{S}_{j}, \bar{S}_{j-v}, \bar{S}_{j-2v}, \ldots ] \\
M_2 & \quad [\bar{S}_{j}, \bar{S}_{j-v}, \bar{S}_{j-2v}, \ldots ] \quad [\bar{S}_{j-1}, \bar{S}_{j-v-1}, \ldots ] \\
\vdots & \quad \quad \vdots \\
M_{v-1} & \quad [\bar{S}_{j-v+3}, \bar{S}_{j-2v+3}, \ldots ] \quad [\bar{S}_{j-v+2}, \bar{S}_{j-2v+2}, \ldots ] \\
M_v & \quad [\bar{S}_{j-v+2}, \bar{S}_{j-2v+2}, \ldots ] \quad [\bar{S}_{j-v+1}, \bar{S}_{j-2v+1}, \ldots ]
\end{align*}
\]

It should now be noted that except for demand sources \( M_1 \) in \( F(j+1,v) \) and \( M_v \) in \( F(j,v) \), the items assigned to the other demand sources \( M_2, \ldots, M_{v-1} \) in \( F(j+1,v) \) and \( M_1, \ldots, M_{v-1} \) in \( F(j,v) \) are the same except for the indexing of the \( M \)'s. Let the total field life for \( M_i \) be denoted by \( Q_{M_i} \), then \( \{Q_{M_1} | F(j+1,v)\} = \{Q_{M_{i-1}} | F(j,v)\} \) for all \( i = 2, \ldots, v \) and

\[
Q_{F(j+1,v)} - \{Q_{M_1} | F(j+1,v)\} = Q_{F(j,v)} - \{Q_{M_v} | F(j,v)\},
\]

and

\[
R_{F(j+1,v)} - R_{F(j,v)} = Q_{F(j+1,v)} - Q_{F(j,v)} - p
\]

\[
= \{Q_{M_1} | F(j+1,v)\} - \{Q_{M_v} | F(j,v)\} - p. \quad (3.1.2)
\]

For simplicity let policy A be the items issued to \( M_1 \) under \( F(j+1,v) \) and let policy B be the items issued to \( M_v \) under \( F(j,v) \). Then
\[
A = [S_{j+1}, S_{j-\nu+1}, S_{j-2\nu+1}, \ldots]
\]
and
\[
B = [S_{j-\nu+1}, S_{j-2\nu+1}, \ldots].
\]

Also let \( Q_A = (Q_{M_1}] \Gamma_{(j+1,\nu)}^* \) and \( Q_B = (Q_{M_\nu}] \Gamma_{(j,\nu)}^* \). Now by Corollary 2.3.1 demand source \( M_1 \) in \( \Gamma_{(j+1,\nu)}^* \) receives
\[
\left\lfloor \frac{1 + \frac{1}{\nu} - 1}{\nu} \right\rfloor + 1 = \left\lfloor \frac{1}{\nu} \right\rfloor + 1
\]
items hence by lemma 3.3
\[
\left\lfloor \frac{1}{\nu} \right\rfloor
\]
\( Q_A - Q_B \geq \left( \frac{3}{K} \right) L(S_{j+1}) \geq 0 \)

but
\[
\left( \frac{1}{K} \right) L(S_{j+1}) \geq \left( \frac{1}{K} \right) L(S_n)
\]
\[
\geq \left( \frac{3}{\nu} \right) L(S_n) \geq p
\]
since \( L(\cdot) \) is nonincreasing, since \( n - 1 \geq j \geq \nu \) and since \( 1 > \frac{1}{K} > 0 \).

Thus
\[
\left\lfloor \frac{1}{\nu} \right\rfloor
\]
\( Q_A - Q_B \geq \left( \frac{1}{K} \right) L(S_{j+1}) \geq p \).

Hence
\[
Q_A - Q_B - p \geq 0
\]
and by (3.1.2)

\[ R_F^{(j+1,\nu)} - R_F^{(j,\nu)} \geq 0 \quad \text{for any } j = \nu, \ldots, n-1. \] 

(3.1.3)

Combining (3.1.1) and (3.1.3)

\[ R_F^{(j+1,\nu)} \geq R_F^{(n,\nu)} \quad \text{for any } j = 1, \ldots, n-1 \]

\[ \therefore R_F^{(n,\nu)} \text{ is optimal.} \qquad \text{q.e.d.} \]

The preceding theorem placed restrictions on \( L^{-}(S) \) which kept
\( L^{-}(S) > -1 \). Theorem 3.3 allows \( L^{-}(S) \geq -1 \) but restricts \( L^{+}(S) \). In
this case, it is not possible to say precisely what the optimal policy
is for general concave nonincreasing \( L(S) \). However, it is possible
to eliminate some of the \( n \) policies which cannot be optimal.

**Theorem 3.3:** Let \( L(S) \) be a concave function with \( L^{-}(S) \geq -1 \) for
\( 0 < S \leq S_0 \) and \( L^{+}(S) \leq -\frac{1}{K} \) for \( 0 \leq S < S_0 \) where \( K > 1 \) is any
finite real number. Let \( \nu = 1 \). If

\[ p \geq \left( \frac{K - 1}{K} \right)^j L(S_{j+1}) > 0 \quad \text{for some } j = 1, \ldots, n-1, \]

then

\[ R_F^{(j,1)} \geq R_F^{(j+i,1)} \quad \text{for all } i = 1, \ldots, n-j. \]
Here again the proof will be made easier with the use of the following lemma.

**Lemma 3.4**: Let $L(S)$ be a concave function with $L^-(S) \geq -1$ for $0 < S \leq S_0$ and $L^+(S) \leq -\frac{1}{K}$ for $0 \leq S < S_0$ where $K > 1$ is any finite real number. Let $S_{J_1}, \ldots, S_{J_i}$ be any $i$ items with $S_{J_k} < S_{J_{k+1}}$ for all $k = 1, \ldots, i-1$ and $S_{J_i} < S_0$. Then if

$$F_i = [S_{J_1}, S_{J_{i-1}}, \ldots, S_{J_i}]$$

and

$$F_{i-1} = [S_{J_{i-1}}, \ldots, S_{J_i}]$$

are two FIFO policies for issuing the $i$ and $i-1$ youngest items respectively, then

$$Q_{F_{i-1}} - Q_{F_i} + (\frac{K-1}{K})^{i-1} L(S_{J_i}) \geq 0.$$  

**Proof of Lemma 3.4**: The proof will be by induction. Consider $i = 2$, it must be shown that

$$Q_{F_1} - Q_{F_2} + \left(\frac{K-1}{K}\right)L(S_{J_2}) \geq 0$$

where $Q_{F_1} = L(S_{J_1})$ and $Q_{F_2} = L(S_{J_2}) + L(S_{J_1} + L(S_{J_2}))$.  

74
Now by lemma 2.1 \( S_{j_1} + L(S_{j_2}) < S_0 \) hence

\[
\frac{L(S_{j_1} + L(S_{j_2})) - L(S_{j_1})}{L(S_{j_2})} \leq -\frac{1}{K} \]

\[ \Rightarrow L(S_{j_1} + L(S_{j_2})) + \frac{1}{K} L(S_{j_2}) \leq L(S_{j_1}). \quad (3.1.4) \]

But

\[
L(S_{j_1} + L(S_{j_2})) + \frac{1}{K} L(S_{j_2}) = L(S_{j_1} + L(S_{j_2})) + \frac{1}{K} L(S_{j_2}) - L(S_{j_2}) + L(S_{j_2})
\]

\[
= Q_{F_2} - \left(\frac{K - 1}{K}\right) L(S_{j_2})
\]

hence in (3.1.4)

\[
Q_{F_2} - \left(\frac{K - 1}{K}\right) L(S_{j_2}) \leq L(S_{j_1}) = Q_{F_1}
\]

\[ \Rightarrow Q_{F_1} - Q_{F_2} + \left(\frac{K - 1}{K}\right) L(S_{j_2}) \geq 0 \]

as required. Now assume the lemma is true for \( i - 1 \) and it will be shown to be true for \( i \).

Let \( x \) be the total field life from the FIFO issuance of the \( i - 2 \) items \([S_{j_1}, \ldots, S_{j_{i-1}}]\) and let \( y \) be the total field life from the FIFO issuance of the \( i - 1 \) items \([S_{j_1}, S_{j_{i-1}}, \ldots, S_{j_2}]\). Then by the inductive assumption
\[ x - y + \left(\frac{K - 1}{K}\right)^{i-2} L(S_{j_1}) \geq 0. \] (3.1.5)

By lemma 2.2,

\[ y - x \geq 0. \]

Now by lemma 2.1, \( S_{j_1} + x < S_o \) and \( S_{j_2} + y < S_o \) thus if \( y > x \)

\[ \frac{L(S_{j_1} + y) - L(S_{j_1} + x)}{y - x} \leq -\frac{1}{K} \]

\[ \Rightarrow L(S_{j_1} + y) - L(S_{j_1} + x) \leq -\frac{1}{K} (y - x). \]

Add \( y - x - \left(\frac{K - 1}{K}\right)^{i-1} L(S_{j_1}) \) to both sides; then

\[ L(S_{j_1} + y) + y - [L(S_{j_1} + x) + x] - \left(\frac{K - 1}{K}\right)^{i-1} L(S_{j_1}) \]

\[ \leq -\frac{1}{K} (y - x) + y - x - \left(\frac{K - 1}{K}\right)^{i-1} L(S_{j_1}) \]

\[ = \left(\frac{K - 1}{K}\right) \left[ y - x - \left(\frac{K - 1}{K}\right)^{i-2} L(S_{j_1}) \right] \]

\[ \leq 0 \] (3.1.6)

by (3.1.5) since \( \frac{K - 1}{K} > 0 \). If \( y = x \) then (3.1.6) still holds.

But

\[ L(S_{j_1} + y) + y = Q_{F_1} \]

and

\[ L(S_{j_1} + x) + x = Q_{F_1} \]

76
and (3.1.6) becomes

\[ Q_{F_1} - Q_{F_{i-1}} - \left( \frac{K - 1}{K} \right)^{i-1} L(S_{j+1}) \leq 0 \]

and by induction the lemma is proved.

q.e.d.

**Proof of Theorem 3.3:** It will be shown that

\[ R_F(j+i,1)^* \geq R_F(j+i+1,1)^* \]

for any \( i = 0, 1, \ldots \), \( n - j - 1 \).

\[ R_F(j+i,1)^* - R_F(j+i+1,1)^* \]

\[ = Q_F(j+i,1)^* - (j + 1)p - [Q_F(j+i+1,1)^* - (j + i + 1)p] \]

\[ = Q_F(j+i,1)^* - Q_F(j+i+1,1)^* + p \]

\[ \geq Q_F(j+i,1)^* - Q_F(j+i+1,1)^* - \left( \frac{K - 1}{K} \right)^{j+i} L(S_{j+i+1}) \]

\[ \geq 0 \] by lemma 3.3, for any \( i = 0, 1, \ldots , n - j - 1 \).

The above inequalities follow from

\[ p \geq \left( \frac{K - 1}{K} \right)^j L(S_{j+1}) \geq \left( \frac{K - 1}{K} \right)^{j+i} L(S_{j+i+1}) \]

\[ \geq \left( \frac{K - 1}{K} \right)^{j+i} L(S_{j+i+1}) \] since \( L(\cdot) \) is nonincreasing and since \( 1 > \frac{K - 1}{K} > 0 \). Thus
\[
R_F(j,1)^* \geq R_F(j+1,1)^* \geq \cdots \geq R_F(n,1)^* \\
\]
q.e.d.

Thus Theorem 3.3 states that all items which have initial age greater than \( S_j \) may be discarded immediately because by Theorem 3.1 and Theorem 3.3 the optimal policy must be one of the \( j \) policies \( F(1,1)^*, F(2,1)^*, \ldots, F(j,1)^* \). It is now possible to generalize Theorem 3.3 to the case of \( v \geq 1 \) demand sources. But before so doing, some useful notation will be presented.

By using corollary 2.3.1 and lemma 2.3 it is possible to re-index the FIFO assigned items to the \( v \) demand sources in the following manner:

<table>
<thead>
<tr>
<th>Demand Source</th>
<th>Items assigned to the demand source using lemma 2.3</th>
<th>New indexing of the items using corollary 2.3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 )</td>
<td>([S_{n}, S_{n-v}, \ldots])</td>
<td>([S^{(1)}<em>{\lceil n-1/v \rceil+1}, S^{(1)}</em>{\lceil n-1/v \rceil}, \ldots, S^{(1)}_1])</td>
</tr>
<tr>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
</tr>
<tr>
<td>( M_j )</td>
<td>([S_{n-j+1}, S_{n-j-v+1}, \ldots])</td>
<td>([S^{(j)}<em>{\lceil n-1/v \rceil+1}, S^{(j)}</em>{\lceil n-1/v \rceil}, \ldots, S^{(j)}_1])</td>
</tr>
<tr>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
</tr>
<tr>
<td>( M_v )</td>
<td>([S_{n-v+1}, S_{n-2v+1}, \ldots])</td>
<td>([S^{(v)}<em>{\lceil n-1/v \rceil+1}, S^{(v)}</em>{\lceil n-1/v \rceil}, \ldots, S^{(v)}_1])</td>
</tr>
</tbody>
</table>
That is, re-index the items assigned to $M_j$ by FIFO from $\left\lceil \frac{n-k}{\nu} \right\rceil + 1$ down to 1 and attach the superscript $(j)$ to indicate that the items are assigned to $M_j$.

**Theorem 3.4**: Let $L(S)$ be a concave function with $L^{-}(S) \geq -1$ for $0 < S \leq S_0$ and $L^{+}(S) \leq -\frac{1}{K}$ for $0 \leq S < S_0$ where $K > 1$ is any finite real number. Let $\nu \geq 1$.

If FIFO maximizes the total field life $Q^*_F(i, \nu)$ for all $i$ and $\nu$ ($i \geq \nu$) and if

$$p \geq (\frac{K-1}{K})^t L(S_{j+1}) > 0$$

for some $j = 1, \ldots, n-1$ where $S_{j+1}$ is the $t + 1^{st}$ item remaining to be issued to some demand source, say $M_k$, (hence $te \left\{ 0, 1, \ldots, \left\lceil \frac{n-k}{\nu} \right\rceil \right\}$), then

$$R^*_F(j, \nu) \geq R^*_F(j+1, \nu)$$

for all $i = 1, 2, \ldots, n-j$.

**Proof of Theorem 3.4**: Using the notation developed above

$s_{j+1} = s_t^{(k)}$

and writing the entire array of items in the new notation:
\[
\begin{align*}
M_1 & \quad [s^{(1)}_{\frac{n-1}{V}+1}, \ldots, s^{(1)}_{t+1}, \ldots, s^{(1)}_1] \\
M_2 & \quad [s^{(2)}_{\frac{n-2}{V}+1}, \ldots, s^{(2)}_{t+1}, \ldots, s^{(2)}_1] \\
& \quad \vdots \\
M_k & \quad [s^{(k)}_{\frac{n-k}{V}+1}, \ldots, s^{(k)}_{t+1}, \ldots, s^{(k)}_1] \\
& \quad \vdots \\
M_v & \quad [s^{(v)}_{\frac{n}{V}+1}, \ldots, s^{(v)}_{t+1}, \ldots, s^{(v)}_1]
\end{align*}
\]

But since \( L(\cdot) \) is nonincreasing and \( 0 < \frac{K - 1}{K} < 1 \)

\[
p \geq \left( \frac{K - 1}{K} \right)^t L(s_{j+1}) \geq \left( \frac{K - 1}{K} \right)^t L(s_{j+1+t}) \quad \text{for all } i = 1, \ldots, n - j - 1
\]

\[
\geq \left( \frac{K - 1}{K} \right)^{t+u} L(s_{j+1+t}) \quad \text{for all } u = 0, 1, \ldots.
\]

Thus

\[
p \geq \left( \frac{K - 1}{K} \right)^t L(s^{(k)}_{t+1}) \geq \left( \frac{K - 1}{K} \right)^t L(s^{(r)}_{t+1}) \quad \text{for } r = 1, \ldots, k
\]

\[
\geq \left( \frac{K - 1}{K} \right)^{t+1} L(s^{(r)}_{t+1})
\]

\[
\geq \left( \frac{K - 1}{K} \right)^{t+1} L(s^{(s)}_{t+2}) \quad \text{for } s = k + 1, \ldots, v.
\]
Using these inequalities then Theorem 3.3 can be applied to each demand source separately. Hence all items older than \( S_j \) may be discarded immediately and

\[
F^*_F(j,v) \geq F^*_F(j+1,v) \quad \text{for all } i = 1, \ldots, n - j.
\]

q.e.d.

3.2 The Case for FIFO When \( L(S) \) is Linear

In the case where \( L(S) \) is a linearly decreasing function with slope \( > -1 \), precise statements can be made concerning the optimality of \( F^*_F(j,v) \) as the issuing policy which maximizes the total return.

The linear function \( L(S) \) allows us to combine the knowledge gained from Theorems 3.2, 3.3, and 3.4 since we now have the condition that

\[
L^+(S) = L^-(S) = -\frac{1}{K} \quad \text{for all } 0 < S < S_0.
\]

Changing the notation slightly let \( L(S) = aS + b \) for \( 0 \leq S \leq S_0 \) where \( b > 0 > a > -1 \). Hence \( a = -\frac{1}{K} \) in the preceding results.

**Lemma 3.5:** Let \( L(S) = aS + b \) for \( 0 \leq S \leq S_0 \) and \( b > 0 > a > -1 \).

Let \( S_{j_1}, \ldots, S_{j_i} \) be any \( i \) items with \( S_{j_k} < S_{j_{k+1}} \) for all \( k = 1, \ldots, i-1 \) and \( S_{j_1} < S_0 \). Then if

\[
F_i = [S_{j_1}, S_{j_1-1}, \ldots, S_{j_1}]
\]

and

\[
F_{i-1} = [S_{j_{i-1}}, \ldots, S_{j_1}]
\]

are two FIFO policies for issuing the \( i \) and \( i-1 \) youngest items respectively, then

81
\[ Q_{F_1} - Q_{F_{i-1}} - (1 + a)^{i-1}L(S_j) = 0. \]

**Proof of Lemma 3.5:** The proof is similar to the proof of lemma 3.4. For completeness it is given below. The proof is by induction.

Consider \( i = 2 \).

\[ Q_{F_1} = L(S_{j_1}) \quad \text{and} \quad Q_{F_2} = L(S_{j_2}) + L(S_{j_1} + L(S_{j_2})) \]

By lemma 2.1 \( S_{j_1} + L(S_{j_2}) < S_0 \) hence

\[
\frac{L(S_{j_1} + L(S_{j_2})) - L(S_{j_1})}{L(S_{j_2})} = a
\]

\[ \Rightarrow L(S_{j_1} + L(S_{j_2})) - L(S_{j_1}) = aL(S_{j_2}) \]

\[ \Rightarrow L(S_{j_1} + L(S_{j_2})) + L(S_{j_2}) - L(S_{j_1}) - (1 + a)L(S_{j_2}) = 0 \]

as required.

Now assume the lemma is true for \( i - 1 \) and it will be shown to be true for \( i \). Let \( x \) be the total field life from the FIFO issuance of the \( i - 2 \) items \([S_{j_{i-2}}, \ldots, S_{j_2}]\) and let \( y \) be the total field life from the FIFO issuance of the \( i - 1 \) items \([S_{j_1}, \ldots, S_{j_2}]\). Then by the inductive assumption
\[ y - x - (1 + a)^{i-2}L(S_{j_1}) = 0 \]  

(3.2.1)

\[ \Rightarrow y - x = (1 + a)^{i-2}L(S_{j_1}) > 0. \]

By lemma 2.1, \( S_{j_1} + x < S_0 \) and \( S_{j_1} + y < S_0 \) thus

\[ \frac{L(S_{j_1} + y) - L(S_{j_1} + x)}{y - x} = a \]

\[ \Rightarrow L(S_{j_1} + y) - L(S_{j_1} + x) = a(y - x) \]

\[ \Rightarrow L(S_{j_1} + y) + y - L(S_{j_1} + x) + x - (1 + a)(y - x) = 0 \]

But \( L(S_{j_1} + y) + y = Q_{F_1} \) and \( L(S_{j_1} + x) + x = Q_{F_{i-1}} \) hence

\[ Q_{F_1} - Q_{F_{i-1}} - (1 + a)(y - x) = 0 \]

\[ \Rightarrow Q_{F_1} - Q_{F_{i-1}} - (1 + a)^{i-1}L(S_{j_1}) + (1 + a)^{i-1}L(S_{j_1}) - (1 + a)(y - x) = 0 \]

(3.2.2)

But

\[ (1 + a)^{i-1}L(S_{j_1}) - (1 + a)(y - x) = -(1 + a)[y - x - (1 + a)^{i-2}L(S_{j_1})] \]

\[ = 0 \]
by (3.2.1) and using this fact in (3.2.2) we obtain

$$Q_{T_1} - Q_{T_1 - 1} - (1 + a)^{i-1}L(S_{j_1}) = 0$$

as required, and by induction the lemma is proved.

q.e.d.

We are now prepared to determine the optimal issuing policy in the linear case. Theorem 3.5 gives the optimal result when $\nu = 1$ and Theorem 3.6 for $\nu \geq 1$. Since Theorem 3.5 is used in the proof of Theorem 3.6 the proof of Theorem 3.5 is presented also.

**Theorem 3.5:** Let $L(S) = aS + b$ for $0 \leq S \leq S_0$ and $b > 0 > a > -1$.

Let $\nu = 1$.

If

(1) $p \geq (1 + a)^jL(S_{j+1})$

and (ii) $p < (1 + a)^{j-1}L(S_{j})$ for some $j = 1, \ldots, n - 1$

then $F_{(j, 1)^*}$ is the optimal policy. That is,

$$R_{F_{(j, 1)^*}} \geq R_{F_{(j+1, 1)^*}}$$ for $i = 1, \ldots, n - j$  \hspace{1cm} (3.2.3)

$$R_{F_{(j, 1)^*}} \geq R_{F_{(j-1, 1)^*}}$$ for $i = 1, \ldots, j - 1$. \hspace{1cm} (3.2.4)

Furthermore,

$$R_{F_{(j, 1)^*}} > R_{F_{(j-1, 1)^*}} > \cdots > R_{F_{(1, 1)^*}}.$$  \hspace{1cm} (3.2.5)
Proof of Theorem 3.5: Inequality (3.2.3) holds by Theorem 3.3. Hence it is only necessary to show (3.2.5); (3.2.4) then follows immediately.

Consider

\[ R^*_F(j-1,1) - R^*_F(j-1,1) \]

for any \( i = 0, 1, \ldots, j - 2 \)

then

\[ R^*_F(j-1,1) - R^*_F(j-1,1) = Q^*_F(j-1,1) - Q^*_F(j-1,1) - p \]

\[ > Q^*_F(j-1,1) - Q^*_F(j-1,1) - (l + a)^{j-1} L(S_j) \]

\[ \geq Q^*_F(j-1,1) - Q^*_F(j-1,1) - (l + a)^{j-1} L(S_{j-1}) \]

since \( L(\cdot) \) is nonincreasing and \( l > 1 + a > 0 \), but

\[ Q^*_F(j-1,1) - Q^*_F(j-1,1) - (l + a)^{j-1} L(S_{j-1}) = 0 \]

by lemma 3.5. Thus \( R^*_F(j-1,1) > R^*_F(j-1,1) \) for all \( i = 0, 1, \ldots, j - 2 \).

q.e.d.

An obvious consequence of Theorem 3.5 is when \( p < (l + a)^{n-1} L(S_n) \), then \( F(n,1) \) is optimal.
As mentioned previously Theorem 3.6 generalizes Theorem 3.5 to the case $\nu \geq 1$. An algorithm for obtaining the optimal policy when $\nu \geq 1$ is presented. For ease of defining the algorithm and for ease of proving that an optimal policy results, it is useful to define an augmented set of items and a search procedure. Recall that assumption (1) of the model states that the process starts initially with $n$ items of initial ages $0 < S_1 < S_2 < \cdots < S_n$. The augmented set of items is the set of $n + \nu$ items $0 < S_1 < S_2 < \cdots < S_n < S_{n+1} < \cdots < S_{n+\nu}$ where $L(S_i) > 0$ for all $i = 1, \ldots, n + \nu$ and where the penalty cost $p$ has

$$p \geq (1 + a)^{\left(\frac{n-\nu}{\nu}\right)} L(S_{n+1}) = (1 + a)^{\left(\frac{n}{\nu}\right)} L(S_{n+1})$$

Note that it is always possible to find items $S_{n+1}', \ldots, S_{n+\nu}$ which satisfy the augmented system since $S_n < S_o$ and $p$ is a fixed positive constant; i.e., for $L(S) = aS + b$ (with $b > 0 > a > -1$) $S_o$ exists and $L(S) \rightarrow 0$ as $S \rightarrow S_o$. It will be shown in the proof of Theorem 3.6 that the optimal policy for the augmented set is the same as for the original set. We now define the search procedure.

**Search Procedure:** Using the re-indexed method of labelling the items issued to each demand source, consider all adjacent pairs of items for each demand source starting with the oldest adjacent pair for $M_1$, viz. $S^{(1)}_{\left[\frac{n-1}{\nu}\right]+2}$ and $S^{(1)}_{\left[\frac{n-1}{\nu}\right]+1}$, then the oldest pair for $M_2$ viz. $S^{(2)}_{\left[\frac{n-2}{\nu}\right]+2}$.
and $S^{(2)}_{\left[\frac{n-2}{v}\right]+1}$ etc. for $M_3$ through $M_v$. Then consider the second oldest adjacent pair for $M_1$ viz. $S^{(1)}_{\left[\frac{n-1}{v}\right]+1}$ and $S^{(1)}_{\left[\frac{n-1}{v}\right]}$, etc. for $M_2$ through $M_v$. Continue in this manner searching all adjacent pairs in order of their age from oldest pair to the youngest pair.

We can now state and prove Theorem 3.6.

**Theorem 3.6:** Let $L(S) = aS + b$ for $0 \leq S \leq S_o$ and $b > 0 > a > -1$. Let $v \geq 1$.

Two cases are possible:

(i) if $p \geq L(S_1)$ then the penalty cost is greater than the value received from any item hence it does not pay to issue any item but if $v$ items must be issued then $F(v, v)^*$ is optimal.

(ii) if $p \not\geq L(S_1)$ then apply the Search Procedure to the augmented set of items. If for some demand source $M_1$, it is the first demand source such that for some $j = 1, \ldots, \left[\frac{n-1}{v}\right] + 1$

\[(1 + a)^{-1}L(S^{(1)}_j) > p \geq (1 + a)JL(S^{(1)}_{j+1}) \quad (3.2.6)\]

then the FIFO policy which issues all items of initial age less than or equal to $S^{(1)}_j$ and discards all items strictly older than $S^{(1)}_j$ is the optimal issuing policy. That is, if $S^{(1)}_j = S_t$ then $F(t, v)^*$ is optimal.
Before beginning the proof of this theorem, it will be enlightening to consider the effect of the theorem on a simple example. Let \( n = 10 \) and \( v = 3 \), then the augmented set of items is \( S_{13}, S_{12}, S_{11}, S_{10}, S_9, S_8, S_7, S_6, S_5, S_4, S_3, S_2, S_1 \) and the policy \( F_{(13,3)} \) assigns:

- \( M_1 \) receives \( [S_{13}, S_{10}, S_7, S_4, S_1] \)
- \( M_2 \) receives \( [S_{12}, S_9, S_6, S_3] \)
- \( M_3 \) receives \( [S_{11}, S_8, S_5, S_2] \)

where in the process of augmentation, \( \lceil \frac{n}{v} \rceil = 3 \) then

\[
p \geq (1 + a)^3 L(S_{11}) \geq (1 + a)^3 L(S_{12}) \geq (1 + a)^3 L(S_{13}) > (1 + a)^4 L(S_{13}) .
\]

Now we look at all adjacent pairs from oldest to youngest i.e., we look at

\[
(S_{13}, S_{10}), (S_{12}, S_9), (S_{11}, S_8), (S_{10}, S_7), (S_9, S_6), (S_8, S_5), (S_7, S_4), (S_6, S_3), (S_5, S_2), (S_4, S_1)
\]

Let us say that \( (S_{10}, S_7) \) is the oldest adjacent pair which satisfies (3.2.6). Then

\[
(1 + a)^2 L(S_7) > p \geq (1 + a)^3 L(S_{10}) .
\]
Furthermore the following relationships hold:

\[ p < (1 + a)^2 L(S_7) \leq (1 + a)^2 L(S_6) \leq (1 + a)^2 L(S_5) \]

and \[ p \geq (1 + a)^3 L(S_0) \geq (1 + a)^3 L(S_9) \geq (1 + a)^3 L(S_{10}) \].

Thus if we apply Theorem 3.5 to maximize the total field life from each demand source, we find that for \( M_1 \) we use \( S_7, S_4, S_1 \) and discard \( S_{10} \) and \( S_{13} \), for \( M_2 \) we use \( S_6, S_3 \) and discard \( S_9 \) and \( S_{12} \), for \( M_3 \) we use \( S_5, S_2 \) and discard \( S_8 \) and \( S_{11} \). But using \( S_7, S_4, S_1, S_6, S_3, S_5 \) and discarding \( S_{13}, S_{12}, S_{11}, S_{10}, S_9, \) and \( S_8 \) is just \( F(7, 3)* \). Now Theorem 3.6 says \( F(7, 3)* \) is optimal hence it is only necessary at this point to show that the sum of the individual maxima results in the maximum total field life. This last step is immediate however, since lemma 2.3 shows that the same items are always grouped together for all \( F(1, \nu)* \) i.e., item \( S_1 \) is issued after item \( S_{1+\nu} = S_4 \) and \( S_4 \) is issued after item \( S_{1+2\nu} = S_7 \) etc. Likewise \( S_2 \) is issued after item \( S_{2+\nu} = S_5 \) etc. Hence we can do no better for the total set of items than to maximize the field life from each individual set.

The proof of Theorem 3.6 follows in the same manner as the example above.

**Proof of Theorem 3.6:**

**Part (i):** \( p \geq L(S_1) \) implies \( p \geq (1 + a)^{j-1} L(S_j^{(i)}) \) for all \( j = 1, \ldots, \left[ \frac{n-1}{\nu} \right] \) and \( i = 1, \ldots, \nu \) since \( L(\cdot) \) is nonincreasing.
Part (ii): Assume there exists an \( S_j^{(i)} \) and \( S_{j+1}^{(i)} \) with the property that upon application of the Search Procedure these two items are the first adjacent pair found to satisfy (3.2.6). Since \( S_j^{(i)} \) is the first item with \( p < (1 + a)^{j-1} L(S_j^{(i)}) \) then for all items \( S_t^{(k)} \) strictly older than \( S_j^{(i)} \) i.e., \( S_j^{(i)} < S_t^{(k)} \) we have

\[
(1 + a)^{t-1} L(S_t^{(k)}) \leq p .
\] (3.2.7)

Now at least \( v \) such \( S_t^{(k)} \)'s exist since the augmented set of items has \( p \geq (1 + a)^{\frac{[n-v]+1}{v}} L(S_{n+1}) \)

\[
\geq (1 + a)^{\frac{[n-j]+1}{v}} L(S_{n+v-j+1}) \quad \text{for all } j = 1, \ldots, v .
\]

Furthermore for all items \( S_u^{(k)} < S_j^{(i)} \),

\[
p < (1 + a)^{j-1} L(S_j^{(i)})
\]

\[
\leq (1 + a)^{j-1} L(S_u^{(k)})
\]

\[
\leq (1 + a)^{u-1} L(S_u^{(k)})
\] (3.2.8)

since \( u \leq j \) and \( 0 < 1 + a < 1 \).

Using (3.2.7) and (3.2.8) we see that for each demand source \( M_k \) if we let \( u_k \) be the subscript of the first item satisfying (3.2.8) then for all \( k = 1, 2, \ldots, v \) \( S_{u_k}^{(k)} \leq S_j^{(i)} \) and \( S_{u_k+1}^{(k)} > S_j^{(i)} \).
\[(1 + a)^{u_k}L(s_{u_k+1}^{(k)}) \leq p < (1 + a)^{u_k-1}L(s_{u_k}^{(k)}) \quad (3.2.9)\]

Hence for each demand source we can apply Theorem 3.5 to maximize the total return for that demand source.

Therefore, if we let \(R_F^{(k)}\) be the return to \(M_k\) starting with item \(S_{u_k}^{(k)}\) and following a FIFO policy, then by Theorem 3.5

\[R_F^{(k)} \geq R_F^{(k)}\quad \text{for all } k = 1, \ldots, v \quad \text{and }\]
\[w = 1, \ldots, \left\lceil \frac{p-k}{2} \right\rceil + 2 - u_k\]

and

\[R_F^{(k)} \geq R_F^{(k)}\quad \text{for all } k = 1, \ldots, v \quad \text{and }\]
\[y = 1, \ldots, u_k - 1.\]

Hence to maximize the total return for each \(M_k\) we merely discard all items older than \(S_{u_k}^{(k)}\) and follow a FIFO policy for the remaining items. But by so doing we are precisely following policy \(F(t, v)^{*}\) since all items \(S_u > S_j^{(1)}\) are discarded and all items \(S_u \leq S_j^{(1)}\) are issued.

We must now show that maximizing the total return for each \(M_k\) is the same as maximizing the total return for all \(M_k\)'s put together.

But this is immediately apparent since by lemma 2.3, the relative item assignment by FIFO for any two policies \(F(f,v)^{*}\) and \(F(g,v)^{*}\) (for say \(g < f\)) is the same for all items common to each policy except that
the demand sources are rotated as to the specific set of items assigned to them. That is under any $F(k,v)^*$ policy $S_1$ is assigned to the same demand source as are items $S_{1+v}, S_{1+2v}, \ldots; S_2$ is assigned to the same demand source as are items $S_{2+v}, S_{2+2v}, \ldots;$ etc. Thus maximizing the individual returns also maximizes the total return. Hence $F(t,v)^*$ is optimal since this policy maximizes all of the individual returns to each $M_k$.

q.e.d.

Before proceeding to the next section it should be pointed out that if the assumption that the process begins with the issuance of $v$ items to $M_1, \ldots, M_v$ is retained, then it is only necessary to find the optimal policy among the $n - v + 1$ policies $F(v,v)^*, \ldots, F(n,v)^*$. This would be the case for the theorems of section 3.1 as well as for the theorems of this section. In addition if we insist that all items with positive field life remaining must be issued, then the optimal policy will be found among the $n - v + 1$ policies $F(v,v)^*, \ldots, F(n,v)^*$ also.

3.3 The Case for LIFO

The results of this section are very similar to the results of the last two sections in that we reduce the search for the optimal policy to the case of searching only $n$ policies. Indeed these $n$ policies bear strong resemblance to the $n$ policies of the previous sections. Here we will be concerned with the $n$ LIFO policies $L(1,v)^*, \ldots, L(n,v)^*$.
where $L_{(i, v)^*}$ is the LIFO issuance of the $i$ youngest items in the stockpile to the $v$ demand sources. Defining $Q_{L_{(i, v)^*}}$ and $R_{L_{(i, v)^*}}$ as the total field life and the total return, respectively, when following policy $L_{(i, v)^*}$ ($R_{L_{(i, v)^*}} = Q_{L_{(i, v)^*}} - i \cdot p$), we can state

**Theorem 3.7**: Let $L(S)$ be a convex nonincreasing function. Let $v \geq 1$. If LIFO is the issuing policy which maximizes the total field life for any $i$ items in inventory, then the optimal issuing policy which maximizes the total return must be one of the $n$ policies

$$L_{(1, v)^*}, L_{(2, v)^*}, \ldots, L_{(n, v)^*}.$$  

We defer the proof of Theorem 3.7 until we have stated and proved the following three lemmas.

**Lemma 3.6**: Let $L(S)$ be a nonnegative convex nonincreasing function defined on $[0, \infty)$. Let $S_1 < S_2$ be any two points on $[0, \infty)$. Then $L^+(S_2 + L(S_1)) \geq -1$.

**Proof of Lemma 3.6**: 

(i) If $L^+(S_2) \geq -1$, then since $L(\cdot)$ is convex nonincreasing $L^+(S_2 + L(S_1)) \geq -1$.

(ii) If $L^+(S_2) < -1$, then let $\overline{S}$ be the point where $L^-(\overline{S}) \leq -1$ and $L^+(\overline{S}) \geq -1$. Now $\overline{S} < \infty$ since $L(S)$ is a nonnegative convex nonincreasing function. Furthermore $S_2 < \overline{S}$ and
\[
\frac{L(\overline{s}) - L(s_2)}{\overline{s} - s_2} < -1
\]

which implies

\[
L(s_2) + s_2 > L(\overline{s}) + \overline{s} \geq \overline{s}.
\]

Then \( L(s_1) + s_2 \geq L(s_2) + s_2 > \overline{s} \) since \( L(\cdot) \) is nonincreasing. We finally obtain

\[-1 \leq L^+(\overline{s}) \leq L^-(s_2 + L(s_1)) \leq L^+(s_2 + L(s_1))
\]

since \( L(s) \) is convex nonincreasing.

q.e.d.

Lemma 3.7: \( L(s) \) be a convex nonincreasing function on \( [0, \infty) \).

Let \( v = 1 \). Let two sets of items with the following characteristics be given:

\[ I = \{s_1, \ldots, s_n | s_1 < s_{i+1} \text{ and } s_n < s_0 \} \]

\[ II = \{\hat{s}_1, \ldots, \hat{s}_n | \hat{s}_1 < \hat{s}_{i+1} \text{ and } \hat{s}_n < s_0 \} \]

and \( s_i \leq \hat{s}_i \) for all \( i = 1, \ldots, n \). Denote by \( Q_L \) and \( \hat{Q}_L \) the total field life by LIFO issuance of the items of Set I and Set II respectively. Then \( Q_L \geq \hat{Q}_L \).
Proof of Lemma 3.7: The proof will be by induction. Let \( n = 1 \). Since \( L(\cdot) \) is nonincreasing and \( S_1 \leq \hat{S}_1 \) then \( L(S_1) \geq L(\hat{S}_1) \) as required.
Assume the lemma is true for \( n = k \) and it will be proved true for \( n = k + 1 \).

Let \( x \) and \( y \) denote the total field lives from the LIFO issuance of the first \( k \) items in Sets I and II respectively. Now \( x \geq y \) by the inductive assumption and \( x > 0 \) and \( y > 0 \) since \( L(S_1) > 0 \) and \( L(\hat{S}_1) > 0 \). We must show

\[
Q_L = x + L(S_{k+1} + x) \geq y + L(\hat{S}_{k+1} + y) = \hat{Q}_L \tag{3.3.1}
\]

If \( x = y \), then (3.3.1) obviously holds. If \( x > y \), then by lemma 3.6
\( L^+(\hat{S}_{k+1} + y) \geq -1 \) and \( L^+(S_{k+1} + x) \geq -1 \). Now since
\( \hat{S}_{k+1} + x > \hat{S}_{k+1} + y \) and \( L(S) \) is convex nonincreasing,

\[
L^{-}(\hat{S}_{k+1} + x) \geq L^+(\hat{S}_{k+1} + y) \geq -1.
\]

We now obtain

\[
\frac{L(S_{k+1} + x) - L(\hat{S}_{k+1} + y)}{x - y} \geq -1
\]

and

\[
L(\hat{S}_{k+1} + y) + y \leq L(\hat{S}_{k+1} + x) + x
\]

\[ \leq L(S_{k+1} + x) + x \]
and (3.3.1) holds. By induction the lemma is proved. Note that if
\( L(\hat{S}_{k+1} + x) = 0 \) and/or \( L(\hat{S}_{k+1} + y) = 0 \) the proof still holds.

q.e.d.

Denote by \( L_{i,v} \) the policy of issuing any \( i \) items to \( v \) demand sources by LIFO.

**Lemma 3.8:** Let \( L(S) \) be a convex nonincreasing function. Let \( v \geq 1 \).

If LIFO is the policy which maximizes the total field life for any \( i \) items in inventory, then
\[
Q_{L(i,v)}^{*} > Q_{L_{i,v}}^{*}.
\]

This lemma is a direct consequence of lemma 3.7 when \( v = 1 \). However for \( v > 1 \) the proof becomes somewhat more complicated.

**Proof of Lemma 3.8:** Choose any \( i \) items from the \( n \) possible items and denote them by \( S_{t_1} < S_{t_2} < \cdots < S_{t_i} \). Assume that the \( i \) items are not the \( i \) newest items. Let \( Q_{L_{1,v}}(S_{t_1}, \ldots, S_{t_i}) \) be the total field life obtained from issuing these \( i \) items by LIFO. We will show that there is a policy which yields a greater total field life, call this policy \( A \).

**Policy A:** Let \( S_{t_j} \) be the youngest item of the \( i \) items such that \( S_{t_j} \neq S_{t_i} \). If \( M_j \) is the demand source which receives item \( S_{t_j} \) under \( L_{1,v}(S_{t_1}, \ldots, S_{t_i}) \) then instead of issuing \( S_{t_j} \) to \( M_j \) issue \( S_j \) in its place. Do not change any of the other items, their order of issue or the demand source to whom they are issued.
By lemma 3.7 the field life contributed by \( M_j \) under policy \( A \) is
greater than or equal to the field life under policy \( L_{i,v}(S_{t_1}, \ldots, S_{t_1}) \).
And the total field life of the other \( M_k \)'s (\( k \neq j \)) is unchanged.
Thus

\[
Q_A = \sum_{k=1 \atop k \neq j}^v Q_{M_k} + Q_{M_j}^{(A)} \geq \sum_{k=1}^v Q_{M_k} = Q_{L_{i,v}}(S_{t_1}, \ldots, S_{t_1}).
\]

But LIFO is optimal for any \( i \) items. Hence

\[
Q_{L_{i,v}}(S_1, \ldots, S_j, S_{t_{j+1}}, \ldots, S_{t_1})
\]

\[
\geq Q_A \geq Q_{L_{i,v}}(S_1, \ldots, S_j, S_{t_{j+1}}, \ldots, S_{t_1})
\]

Now apply policy \( A \) again to the new set of items
\( S_1, S_2, \ldots, S_j, S_{t_{j+1}}, \ldots, S_{t_1} \) and continue this process until
\( Q_{L_{i,v}}^{(i,v)*} \geq Q_{L_{i,v}}(S_{t_1}, \ldots, S_{t_1}) \) as required. The process terminates
at policy \( L_{i,v}^{(i,v)*} \) since there are only a finite number of items (at
most \( n \)) to replace.

q.e.d.

We are now ready to prove Theorem 3.7.

Proof of Theorem 3.7: Let \( A_{i,v} \) be any arbitrary policy of issuing
any \( i \) items to the \( v \) demand sources. Now since LIFO is optimal for
any \( i \) then \( Q_{L_{i,v}}^{(i,v)*} \geq Q_{A_{i,v}} \) and by lemma 3.8 we have
\( Q_{L_{i,v}}^{(i,v)*} \geq Q_{L_{i,v}}^{(i,v)} \geq Q_{A_{i,v}} \). Now \( L_{i,v}^{(i,v)*} \) issues at most \( i \) items (less
than \( i \) if some have zero field life). Thus,

\[
R_{L(i,v)^*} \geq Q_{L(i,v)^*} - ip \geq Q_{A_{i,v}} - ip = R_{A_{i,v}}.
\]

And since \( A_{i,v} \) was any arbitrary policy, the theorem is proved.

q.e.d.

**Corollary 3.3:** Let \( L(S) \) be linear on \([0, S_0]\) with \( L'(S) = -1 \) on \([0, S_0]\). Let \( v \geq 1 \). The optimal policy which maximizes the total return must be one of the \( n \) policies \( L(1,v)^*, \ldots, L(n,v)^* \).

**Proof of Corollary 3.3:** By lemma 5.2 (ff.) LIFO maximizes the total field life for any \( i \) items assigned to the \( v \) demand sources. Therefore, the application of Theorem 3.7 proves the corollary.

q.e.d.

**Theorem 3.8:** Let \( L(S) \) be a convex or a concave differentiable function on \([0, S_0]\) with \( L'(S) < -1 \) on \([0, S_0]\). Let \( v \geq 1 \). The optimal policy which maximizes the total return must be one of the \( v \) policies \( L(1,v)^*, \ldots, L(v,v)^* \).

**Proof of Theorem 3.8:** By Zehna [11] Theorems 2.4, 2.6, 4.2 and 4.3, LIFO maximizes the total field life for any \( i \) items and \( v \) demand sources. Since this condition satisfies the hypothesis of Theorem 3.7, we merely apply Theorem 3.7 and we have proved Theorem 3.8 in the convex case. However the following proof holds for both the convex and concave cases. If only \( i \) items where \( i < v \) have the property that \( L(S_j) > p \) for \( j = 1, \ldots, i \) then it is never optimal to issue more than the \( i \)
newest items since if more than \( i \) items are issued the penalty costs exceed the value of the additional items or if the \( i \) items issued are not the \( i \) newest items then the total return can be improved by issuing the \( i \) newest items. Henceforth assume \( i \geq v \) items have the property that \( L(S_j) > p \) for \( j = 1, \ldots, i \). Now for any \( S < S_o \) and since \( L'(S) < -1 \) we have

\[
\frac{L(S_o) - L(S)}{S_o - S} < -1
\]

which implies

\[
S_o < L(S) + S. \tag{3.3.2}
\]

As pointed out above Zehna's theorems prove that LIFO maximizes the total field life. Hence for any policy \( A_{i,v} \quad Q_L \geq Q_{A_{i,v}} \). But under LIFO after the first \( v \) items are issued to start the process, all other items \( S_k > S_v \) have field life of zero when they are to be issued since

\[
S_k + L(S_v) > S_v + L(S_v) > S_o
\]

by (3.3.2). Hence LIFO issues only \( v \) items. Therefore

\[
R_L = Q_L - v \cdot p \geq Q_{A_1} - v \cdot p
\]

\[
\geq Q_{A_1} - i \cdot p = R_{A_1}
\]

and since \( A_1 \) was arbitrary, the theorem is proved. \( \Box \)
Just as in the case for FIFO, if we retain the assumption that \( v \) items must be issued to start the process, then the optimal policy will be found among the \( n - v + 1 \) policies \( L(v,v)^* , \ldots , L(n,v)^* \) in the general case of Theorem 3.7 and Corollary 3.3 and will be \( L(v,v)^* \) in the case of Theorem 3.8. Furthermore it should be noted that if the assumption is retained that any item will be issued provided it has positive field life, then LIFO is optimal for Theorems 3.7 and 3.8 and Corollary 3.3. By LIFO it is meant that the newest item is always issued to a demand source provided the item has positive field life (in Theorem 3.8 \( \text{LIFO} = L(v,v)^* \)).

It is also interesting to point out that if only \( i \) items where \( 1 \leq v \) have the property that \( L(S_j) > p \) for \( j = 1, \ldots , i \) then \( L(i,v)^* \) is the optimal policy for the general case of Theorems 3.7 and 3.8 and for Corollary 3.3. The proof of this statement is the same as is given in Theorem 3.8.
Chapter 4

Field Life Functions Which Are Not Convex or Concave

It may be the case that for a certain type of inventory item, the actual field life function may not be convex or concave but would be an S-shaped type of function. For example,

\[
L(S) = \begin{cases} 
\frac{1}{3} (-S + 3)^{\frac{1}{3}} + 2 & \text{for } 0 \leq S \leq 11 \\
0 & \text{otherwise}
\end{cases}
\]

yields an S-shaped function of this type

Unfortunately, it is not possible to find a specific policy which is optimal for the general S-shaped function. It is possible, however, to define a particular type of S-shaped function which has the property that when there are \( n \) items in inventory, the optimal policy must be one of \( n \) policies. It has the added property that it can be used as an approximation to the general S-shaped function. The particular S-shaped function referred to above is: \( L(S) \) is concave nonincreasing...
for all \( S \in [0, t] \) where \( t > 0 \) and \( L(S) = c \) for all \( S = [t, \infty) \).

In addition will usually be assumed that \( L'(S) \geq -1 \) for all \( S \in (0, t] \).

Diagrammatically,

\[c\]
\[0\]
\[t\]

The more specialized field life function \( L(S) = aS + b \) for all \( S \in [0, t] \) and \( L(S) = c \) for all \( S \in [t, \infty) \) where \( b > c > 0 > a > -1 \) is also examined. In this case specific statements about the optimal policy can be made.

Because it will continually be of interest in this chapter, we will define two models for the field life function, \( L(S) \).

**Model I**

\( L'(S) \) is concave nonincreasing for all \( S \in [0, S_o] \),

\( L(S) = 0 \) for all \( S \in [S_o, \infty) \) and

\( L'(S) \geq -1 \) for all \( S \in (0, S_o] \).

\[L(S)\]
\[L'(S)\]
\[0\]
\[S_o\]
\[S\]
Model II

$L_1(S)$ coincides with $L_1(S)$ in Model I for all $S \in [0, t]$ where $t < S_o$ and $L_2(S) = c$ for all $S \in [t, \infty)$.

The results concerning the optimal policies for Model II are presented in Section 4.2. Section 4.1 contains a series of lemmas which aid in the proofs of the theorems of Section 4.2.

4.1 Lemmas

Lemma 4.1: Let $v = 1$. If a FIFO issuing policy is used in both Model I and Model II, then

$$Q_{F II} > Q_{F I}.$$

Proof of Lemma 4.1: The proof will be by induction. $n = 1$ is trivially true. Now assume the lemma is true for $n = k$ and it will be proved for $n = k + 1$.

Let $x$ and $y$ denote the total field lives from the first $k$ items issued by FIFO in Model II and Model I respectively. Then by the
inductive assumption $x \geq y$. We consider the five mutually exclusive and exhaustive cases:

**Case 1**

$t \leq S_1 + y \leq S_1 + x$

Then $Q_{II} = x + L_2(S_1 + x) = x + c \geq y + L_1(S_1 + y) = Q_{I}$

**Case 2**

$S_1 + y < t \leq S_1 + x$ (hence $x > y$)

Then since $L_1(S) \geq -1$ for all $S \in [0, S_0]$

$$\frac{L_1(S_1 + y) - L_2(S_1 + x)}{y - x} \geq -1$$

implies $Q_{III} = x + L_2(S_1 + x) = x + c \geq y + L_1(S_1 + y) + y = Q_{I}$

**Case 3**

$S_1 + y < S_1 + x \leq t$ (hence $x > y$)

Then

$$\frac{L_1(S_1 + x) - L_1(S_1 + y)}{x - y} \geq -1$$

implies $Q_{III} = x + L_1(S_1 + x) \geq y + L_1(S_1 + y) = Q_{I}$

**Case 4**

$S_1 + y = S_1 + x \leq t$ (hence $x = y$)

Then

$$Q_{III} = x + L_1(S_1 + x) = y + L_1(S_1 + y) = Q_{I}$$
Case 5
\[ S_1 + y \leq t < S_1 + x \quad \text{(hence } x > y) \]

Then
\[
\frac{L_1(S_1 + y) - L_2(S_1 + x)}{y - x} \geq -1
\]
implies
\[
Q_{II} = x + L_2(S_1 + x) = x + c \geq y + L_1(S_1 + y) = Q_{I}
\]
And in all cases \( Q_{II} \geq Q_{I} \).

q.e.d.

Lemma 4.2: (Generalization of Lemma 4.1 to \( \nu \geq 1 \)) Let \( \nu \geq 1 \). If a FIFO issuing policy is used in both Model I and Model II, then
\( Q_{II} \geq Q_{I} \).

Proof of Lemma 4.2: By lemma 2.3 each demand source receives the same indexed items (and in the same order) under both models. Hence we can consider each demand source separately. Then \( Q_{II} = \sum_{i=1}^{\nu} Q_{II_{Mi}} \) and
\[
Q_{I} = \sum_{i=1}^{\nu} Q_{I_{Mi}} \geq Q_{II_{Mi}} \quad \text{for all } i = 1, \ldots, \nu ;
\]
therefore \( Q_{II} \geq Q_{I} \).

q.e.d.

Lemma 4.3: Let \( L(S) = aS + b \) for all \( S \in [0, S_0] \) where \( b > c > a > -1 \) and \( S_0 = -\frac{b}{a} \). Let \( \nu = 1 \). If a FIFO issuing policy is used then the total field life, \( Q_{fn} \), is
\[ Q_n = a \sum_{i=1}^{n} (1 + a)^{i-1} S_i + \frac{b}{a} [(1 + a)^n - 1]. \]

**Proof of Lemma 4.3:** By lemma 2.1, the field life of any item on issuance is positive. The proof proceeds by induction. Let \( n = 1 \) then

\[ Q_1 = aS_1 + \frac{b}{a} (1 + a - 1) = aS_1 + b \]

as required. Assume the lemma is true for \( n = k \). Then

\[ Q_{k+1} = aS_{k+1} + b + a \sum_{i=1}^{k} (1 + a)^{i-1}(S_i + aS_{k+1} + b) + \frac{b}{a} [(1 + a)^k - 1] \]

\[ = aS_{k+1} \left[ 1 + a \sum_{i=1}^{k} (1 + a)^{i-1} \right] + a \sum_{i=1}^{k} (1 + a)^{i-1}S_i + b \]

\[ + ba \sum_{i=1}^{k} (1 + a)^{i-1} + b \sum_{i=1}^{k} (1 + a)^{i-1} \]

\[ = aS_{k+1} \left[ 1 + a \left( \frac{(1 + a)^k - 1}{1 + a - 1} \right) \right] + a \sum_{i=1}^{k} (1 + a)^{i-1}S_i + b \]

\[ + b \sum_{i=1}^{k} (1 + a)^{i} \]

\[ = aS_{k+1} (1 + a)^k + a \sum_{i=1}^{k} (1 + a)^{i-1}S_i + b \sum_{i=1}^{k+1} (1 + a)^{i-1} \]

\[ = a \sum_{i=1}^{k+1} (1 + a)^{i-1}S_i + \frac{b}{a} [(1 + a)^{k+1} - 1]. \]

And by induction the lemma is proved. \[ \text{q.e.d.} \]
Lemma 4.4: Let $c, b, a$ be given real numbers such that $b > c > 0 > a > -1$. Then the function

$$L_i = \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} \quad \text{for } i = 1, 2, 3, \ldots$$

is a strictly decreasing function of $i$.

Proof of Lemma 4.4: Consider $i = k$ and $i = k + 1$ and form

$$L_k - L_{k+1} = \frac{c - b(1 + a)^{k-1}}{a(1 + a)^{k-1}} - \frac{c - b(1 + a)^k}{a(1 + a)^k}$$

$$= \frac{c}{a(1 + a)^{k-1}} - \frac{c}{a(1 + a)^k} = \frac{b + b}{a}$$

$$= \frac{c}{a(1 + a)^k} - \frac{1}{(1 + a)^k} = \frac{c}{a} \left[ \frac{1}{(1 + a)^k} - \frac{1}{(1 + a)^k} \right] = \frac{c}{a} \left[ \frac{1 + a - 1}{(1 + a)^k} \right]$$

$$= \frac{c}{(1 + a)^k} > 0 \quad \text{since } 1 + a > 0 \text{ and } c > 0.$$

Therefore $L_k > L_{k+1}$ and since $k$ was arbitrary the lemma is proved.

q.e.d.

Lemma 4.5: Let $L(S) = aS + b$ for all $S \in [0, t]$ and $L(S) = c$ for all $S \in [t, \infty)$ where $b > c > 0 > a > -1$. [Thus $t = \frac{c - b}{a}$.] Let $v = 1$.

(i) If $S_i \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}$ for some $i = 2, \ldots, n$ and if a FIFO issuing policy is used for $S_i, S_{i-1}, \ldots, S_1$ then the field life on issuance of each of the items $S_i, S_{i-1}, \ldots, S_1$ is strictly greater than $c$.
(ii) If $S_1 < \frac{c-b}{a}$ then the field life of $S_1$ is strictly greater than $c$ and if $S_1 > \frac{c-b}{a}$ then the field lives of all items on issuance are equal to $c$.

Proof of Lemma 4.5: We first prove part (ii). $S_1 < \frac{c-b}{a} = t$ implies $L(S_1) > a \left( \frac{c-b}{a} \right) + b = c$. $S_1 > \frac{c-b}{a} = t$ implies $S_1 > \frac{c-b}{a} = t$, therefore $L(S_1) = c$ for all $i$. Hence if $S_1$ is issued $x_1$ time units after the process starts

$$L(x_1 + S_1) = L(S_1) = c.$$ 

We now prove part (i). By lemma 4.4 for $i = 2, \ldots, n$

$$S_i < \frac{c-b}{a} \frac{1}{(1+a)^{i-1}} < \frac{c-b}{a} = t; \text{ hence } L(S_i) > c.$$ 

Assume item $S_{i-j}$ has field life $> c$ on issuance. It will be proved that item $S_{i-j-1}$ has field life $> c$ on issuance $(j = 1, \ldots, i-2)$. Let $x_{i-j}$ be the total field life from the FIFO issuance of $S_i, S_{i-1}, \ldots, S_{i-j}$. Then we must prove $L(S_{i-j-1} + x_{i-j}) > c$. It suffices to show that

$$S_{i-j-1} + x_{i-j} < t. \quad (4.1.1)$$

Now by lemma 4.3 and the inductive assumption

$$x_{i-j} = a \sum_{k=1}^{j+1} (1+a)^{k-1} S_{i+k-j-1} + \frac{b}{a} [(1+a)^{j+1} - 1].$$
Since $0 > a > -1$ and since $S_{i-j-1} < S_{i+k-j-1}$ for all $k = 1, \ldots, j + 1$

\[
x_{i-j} < aS_{i-j-1} \sum_{k=1}^{j+1} (1 + a)^{k-1} + \frac{b}{a} [(1 + a)^{j+1} - 1]
= S_{i-j-1} [(1 + a)^{j+1} - 1] + \frac{b}{a} [(1 + a)^{j+1} - 1].
\]

Then

\[
S_{i-j-1} + x_{i-j} < S_{i-j-1} (1 + a)^{j+1} + \frac{b}{a} [(1 + a)^{j+1} - 1]
< S_{i} (1 + a)^{j+1} + \frac{b}{a} [(1 + a)^{j+1} - 1]. \quad (4.1.2)
\]

Now by lemma 4.4 and the hypothesis of this lemma

\[
S_{i} \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} = L_{i} < L_{i-1} < \ldots < L_{j+2} = \frac{c - b(1 + a)^{j+1}}{a(1 + a)^{j+1}}
\]

where $j = 0, 1, \ldots, i - 2$ then from (4.1.2)

\[
S_{i-j-1} + x_{i-j} < \left( \frac{c - b(1 + a)^{j+1}}{a(1 + a)^{j+1}} \right) (1 + a)^{j+1} + \frac{b}{a} [(1 + a)^{j+1} - 1]
= \frac{c}{a} - \frac{b}{a} (1 + a)^{j+1} + \frac{b}{a} (1 + a)^{j+1} - \frac{b}{a}
= \frac{c - b}{a} = t.
\]

Hence (4.1.1) holds and by induction the lemma is proved.

q.e.d.
Lemma 4.6: Let \( L(S) = aS + b \) for all \( S \in [0, t] \) and \( L(S) = c \) for \( S \in [t, \infty) \) where \( b > c > 0 > a > -1 \). Let \( v = 1 \). If

\[
S_i \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} = L_i
\]

and

\[
S_{i+1} \geq \frac{c - b(1 + a)^i}{a(1 + a)^i} = L_{i+1}
\]

for some \( i = 1, \ldots, n - 1 \) and if a FIFO policy is used on \( S_1, S_{i-1}, \ldots, S_1 \) then the age of item \( S_{i+1} \) (and hence all \( S_k > S_{i+1} \)) is greater than or equal to \( t \) after these first \( i \) items \( S_1, \ldots, S_i \) are issued by FIFO.

Proof of Lemma 4.6: Note in the hypothesis and by lemma 4.4 that \( S_i < S_{i+1} \) and \( L_{i+1} < L_i \). These relationships are often used in this proof.

Case 1

\[
S_i \leq L_{i+1}
\]

By lemma 4.3 and lemma 4.5 the total field life for the first \( i \) items issued by FIFO is

\[
Q_{F_i} = a \sum_{k=1}^{i} (1 + a)^{k-1} S_k + \frac{b}{a} [(1 + a)^i - 1]
\]

and since \( 0 > a > -1 \) and \( 1 + a > 0 \) and \( S_k \leq L_{i+1} \forall k \leq i \)
\[ Q_{F_1} \geq a \sum_{k=1}^{i} (1 + a)^{k-1} \left[ \frac{c - b(l + a)^i}{a(l + a)^i} \right] + \frac{b}{a} [(1 + a)^i - 1] \]

\[ = a \left[ \frac{(1 + a)^i - 1}{1 + a - 1} \right] \left[ \frac{c - b(l + a)^i}{a(l + a)^i} \right] + \frac{b}{a} [(1 + a)^i - 1] \]

\[ = [(1 + a)^i - 1] \left[ \frac{c}{a(l + a)^i} - \frac{b}{a} + \frac{b}{a} \right] \]

\[ = \frac{c}{a} - \frac{c}{a(l + a)^i} \]

Now

\[ S_{i+1} + Q_{F_1} \geq \frac{c - b(l + a)^i}{a(l + a)^i} + Q_{F_1} \]

\[ \geq \frac{c - b(l + a)^i}{a(l + a)^i} + \frac{c}{a} - \frac{c}{a(l + a)^i} \]

\[ = \frac{c}{a(l + a)^i} - \frac{b}{a} + \frac{c}{a} = \frac{c}{a(l + a)^i} \]

\[ = \frac{c - b}{a} = t \]

\[ \therefore \text{for } S_i < \frac{c - b(l + a)^i}{a(l + a)^i} \]

\[ S_{i+1} + Q_{F_1} \geq t \]

as required.
Case 2

\[ L_{i+1} = \frac{c - b(l + a)^i}{a(l + a)^i} \leq S_i = \frac{c - b(l + a)^{i-1}}{a(l + a)^{i-1}} = L_i \]

Let \( 0 \leq \beta \leq 1 \) and let

\[ S_i = P_0 = \beta L_i + (1 - \beta)L_{i+1} \]

for some \( \beta \). Then

\[
P_0 = \beta \left[ \frac{c}{a(l + a)^{i-1}} - \frac{b}{a} \right] + (1 - \beta)\left[ \frac{c}{a(l + a)^i} - \frac{b}{a} \right]
\]

\[
= \frac{\beta c}{a(l + a)^{i-1}} + \frac{c}{a(l + a)^i} - \frac{\beta c}{a(l + a)^i} - \frac{b}{a}
\]

\[
= \frac{\beta c}{(l + a)^i} + \frac{c}{a(l + a)^i} - \frac{b}{a}
\]  \hspace{1cm} (4.1.3)  

since

\[
\frac{\beta c}{a(l + a)^{i-1}} \left[ 1 - \frac{1}{l + a} \right] = \frac{\beta c}{a(l + a)^{i-1}} \left( \frac{1 + a - 1}{l + a} \right),
\]

Again by lemma 4.3 and 4.5

\[
Q_{P_i} = a \sum_{k=1}^{i} (l + a)^{k-1}S_k + \frac{b}{a} [((l + a)^i - 1]
\]

and again since \( 0 > a > -1 \) and \( l + a > 0 \) and \( S_k \leq P_0 \) \( \forall k \leq i \)
\[ Q_{F_i} \geq a \sum_{k=1}^{i} (1 + a)^{k-1} \left[ \frac{\beta c}{(1 + a)^i} + \frac{c}{a(1 + a)^i} - \frac{b}{a} \right] + \frac{b}{a} [(1 + a)^i - 1] \]

\[ = [(1 + a)^i - 1] \left[ \frac{\beta c}{(1 + a)^i} + \frac{c}{a(1 + a)^i} - \frac{b}{a} \right] + \frac{b}{a} [(1 + a)^i - 1] \]

\[ = \beta c + \frac{c}{a} - \frac{\beta c}{(1 + a)^i} - \frac{c}{a(1 + a)^i} \]

Now \( S_{i+1} > S_i \Rightarrow S_{i+1} > P_o \) hence

\[ S_{i+1} + Q_{F_i} > P_o + Q_{F_i} \]

and using (4.1.3) and (4.1.4)

\[ > \frac{\beta c}{(1 + a)^i} + \frac{c}{a(1 + a)^i} - \frac{b}{a} + \beta c + \frac{c}{a} - \frac{\beta c}{(1 + a)^i} - \frac{c}{a(1 + a)^i} \]

\[ = \frac{c - b}{a} + \beta c \]

\[ = t + \beta c \]

\[ \geq t \]

since \( l \geq \beta \geq 0 \) and \( c > 0 \).

Thus for all \( S_i \leq L_1 \) we have

\[ S_{i+1} + Q_{F_i} \geq t \]

for any \( i = 1, \ldots, n - 1 \) satisfying the hypothesis.

q.e.d.

4.2 Optimal Policies

In this section the preceding lemmas will be used to obtain (i) the set of \( n \) policies which contains the optimal policy for Model II and (ii) the specific optimal policy when \( L_1(S) \) is linear.
Let us define $F_i A$ as the policy which issues the $i$ youngest items by FIFO first and then the remaining $n - i$ items by any arbitrary policy $A$.

**Theorem 4.1:** Let $L(S)$ be concave nonincreasing for all $S \in [0, t]$ and $L(S) = c$ for all $S \in [t, \infty)$. Let $L''(S) \geq -1$ for all $S \in (0, t]$. Let $\nu \geq 1$. If $B$ is any arbitrary policy which results in exactly $i$ items having field life $> c$ on issuance and the remaining $n - i$ items having field life $= c$ on issuance and if FIFO is optimal for Model I, then in Model II

$$Q_{F_i A} \geq Q_B.$$

**Proof of Theorem 4.1:** Denote the $i$ items in policy $B$ which have field life $> c$ on issuance and the demand sources to which the $i$ items are assigned by (abbreviate the words "field life" by "f.l.")

<table>
<thead>
<tr>
<th>Demand Source</th>
<th>Items with $f.l. &gt; c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>$S_{11}, S_{12}, \ldots, S_{1k_1}$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$S_{21}, S_{22}, \ldots, S_{2k_2}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$M_v$</td>
<td>$S_{v1}, S_{v2}, \ldots, S_{vk_v}$</td>
</tr>
</tbody>
</table>

where $i = \sum_{j=1}^{v} k_j$. 

114
Now for any \( M_j \) we can locate \( k_j \) ages \( \overline{S}_{j1}, \overline{S}_{j2}, \ldots, \overline{S}_{jk_j} \) in Model I such that the field life from each of the \( k_j \) items in Model I is the same as the field life of each of the \( k_j \) items in Model II under policy B and in the same order. This is done by: If \( \eta_1 \) items with field life \( c \) are issued first to \( M_j \) and then item \( S_{j1} \) is issued we define \( \overline{S}_{j1} = S_{j1} + \eta_1c \); if \( \eta_2 \) items of field life \( c \) are issued between \( S_{j1} \) and \( S_{j2} \) then \( \overline{S}_{j2} = L_1(S_{j1}) + (\eta_1 + \eta_2)c + S_{j2} \); etc., for the other \( \overline{S}_{j3}, \ldots, \overline{S}_{jk_j} \). Now since this relocating of items in Model I can be done for all \( M_j \)'s we have all \( i \) items so relocated.

Denote the total field life of the \( i \) items in Model II under policy B by \( x_{B_1}^i \). Denote the total field life of the \( i \) relocated items in Model I by \( Q_1i \). By the construction above \( x_{B_1}^i = Q_1i \).

Furthermore, denote by \( Q_{F_1}^i \) and \( Q_{F_1}^i \) the total field lives of the \( i \) relocated items in Model I and the \( i \) youngest items \( (S_1, \ldots, S_i) \) respectively where in both cases FIFO is used. [We know that the \( i \) youngest items must have \( S_1 < S_o \) or else in Model II under policy B there could not be \( i \) items with \( f, l, > c \).]

Since FIFO is optimal in Model I then

\[
Q_{F_1}^i \geq Q_1i
\]

and by lemma 2.5

\[
Q_{F_1}^i \geq Q_{F_1}^i.
\]

But \( Q_{F_1}^i = Q_{F_1}^i \) of lemma 4.2; thus

115
\[ Q^{II} \geq Q^{I} \equiv Q^{*} \geq Q^{I} \geq Q = x_{B} \]

where \( Q^{II} \) is the total f.l. from the FIFO issuance of the \( i \)
youngest items in Model II. If we denote the total field life from the
remaining \( n - i \) items in policy \( F_A \) by \( Q_{A_{n-i}} \) then

\[ Q_{F_A} = Q^{II} + Q_{A_{n-i}} \geq x_{B} + (n - i)c = Q_B \]

And since \( B \) was any arbitrary policy with exactly \( i \) items with
f.l. > \( c \) then \( F_A \) dominates any policy with this characteristic.

q.e.d.

Theorem 4.1 reduces the search for the optimal policy to the
policies \( F_A, \ldots, F_A \). But when \( v = 1 \), then \( F_A \), itself,
consists of \( (n - i)! \) policies and it appears that Theorem 4.1 has not
greatly reduced the set of possible policies. But we need only to apply
Theorem 4.1 over and over again to see that the optimal policy must have
the property that the \( n - i \) items issued by \( A \) all have field life
= \( c \) on issuance. For example, let \( F_A \) be a policy with \( j + k \) items
with f.l. > \( c \) on issuance for some \( k = 1, 2, \ldots, n - j \). Now none
of the first \( j \) items issued can have f.l. = \( c \) on issuance or else all
of the items initially older than \( S_j \) viz. \( S_{j+1}, S_{j+2}, \ldots, S_n \) would
also have f.l. = \( c \) and then less than \( j \) of the \( n \) items would have
f.l. > \( c \) on issuance. By Theorem 4.1 we then have that \( Q_{F_{j+k}} > Q_{F_A} \)
Thus by repeated application of the theorem we see that the optimal
policy must have the property that all of the items issued by \( A \) have
f.l. = \( c \) on issuance. But then \( A \) no longer need consist of the
(n - 1)! policies but can be reduced to any fixed policy. Hence we arbitrarily let $A = \text{LIFO}$ and we only need to search the $n$ policies $F_1L, F_2L, \ldots, F_nL$ (actually we only need to search the less than $n$ policies such that the LIFO issued items have $f.l. = c$ on issuance).

The next theorem reduces the search even further. It states that the optimal policy can be found among the $F_iL$'s which have the additional property that all of the $i$ items issued by FIFO have field life $> c$ on issuance. That the search cannot be narrowed even further is shown by an example following Theorem 4.2.

**Theorem 4.2:** Let $L(S)$ be a concave nonincreasing function for all $S \in [0, t]$ and $L(S) = c$ for all $S \in [t, \infty)$. Let $L'(S) \geq -1$ for all $S \in (0, t]$. Let $v \geq 1$. Assume FIFO is optimal for Model I.

If the policy $F_kA$ ($k = 2, \ldots, n$) has the property that only $j$ items (where $1 \leq j < k$) have field life $> c$, then there exists a policy $F_jA$ with $1 \leq j < k$ and with the property that all $i$ of the first items issued (by FIFO) have field life $> c$ on issuance and such that $Q_{F_iA} > Q_{F_jA} > Q_{F_kA}$.

**Proof of Theorem 4.2:** Note that $j \geq 1$ implies $S_1 < t$ which implies $L(S_1) > c$. Therefore the set of $F_iA$ policies such that the first $i$ items have field life $> c$ on issuance is not the empty set.

Let $F_kA = B$ in Theorem 4.1 then by application of Theorem 4.1

$Q_{F_jA} > Q_{F_kA} = Q_B$. Now if $F_jA$ has the first $j$ items with field life $> c$, this theorem is proved. Therefore assume $j_1 < j$ of the first $j$ items have $f.l. > c$; we will show only $j_1$ of all the $n$ items issued by $F_jA$ have $f.l. > c$.
Case 1

\[ S_j \geq t \]

Then \( S_{j+p} \geq t \) for all \( p = 0, 1, \ldots, n - j \) hence there are only \( j \cdot n \) items with \( f.l. > c \).

Case 2

\[ S_j < t \text{ and } j \leq \nu \]

Then \( L(S_{j-p}) > c \) for all \( p = 0, \ldots, j - 1 \) but since \( j \leq \nu \) all the \( j \) items are issued immediately to start the process; hence all \( j \) items have \( f.l. > c \) on issuance contrary to our assumption that \( j \cdot \nu < j' \).

Case 3

\[ S_j < t \text{ and } j > \nu \]

We must show that when \( S_{j+1}, \ldots, S_n \) are issued, they will have \( f.l. = c \).

Let \( S_{j-p} \) for some \( p = 1, \ldots, j - 1 \) be the oldest item among the \( S_j, \ldots, S_1 \) such that when \( S_{j-p} \) is issued it has \( f.l. = c \).

We know \( S_{j-p} \) exists since \( j \cdot \nu < j' \). Let \( S_{j-p} \) be assigned to demand source \( M_\alpha \), and let the total \( f.l. \) of all the items assigned to \( M_\alpha \) up to but not including item \( S_{j-p} \) be denoted by \( x_\alpha \). Thus \( S_{j-p} + x_\alpha = t \) since \( L(S_{j-p} + x_\alpha) = c \).

But \( S_{j-p} < S_{j+1} < \ldots < S_n \) hence \( t \leq S_{j-p} + x_\alpha < S_{j+1} + x_\alpha < \ldots < S_n + x_\alpha \). Thus if any of the items \( S_{j+1}, \ldots, S_n \) are issued to \( M_\alpha \) in the A stage of \( F_j A \), they will have \( f.l. = c \) on issuance.

Now denote the \( f.l. \) of the other demand sources at the time of issuance of \( S_{j-p} \) to \( M_\alpha \) by \( x_1, x_2, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_\nu \). Now since each demand source is busy from the time the process starts then
\[ x_1 = x_2 = \cdots = x_{\alpha-1} = x_{\alpha+1} = \cdots = x_v = x_\alpha \] at the instant \( S_{j-p} \) is issued to \( M_\alpha \). But then \( t < S_{j+1} + x_\alpha = S_{j+1} + x_1 = \cdots = S_{j+1} + x_v \), which implies that all items \( S_{j+1}, \cdots, S_n \) have f.l. \( = c \) on issuance.

Therefore in all cases there are only \( j_1 \) of the \( n \) items which have f.l. \( > c \) on issuance and these \( j_1 \) items belong to the first \( j \) items.

But then application of Theorem 4.1 again gives

\[ Q_{T_{j_1}} \geq Q_{T_{j_1}}. \]

We repeatedly use Theorem 4.1 until we achieve an \( F_iA \) policy (\( 1 \leq j < k \)) with all of the first \( i \) items issued (by FIFO) having f.l. \( > c \). This \( F_iA \) is achieved since

1. at least one such policy exists \( \text{viz.} \ F_iA \) and
2. the number of possible \( F_iA \)'s is finite.

q.e.d.

At this point is is worth noting that if \( v = 1 \) or if \( L_1(S) \) is linear then the assumption that FIFO is optimal for Model I can be removed in both theorems 4.1 and 4.2.

Now the results of Theorem 4.2 do not imply that there does not exist an \( F_rA \) policy such that the first \( r \) items have f.l. \( > c \) on issuance and \( r > 1 \). But if such an \( F_rA \) policy does exist then under the conditions of Theorem 4.2, we must have \( r < k \). This last statement is proved as follows: assume \( r \) exists and \( r \geq k \). Clearly \( r \neq k \) since the hypothesis of Theorem 4.2 is then violated. Hence consider \( r > k \). But by lemma 2.2 applied to each demand source, each
of the first \( k \) items of \( F_k A \) has age less than or equal to the age of these same \( k \) items upon issuance under \( F_r A \). But under \( F_r A \) these \( k \) items have \( f.l. > c \) on issuance; hence under \( F_k A \) these \( k \) items must also have \( f.l. > c \) on issuance. We have achieved a contradiction to the hypothesis that \( j < k \); hence it must be true that \( r < k \).

As mentioned before, Theorems 4.1 and 4.2 reduce the search for the optimal policy to those \( F_1 L \)'s with the property that the first \( 1 \) items have \( f.l. > c \) and the last \( n - 1 \) items have \( f.l. = c \) on issuance. That we cannot go further is shown by the following example:

\[
L(S) = \begin{cases} 
1.5 & \text{for } 0 \leq S \leq 1.5 \\
\frac{1}{3} S + 2 & \text{for } 1.5 \leq S \leq 4.5 \\
0.5 & \text{for } 4.5 \leq S \\
\end{cases}
\]

\[
v = 1
\]

For \( S_1 = 2.0 \) \( Q_{F_1 L} = 2.833 \)

\( S_2 = 4.0 \) \( Q_{F_2 L} = 2.777 \)

\( S_3 = 5.0 \) \( Q_{F_3 L} = 2.500 \)

\( S_4 = 6.0 \) \( Q_{F_4 L} = 2.333 \)
and $F_1L = \{S_1, S_2, S_3, S_4\}$ is optimal. But both $F_1L$ and $F_2L$ have the property that the FIFO issued items have $f.l. > c$ and the LIFO issued items have $f.l. = c$ on issuance. Hence we cannot always locate in the set of $\{F_1L\}$ policies, a unique $F_1L$ with the requisite properties. However, in Model II when we let $L_1(S) = aS + b$ with $b > c > 0 > a > -1$, we are able to isolate the unique optimal $F_1L$ policy. In addition we are able to show that if $L_1(S)$ is concave or convex and $L_1'(S) \leq -1$, then $F_1L$ is optimal. But before doing either we will prove the following lemma.

**Lemma 4.7:** Let $L(S) = aS + b$ for all $S \in [0, t]$ and $L(S) = c$ for all $S \in [t, \infty)$ where $b > c > 0 > a > -1$. Let $v = 1$. If

$$S_i < \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} = L_i$$

and

$$S_{i+1} > \frac{c - b(1 + a)^{i}}{a(1 + a)^{i}} = L_{i+1}$$

for some $i = 1, \ldots, n - 1$ then for any $F_jL$ policy with $j \leq i$, the age of item $S_{i+1}$ when it is issued is $\geq t$. Hence item $S_{i+1}$ has field life $= c$ on issuance. Consequently all $S_{i+1+k}$ for $k = 0, 1, \ldots, n - (i + 1)$ have field life = c on issuance.

**Proof of Lemma 4.7:** If $i = j$ then by lemma 4.6, this lemma holds. If $j < i$, then we consider two cases:

**Case 1:** Some item $S_k$, where $j < k \leq i$, under policy $F_jL$ has $f.l. = c$ on issuance. But then all items $S_{k+p}$ for $p = 1, \ldots, n - k$
must have \( f.l. = c \) on issuance since their initial ages are \( > S_k \) and they are issued after \( S_k \) is issued. Therefore \( S_{i+1} \) has \( f.l. = c \) on issuance since \( i + 1 > k \).

**Case 2:** All items \( S_k \) for \( j < k \leq i \) under policy \( P_{j}L \) have \( f.l. > c \) on issuance.

First note that all of the first \( j \) items issued by FIFO have \( f.l. > c \) on issuance since by lemma 4.5 \( P_{i}L \) has its first \( i \) items with \( f.l. > c \). Then by application of lemma 2.2 each of the first \( j \) items of \( P_{j}L \) must have \( f.l. > c \) on issuance.

Now by lemma 4.3 the first \( j \) items issued have total field life

\[
B_j = a \sum_{p=1}^{j} (1 + a)^{p-1}S_p + \frac{b}{a} [(1 + a)^{j} - 1]. \tag{4.2.1}
\]

Since \( P_{j}L \) says to issue in the order \( S_j, S_{j-1}, \ldots, S_2, S_1, S_{j+1}, S_{j+2}, \ldots, S_{i}, \ldots, S_n \) then by induction we will show that the total field life for items \( S_{j+1}, \ldots, S_i \) is given by

\[
C_i = a \sum_{p=1}^{i-j} (1 + a)^{p-1}S_{i-p+1} + B_j[(1 + a)^{i-j} - 1] + \frac{b}{a} [(1 + a)^{i-j} - 1]. \tag{4.2.2}
\]

First note that for \( i = j + 1 \)

\[
C_{j+1} = L(S_{j+1} + B_j) = a(S_{j+1} + B_j) + b = aS_{j+1} + aB_j + b
\]

as required. Now assume (4.2.2) holds for \( i = k - 1 \), then...
\[ C_k = L(S_k + C_{k-1} + B_j) + C_{k-1} = a(S_k + C_{k-1} + B_j) + b + B_{k-1} \]
\[ = aS_k + (1 + a)C_{k-1} + aB_j + b \]
\[ = aS_k + (1 + a)\left[ \sum_{p=1}^{k-j-1} (1 + a)^{p-1}S_{k-p} + \left( B_j + \frac{b}{a} \right) \left( (1 + a)^{k-j-1} - 1 \right) \right] \]
\[ + aB_j + b \]
\[ = a \sum_{p=1}^{k-j} (1 + a)^{p-1}S_{k-p+1} + B_j[(1 + a)^{k-j} - 1] + \frac{b}{a}[(1 + a)^{k-j} - 1] \]

which is (4.2.2) as required. Combining (4.2.1) and (4.2.2) we obtain

the total field life of the first \( i \) items issued by \( F_j \).

\[ B_j + C_i = a \sum_{p=1}^{i-j} (1 + a)^{p-1}S_{i-p+1} + B_j(1 + a)^{i-j} + \frac{b}{a}[(1 + a)^{i-j} - 1] \]
\[ = a \sum_{p=1}^{i-j} (1 + a)^{p-1}S_{i-p+1} + \frac{b}{a}[(1 + a)^{i-j} - 1] \]
\[ + \left[ a \sum_{p=1}^{j} (1 + a)^{p-1}S_p + \frac{b}{a} \left( (1 + a)^j - 1 \right) \right](1 + a)^{i-j} \]
\[ = a \sum_{p=1}^{i-j} (1 + a)^{p-1}S_{i-p+1} + a(1 + a)^{i-j} \sum_{p=1}^{j} (1 + a)^{p-1}S_p \]
\[ + \frac{b}{a}[(1 + a)^{i-j} - 1] \]

(4.2.3)

We now establish an inequality for (4.2.3) since \( 0 > a > -1 \), \( 1 + a > 0 \)

and \( S_{i+1} > S_p \) for all \( p = 1, \ldots, i \) we have

123
\[ B_j + C_i > a \sum_{p=1}^{i-j} (1 + a)^{p-1} S_{i+1} + a(1 + a)^{i-j} \sum_{p=1}^{j} (1 + a)^{p-1} S_{i+1} \]
\[ + \frac{b}{a} [(1 + a)^i - 1] \]
\[ = a S_{i+1} \left[ \sum_{p=1}^{i-j} (1 + a)^{p-1} + (1 + a)^{i-j} \sum_{p=1}^{j} (1 + a)^{p-1} \right] \]
\[ + \frac{b}{a} [(1 + a)^i - 1] \]
\[ = S_{i+1} [(1 + a)^{i-j} - 1 + (1 + a)^i - (1 + a)^{i-j}] + \frac{b}{a} [(1 + a)^i - 1] \]
\[ = \left( S_{i+1} + \frac{b}{a} \right) [(1 + a)^i - 1]. \] \hfill (4.2.4)

Now since
\[ S_{i+1} > \frac{c - b(1 + a)^i}{a(1 + a)^i} \]

we have
\[ S_{i+1} + B_j + C_i > S_{i+1} + S_{i+1} [(1 + a)^i - 1] + \frac{b}{a} [(1 + a)^i - 1] \]
\[ = S_{i+1} [(1 + a)^i + \frac{b}{a} [(1 + a)^i - 1] \]
\[ > \frac{c - b(1 + a)^i}{a(1 + a)^i} (1 + a)^i + \frac{b}{a} [(1 + a)^i - 1] \]
\[ = \frac{c - b}{a} (1 + a)^i + \frac{b}{a} (1 + a)^i - \frac{b}{a} \]
\[ = \frac{c - b}{a} = t \]

Therefore item \( S_{i+1} \) has f.l. = c on issuance.

q.e.d.
Theorem 4.3: Let $L(s) = as + b$ for all $s \in [0, t]$ and $L(s) = c$ for all $s \in [t, \infty)$ where $b > c > 0 > a > -1$. Let $v \geq 1$. Using the item indexing notation of Chapter 3 (cf. Theorem 3.6)

(a) If

$$S_j = S(j) \leq \frac{c - b(1 + a)^{\frac{n-j}{v}}}{a(1 + a)^{\frac{n-j}{v}}},$$

and

$$S(j-1) \geq \frac{c - b(1 + a)^{\frac{n-j+1}{v}}}{a(1 + a)^{\frac{n-j+1}{v}}}$$

for some $j = 1, \ldots, v$, then $F_jL$ is the optimal policy.

(b) If $S_j = S(j) \geq \frac{c - b}{a}$ for some $j = 1, \ldots, v$ then $F_jL$ is the optimal policy. [In this case $F_jL = LIFO$.]

(c) If neither (a) nor (b) is satisfied then use the Search Procedure defined in Chapter 3 and consider all adjacent pairs of items for each demand source starting with the oldest adjacent pair and ending with the newest, then if $M_j$ is the first demand source such that for two adjacent items $S_i = S_i$ and $S_{i+1} = S_{i+1}$ assigned to $M_j$

$$S_i \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} \quad (4.2.5)$$

and

$$S_{i+1} \geq \frac{c - b(1 + a)^{i}}{a(1 + a)^{i}} \quad (4.2.6)$$

for some $i \in \{v + 1, \ldots, n - v - 1\}$ then $F_iL$ is the optimal policy.
Proof of Theorem 4.3: Note that by lemma 4.4 (a), (b) and (c) are mutually exclusive and exhaustive.

We defer the proof of (a) until after we have proved (b) and (c).

Part (b): \( S_j = S_1^{(j)} \geq \frac{c - b}{a} = t \) implies all \( S_i > S_1^{(j)} > t \) and \( L(S_i) = c \) for all \( i \geq J \). But then less than \( v \) items have initial field life \( > c \) and all \( n - v \) or more items have initial f.l. = c. It is optimal to issue immediately the \( J - 1 \) or less items with f.l. \( > c \) and then issue the remaining items by any policy. But policy \( F_{j, L} \) does precisely this. Hence \( F_{j, L} \) is optimal.

Part (c): Since \( S_1^{(j)} \) is the first (in the sense of oldest) item for which (4.2.5) and (4.2.6) hold, then for all \( 0 < j - k < j \) where \( (k = 1, \ldots, j - 1) \)

\[
S_{i}^{(j-k)} > \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}
\]

(4.2.7)

and for all \( j + k > j \) where \( (k = 1, \ldots, v - j) \)

\[
S_{i+1}^{(j+k)} > \frac{c - b(1 + a)^{i}}{a(1 + a)^{i}}
\]

(4.2.8)

In addition for all \( S_p \leq S_1^{(j)} \) we have

\[
S_p \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}
\]

(4.2.9)
In (4.2.9) we consider in particular

\[ S_{i}^{(j+k)} < \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} \text{ for all } k = 1, \ldots , v - j \]  

(4.2.10)

and

\[ S_{i-1}^{(j-k)} < \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} < \frac{c - b(1 + a)^{i-2}}{a(1 + a)^{i-2}} \]  

(4.2.11)

for \( k = 1, \ldots , j - 1 \), by lemma 4.4. Hence combining (4.2.7) with (4.2.11) and (4.2.8) with (4.2.10) we have the case that all \( v - 1 \) pairs of items following the first pair, \( S_{i}^{(j)} \) and \( S_{i+1}^{(j)} \), also satisfy conditions (4.2.5) and (4.2.6) of the theorem. (Since \( \frac{n}{v} > i > 1 \) we know (4.2.7), (4.2.8), (4.2.10), and (4.2.11) exist for all \( j = 1, \ldots , v \).) We will now show

\[ Q_{F_{1}L}^{I-k} \geq Q_{F_{1}L}^{I-k} \text{ for all } k = 1, \ldots , I - 1 \]  

(4.2.12)

and

\[ Q_{F_{1}L}^{I-k} \geq Q_{F_{1}L}^{I-k} \text{ for all } k = 1, \ldots , n - I \]  

(4.2.13)

We first prove (4.2.12).

By lemma 4.5 the first I items issued under \( F_{1}L \) have f.1. > c since (4.2.5), (4.2.6), (4.2.7), (4.2.8), (4.2.10) and (4.2.11) hold for all \( M_{q} \) (\( q = 1, \ldots , v \)). We wish to show that the remaining \( n - I \) items \( S_{i+1}^{I}, \ldots , S_{n}^{I} \) under \( F_{1}L \) have f.1. = c on issuance. Then by lemma 4.7 any \( F_{I-k}L \) policy has \( S_{i+1}^{I}, \ldots , S_{n}^{I} \) with f.1. = c on issuance.
Consider item $S_{I+1}$ in $F_{I,L}$. Item $S_{I+1}$ is the first item to be issued under the LIFO part of $F_{I,L}$. We will show that item $S_{I+1}$ has $f_{L} = c$ on issuance under $F_{I,L}$. If $j = 1$ in (4.2.5) and (4.2.6) then by (4.2.8)

$$S_{I+1} = S_{I+1}^{(v)} > \frac{c - b(1 + a)^i}{a(1 + a)^i}$$

but by (4.2.9)

$$S_{I+1} - v = S_{I}^{(v)} \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}$$

hence by lemma 4.7, $S_{I+1}$ has age $\geq t$ on issuance to any $M_q$ $(q = 1, \ldots, v)$. If $j > 1$ in (4.2.5) and (4.2.6) then $S_{I+1}$ can be represented by (4.2.7) or (4.2.8). If $S_{I+1}$ is represented by (4.2.7) then

$$S_{I+1} = S_{I}^{(j-k)} > \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} > \frac{c - b(1 + a)^i}{a(1 + a)^i} \quad (4.2.14)$$

by lemma 4.4. If $S_{I+1}$ is represented by (4.2.8) then

$$S_{I+1} = S_{I}^{(j+k)} > \frac{c - b(1 + a)^i}{a(1 + a)^i}. \quad (4.2.15)$$

And in either case (4.2.9) still holds for $S_{I+1} - v$. Thus applying lemma 4.7 again the age of $S_{I+1}$ on issuance is $\geq t$ for any $M_q$ $(q = 1, \ldots, v)$. Hence for $F_{I,L}$ item $S_{I+1}$ (and all $S_k > S_{I+1}$) has $f_{L} = c$ on issuance. Now by repeated applications of lemma 4.7 to each
demand source under $F_{I-k}L$ we have that regardless of which demand
source receives item $S_{I+1}$, $S_{I+1}$ will have $f.l. = c$ on issuance.
Thus all $S_{I+1}, \ldots, S_n$ will have $f.l. = c$ on issuance. Hence policy
$F_{I-k}L$ can have at most $I$ items with $f.l. > c$ on issuance. Now
$F_{I-k}L$ cannot have less than $I-k$ items with $f.l. > c$ (namely the
first $I-k$ items). This last statement follows by applying lemma 4.5
and lemma 2.2 to each demand source and noting that the first $I$ items
issued by $F_{I}L$ have $f.l. > c$ on issuance. Now since $L(K)$ is linear
then by Zehna [11], FIFO is optimal for Model I. Thus we can apply the
results of Theorem 4.1 and Theorem 4.2 which allow us to restrict our
search for the optimal policy to those $F_{K}L$'s which have the properties
(i) the first $K$ items have $f.l. > c$ on issuance and (ii) the remain-
ing $n-K$ items have $f.l. = c$ on issuance.

Let $F_{I-k}L$ be any policy with these properties where
$k = 0, 1, \ldots, I-1$. Form

$$Q_{F_{I}L} - Q_{F_{I-k}L} \quad \text{for } k > 0 \quad (4.2.16)$$

If $k = 0$, then $F_{I}L$ is the only policy satisfying the above properties
and by the argument given in the preceding paragraph, $F_{I}L$ is then
optimal.

It will be convenient to change our notation in regard to the items
assigned to $M_q$ under any policy $F_{I}L$. By lemma 2.3 we stated that
$M_q$ receives items indexed by $(n-hv = q + 1)$, we could have
relabelled the $M_j$'s to say $M_q$ receives items indexed by $q + hv$ for
$h = 0, 1, 2, \ldots$ where $q + hv \leq n$. We will now use this second
method. Then for any two policies say $F_{L}^{q}$ and $F_{L-g}^{q}$ demand source
$M_{q}$ receives the same indexed items except under $F_{L-g}^{q}$, $M_{q}$ perhaps
receives more items of higher indexing.

Under $F_{L}^{q}$, $M_{q}$ receives $i$ (or $i-1$) items in the FIFO part
of the policy and under $F_{L-k}^{q}$, $M_{q}$ receives, say $i-k$ items in
the FIFO part of the policy where $\sum_{r=1}^{V} k_r = k$. Then if we denote by
$Q_{M_{q},i}$ and $Q_{M_{q},i-k}$ the total field life of the first $i$ items and
the first $i-k$ items issued to $M_{q}$ by FIFO under $F_{L}^{q}$ and $F_{L-k}^{q}$
respectively then we will show

$$Q_{M_{q},i} - Q_{M_{q},i-k} > k_q (1 + a)^{i-1} L(S_i) \quad (4.2.17)$$

as follows:

Apply lemma 3.5 to each pair in the right hand side of

$$Q_{M_{q},i} - Q_{M_{q},i-k} = Q_{M_{q},i} - Q_{M_{q},i-1} + Q_{M_{q},i-1} - \cdots + Q_{M_{q},i-k+1} - Q_{M_{q},i-k}$$

$$= (1 + a)^{i-1} L(S_{i}^{(q)}) + (1 + a)^{i-2} L(S_{i-1}^{(q)}) + \cdots$$

$$+ (1 + a)^{i-k} L(S_{i-k+1}^{(q)}) \quad (4.2.18)$$

but $1 + a > 0$ and $L(S_{i-p}^{(q)}) \geq L(S_i)$ for all $p = 0, 1, \ldots, k_q - 1$. 

130
\[ Q_{M, q, i} - Q_{M, q, i-k_q} \geq (1 + a)^{i-1} L(S_i) + (1 + a)^{i-2} L(S_i) + \ldots + (1 + a)^{i-k} q L(S_i) \]
\[ > k_q (1 + a)^{i-1} L(S_i) \]

which is (4.2.17) since \((1 + a)^{i-1} < (1 + a)^{i-2} < \ldots < (1 + a)^{i-k} q\).

But then in (4.2.16) we have

\[ Q_{F, I-L} - Q_{F, I-k-L} > (1 + a)^{i-1} L(S_i) \sum_{r=1}^{v} k_r - kc \]

where \(-kc\) appears since \(Q_{F, I-L}^L\) has \(k\) more items with \(f.l. = c\) than does \(Q_{F, I-L}^{I-k}\). Thus

\[ Q_{F, I-L} - Q_{F, I-k-L} > (1 + a)^{i-1} L(S_i) k - kc \]
\[ = k[(1 + a)^{i-1} L(S_i) - c] \]
\[ = k[(1 + a)^{i-1}(aS_i + b) - c] \]
\[ \geq k[(1 + a)^{i-1}\left\{a\left[\frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}\right] + b\right\} - c] \]

since \(aS_i k(1 + a)^{i-1} < 0\)

and \(0 < S_i \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}\)

\[ = k[(1 + a)^{i-1}b + c - b(1 + a)^{i-1} - c] \]
\[ = 0. \]
Therefore

\[ Q_{F_{I+L}} > Q_{F_{I-k}} \]

where \( k = 1, \ldots, I - 1 \),

and (4.2.12) holds.

We now prove (4.2.13).

By Theorem 4.2 we only need to consider policies where the first \( I + k \) items have \( f.1 > c \) on issuance.

If none exists then since (4.2.12) holds and by Theorem 4.2, \( F_{I+L} \)
is optimal. Let \( F_{I+k} \) be such a policy for \( k > 0 \). Now item \( S_{I+1} \)has \( f.1 = c \) on issuance under \( F_{I+L} \). Let us look at item \( S_{I+k+1} \)under \( F_{I+k} \). It also has \( f.1 = c \) on issuance since by lemma 2.2
the total field life of the first \( I + k \) items issued is (if we
rearrange the labelling of the \( M_q 's \) at least as large for each \( M_q \)
as it is for the first \( I \) items issued under \( F_{I+L} \). Thus
\( S_{I+k+1}, \ldots, S_n \) have \( f.1 = c \) on issuance. Now under the FIFO part
of \( F_{I+k} \) each demand source will have \( k_q \) more items assigned than
under \( F_{I+L} \), where \( k_q > 0 \) and \( \sum_{i=1}^{v} k_i = k \). Then by applying lemma 3.5
to each pair on the right hand side of

\[ Q_{M_q,i+k-1} - Q_{M_q,i} = Q_{M_q,i+k} - Q_{M_q,i+k-1} + Q_{M_q,i+k-1} - \cdots - Q_{M_q,i} \]

(4.2.19)

we obtain
\[ Q_{q, i+k} - Q_{q, i} = (1 + a)^{i+k-1} L(S_{i+k}) + \ldots + (1 + a)^i L(S_{i+1}) \]

\[ \leq [(1 + a)^{i+k-1} + (1 + a)^{i+k-2} + \ldots + (1 + a)^i] L(S_{i+1}) \]

since \( L(S_{i+1}) \geq L(S_{i+j}) \) for all \( j = 1, \ldots, k \)

\[ \leq [(1 + a)^{i+k-1} + \ldots + (1 + a)^i] L(S_{i+1}) \]

since \( L(S_{i+1}) \geq L(S_{i+j}) \)

\[ < k q (1 + a)^i L(S_{i+1}). \]  \hspace{1cm} (4.2.20)

That is

\[ Q_{q, i} - Q_{q, i+k} > - k q (1 + a)^i L(S_{i+1}) \]  \hspace{1cm} (4.2.21)

and applying this to

\[ Q_{F, i} - Q_{F, i+k} > - \sum_{r=1}^{\nu} k r (1 + a)^i L(S_{i+1}) + (I + k - I) c \]

\[ = -k(1 + a)^i L(S_{i+1}) + kc \]

\[ = -k(1 + a)^i(a S_{i+1} + b) + kc \]

\[ = -ak(1 + a)^i S_{i+1} - kb(1 + a)^i + kc \]

but \( S_{i+1} \geq \frac{c - b(1 + a)^i}{a(1 + a)^i} \) as shown in

in (4.2.14) and (4.2.15). And since \(-a > 0\), \n
\[ k > 0, \text{ and } (1 + a) > 0 \]
\[ > -ak(1 + a)^{i} \left[ \frac{c - b(l + a)^{i}}{a(l + a)^{i}} \right] - kb(l + a)^{i} + kc \]
\[ = -kc + kb(l + a)^{i} - kb(l + a)^{i} + kc \]
\[ = 0 . \quad (4.2.22) \]

Therefore \( Q_{T_{1} L} > Q_{T_{1+k} L} \) hence (4.2.13) holds since \( k > 0 \) was arbitrary.

Thus for part (c) \( F_{1 L} \) is optimal since (4.2.12) and (4.2.13) hold.

We now prove part (a). But (a) is just a special case of the proof of (4.2.12) of part (c) above. Since if

\[
S'(j) \leq \frac{c - b(l + a)^{i}}{a(l + a)^{i}} \quad \left( \frac{[n-j]}{v} \right)
\]

is the first item and

\[
S'(j-1) > \frac{c - b(l + a)^{i}}{a(l + a)^{i}} \quad \left( \frac{[n-j+1]}{v} \right)
\]

and let \( S'(j) = S_{1} \) in the proof of (4.2.12) and \( S'(j-1) = S_{I+1} \) in the proof of (4.2.12) and then \( F_{j L} = F_{1 L} \) is optimal for part (a) (where \( J = I \)).

q.e.d.

**Theorem 4.4:** Let \( L(S) \) be a concave or convex decreasing function with \( L(S) \leq -1 \) and \( L'(0) \leq -1 \) for all \( S \in [0, t] \). Let \( L(S) = c \) for all \( S \in [t, \infty) \). Let \( v \geq 1 \). Then LIFO is the optimal policy.
Proof of Theorem 4.4: First, the theorem will be proved for \( v = 1 \).

Let \( A \) be any issue policy other than LIFO and let \( A \) have \( j \) items with \( f.l. > c \) on issuance, \( j = 0, 1, \ldots, n \). We will now show that once any item \( S_i \) has been issued all items \( S_k > S_i \) which are still unissued have \( f.l. = c \). For any \( S_k > S_i \) and \( S_k \) issued after \( S_i \) let \( S_i + x \) and \( S_k + y \) be the age of items \( S_i \) and \( S_k \) respectively when they are issued to the field. Then

\[
y \geq x + L(S_i + x) > 0
\]

If \( S_i + x \geq t \) then \( L(S_i + x) = c \) and \( L(S_k + y) = c \). If \( S_i + x < t \) then \( L(S_i + x) > c \)

(i) if \( x = 0 \) then \( \frac{L(S_i) - L(t)}{S_i - t} \leq -1 \) implies

\[
L(S_i) + S_i \geq L(t) + t
\]

\[
> t \quad \text{since } L(t) = c > 0;
\]

but then

\[
y + S_k \geq L(S_i) + S_k > L(S_i) + S_i > t
\]

and

\[
L(S_k + y) = c.
\]

(ii) if \( x > 0 \) then \( \frac{L(S_i + x) - L(t)}{S_i + x - t} \leq -1 \) implies

\[
L(S_i + x) + S_i + x \geq L(t) + t > t
\]

135
and

\[ y + S_k > y + S_1 \geq L(S_1 + x) + x + S_1 > t \]

and

\[ L(S_k + y) = c \]

We are now able to apply the above result to the following three cases concerning policy A.

**Case 1**

\[ J = 0 \]

Then

\[ Q_L = L(S_1) + (n - 1)c \geq nc = Q_A \]

since \( L(S_1) \geq c \).

**Case 2**

\[ j = 1 \]

Then

\[ Q_L = L(S_1) + (n - 1)c \]

\[ Q_A = L(S_1 + x) + (n - 1)c \]

Now if

(1) \( x > 0 \) then \( \frac{L(S_1) - L(S_1 + x)}{-x} \leq -1 \) implies

\[ L(S_1) \geq L(S_1 + x) + x > L(S_1 + x) \]

but \( L(S_1) \geq L(S_1) \) therefore \( Q_L \geq Q_A \).
(ii) if $x = 0$ then $L(S_1) \geq L(S_1)$ and $Q_L \geq Q_A$.

**Case 3** \hspace{2cm} $1 \leq j \leq n$

Let $S_1$ denote the initial age of the **youngest** item issued under policy $A$ such that upon issuance of $S_1$, it has $f.l. > c$. Then all items issued after $S_1$ have $f.l. = c$, since for all $S_k > S_1$ issued after $S_1$, $S_k$ has $f.l. = c$ on issuance and for all $S_k < S_1$ issued after $S_1$, $S_k$ has $f.l. = c$ since $S_1$ is **youngest** item with $f. l. > c$ on issuance. But this implies that $S_1$ is the last item issued such that it has $f.l. > c$. Thus if $S_1 + x$ is the age of $S_1$ on issuance then all of the field life of the other $j - 1$ items with $f.l. > c$ is included in $x$. Now $j > 1$ hence there is at least one item with $f.l. > c$ issued before $S_i$ hence $x > 0$. Thus

$$\frac{L(S_1) - L(S_1 + x)}{-x} \leq -1$$

implies

$$L(S_1) \geq L(S_1 + x) + x , \hspace{2cm} (4.2.23)$$

But

$$Q_L = L(S_1) + (n - 1)c$$

and

$$Q_A = x + L(S_1 + x) + (n - p)c \hspace{0.5cm} \text{where} \hspace{0.5cm} p \geq j > 1.$$ since $x$ may include some items with $f.l. = c$. 

137
Using (4.2.23) then \( Q_L > Q_A \) since \( (n - 1)c > (n - p)c \). Now consider \( v = 2 \). If LIFO is optimal for \( v = 2 \) also, then by Zehna [11]

Theorem 4.3, LIFO is optimal for all \( 1 \leq v \leq n \), \( v \) integer.

Let A be any policy, not LIFO, and let \( j \) items under policy A have \( f.l. > c \) on issuance.

**Case 1**

\[ j \leq 2 = v \]

Then

\[ Q_L = L(S_1) + L(S_2) + (n - 2)c \]

\[ Q_A = L(S_1 + x) + L(S_j + y) + (n - 2)c \]

where \( L(S_1 + x) \geq c \), \( L(S_j + y) \geq c \), \( x \geq 0 \), \( y \geq 0 \) and without loss of generality assume \( S_1 < S_j \).

Then \( L(S_1) \geq L(S_1 + x) \) and \( L(S_2) \geq L(S_j + y) \) hence \( Q_L \geq Q_A \).

**Case 2**

\[ 2 < j \leq n \]

Let \( j_1 \) items issued to \( M_1 \) have \( f.l. > c \) and \( j_2 \) items issued to \( M_2 \) have \( f.l. > c \). Then \( j = j_1 + j_2 \) and \( j_1 \geq 0 \), \( j_2 \geq 0 \).

Let \( S_i \) and \( S_j \) denote the youngest items issued by A to \( M_1 \) and \( M_2 \) respectively such that upon issuance items \( S_i \) and \( S_j \) have \( f.l. > c \). Assume \( S_i < S_j \). Then by the same argument as in **Case 3**

\( v = 1 \) above we have

\[ L(S_1) \geq L(S_1 + x) + x \]

\[ L(S_2) \geq L(S_j + y) + y \]
and

\[ Q_A = L(S_i + x) + x + L(S_j + y) + y + (n - p)c \quad \text{where} \quad p \geq j \]

and

\[ Q_L \geq Q_A \]

as required.

q.e.d.
Chapter 5
The Dynamic Inventory-Depletion Model

As mentioned in the Introduction, we have always been assuming that new items are never added to the inventory after the process has started. In many instances this is an essential assumption of the model. However, a static model of this nature is not representative of actual inventory systems and it is interesting to ask "What sufficient conditions can be formulated when the model is dynamic, i.e., new items are continually added to the inventory, with the result that LIFO or FIFO is the optimal issuing policy?" The following sections give some answers to this question.

For the results of the following sections, we remove the assumption that new items are never added and replace it with this new assumption:

"If a new item is added to inventory, it has age $S = 0$ and initial field life $L(0)$ immediately upon entry to the inventory."

All of the other assumptions of the model, given in the introductory chapter, remain unchanged.

In addition to the new assumption we will assume that only a finite number, $N$, of new items are ever added to the inventory. The ages of the new items are all assumed to be different and we will denote the ages by $F_1, F_2, \ldots, F_N$ where $F_i > F_{i+1}$ means that item $F_i$ arrives at the inventory before item $F_{i+1}$.
The assumption that $N$ is finite is not necessarily restrictive since $N$ can be chosen so large as to encompass the "going life" of any business concern.

We now construct two different problems called (i) the "original" problem and (ii) the "extended" problem. The original problem is the dynamic inventory problem of finding the optimal issuing policy for the $n$ items $S_1 < S_2 < \cdots < S_n < S_0$ which are originally in the stockpile and the $N$ items $F_1 > F_2 > \cdots > F_N$ which are added at arbitrary times in the future. The time of arrival of item $F_N$ will be denoted by $T$ (we are presently at time zero). The extended problem is the static inventory problem of finding the optimal issuing policy for the $N+n$ items $F_N < F_{N-1} < \cdots < F_1 < S_1 < \cdots < S_n$ where all $n+N$ items are originally in the stockpile and no new items are ever added. If we consider $S < 0$ as future time in the original problem, then the extended problem can be thought of as the original problem under the transformation $L(S) = L(S + T)$ i.e., shift the ordinate axis left to the point $-T$ of the original problem. Reference figures 1 and 2 below.

"Original Problem"

![Diagram](image.png)

Figure 1
5.1 $L(S)$ Concave with $0 > L'(S) > -1$

**Theorem 5.1:** Let $L(S)$ be a concave nonincreasing function with $L'(S) \geq -1$ for all $S \leq S_o$. Let $\nu \geq 1$. If

(i) FIFO is optimal in the extended problem and

(ii) the arrival of items $F_1, \ldots, F_N$ in the original problem are timed so that no stockouts occur

then FIFO is the optimal issuing policy in the original problem, i.e., in the dynamic inventory problem.

**Proof of Theorem 5.1:** Since no stockouts occur, then the FIFO issuance of $S_n, \ldots, S_1$ and $F_1, \ldots, F_N$ in the original problem results in the same total field life as FIFO in the extended problem.

Now let us assume that there exists a policy $A$ in the original problem which gives a greater total field life than FIFO. Then policy $A$ must give a greater total field life than FIFO in the extended problem. This last statement follows from the fact that the set of all possible
policies in the extended problem includes all policies of the original problem. But FIFO is optimal in the extended problem hence we have a contradiction. Therefore there cannot exist a policy \( A \) in the original problem which has a greater total field life than FIFO. Hence FIFO is optimal for the dynamic inventory problem.

\[ \text{q.e.d.} \]

**Corollary 5.1:** Let \( L(S) = aS + b \) for all \( S \leq S_0 \) and with \( b > 0 > a > -1 \). Let \( v \geq 1 \). If no stockouts occur in the original problem, then FIFO is optimal for this dynamic inventory model (the original problem).

**Proof of Corollary 5.1:** By Zehna [11] Theorems 4.1 and 4.3, FIFO is optimal for the extended problem; hence, by Theorem 5.1 above FIFO is optimal for the original problem.

\[ \text{q.e.d.} \]

**Corollary 5.2:** Let \( L(S) \) be a concave nonincreasing function with \( L'(S) \geq -1 \) for all \( S \leq S_0 \). Let \( v = 1 \).

If no stockouts occur in the original problem, then FIFO is optimal for this dynamic inventory model (the original problem).

**Proof of Corollary 5.2:** By Lieberman [9] Theorem 3, FIFO is optimal for the extended problem; hence by Theorem 5.1 above, FIFO is optimal for the original problem.

\[ \text{q.e.d.} \]

**Corollary 5.3:** Let \( L(S) \) be concave nonincreasing with \( L'(S) \geq -1 \) for all \( S \leq S_0 \). Denote by \([x]\) the largest integer \( \leq x \) where \( x \)
is a real number. Then if the number of demand sources $v$ has
\[ \left\lfloor \frac{1}{2} (N + n + 1) \right\rfloor \leq v \leq n \]
and if no stockouts occur in the original problem, then FIFO is optimal for the original problem.

Proof of Corollary 5.3: By Theorem 2.6 FIFO is optimal for the extended problem; hence, by Theorem 5.1 above, FIFO is optimal for the original problem.

q.e.d.

The preceding theorem and corollaries were concerned with $L(S)$ concave with slope $\geq -1$ and in the linear case with $L'(S) > -1$. We now consider the linear case for $L'(S) = -1$, and show that FIFO is optimal for this case also. It is only necessary to prove that FIFO is optimal for the extended problem and then apply Theorem 5.1.

As was done in all of the preceding work, we assume that the stockpile has $n$ items of initial ages $S_1 < \cdots < S_n < S_o$ at the start. And for the time being we do not consider adding any items to the stockpile.

Lemma 5.1: Let $L(S)$ be linear for all $S \leq S_o$ and $L'(S) = -1$ for all $S < S_o$. Let $v = 1$. Then any issuing policy is optimal and the total field life of the stockpile for any issuing policy is

\[ Q^* = S_o - S_1 = L(S_1). \]

Proof of Lemma 5.1: Remember that we are only interested in the static model with $n$ items $S_1 < \cdots < S_n < S_o$.

We first note that for any two items in inventory with current age $S_i < S_j (< S_o)$ that
If \( S_1 \) is chosen to be issued first, then at the expiration of the field life of \( S_1 \), \( S_j \) will have no field life remaining:

\[
\frac{L(S_0) - L(S_1)}{S_0 - S_1} = -1, \quad \text{where} \quad L(S_0) = 0
\]

\[
\Rightarrow -L(S_1) = -S_0 + S_1
\]

\[
\Rightarrow S_0 = S_1 + L(S_1) < S_j + L(S_1)
\]

\[
\Rightarrow L(S_j + L(S_1)) = 0 \quad \text{since for all} \quad S > S_0, \quad L(S) = 0,
\]

and

if \( S_j \) is chosen to be issued first, then at the expiration of the field life of \( S_j \), \( S_i \) will still have positive field life remaining:

\[
\frac{L(S_0) - L(S_i)}{S_0 - S_i} = -1
\]

\[
\Rightarrow S_0 = S_i + L(S_i) > S_1 + L(S_i)
\]

\[
\Rightarrow L(S_i + L(S_i)) > 0 \quad \text{since for all} \quad S < S_0, \quad L(S) > 0.
\]

We now use the above two properties to show: in any issue policy

\[ A = [S_{i_1}, \ldots, S_{i_n}] \]

we can omit any items \( S_{i_j} \) for which there is
some $S_{i_j}^{(k = 1, \ldots, j - 1)}$ such that $S_{i_j}^{(k = 1)} < S_{i_j}^{(j - k)}$, since for these $S_{i_j}^{(j - k)}$, they will have no field life remaining when they are ready to be issued.

By statement (5.1.1) above when $S_{i_j}^{(j - k)}$ is issued it has current age, say $S_t$, and $S_{i_j}^{(j - k)}$ has current age $S_u^{(i.e., \text{if the total field life of the items up to } S_{i_j}^{(j - k)} \text{ is } Q, \text{ then } S_t = S_{i_j}^{(j - k)} + Q \text{ and } S_u = S_{i_j}^{(j - k)} + Q \text{ but } S_t < S_u^{(< S_0)} \text{ hence } S_u^{(j)} \text{ has no field life remaining after } S_t^{(j)} \text{ is issued.}}$

Thus of all possible policies, we only have to consider policies where each succeeding item is younger than the previously issued item since any other policy will have total field life equivalent to one of these oldest to youngest ordered policies (where all items with field life of zero have been discarded).

Now by statement (5.1.2) since $S_i < S_i^{(j)}$ for all $i = 2, \ldots, n$, we must have that upon issue at any time, $S_i^{(j)}$ will have positive field life. But as shown above any item issued after $S_i^{(j)}$ has field life of zero and can be discarded without issuance, hence $S_i^{(j)}$ is the last item to be issued under all policies which we need to consider.

It now remains to be shown that for any policy $B = [S_{i_1}^{(j)}$, \ldots, $S_{i_n}^{(j)}$] where $S_{i_j}^{(j)} > S_{i_{j+1}}^{(j)}$, for all $j$ in the policy, that $B$ has a total field life of $S_o - S_{i_n} = L(S_{i_n})$.

Let policy $B$ contain the issuance of $k$ items ($k = 1, \ldots, n$). Obviously if $k = 1$, then $B$ is LIFO and

$$\frac{L(S_o) - L(S_{i_1})}{S_o - S_{i_1}} = -1$$
\[ \Rightarrow L(S_1) = S_o - S_1 = Q_{\text{LIFO}} = Q^* . \quad (5.1.3) \]

Let \( k > 1 \) and let the total field life of the \( k - 1 \) items up to but not including \( S_1 \) be denoted by \( x \), then \( S_1 + x < S_o \) and

\[ \frac{L(S_1 + x) - L(S_o)}{S_1 + x - S_o} = -1 \]

\[ \Rightarrow L(S_1 + x) = S_o - S_1 - x . \quad (5.1.4) \]

But the total field life from policy B is

\[ Q_B = L(S_1 + x) + x \]

hence

\[ Q_B = L(S_1 + x) + x = S_o - S_1 = L(S_1) \]

by (5.1.3) and (5.1.4). Now policy B was arbitrary; thus any issue policy has total field life \( S_o - S_1 \); hence all issue policies are optimal.

**NOTE:** This result means

\[ Q_{\text{FIFO}} = Q_{\text{LIFO}} = S_o - S_1 = L(S_1) . \]

q.e.d.

**Corollary 5.4:** Let \( L(S) \) be linear, with \( L'(S) = -1 \) for all \( S < S_o \).

Let \( v = 1 \). Then any issue policy
\[ A = [S_{i1}, \ldots, S_{ij}] \text{ where } S_{ik} > S_{i(k+1)} \]

\[(k = 1, \ldots, j - 1)\]

has a total field life of

\[ Q_A = L(S_{ij}) = S_o - S_{ij} \]

**Proof of Corollary 5.4:** Let \( x \) denote the total field life up to but not including the issue of item \( S_{ij} \). Then

\[ Q_A = L(S_{ij} + x) + x \]

and by lemma 2.1 \( S_{ij} + x < S_o \).

Hence

\[
\frac{L(S_{ij} + x) - L(S_o)}{S_{ij} + x - S_o} = -1
\]

\[ \Rightarrow Q_A = L(S_{ij} + x) + x = S_o - S_{ij} \]

but

\[
\frac{L(S_{ij}) - L(S_o)}{S_{ij} - S_o} = -1
\]

\[ \Rightarrow L(S_{ij}) = S_o - S_{ij} \]

148
hence

\[ Q_A = S_o - S_{ij} = L(S_{ij}) \]

\[ q.e.d. \]

**Lemma 5.2:** Let \( L(S) \) be linear with \( L'(S) = -1 \) for all \( S \leq S_o \).

Let \( \nu \geq 1 \). Then any issuing policy which issues items \( S_1, S_2, \ldots, S_{\nu} \) (i.e. the \( \nu \) youngest items) each to a different demand source is optimal and the total field life from an optimal policy, \( Q^* \), is given by

\[ Q^* = \sum_{i=1}^{\nu} L(S_i) = \nu S_o - \sum_{i=1}^{\nu} S_i. \]  \hspace{1cm} (5.1.5)

Furthermore

\[ Q_{FIFO,\nu} = Q_{LIFO,\nu} = Q^* . \]

**Proof of Lemma 5.2:** We will first show that any policy which issues items \( S_1, S_2, \ldots, S_{\nu} \) each to different demand sources has total field life given by (5.1.5). We will then show that any other policy not of this form has field life less than (5.1.5). Finally we will show

\[ Q_{FIFO,\nu} = Q_{LIFO,\nu} = Q^* . \]

Consider any policy which issues \( S_1, \ldots, S_{\nu} \) each to different demand sources say \( M_1, \ldots, M_{\nu} \) respectively. Hence if demand source \( M_j \) receives the \( c \) items \( [S_{j_1}, \ldots, S_{j_c}] \) then by the same argument as given in (5.1.1) and (5.1.2) of lemma 5.1 we only need to consider the ordering...
\[ A_{M_j} = [S_{j_1}, \ldots, S_{j_k}] \text{ where } S_{j_1} > S_{j_k} \]

Now by corollary 5.4, the total field life obtained from policy \( A_{M_j} \)
is \( Q_{A_{M_j}} = L(S_j) = S_0 - S_j \). Since \( M_j \) was picked arbitrarily, then for all \( j = 1, \ldots, v \)

\[ Q_{A_{M_j}} = L(S_j) = S_0 - S_j \]

and

\[ Q = \sum_{j=1}^{v} Q_{A_{M_j}} = \sum_{j=1}^{v} L(S_j) = \sum_{j=1}^{v} (S_0 - S_j) = vS_0 - \sum_{j=1}^{v} S_j , \]

which is (5.1.5) as required.

Now let \( B \) be any policy which does not issue \( S_1, \ldots, S_v \) each to different \( M_1, \ldots, M_v \). Hence \( B \) must issue at least two of the items \( S_1, \ldots, S_v \) to the same demand source, say \( S_1 \) and \( S_j \) are issued to \( M_k \) where \( S_1 \leq S_i < S_j \leq S_v \). Now by (5.1.1), (5.1.2) and corollary 5.4 we have

\[ Q_{B_{M_k}} = L(S_1) = S_1 - S_0 \]

And since \( S_1 \) and \( S_j \) are issued to \( M_k \) then there is at least one \( M_t \) such that the youngest item issued to \( M_t \) has initial age \( S_t > S_v \).
Hence

\[ Q_{B_{M_t}} = L(S_t) = S_o - S_t \]

by corollary 5.4 and (5.1.1) and (5.1.2).

Thus the total field life for policy B is at most

\[
Q_B \leq \sum_{i=1}^{v} \left[ L(S_i) + L(S_t) \right] = vS_o - S_t - \sum_{i=1}^{v} S_i
\]

\[< vS_o - \sum_{i=1}^{v} S_i \quad \text{since} \quad S_i < S_t \]

= \(Q\)

in (5.1.6). Thus \(Q = Q^*\) since \(B\) was any arbitrary policy. Now \(Q_{LIFO,v}\) issues only \(S_1, \ldots, S_v\) and each to different demand sources since for all \(k > v\)

\[ S_k + L(S_v) > S_v + L(S_v) = S_o \]

\[ \Rightarrow L(S_k + L(S_v)) = 0. \]

Hence

\[ Q_{LIFO,v} = \sum_{i=1}^{v} L(S_i) = Q^* \]
Furthermore by lemma 2.3, we note that FIFO belongs to the class of policies such that items $S_1, \ldots, S_v$ are each issued to different $M_1, \ldots, M_v$; hence

$$Q_{\text{FIFO},v} = \sum_{i=1}^{v} L(S_i) = Q^*.$$  

q.e.d.

We are now able to state:

**Theorem 5.2:** Let $L(S)$ be linear with $L'(S) = -1$ for all $S \leq S_o$. Consider the original problem and the extended problem given in pages 141-142. Let $v \geq 1$. If no stockouts occur in the original problem, then FIFO is optimal for the original problem.

**Proof of Theorem 5.2:** By lemma 5.2, FIFO is optimal for the extended problem; hence by Theorem 5.1 FIFO is optimal for the original problem, the dynamic inventory model.

q.e.d.

Note that by lemma 5.2 if the $F_i$ are known for all $i = N - v + 1, \ldots, N$ then the total field life for the model of Theorem 5.2 is

$$Q^* = Q_{\text{FIFO},v} = \sum_{i=0}^{v-1} L(F_{N-i}).$$

5.2 $L(S)$ Concave or Convex with Slope $< -1$

In this section we seek the optimal issuing policy for the dynamic inventory model when $L(S)$ is concave or convex and has slope $< -1$ for all $0 \leq S \leq S_o$. The optimal policy is found for the case $v \geq 1$
demand sources; however it will be instructive to state the case \( v = 1 \) first and subsequently to state and prove the case for all \( v \) (1 \( \leq \) \( v \) \( \leq \) n). It is interesting to note that we no longer need the assumption that no stockouts occur; the reason for this will be discussed later.

We define a modified-LIFO policy (ML) for the case \( v = 1 \) in the following way: Use LIFO until a new item arrives, then discard the item currently in use and use the new item immediately.

In addition, it will be assumed that there is no penalty cost for the installation or removal of an item in the field.

Theorem 5.3: Let \( L(S) \) be a convex or concave differentiable function on \([0, S_0]\) with \( L'(S) < -1 \) on \([0, S_0]\). Let \( v = 1 \). Then modified-LIFO is the optimal issuing policy for the original problem i.e. the dynamic inventory model.

Since this theorem is a special case \((v = 1)\) of Theorem 5.4, it will not be proved here. It was presented here in order to introduce the concept of a modified-LIFO policy and some sufficient conditions under which ML is optimal. For Theorem 5.4 it will be necessary to generalize the ML concept. But before so doing we present the following useful lemma.

Lemma 5.3: Let \( L(S) \) be a convex or concave differentiable function on \([0, S_0]\) with \( L'(S) < -1 \) on \([0, S_0]\). Let there be \( n \) items \( 0 < S_1 < S_2 < \cdots < S_n < S_0 \) in inventory and no new items are ever added to the inventory. Let \( v \geq 1 \). If \( x_1 \) is the total field life
contributed by demand source \( M_i \) under any arbitrary policy \( A \) and if the \( x_i \) are ordered \( x_1 \geq x_2 \geq \ldots \geq x_v \), then

\[ x_i \leq L(S_i) \quad \text{for all } i = 1, \ldots, v. \]

By Zehna [11] Theorems 4.2 and 4.3 we know that LIFO maximizes the total field life. This lemma states that not only is that fact true but also each demand source under LIFO receives more field life than from any other policy.

**Proof of Lemma 5.3:** Assume to the contrary that \( x_i > L(S_i) \) for some \( i = 1, \ldots, v \). Then \( x_i \) must contain one or more items \( S_j < S_i \) for if all items \( S_k \) assigned to \( M_i \) under policy \( A \) are such that \( S_k > S_i \), then since LIFO is optimal for \( v = 1 \) (cf. Zehna [11] Theorems 2.4 and 2.6) we would have \( L(S_i) \geq x_i \) contrary to the assumption \( x_i > L(S_i) \).

But if \( S_j < S_i \) is assigned to \( M_i \) then there are at most \( i - 2 \) \( S \)'s which have \( S_k < S_i \), \( k \neq j \), available for assignment to the \( i - 1 \) \( M_i \)'s viz. \( M_1, \ldots, M_{i-1} \). Hence some \( M_t \), \( t = 1, \ldots, i - 1 \), does not receive any \( S_k < S_i \). Therefore as stated in the preceding paragraph we must have \( x_t \leq L(S_i) \) and

\[ x_i > L(S_i) \geq x_t \quad \text{where } t < i. \]

But \( x_i > x_t \) for \( t < i \) contradicts the hypothesis of the lemma.

Therefore

\[ x_i \leq L(S_i) \quad \text{for all } i = 1, \ldots, v. \]

q.e.d.
Let \( A \) be any arbitrary policy for issuing the \( n \) items originally in the inventory and the \( N \) items added to the inventory in the future. We define a generalized-modified-A policy, GMA, for issuing items to the \( \gamma \geq 1 \) demand sources in the following way: Use policy \( A \) until a new item arrives, then discard the oldest item currently in use in the field and immediately replace it with the new item. When \( A = \text{LIFO} \) we denote GMA by GML.

**Theorem 5.4:** Let \( L(S) \) be a convex or concave differentiable function on \([0, S_0]\) with \( L'(S) < -1 \) on \([0, S_0]\). Let there be no penalty costs for the removal or the installation of an item in the field. Let \( \gamma \geq 1 \). Then GML is the optimal issuing policy for the original problem, i.e., the dynamic inventory model.

**Proof of Theorem 5.4:** The proof will be by induction on \( N \). Let \( N = 1 \). And let the time of arrival of the new item be denoted by \( t \).

(We are initially at time zero.)

We first show that under any policy \( A \) it is always better to discard some item currently in use and use the new item immediately.

Let \( T \) be the field life remaining to demand source \( M_i \) when the new item arrives. There are three cases:

**Case (i):** \( 0 < T < S_0 \) then \( \frac{L(0) - L(T)}{T} < -1 \) implies \( L(0) > L(T) + T \) and it is better to use the new item immediately. For \( j \neq i \) the field lives of the other \( M_j \)'s are not affected by this change.
Case (ii): \( S_0 \leq T \) which implies \( L(T) = 0 \). Then
\[ L(0) \geq L(S) \geq T = T + L(T) \]
and again it is better to use the new item immediately.

Case (iii): \( T = 0 \) then \( L(0) = T + L(T) \) and the new item should be installed immediately on arrival.

In the above we have implicitly assumed for \( j \neq i \) that all \( M_j \)'s have items currently in use. If some \( M_j \) did not have any items left and if \( T > 0 \) for \( M_i \) the new item would be assigned to \( M_j \). This last remark is contained in the next two paragraphs.

We will show that the policy of assigning the new item to the demand source \( M_i \) which loses the least life by discarding the items currently assigned to it (they all have life zero after time \( L(0) \)) is better than assigning the new item to some other \( M_j, j \neq i \).

Let \( M_i \) be the demand source with the least field life remaining at time \( t \). Denote this remaining field life to \( M_i \) by \( T_{\text{min}} \).
\[ T_{\text{min}} > 0 \]. For any \( j \neq i \), let \( T_j \) be the field life remaining to \( M_j \). Then \( T_j > T_{\text{min}} \). Let \( Q \) be the total field life obtained by all the \( M_k \)'s, \( k = 1, \ldots, v \), if the new item is not issued until the current items issued to \( M_i \) and \( M_j \) expire. Then
\[ Q + L(0) - T_{\text{min}} \geq Q + L(0) - T_j, \quad \text{for any } j \neq i \].

Hence under any policy \( A \) we obtain

Statement (1): The new item should be issued immediately upon its arrival to the demand source which must discard the least field life.
Thus statement (1) says that the optimal policy for the case \( N = 1 \) must belong to the class of generalized-modified policies.

We now show GML is optimal for \( N = 1 \). Consider any policy GMA with GMA \( \neq \) GML. Let \( x_1 \geq x_2 \geq \cdots \geq x_v \) be the field life contributed by \( M_1, \ldots, M_v \) under policy A when the new item is not considered. By GMA the new item will be assigned to \( M_v \) since \( x_v \) is the smallest field life. We consider five mutually exclusive and exhaustive cases. Recall by lemma 5.3 that \( x_i \leq L(S_i) \) for all \( i \).

**Case 1** \( x_v = L(S_v) \) and \( t \leq L(S_v) \) \( t \) is the arrival time of the new item.

Then

\[
Q_{GML} = \sum_{i=1}^{v-1} L(S_i) + t + L(0) \geq \sum_{i=1}^{v-1} x_i + t + L(0) = Q_{GMA}
\]

**Case 2** \( x_v = L(S_v) \) and \( t > L(S_v) \)

Then

\[
Q_{GML} = \sum_{i=1}^v L(S_i) + L(0) \geq \sum_{i=1}^v x_i + L(0) = Q_{GMA}
\]

**Case 3** \( x_v \leq L(S_v) \) and \( t \leq x_v \)

Then

\[
Q_{GML} = \sum_{i=1}^{v-1} L(S_i) + t + L(0) \geq \sum_{i=1}^{v-1} x_i + t + L(0) = Q_{GMA}
\]
Case 4 \[ x_v < L(S_v) \text{ and } x_v < t \leq L(S_v) \]

Then \[ Q_{GML} = \sum_{i=1}^{v-1} L(S_i) + t + L(0) > \sum_{i=1}^{v} x_i + L(0) = Q_{GMA} \]

Case 5 \[ x_v < L(S_v) \text{ and } t > L(S_v) \]

Then \[ Q_{GML} = \sum_{i=1}^{v} L(S_i) + L(0) > \sum_{i=1}^{v} x_i + L(0) = Q_{GMA} \]

In all cases \( Q_{GML} \geq Q_{GMA} \) and since \( A \) was any arbitrary policy GML is optimal for \( N = 1 \).

Assume GML is optimal for adding \( N = k \) items to inventory and it will be proved that GML is optimal for adding \( N = k + 1 \) items.

We will first establish that the optimal policy must belong to the class of generalized-modified policies. Let \( T \) be the field life remaining to \( M_i \) when the \( k + 1 \)st item arrives. Let \( t_k \) and \( t_{k+1} \) denote the time of arrival of the \( k \)th and \( k + 1 \)st items respectively. Since arrivals are distinct events \( t_k < t_{k+1} \) and all items in use or in the stockpile, except the \( k + 1 \)st item, have age greater than zero at time \( t_{k+1} \), then \( L(0) > T > 0 \), and we have the same three cases as before:

Case (i) \[ 0 < T < S_0 \]

Then \[ \frac{L(0) - L(T)}{-T} < -1 \]
implies

\[ \text{L(0)} > \text{L(T)} + \text{T} \]

**Case (ii)**

\[ S_0 \leq T \]

Then

\[ \text{L(T)} = 0 \]

and

\[ \text{L(0)} > T + \text{L(T)} \]

**Case (iii)**

\[ 0 = T \]

Then

\[ \text{L(0)} = T + \text{L(T)} \]

Hence the new item should always be installed immediately on arrival. Now by the same argument given in \( N = 1 \), the new item should be assigned to the demand source which loses the least field life. Thus Statement (i) applies also to the case of \( N = k + 1 \) and the optimal policy belongs to the class of generalized-modified policies.

In order to proceed further it is necessary to develop some additional notation. Let \( Q_{M_i,N} \) and \( x_{i,N} \) denote the total field life for \( M_i \) under GML and GMA respectively. In addition relabel the \( M_i \)'s in GML and in GMA such that \( Q_{M_i,N} \geq Q_{M_{i+1},N} \) and \( x_{i,N} \geq x_{i+1,N} \) for all \( i = 1, \ldots , v - 1 \). It is possible that \( M_1 \) under GML is not the same \( M_1 \) as under GMA but this fact is of no importance in the following.

In the case \( N = 1 \) we showed \( Q_{GML} \geq Q_{GMA} \) but also in conjunction with lemma 5.3 we showed that the total field life for each \( M_i \) under GML is greater than under GMA i.e., using the notation above
\[ Q_{M_i,1} \geq x_{i,1} \text{ for all } i = 1, \ldots, v \]  
(5.2.1)

It will now be proved that

\[ Q_{M_i,k+1} \geq x_{i,k+1} \text{ for all } i = 1, \ldots, v \]  
(5.2.2)

where we inductively assume

\[ Q_{M_i,k} \geq x_{i,k} \text{ for all } i = 1, \ldots, v \]  
(5.2.3)

Now (5.2.3) and Statement (1) inform us that the \( k + 1 \)st arrival is immediately assigned to \( M_v \). We only need to show \( Q_{M_v,k+1} \geq x_{v,k+1} \).

Let \( T_{GML} \) and \( T_{GMA} \) be the total field life remaining to \( M_v \) at time \( t_{k+1} \) when GML and GMA are being followed respectively. By (5.2.3)

\[ T_{GML} \geq T_{GMA} \geq 0 \]. We again consider the five mutually exclusive and exhaustive cases:

**Case 1**

\( x_{v,k} = Q_{M_v,k} \) and \( t_{k+1} \leq Q_{M_v,k} \)

Then \( x_{v,k} = Q_{M_v,k} \) implies \( T_{GML} = T_{GMA} \) and

\[ Q_{M_v,k+1} = Q_{M_v,k} - T_{GML} + I(0) = x_{v,k} - T_{GMA} + I(0) = x_{v,k+1} \]

**Case 2**

\( x_{v,k} = Q_{M_v,k} \) and \( t_{k+1} > Q_{M_v,k} \)

Then \( Q_{M_v,k+1} = Q_{M_v,k} + I(0) = x_{v,k} + I(0) = x_{v,k+1} \)

160
Case 3 \[ x_{v,k} \leq Q_{v,k} \quad \text{and} \quad t_{k+1} \leq x_{v,k} \]

Then \[ Q_{v,k} - x_{v,k} = T_{GML} - T_{GMA} \]

and \[ Q_{v,k+1} = Q_{v,k} - T_{GML} + L(0) = x_{v,k} - T_{GMA} + L(0) = x_{v,k+1} \]

Case 4 \[ x_{v,k} < Q_{v,k} \quad \text{and} \quad x_{v,k} + t_{k+1} < Q_{v,k} \]

Then \[ T_{GMA} = 0 \quad \text{and} \quad Q_{v,k} - x_{v,k} > T_{GML} - T_{GMA} = T_{GML} \]

\[ Q_{v,k+1} = Q_{v,k} - T_{GML} + L(0) > x_{v,k} + L(0) = x_{v,k+1} \]

Case 5 \[ x_{v,k} < Q_{v,k} \quad \text{and} \quad t_{k+1} > Q_{v,k} \]

Then \[ Q_{v,k+1} = Q_{v,k} + L(0) > x_{v,k} + L(0) = x_{v,k+1} \]

Hence in all cases

\[ Q_{v,k+1} \geq x_{v,k+1} \quad (5.2.4) \]

Now since the field life for the other \( M_i \), \( i = 1, \ldots, v - 1 \) are unchanged then by (5.2.3)

\[ Q_{M_i,k} = Q_{M_i,k+1} \geq x_{i,k+1} = x_{i,k} \quad (5.2.5) \]

for all \( i = 1, \ldots, v - 1 \).
Combining (5.2.4) and (5.2.5) we see that (5.2.2) holds for all 

\[ i = 1, \ldots, v. \]

But GML for \( N = k + 1 \) yields

\[
Q_{\text{GML}} = \sum_{i=1}^{v-1} Q_{i,k} + Q_{v,k+1}
\]

and GMA for \( N = k + 1 \) yields

\[
Q_{\text{GMA}} = \sum_{i=1}^{v-1} x_{i,k} + x_{v,k+1}
\]

Hence by (5.2.4) and (5.2.5) \( Q_{\text{GML}} \geq Q_{\text{GMA}} \) where \( A \) was any arbitrary policy. Therefore by induction GML is optimal for all \( N \).

q.e.d.

5.3 The Problem of Stockouts

In the results of section 5.1, it was assumed that the ordering schedule for new items was arranged so that stockouts did not occur. This assumption was essential for FIFO optimality as the following example shows:

\[
L(S) = -\frac{1}{3} S + 3 \quad \text{for} \quad S \in [0, 9]
\]

\[
= 0 \quad \text{for} \quad S \in [9, \infty)
\]

\( v = 2 \)
Then \( \text{FIFO} = [S_5, S_3, S_1, F_2; S_4, S_2, F_1] \) which yields \( Q_F = 10.9883 \) as compared to \( A = [S_5, S_4, S_3, S_2, F_1; S_1, F_2] \) which yields \( Q_A = 11.0623 \) is definitely not optimal.

In the case of FIFO, however, a stockout occurred because the total field life for \( [S_4, S_2] \) is 1.7781 whereas item \( F_1 \) does not arrive until \( t_1 = 2.3956 \).

It is interesting to note, however, that the results of section 5.2 do not require the assumption of no stockouts. The reason for this is essentially contained in lemma 5.3 which states that each demand source receives more field life under LIFO than from any other policy. Thus if we followed a non-LIFO policy, say GMA, we could expect stockouts to be more frequent and of a much longer duration. But, the new arriving item under any generalized modified policy is used to its fullest extent. Therefore the policy which minimizes the total stockout duration will maximize the total field life; and as shown in section 5.2 this optimal policy is GML for the dynamic depletion model.

As the concluding statement in this chapter, it should be noted that results were not presented for the case \( L(S) \) convex decreasing.
with slope $L'(S) \geq -1$. Even if we assumed that LIFO was optimal for the static depletion model, there are numerous counterexamples for $v = 1$, $n = 2$, and $N = 1$ in the dynamic model where neither LIFO nor modified LIFO nor any of the other possible policies is optimal in all cases. If we desire to find the conditions in this simple case where LIFO or ML is optimal, it is necessary to make very restrictive assumptions on $S_1$, $S_2$, and $F_1$. We have not done this because the transition to general $n$ and $N$ even keeping $v = 1$ does not appear to be interesting from a practical point of view.
Chapter 6
A Stochastic Field Life Function

All of the previous results depend upon \( L(S) \) being a known function. In practice, this assumption is often not satisfied. However, it may be the case that although the specific field life function for a given item may not be known, it may be known that any given item must obey one of a finite family of field life functions. More specifically, we will assume that the field life of an item of age \( S \) on issuance is a nonnegative random variable, \( X(S) \). \( X(S) \) takes on a value \( L_i(S) \) with probability \( p_i \) (\( i = 1, \ldots, M \), \( \sum_{i=1}^{M} p_i = 1 \), \( p_i > 0 \)). Furthermore, it will be assumed that \( 0 < L_1(S) < L_2(S) < \ldots < L_M(S) \) for all \( S \in [0, S_0) \) and \( L_i(S) = L_{i+1}(S) = 0 \) for all \( S \in [S_0, \infty) \) and for all \( i = 1, \ldots, M - 1 \).

Diagrammatically for \( M = 2 \) and \( L_1(S) \) concave for \( S \in [0, S_0] \), we have
Thus with the other assumptions of the model (cf. Chapter 1) holding, the total field life, \( Q_{A_n} \), for any given realization of the process and any policy \( A \) is a function of \( n \) dependent variables. A realization of the process is e.g. \( S_1 \) lies on \( L_2 \) when issued, \( S_2 \) lies on \( L_2 \) when issued, \( S_3 \) lies on \( L_1 \) when issued, etc. for \( S_n \) through \( S_n \).

The total expected return from policy \( A \), \( U_{A_n} \), is given by
\[
U_{A_n} = \mathbb{E}[Q_{A_n}].
\]
The objective of this stochastic process is to find the issue policy which maximizes \( U_n \), the total expected field life.

The interpretation of this stochastic model follows easily. We have \( n \) items in inventory each of which deteriorates in accordance with one of \( M \) field life functions. It is not known which specific function an item will follow; however, the probability of it following a given function is fixed and independent of the age of the item. When an item is young, its range of possible values is large with respect to its actual field life, but as the items age those with lower quality initially, do not deteriorate as fast. The process of deterioration continues until all of the field life functions approach the same truncation point. When \( p_i = 1 \) for some \( i \), this model reduces to the previous deterministic model.

**Theorem 6.1:** Let \( X(S) \) be a random variable which takes on any one value of the \( M \) values \( L_i(S) \) with probability \( p_i \) (\( i = 1, 2, \ldots, M \)), \( \sum_{i=1}^{M} p_i = 1 \), \( p_i \geq 0 \). Let \( v = 1 \). Let \( L_i(S) \) have the following properties:
(1) \( L_i(S) = a_i S + b_i \) for all \( S \geq 0 \), \( S \in [0, S_0] \) and \( L_i(S) = 0 \) for all \( S < S_0 \), and all \( i = 1, \ldots, M \).

(ii) \( b_M > b_{M-1} > \cdots > b_1 > 0 > a_1 > a_2 > \cdots > a_M > -1 \)

Then FIFO maximizes the total expected return for all \( n \geq 2 \).

Proof of Theorem 6.1: For any fixed policy there are \( M^n \) possible realizations of the total field life from the \( n \) items in the stockpile. Let \( Q_{F_n}^j \) and \( Q_{A_n}^j \) be the \( j^{th} \) such possible realization by FIFO and by an arbitrary policy \( A \) respectively, where the realizations are ordered from 1 to \( M^n \) on the basis of their probability of occurrence. Thus the probability of occurrence of \( Q_{F_n}^j \), denoted by \( P(j) \) is identical to the probability of occurrence of \( Q_{A_n}^j \). Thus

\[
U_{F_n} - U_{A_n} = \sum_{j=1}^{M^n} (Q_{F_n}^j - Q_{A_n}^j) P(j) = E[Q_{F_n} - Q_{A_n}] . \tag{6.1.1}
\]

We must prove \( U_{F_n} - U_{A_n} > 0 \). It is sufficient to show

\[
Q_{F_n}^j - Q_{A_n}^j > 0 \quad \text{for all} \quad j = 1, \ldots, M^n . \tag{6.1.2}
\]

Let \( n = 2 \). Then for two items each realization must take one of these three cases:

1. \( S_1 \) lies on \( a_j S + b_j \) and \( S_2 \) lies on \( a_j S + b_j \) for some \( j = 1, \ldots, M \).

2. \( S_1 \) lies on \( a_j S + b_j \) and \( S_2 \) lies on \( a_k S + b_k \) for some \( j < k \).
(3) \( S_1 \) lies on \( a_k S + b_k \) and \( S_2 \) lies on \( a_j S + b_j \) for some \( j < k \).

Note that since \( L_j(S_o) = L_k(S_o) = 0 \)

\[
S_o = -\frac{b_j}{a_j} - \frac{b_k}{a_k}
\]

\[
\Rightarrow a_j b_k = a_k b_j \text{ for all } j, k = 1, \ldots, M.
\]  

In addition, by lemma 2.1

\[
L_k(S_1 + L_j(S_2)) > 0 \quad (6.1.4)
\]

for all \( j, k = 1, \ldots, M \).

**Case (1)**

(i) If \( L_j(S_2 + L_j(S_1)) = a_j(S_2 + a_j S_1 + b_j) + b_j = 0 \) then

\[
Q_F^{(1)} - Q_L^{(1)} = a_j S_2 + b_j + a_j(S_1 + a_j S_2 + b_j) + b_j - a_j S_1 = b_j
\]

\[
= a_j S_2 + b_j + a_j^2 S_2 + a_j b_j
\]

\[
= (1 + a_j)(a_j S_2 + b_j) > 0
\]

since \( 1 + a_j > 0 \) and \( a_j S_2 + b_j > 0 \) by assumption (7) of the model.
(11) If \( L_j(S_2 + L_j(S_1)) > 0 \) then

\[
Q_{F_2}^{(1)} - Q_{L_2}^{(1)} = a_jS_2 + b_j + a_j(S_1 + a_jS_2 + b_j) + b_j
\]

\[
- [a_jS_1 + b_j + a_j(S_2 + a_jS_1 + b_j) + b_j]
\]

\[
= a_j^2(S_2 - S_1) > 0 \quad \text{since } S_2 > S_1.
\]

Therefore \( Q_{F_2}^{(1)} > Q_{L_2}^{(1)} \) in both subcases.

**Case (2)**

(i) If \( L_k(S_2 + L_j(S_1)) = a_k(S_2 + a_jS_1 + b_j) + b_k = 0 \) then

\[
Q_{F_2}^{(2)} - Q_{L_2}^{(2)} = a_kS_2 + b_k + a_j(S_1 + a_kS_2 + b_k) + b_j - a_jS_1 - b_j
\]

\[
= (1 + a_j)(a_kS_2 + b_k) > 0
\]

since \( 1 + a_j > 0 \) and \( a_kS_2 + b_k > 0 \) by assumption (7).

(ii) If \( L_k(S_2 + L_j(S_1)) > 0 \) then using (6.1.3)

\[
Q_{F_2}^{(2)} - Q_{L_2}^{(2)} = a_kS_2 + b_k + a_j(S_1 + a_kS_2 + b_k) + b_j
\]

\[
- [a_jS_1 + b_j + a_k(S_2 + a_jS_1 + b_j) + b_k]
\]

\[
= a_ja_k(S_2 - S_1) + a_jb_k = a_kb_j
\]

\[
= a_ja_k(S_2 - S_1) > 0
\]

since \( a_ja_k > 0 \) and \( S_2 - S_1 > 0 \).

Therefore \( Q_{F_2}^{(2)} > Q_{L_2}^{(2)} \) in both subcases.
Case (3):

(i) If $L_{j}(S_{2} + L_{k}(S_{1})) = 0$ then

$$q_{F_{2}}^{(3)} - q_{L_{2}}^{(3)} = a_{j}S_{2} + b_{j} + a_{k}(S_{1} + a_{j}S_{2} + b_{j}) + b_{k} - a_{k}S_{1} - b_{k}$$

$$= (1 + a_{k})(a_{j}S_{2} + b_{j}) > 0.$$

(ii) If $L_{j}(S_{2} + L_{k}(S_{1})) > 0$ then using (6.1.3)

$$q_{F_{2}}^{(3)} - q_{L_{2}}^{(3)} = a_{j}S_{2} + b_{j} + a_{k}(S_{1} + a_{j}S_{2} + b_{j}) + b_{k}$$

$$- [a_{k}S_{1} + b_{k} + a_{j}(S_{2} + a_{k}S_{1} + b_{k}) + b_{j}]$$

$$= a_{j}a_{k}(S_{2} - S_{1}) + a_{k}b_{j} - a_{j}b_{k}$$

$$= a_{j}a_{k}(S_{2} - S_{1}) > 0.$$

Therefore $q_{F_{2}}^{(3)} > q_{L_{2}}^{(3)}$ in both subcases.

Hence in all three cases (6.1.2) holds and

$$U_{F_{2}} > U_{L_{2}}.$$

Assume (6.1.2) holds for $n$, it will be shown that (6.1.2) holds for $n + 1$. That is we must show

$$q_{n+1}^{k} - q_{A_{n+1}}^{k} > 0 \text{ for all } k = 1, \ldots, M^{n+1}.$$

Let $S^{*}$ denote the initial age of the last item issued under policy $A$ and let $S^{*}$ have realization $L_{r}(S) = a_{r}S + b_{r}$ under outcome $k$. 

170
Moreover, let $x$ denote the total field life from the first $n$ items issued under outcome $k$. Then

$$Q^k_{A^{n+1}}(x) = x + a_r(S^* + x) + b_r$$

and

$$\frac{dQ^k_{A^{n+1}}(x)}{dx} = 1 + a_r > 0.$$ 

Hence $Q^k_{A^{n+1}}(x)$ is an increasing function of $x$, and $Q^k_{A^{n+1}}$ is maximized by making $x$ as large as possible. But by the inductive assumption $x$ is maximized by FIFO; therefore $Q^k_{A^{n+1}}$ is maximized by using FIFO on the first $n$ items issued.

Now if $S^* \neq S_1$ then $S_1$ is issued next to last and since $Q^j_{F^{n+1}} - Q^j_{L^2} > 0$ for all $j = 1, \ldots, M^2$ then the total field life of $Q^k_{A^{n+1}}$ can be improved by interchanging the order of issue of the last two items. Hence $S_1$ is issued last and the first $n$ items are issued by FIFO by the application, again, of the inductive assumption. Since $k$ was arbitrary

$$Q^k_{F^{n+1}} - Q^k_{A^{n+1}} > 0 \text{ for all } k = 1, \ldots, M^{n+1}.$$ 

Therefore $U_{F^{n+1}} - U_{A^{n+1}} > 0$ since $A$ was any arbitrary policy. Thus FIFO maximizes the total expected return.

q.e.d.

Theorem 6.1 can be generalized to a countably infinite family of $L_1(S) = a_S + b_1$ where $X(S)$ takes on any $L_1(S)$ with probability $p_1$.  

171
and \( \sum_{i=1}^{\infty} p_i = 1, \quad p_i \geq 0 \) for all \( i \). This generalization follows from the fact that \( 0 < Q_{A_n}^i \leq K < \infty \) for any policy \( A \) and all \( i \), where \( K \) is a constant upper bound. Thus,

\[
U_{A_n} = \sum_{i=1}^{\infty} Q_{A_n}^i \mathbb{P}(i)
 = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} Q_{A_n}^{k_1 k_2 \cdots k_n} \frac{p_{k_1} p_{k_2} \cdots p_{k_n}}{
 \leq K \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} p_{k_1} \cdots p_{k_n} = K < \infty
\]

and

\[
U_{F_n} - U_{A_n} = \sum_{i=1}^{\infty} (Q_{F_n}^i - Q_{A_n}^i) \mathbb{P}(i) < \infty
\]

since also

\[
0 < Q_{F_n}^i - Q_{A_n}^i < K \quad \text{for all } i.
\]

It was hoped that Theorem 6.1 could be extended to the general case for \( L_1(S) \) concave nonincreasing. Repeated attempts to do this have not been successful. For one thing \( Q_{F_n}^j - Q_{A_n}^j \neq 0 \) for all \( j \). However it is possible to establish a form of Theorem 6.1 for \( L_1(S) \) concave nonincreasing when there are but two items in the stockpile.

**Theorem 6.2:** Let \( X(S) \) be a random variable which takes on any one value of the \( M \) values \( L_1(S) \) with probability \( p_i \).
\((i = 1, 2, \ldots, M, \sum_{i=1}^{M} p_i = 1, p_i \geq 0)\). Let \(v = 1\). Let \(n = 2\) items in the stockpile. Let \(L_i(S)\) have the following properties:

(i) \(L_i(S)\) is concave for all \(S \in [0, S_0]\)

(ii) \(L_i(S) = 0\) for all \(S \in [S_0, \infty)\) and \(L_i(S) > 0\) for all \(S \in [0, S_0]\)

(iii) \(0 > L'_1(S) > L'_2(S) > \cdots > L'_M(S) \geq -1\) for \(S \in [0, S_0]\).

Then FIFO maximizes the total expected return.

**Proof of Theorem 6.2:** We must show \(U_{p_2} - U_{l_2} \geq 0\). Define

\[
Q_{T,j,k} = Q_{L,j,k} = L_j(S_2) + L_k(S_1 + L_j(S_2)) - [L_k(S_1) + L_j(S_2 + L_k(S_1))]
\]

Then

\[
U_{p_2} - U_{l_2} = \sum_{i=1}^{M^2} (Q_{T,i} - Q_{L,i}) P(i) = E[Q_T - Q_L]
\]

\[
= \sum_{k=1}^{M} \sum_{j=1}^{M} (Q_{T,j,k} - Q_{L,j,k}) P_{j,k} P_k
\]

\[
= \sum_{k=1}^{M} (Q_{T,k,k} - Q_{L,k,k})^2 P_k^2 + \sum_{k=2}^{M} \sum_{j=1}^{k-1} [Q_{T,j,k} - Q_{T,j,k} + Q_{L,j,k} - Q_{L,j,k}] P_{j,k} P_k.
\]

(6.1.5)

We will show

\[
Q_{T,k,k} - Q_{L,k,k} > 0
\]

(6.1.6)

and

173
\[ Q_{jk}^F - Q_{L^1_{jk}} + Q_{L^2_{jk}} - Q_{L^1_{jk}} \geq 0 \quad \text{for all } j, k; \ j \neq k. \]

(6.1.7)

Then (6.1.5) has \( U_{L_2} - U_{L_2} \geq 0 \) since \( p_i \geq 0 \) for all \( i = 1, \ldots, M \).

Now (6.1.6) follows immediately from Lieberman [9] Theorem 3. Hence it is only necessary to prove (6.1.7). Let any \( j, k \) be given such that \( j \neq k \), then

\[ Q_{jk}^F = Q_{L_{jk}}^1 + Q_{L_{jk}}^2 - Q_{L_{jk}} \]

\[ = L_j(S_2) + L_k(S_1 + L_j(S_2)) - L_k(S_1) \]

\[ - L_j(S_2 + L_k(S_1)) + L_k(S_2) + L_j(S_1 + L_k(S_2)) \]

\[ - L_j(S_1) - L_k(S_2 + L_j(S_1)) \]  

(6.1.8)

To prove (6.1.8) is nonnegative, we first establish two relationships:

1. By lemma 2.1, \( S_1 + L_j(S_2) < S_o \) and \( S_1 + L_k(S_2) < S_o \).

2. \[ \frac{L_j(S_o) - L_j(S_2)}{S_o - S_2} \geq -1 \]

\[ \Rightarrow S_o \geq S_2 + L_j(S_2). \]

Similarly

\[ S_o \geq S_2 + L_k(S_2). \]

Case la: \( S_2 + L_k(S_1) < S_o \)

Then since \( L_j(\cdot) \) is concave nonincreasing

\[ \frac{L_j(S_2) - L_j(S_1)}{S_2 - S_1} \geq \frac{L_j(S_2 + L_k(S_1)) - L_j(S_1 + L_k(S_1))}{S_2 - S_1} \]
\[ \Rightarrow L_j(S_2) - L_j(S_1) \geq L_j(S_2 + L_k(S_1)) - L_j(S_1 + L_k(S_1)) \]
\[ \geq L_j(S_2 + L_k(S_1)) - L_j(S_1 + L_k(S_2)) \]

since \( S_2 - S_1 > 0 \) and \( L_j(S_1 + L_k(S_2)) \geq L_j(S_1 + L_k(S_1)) \).

Hence we obtain

\[ L_j(S_2) + L_j(S_1 + L_k(S_2)) - L_j(S_1) - L_j(S_2 + L_k(S_1)) \geq 0. \quad (6.1.9) \]

**Case 1b:** \( S_2 + L_k(S_1) \geq S_0 \)

Then

\[ \frac{L_j(S_2) - L_j(S_1)}{S_2 - S_1} \geq \frac{L(S_0) - L_j(S_1 + L_k(S_2))}{S_0 - (S_1 + L_k(S_2))} \]
\[ \geq \frac{L(S_0) - L_j(S_1 + L_k(S_2))}{S_2 - S_1} \]

since by (2) above \( S_0 - L_k(S_2) \geq S_2 \); hence \( S_0 - L_k(S_2) \geq S_1 \)

\[ \geq S_2 - S_1 > 0 \]

and since \( L(S_0) - L_j(S_1 + L_k(S_2)) < 0 \). But then we have

\[ L_j(S_2) + L_j(S_1 + L_k(S_2)) - L_j(S_1) \geq 0. \quad (6.1.10) \]

**Case 2a:** \( S_2 + L_j(S_1) < S_0 \)

Then just as in case 1a with \( j \) and \( k \) interchanged, we obtain

\[ L_k(S_2) + L_k(S_1 + L_j(S_2)) - L_k(S_1) - L_k(S_2 + L_j(S_1)) \geq 0. \quad (6.1.11) \]
Case 2b: \[ S_2 + L_j(S_1) > S_0 \]

Then just as in case 1b with \( j \) and \( k \) interchanged, we obtain

\[ L_k(S_2) + L_k(S_1 + L_j(S_2)) - L_k(S_1) > 0 . \quad (6.1.12) \]

Then adding (6.1.9) to (6.1.11) or (6.1.12) we see that (6.1.7) holds.
Or adding (6.1.10) to (6.1.11) or (6.1.12), again (6.1.7) holds. But
we have exhausted all possibilities for (6.1.8). Hence (6.1.7) holds for
all \( j, k, j \neq k \) since \( j \) and \( k \) were arbitrary. Therefore

\[ U_{f_2} - U_{t_2} > 0 . \]

q.e.d.

This theorem (6.2) also generalizes to the case where \( M \) is countably
infinite.

A further generalization of Theorem 6.1 can be stated immediately.

Corollary 6.1: Let the hypotheses of theorem 6.1 hold. If no stockouts
occur, then FIFO is optimal for the dynamic inventory depletion model.

Proof of Corollary 6.1: As shown in the proof of theorem 6.1, FIFO is
optimal for any realization in the extended problem. Since no stockouts
occur then FIFO for any realization in the original problem equals FIFO
for the same realization in the extended problem. Now assume there
exists some policy \( A \) which has a greater field life than FIFO for some
realization of the original problem. But then policy \( A \) yields a greater
total field life than FIFO in the extended problem since all policies of
the original problem are contained in the set of all policies for the
extended problem. Since FIFO is optimal for the extended problem we obtain a contradiction hence A cannot be optimal for the original problem. But A was arbitrary for any realization hence FIFO is optimal for the dynamic model.

q.e.d.
Chapter 7

Batches of Items of the Same Age in the Stockpile

It has always been assumed that the \( n \) items in inventory all have different initial ages. In general, this assumption is not necessary. With minor modifications, the theorems, lemmas and corollaries of the preceding chapters as well as most of the results of the papers referenced in the Bibliography can be stated for batches of items of the same age. More specifically we modify assumption (1) of the model as follows:

Assumption (1)': At the start of the process a stockpile has \( N \) sets of indivisible identical items where the items in the \( i^{th} \) set all have the same initial age, \( S_i \), for \( i = 1, \ldots, N \). The initial age of the items in any set, say the \( i^{th} \) set, is different than the initial age of the items in any other set. Assume

\[ 0 \leq S_1 < S_2 < \cdots < S_N < S_0 \]

and that the \( i^{th} \) set contains \( n_i \) items for \( i = 1, \ldots, N \). Then \( \sum_{i=1}^{N} n_i = n \).

For ease of adapting the previous results to the batch problem we make the following ordering of the \( n \) items.

Let the first \( n_1 \) items be numbered from 1 to \( n_1 \), the next \( n_2 \) items from \( n_1 + 1 \) to \( n_1 + n_2 \), the next \( n_3 \) items from \( n_1 + n_2 + 1 \) to \( n_1 + n_2 + n_3 \), etc. until all the items possess a number from 1 to \( n \) and such that \( S_i \leq S_{i+1} \) for all \( i = 1, \ldots, n - 1 \). We note that there are \( \prod_{i=1}^{N} (n_i)! \) ways to complete the above ordering; hence choosing
any one of these ways is somewhat arbitrary. However the total field life realized from the \( n \) items by any policy under any one of the \( \prod (n_i)! \) ways is the same. Thus we choose an ordering and define FIFO (LIFO) as the policy which issues the highest (lowest) indexed item in the stockpile each time an item is issued.

We now prove a general theorem which applies to most of the previous results in inventory depletion theory.

**Theorem 7.1:** Let \( L(S) \) be a continuous function. If FIFO (LIFO) is optimal when assumption (1) of the model holds, i.e., when there are no batches in the inventory, then FIFO (LIFO) is optimal when assumption (1)' holds, i.e., when batches are allowed.

**Proof of Theorem 7.1:** We will prove the theorem for FIFO; the theorem for LIFO follows *mutatis mutandis.*

Let \( \epsilon_0 = \min_{1 \leq i \leq N-1} [S_{i+1} - S_i, S_0 - S_{i}] > 0 \). \( \epsilon_0 \) exists since \( 0 \leq S_1 < S_2 < \ldots < S_N < \infty \). Consider any \( \epsilon \) such that

\[
\epsilon_0 > \epsilon > 0 \quad (7.1.1)
\]

and consider the \( n \) items defined by

\[
T_{ij} = S_i + \frac{\epsilon}{n_i - j + 1} \quad (7.1.2)
\]

for all \( j = 1, \ldots, n_i \) and \( i = 1, \ldots, N \). Then from (7.1.2) we have

179
\[ 0 < T_{11} < T_{12} < \cdots < T_{1n_1} < T_{21} < \cdots < T_{n_1} < \cdots < T_{nn_N} < S_0. \]

(7.1.3)

Denote by \( Q_F(\varepsilon) \) and \( Q_A(\varepsilon) \) the total field life from the issuance of the \( n \) items of (7.1.3) by FIFO and by an arbitrary policy \( A \), respectively. Denote by \( Q_{FB} \) and \( Q_{AB} \) the total field life from the issuance of the \( n \) items in batches by FIFO and by an arbitrary policy \( A \), respectively.

Since \( L(\cdot) \) is a continuous function, then \( Q_A \) is also a continuous function for any policy \( A \).

Hence for any \( \delta > 0 \) and \( \delta \) sufficiently small there is an \( \varepsilon > 0 \) such that \( \varepsilon \) satisfies (7.1.1) and such that

\[ |Q_{FB} - Q_F(\varepsilon)| < \delta \]

and

\[ |Q_{AB} - Q_A(\varepsilon)| < \delta. \]

Then

\[ Q_{FB} + \delta > Q_F(\varepsilon) \]

(7.1.4)

and

\[ Q_A(\varepsilon) > Q_{AB} - \delta. \]

(7.1.5)
By hypothesis however
\[ Q_T(\epsilon) \geq Q_A(\epsilon) \] (7.1.6)

for all \( \epsilon \) satisfying (7.1.1). Thus
\[ Q_{TB} > Q_{AB} - 2\epsilon. \] (7.1.7)

Now (7.1.4), (7.1.5), and (7.1.6) hold for all \( \delta > 0 \) where \( Q_B(\epsilon) \) and
\( Q_A(\epsilon) \) are defined by the issuance of the items in (7.1.3). And since
\( \delta > 0 \) can be made arbitrarily small, we have
\[ Q_{TB} > Q_{AB} \]

for any arbitrary policy \( A \).

q.e.d.

The foregoing proof also holds when we consider the stochastic case
since \( U_A \) is the sum of continuous functions, \( Q_{A_i} \); hence \( U_A \) is also
continuous. In the case of Chapter 6, the continuity of \( U_A \) in the
countably infinite case follows from the dominated convergence theorem,
i.e.
\[ |U_{AB} - U_A(\epsilon)| \leq \sum_{i=1}^{\infty} \left| Q_{AB,1} - Q_{A_i}(\epsilon) \right|_{p(1)} \]

but since \( \sum_{i=1}^{\infty} \left| Q_{AB,1} - Q_{A_i}(\epsilon) \right|_{p(1)} \leq 2K \) where \( K \) is an upper bound for
all \( Q_A \) and any \( A \), then there exists an \( N \) such that
\[
\sum_{i=N+1}^{\infty} p(i) < \frac{\epsilon}{4K} \quad \text{and we have}
\]

\[
|U_A - U_A(\epsilon)| \leq \sum_{i=1}^{\infty} |Q_{A_{B,i}} - Q_{A_1}(\epsilon)|p(i)
\]

\[
\leq \sum_{i=1}^{N} |Q_{A_{B,i}} - Q_{A_1}(\epsilon)|p(i) + 2K \left( \frac{\epsilon}{4K} \right).
\]

But \( Q_A \) is continuous hence we can choose

\[
|Q_{A_{B,i}} - Q_{A_1}(\epsilon)| < \frac{\epsilon}{2N}
\]

then

\[
|U_A - U_A(\epsilon)| \leq \sum_{i=1}^{N} |Q_{A_{B,i}} - Q_{A_1}(\epsilon)|p(i) + \frac{\epsilon}{2}
\]

\[
\leq \sum_{i=1}^{N} p(i) \frac{\epsilon}{2N} + \frac{\epsilon}{2}
\]

\[
< N \left( \frac{\epsilon}{2N} \right) + \frac{\epsilon}{2} = \epsilon.
\]

We will now consider the batch problem relative to the results of each of the preceding chapters. As mentioned previously certain modifications to the hypotheses are in order. In the theorems, lemmas and corollaries where it was assumed that \( L^*(S) \geq -1 \) for all \( S \in (0, S_o] \)
we will now assume

\[
\frac{L(S) - L(S_o)}{S - S_o} > -1 \quad \text{for all} \quad S \in (0, S_o].
\]

(7.1.8)
This assumption guarantees that none of the items under FIFO deteriorate to zero field life prior to issuance. Thus Lemma 2.1 holds for the batch problem. Rather than restating and proving all of the results of the preceding chapters, we will just prove Lemma 2.1 under assumption (7.1.8) and for the remainder of the theorems and lemmas we will merely point out certain changes which are required in order to complete their proofs for the batch case. Some of the changes will not be listed since they are quite obvious when going through the proofs.

Proof of Lemma 2.1 assuming batches and (7.1.8): The proof will be by induction. Assume \( N = 1 \). Thus there are \( n_1 = n \) items of the same age \( S_1 \), \( 0 \leq S_1 < S_0 \). Now \( L(S_1) > 0 \) for the first item issued.

Let \( x \) denote the total field life from the first \( k - 1 \) items issued by FIFO. Assume the lemma is true for the first \( k \) items issued, then \( L(S_1 + x) > 0 \) by the inductive assumption. By (7.1.8) we have

\[
\frac{L(S_0) - L(S_1 + x)}{S_0 - S_1 - x} > -1
\]

which implies \( S_1 + x + L(S_1 + x) < S_0 \). But \( S_1 + x + L(S_1 + x) \) is the age of the \( k + 1 \)st item upon issuance; hence

\( L(S_1 + x + L(S_1 + x)) > 0 \).

Now assume the lemma is true for \( N = 1 \). It will be proved true for \( N \) batches. Let \( y \) denote the total field life from the first \( N - 1 \) batches issued by FIFO but not including the last item issued in the last batch, i.e., \( y \) is the total field life from the FIFO issuance of the items indexed from \( n \) down through item \( n_1 + 2 \).

183
Then by the inductive assumption
\[ L(S_2 + y) > 0 \]
for item \( n_1 + 1 \) in the \( n_2 \)th batch. By (7.1.8)
\[
\frac{L(S_0) - L(S_2 + y)}{S_0 - S_2 + y} > -1
\]
implies
\[
S_0 > S_2 + y + L(S_2 + y) > S_1 + y + L(S_2 + y) . \quad (7.1.9)
\]
But \( S_1 + y + L(S_2 + y) \) is the age of the first item in the \( n_1 \)st set issued by FIFO; hence \( L(S_1 + y + L(S_2 + y)) > 0 \). In addition (7.1.9) says that at the time of issue of the first item in the \( n_1 \)st set all of the items in the \( n_1 \)st set have age strictly less than \( S_0 \). Hence by the same argument given in the first paragraph of this proof all of the items in the \( n_1 \)st set have positive field life on issuance by FIFO.

q.e.d.

All of the theorems, lemmas, and corollaries of the preceding chapters can be adapted to the batch case using (7.1.8) when appropriate. If a theorem, lemma, or corollary is not mentioned in the following paragraphs it is because either the result needs no modification or else the modification is very slight and quite obvious. We will proceed chapter by chapter.
In lemmas 2.4 and 2.5 it must be assumed that the number of items in each batch for set I is the same as the number of items in the corresponding batch for set II. In theorem 2.5, it is necessary to consider several more cases since \( S_j \leq S_{k+2} \). And in this theorem as well as elsewhere whenever the fact \( S_i < S_j \) is used in

\[
\frac{L(S_j + x) - L(S_i + x)}{S_j - S_i} \geq -1 \tag{7.1.10}
\]

to obtain \( L(S_j + x) + S_j \geq L(S_i + x) + S_i \), then this latter inequality will hold even when \( S_i = S_j \) and it is not necessary to use (7.1.10).

Most of the results in Chapter 3 are valid only with the addition of (7.1.8). However in lemmas 3.3, 3.4 and 3.5, it is necessary to allow \( S_{j_k} \leq S_{j_{k+1}} \) for all \( k = 1, \ldots, i - 1 \). For theorem 3.6 the augmented set will now be \( 0 \leq S_1 \leq S_2 \leq \cdots \leq S_n < S_{n+1} < \cdots < S_{n+\nu} \), etc. In lemma 3.6 we can permit \( S_1 \leq S_2 \) and the conclusion \( L^+(S_2 + L(S_1)) \geq -1 \) is still valid. Finally, as in lemmas 2.4 and 2.5, we must assume for lemma 3.7 that the number of items in each batch for set I is the same as the number of items in the corresponding batch for set II.

For Chapter 4 we make the following changes. In lemmas 4.5 and 4.7 and in theorem 4.3 assume \( S_i < \frac{c - b(l + a)^{i-1}}{a(l + a)^{i-1}} \). Furthermore in lemma 4.7 and part (c) of theorem 4.3 let \( S_{i+1} > \frac{c - b(l + a)^i}{a(l + a)^i} \) and in part (a) of theorem 4.3 let \( S_j < \frac{c - b(l + a)^{\frac{n}{[n-1]}}}{a(l + a)^{\frac{n}{[n-1]}}} \).
In Chapter 5 we modify (7.1.8) to

\[
\frac{L(S) - L(S_0)}{S - S_0} > -1 \quad \text{for all } S \in (-\infty, S_0].
\]

(7.1.11)

Also batch additions to the inventory are permitted where each item in
a given batch has field life \( L(0) \) upon arrival at the stockpile. In
the proof of lemma 5.1 note that if \( S_i = S_j \) then \( \frac{L(S_i) - L(S_j)}{S_i - S_j} = -1 \)
implies \( L(S_i) + S_i = L(S_j) + S_j = S_0 \); hence all items older than or
the same age as \( S_i \) and issued after \( S_i \) has been issued can be omitted
from consideration. The proof then goes through. In corollary 5.4
assume \( S_i > S_k \) for all \( k = l, \ldots, j - 1 \). Theorems 5.3 and 5.4
need a slight change to the definition of generalized modified policies.

GMA now becomes "when a batch of items of size \( n_i \) arrives at the
stockpile, then immediately issue all \( n_i \) items to those demand sources
which have the least useful life remaining in their items currently in
use; if \( n_i > v \) then immediately issue \( v \) of the new items, one to
each demand source."

For Chapter 6, it is only necessary to assume (7.1.8). The two
theorems and their extensions to the countably infinite case then follow.
Chapter 8
Summary and Conclusions

In the preceding chapters we have presented a considerably more general model than was originally formulated in earlier papers.

In Chapter 2 we have carried on the modification of assumption (6) which was started by Zehna [11] and Eilon [4]. For $L(S)$ concave non-increasing and $L^-(S) \geq -1$ for $S \leq S_o$, common upper and lower bounds for FIFO and for the optimal policy were obtained when there are several demand sources. And if $\left[\frac{1}{2} (n + 1)\right] \leq v \leq n$, then FIFO is optimal.

In Chapter 3 the assumption (4) of no penalty costs was removed and optimal policies were presented. For $L(S)$ convex nonincreasing, if LIPO maximizes the total field life for any $i$ items in inventory then the optimal policy must be one of the $n$ policies $I(i,v)^*$ $i = 1, \ldots, n$ which says issue the $i$ youngest items by LIPO and discard the remaining $n-i$ items. For $L(S)$ concave nonincreasing, if $L^-(S) \geq -1$ for $S \leq S_o$ and if FIFO is optimal for $i$ items in inventory then the optimal policy must be one of the $n$ policies $F(i,v)^*$. When more restrictions were placed on $L^-(S)$ and on $L^+(S)$ we were able to reduce the search even further. In particular, if $L(S) = aS + b$ where $b > 0 > a > -1$ for $S \leq S_o$ then the specific optimal policy was obtained.

In Chapter 4 we considered an S-shaped field life function, $L(S)$ concave nonincreasing for $S \leq t$, $L(S) = L(t) = c > 0$ for $S \geq t$, 187
and $L^*(S) \geq -1$ for $S \leq t$. In this case the optimal policy is one of the $n$ policies $F_i L$, $i = 1, \ldots, n$, where $F_i L$ says to issue the $i$ youngest items by FIFO and the remaining $n - i$ items by LIFO. If $L(S)$ for $S \leq t$ is linear, then the specific optimal policy was obtained. Also we showed, if $L(S)$ is convex or concave decreasing, $L^*(S) < -1$ for all $S \in (0, t]$ and $L(S) = L(t) = c > 0$ for all $S \geq t$, then LIFO is optimal.

In Chapter 5 the assumption (5) of a static inventory depletion model was removed and a dynamic model was introduced. It was seen that if $L(S)$ is concave nonincreasing, $L^*(S) \geq -1$ for $S \leq S_0$, if stockouts did not occur then if FIFO was optimal in the static model, FIFO was optimal in the dynamic model. And if $L(S)$ is concave or convex decreasing and $L^*(S) < -1$ for $S \leq S_0$, then GML (generalized-modified-LIFO) is optimal.

In Chapter 6 we again looked at the field life function but now from the viewpoint of a stochastic model. It was proved that if $X(S)$ is a nonnegative random variable which takes on any one of a countable number of nonincreasing $L_i(S)$ with probability $p_i$ and if $0 \geq L_i(S) > L_{i+1}(S) \geq -1$ for all $i$ and any $S < S_0$, then (i) if $L_i(S) = a_i S + b_i$, FIFO is optimal for $n \geq 2$ items and (ii) if $L_i(S)$ is concave nonincreasing, FIFO is optimal for $n = 2$ items.

In Chapter 7 the batch problem (where one or more of the $n$ items have the same initial age) was considered and the general result was presented that if $L(S)$ is a continuous function and if FIFO (LIFO) is optimal without batches, then FIFO (LIFO) is optimal with batches.
Consideration of the batch case led to the modification of assumption (1) of the model.

This work has presented changes to assumptions (1), (2), (4), (5) and (6). We did not consider field life functions which become negative since they do not seem to have a nice interpretation in the inventory context of the model. However it may be possible that negative $L(S)$ would be an interesting representation of some other model based on costs or profits.

Also we have not sought to change assumption (3). Assumption (3) in conjunction with assumption (6) defines the demand function on the inventory and this assumption is what gives this particular model its interest and its problems. It makes any current decision as to the optimal issuing policy entirely dependent upon all past decisions. Other authors have removed assumption (3) and assumption (6) to make the demands on the stockpile independent of the ages of the items in the stockpile. They have obtained some very interesting results (ref. [2], [3], [5], [10], and [11] in the Bibliography).

It should be noted that the theorems presented in the preceding chapters do not exhaust the possible changes to the model nor do they exhaust the possible variations on the changes already suggested. For example, the stochastic models presented in Chapter 5 of Zehna [11] and in Chapter 6 of this work are stated only for the case $\nu = 1$ and for special probability density functions and $L_i(S)$. Since the stochastic case is very important from a practical point of view, extensions to $\nu \geq 1$ and other field life functions would be desirable. It should be
mentioned that these extensions appear to be difficult since the truncation point $S_0$ makes most proofs rather complicated.

Some other areas of future research should be

(i) for the case $v > 1$ find the optimal policy for various $L(S)$'s and/or find more sufficient conditions for LIFO or FIFO optimality,

(ii) investigate other types of S-shaped functions,

(iii) consider the dynamic depletion model in relation to removing other assumptions of the model,

(iv) investigate appreciating field life functions, and

(v) look at a fixed (or minimum) spacing, $\delta$, between the initial ages of the items, i.e., $S_{j+1} \geq S_j + \delta$.

This last area would correspond to the case when inventory items arrived at the stockpile in some fixed pattern as would be the case, e.g., if the production facility was turning out one item every time period.

As a final consideration some research should be devoted to different types of objective functions. An important case of this is pointed out in Eilon [6] and Eilon [5]. He suggested that if we are a seller (rather than consumer) of the items in the stockpile and if the holding costs per item are high, then we may wish to minimize the time the items spend in inventory. This type of objective function is, in a sense, the polar image of the objective function considered in the preceding chapters since we now wish to minimize the total field life of the stockpile (at least of the first $n - v$ items issued from the stockpile). However it is not, in general, true that if FIFO (LIFO) is optimal for the one objective function that LIFO (FIFO) is optimal for the other.
Bibliography


