OPTIMAL ORDERING AND RATIONING POLICIES IN A NONSTATIONARY DYNAMIC INVENTORY MODEL WITH \( n \) DEMAND CLASSES

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0. Introduction

The theory of inventory control has dealt extensively with problems in which there exists only a single class of demand for the item under consideration. However, it is not uncommon to deal with items for which there are several "classes" of demand. That is, items for which the penalty for not being able to satisfy a demand depends on the source from which that demand is generated.

One example is a spare part which is more critical in some locations than in others.

Another example occurs when different customers for the same product yield different profits per unit purchased. This is the case, for instance, if a company has a national price list for its products so that, with transportation costs being different to different parts of the country, the profit per unit sold is a function of the geographic origin of the demand.

We consider a situation in which there are $n$ classes of demand for a single product, demands arrive stochastically, and additional stock only becomes available at some fixed time(s) in the future. Roughly, whenever a unit of a low class demand (i.e., less important and thus with a low penalty cost for being left unsatisfied) arrives one is faced with the decision of whether to satisfy that demand and thus avoid a small penalty cost or to leave that demand unsatisfied in the hope of later using that extra unit of stock to satisfy a higher class of demand thereby avoiding a high penalty cost. Thus one must ration stock in order to balance a smaller but certain cost against the possibility of a greater though uncertain cost.
The costs considered are penalty costs for leaving demand unsatisfied, holding costs, ordering costs, and salvage value (a negative cost). Of course, the penalty cost for not satisfying a demand at a given time is greater (or no less) than that for a less important demand. The objective in all cases will be to minimize the expected value of all costs.

The analysis is facilitated by breaking down the period until the next ordering opportunity into a finite number of intervals. It is assumed that the decision-maker observes all demands in a given interval before having to decide which demands to satisfy in that interval. Even if in fact demands arrive continuously in time and the decision regarding satisfying a demand must be made immediately upon its arrival, this assumption still might be expected to be a good approximation to reality if the intervals are made "small enough." No stationarity is assumed from one interval or one ordering period to another, but the demand distributions in different intervals and in different ordering periods must be independent. Most of our results will require that in any given interval unsatisfied demand is either entirely backlogged or entirely lost.

In Section 1, the optimal rationing policy is characterized under modest assumptions in the case in which no procurement opportunities remain but one incurs a final salvage cost after a finite number of intervals. In any given interval there is a non-negative (possibly infinite) critical rationing level associated with each demand class such that one should satisfy as much demand of a given class as is possible with existing stock as long as there is no unsatisfied demand of a higher class remaining and the stock level does not drop below
the critical rationing level for that class. In each interval these critical rationing levels are non-increasing functions of their associated demand class. We remark that the desirability of satisfying a marginal amount of demand of a given class does not depend on the level of unsatisfied demand of that class or of lower classes. Because in any interval the expected value of all future costs is shown to be a convex function of the stock level, if the ordering cost is convex then at the last procurement opportunity before these intervals the optimal ordering quantity at that time is determined by minimizing a convex function over \([0, \infty)\). If the ordering cost is linear at that time, then one will have the usual single critical level ordering policy. Some qualitative properties of the relevant cost functions are indicated, and when there is no backlogging a condition is given that implies that the critical rationing levels for a given demand class are non-increasing in time.

In Section 2, some computational improvements are made on the basic dynamic programming recursion for the model of Section 1. Conditions are given that, if satisfied, assure that certain critical rationing levels are 0. For instance, in most reasonable cases this result shows that the critical levels for the highest class of demand are 0. The computational scheme suggested for finding the critical rationing levels in the no backlogging case only involves recursively calculating a function of one variable. However when backlogging is allowed one must calculate functions of many variables, and much more effort is required. This effort may be reduced somewhat (especially when \(n\) is "small") when there is backlogging in each interval because then the critical levels for a given class of demand will depend neither on the
distributions of that and lower classes of demand nor on the penalty costs for lower classes.

In Section 3, the case of many ordering opportunities is considered. Here it is assumed that all unsatisfied demand remaining at the time of a procurement opportunity must then be immediately satisfied and all ordering costs are linear. The optimal rationing policy within each procurement period is characterized by a set of critical rationing levels as in Section 1, and the ordering policy is always determined by a single critical number in the usual manner. When there is backlogging in every interval of a period then the rationing policy in that period will depend only on the holding and penalty costs within that period and so these policies may be determined in a myopic single period context. If the minima of the appropriately defined single period cost functions are non-decreasing in time, then these minima will be the optimal critical ordering levels in their respective periods and if the initial stock level is small enough the single period rationing policies will also be optimal.

In a model formulated similar to ours Veinott [8], [9] also considered a problem with n demand classes, but he required the specific rationing policy that, using our terminology, has all the critical rationing levels equal to 0. In [8] he noted that the conditions he imposed on the costs would ensure that this particular rationing policy is indeed optimal.

The content of this paper represents a modification and extension of an earlier paper of the author [6] which was limited to the case n = 2. After the appearance of [6] its results were independently partially duplicated in the case n = 2 by Evans [2] and Kaplan [3].
1. **Dynamic Rationing Following a Single Procurement — The Optimal Policy**

In this section we consider the optimal policy for a single product, single period inventory problem in which there are \( n \) distinct classes of demand. Enough generality will be included so that the results of this section can easily be extended to the multi-period case in Section 3.

The period is divided into \( k \) intervals, not necessarily of equal length. The joint probability distribution for class \( 1, \ldots, n \) demands is known and has finite means for each class of demand in each interval. This distribution need not be stationary. Demands in different intervals are independent. At the beginning of each interval the demands for that interval are observed, and a decision must then be made to determine how much outstanding demand (demand arriving in that interval plus any demand backlogged from previous intervals) of classes \( 1, \ldots, n \) is to be satisfied with existing stock.

An interval is given the index \( t \in \{1, \ldots, k\} \) if \( k-t \) intervals precede it in the period. Let \( d_t = (d_1^t, \ldots, d_n^t)^T \) be the vector of new demands in interval \( t \), \( u_t = (u_1^t, \ldots, u_n^t)^T \) the vector of unsatisfied outstanding demand at the end of interval \( t \), \( b_t = (b_1^t, \ldots, b_n^t)^T \) the vector of backlogged demand at the beginning of interval \( t \), and \( B_t = b_t + d_t \) the total demand vector one faces in interval \( t \). It will be assumed that there are constants \( [a_t]_{t=1}^n \), \( a_t \geq 0 \), such that \( b_{t-1} = a_t u_t \). There is complete backlogging in interval \( t \) if \( a_t = 1 \).
and no backlogging if \( a_t = 0 \).

At the beginning of interval \( k \) the stock level \( z \) can be changed by an amount \( w \geq 0 \) for a cost \( c(w) \). This is the only ordering opportunity. At the end of interval \( l \) one incurs a cost \( v_1(z) + v_2(z-a_l \sum_{i=1}^{n} u_l^i) \) where \( z \) is the stock remaining at the end of interval \( l \). Letting \( p_t = (p_1^t, \ldots, p_n^t) \), at the end of interval \( t \) there is a penalty cost \( p_t \cdot u_t \) and a holding cost \( h_t(\cdot) \) for stock on hand at the end of interval \( t \).

We will assume

(A) \( h_t(\cdot) \) is continuous and convex on \( [0, \infty) \),

(B) \( v_1(z) \) and \( v_2(w) \) are convex and continuous on \( [0, \infty) \) and \( (-\infty, \infty) \), respectively, and \( \lim_{w \to -\infty} v_2(w) = -\infty \)

(C) \( 0 \leq p_t^1 \leq p_t^2 \leq \cdots \leq p_t^n \) for each \( t \),

and

(D) \( c(\cdot) \) is continuous on \( [0, \infty) \) and \( \lim_{w \to \infty} D^+[c(w) + \sum_{t=1}^{k} h_t(w) + v_1(w) + v_2(w)] > 0 \).

We shall proceed to characterize the optimal policy and relevant cost functions. It will be shown that the optimal policy within the period is characterized by an \( n \times k \) dimensional matrix \( \{z_t^j : t = 1, \ldots, k; j = 1, \ldots, n\} \), such that in interval \( t \) one satisfies as much class \( j \) demand as possible as long as the inventory level is not below \( z_t^j \) and there is no remaining unsatisfied class \( i \) demand for any \( i > j \). Also \( z_t^1 \geq z_t^2 \geq \cdots \geq z_t^n \) for all \( t \), and additional assumptions will be given that imply \( z_t^j \leq z_{t+1}^j \). Under the optimal policy the total expected costs for the last \( t \) interval is a convex
function of the amount of stock on hand and the vector of backlogged demand at the beginning of interval $t$. The dynamic programming formulation of the problem will indicate computational methods for determining the optimal policies and cost functions, and further simplifications of this will be noted in Section 3.

Define $f_t(z,B)$ to be the infimum over all policies of the total expected costs in the last $t$ intervals given a stock level $z$ and a vector of still backlogged and newly arrived demands $B$ at the beginning of interval $t$. Let $g_t(z,b) = E f_t(z,b+d_t)$. We represent $1 \times n$ vectors of 1's and 0's by 1 and 0, respectively. Then clearly we have

$$f_t(z,B) = \inf_{0 \leq u \leq B} \left[ p_t \cdot u + h_t(w) + g_{t-1}(w,a_t u) \right],$$

where $g_0(z,b) = v_1(z) + v_2(z-1 \cdot b)$.

The following well known [1] result will be of use in characterising $g_t(z,b)$ and $f_t(z,B)$.

**Lemma 1**: If $f(u,w)$ is a convex real-valued function on the convex set $A$, then $g(u) = \inf_{w:(u,w)\in A} f(u,w)$ is convex on $\{u : \text{there exists } (u,w)\in A\}$.

**Lemma 2**: $g_t(z,b)$ and $f_t(z,B)$ are convex and continuous on $n+1 \times [0,\infty)$ for all $t$. Thus the "infimum" on the right hand side of (1) and in the definition of $f_t(z,B)$ can be replaced by "minimum."

**Proof**: It suffices to show that this holds on the set
\[ [0,M] \times \times [0,\infty) \] for arbitrary finite \( M > 0 \). For \( z \in [0,M) \),

\[
\min_{0 \leq w \leq M} (v_1(w) + v_2(w)) - l \cdot b D^+ v_2(M) \leq
\]

\[
v_1(z) + v_2(z) - l \cdot b D^+ v_2(M) \leq g_o(z,b) \leq v_1(z) + v_2(z) - l \cdot b \lim_{w \to -\infty} D^+ v_2(w)
\]

\[
\leq \max_{0 \leq w \leq M} (v_1(w) + v_2(w)) - l \cdot b \lim_{w \to -\infty} D^+ v_2(w).
\]

\( g_o(z,b) \) is continuous by assumption. Now assume that for \( z \in [0,M] \)
\( g_{t-1}(z,b) \) is continuous and is bounded above and below by a constant
plus a linear function of \( b \). It follows immediately from (1) that
\( f_t(z,B) \) is continuous. Observe that from (1)

\[
\min_{0 \leq w \leq M} h_t(w) + \min_{0 \leq w \leq M} g_{t-1}(w,a_t u) \leq f_t(z,B)
\]

\[
0 \leq u \leq B
\]

\[
\leq p_t \cdot B + \max_{0 \leq w \leq M} h_t(w) + \max_{0 \leq w \leq M} g_{t-1}(w,a_t B)
\]

Thus \( f_t(z,B) \) is bounded above and below by a constant plus a linear
function of \( B \), so by the Lebesgue Convergence Theorem [5] and the
finite means of the demands it follows that \( g_t(z,b) \) is continuous and
also is bounded above and below by a constant plus a linear function of
\( b \).

Convexity of \( g_t(z,b) \) and \( f_t(z,B) \) then follows from Lemma 1 and
the fact that the expected value of a convex function is convex.

Let \( \delta_1 \) be an \( n \times 1 \) vector with 1 as its \( i^{th} \) element and 0's
as the other elements. Define \( y_1 \lor y_2 = \max(y_1, y_2) \), \( y_1 \land y_2 = \min(y_1, y_2) \), and \( y^+ = y \lor 0 \).

**Lemma 3:** Suppose that \( i > j \) and \( b^i \land b^j \geq \varepsilon > 0 \). Then

\[
(2) \quad f_t(z, B) \geq f_t(z, B - \varepsilon(\delta_i - \delta_j))
\]

and

\[
(3) \quad g_t(z, b) \geq g_t(z, b - \varepsilon(\delta_i - \delta_j))
\]

**Proof:** Clearly (3) holds for \( t = 0 \), and if (2) held for some \( t \) then (3) would hold for that \( t \). Thus to complete an induction proof it would suffice to show that if (3) held for \( t-1 \) then (2) would hold for \( t \). Now assume that (3) holds for \( t-1 \). Suppose that for some \( i > l, B^i \geq \varepsilon \geq 0 \). Let \( \hat{u} \) be the optimal feasible solution for the minimization problem on the right hand side of (1), given \( z \) and \( B \). For some \( j < i \), define \( \tilde{u} = \hat{u} - (\delta_i - \delta_j)(\tilde{u}^i + \varepsilon \cdot B^i)^+ \). Then \( z \geq l \cdot (B - \hat{u}) = l \cdot (B - \varepsilon(\delta_i - \delta_j) - \tilde{u}) \) and \( 0 \leq \tilde{u} \leq B - \varepsilon(\delta_i - \delta_j) \), so

\[
f_t(z, B) - f_t(z, B - \varepsilon(\delta_i - \delta_j))
\]

\[
= p_t \hat{u} + h_t(z - l \cdot (B - \hat{u})) + g_{t-1}(z - l \cdot (B - \hat{u}), a_t \hat{u})
\]

\[- p_t \tilde{u} - h_t(z - l \cdot (B - \varepsilon(\delta_i - \delta_j) - \tilde{u}))
\]

\[- g_{t-1}(z - l \cdot (B - \varepsilon(\delta_i - \delta_j) - \tilde{u}), a_t \tilde{u})
\]

\[= p_t \tilde{u}(\tilde{u}^j - \tilde{u}^i) + p_t^i(u^i - \tilde{u}^i) + g_{t-1}(z - l \cdot (B - \hat{u}), a_t \hat{u})
\]

\[- g_{t-1}(z - l \cdot (B - \hat{u}), a_t (\hat{u} - (\tilde{u}^i - \tilde{u}^i)(\delta_i - \delta_j)))
\]

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\[ \geq 0 \text{ because } p^i_t \geq p^j_t, \quad \tilde{u}^i_t - \tilde{u}^i_t = -\tilde{u}^j_t + \tilde{u}^j_t \geq 0, \quad a_t \geq 0, \]

and by the induction assumption.

The above lemma may be intuitively obvious, but one should note the critical manner in which the assumption \( b_{t-1} = a_t u_t \) comes into play.

For any \( y = (y^1, \ldots, y^n) \), define \( y^{(j)} = \sum_{i=j}^n y^i \).

For \( w \in [(2-B(1)^+), z] \), define

\[
(4) \quad f_t(w;z,B) = \min_{0 \leq u \leq B} \left[ p_t \cdot u + g_{t-1}(w, a_t u) \right] + h_t(w)
\]
\[
= \min_{1 \cdot (B-u)=z-w} \left[ p_t \cdot u + g_{t-1}(w, a_t u) \right] + h_t(w)
\]

and

\[
u(z-w,B) = (u^1(z-w,B), \ldots, u^n(z-w,B))^T
\]

where

\[
(5) \quad u^j(z-w,B) = (B(1)^-z+w)^+ \wedge B^j.
\]

\( f_t(w;z,B) \) may be interpreted as the optimal \( t \) interval expected cost function given an initial vector of demands \( B \) waiting to be satisfied in interval \( t \) and an initial stock level \( z \) of which exactly \( z-w \) units are used to satisfy demand in interval \( t \). \( u(z-w,B) \) is that vector of backlog remaining at the end of an interval given that there was a backlog vector \( B \) directly following the arrival of demand in the interval, that \( z-w \) units of demand were satisfied during the interval, and that no demand of a given class is satisfied unless all higher class demand has been satisfied (i.e., \( u^j < B^j \) implies \( u^i = 0 \) for all \( i > j \)). Clearly
\[ f_t(z,B) = \min_{w \in [(z-B(1))^+,z]} f_t(w;z,B) \]

Lemma 4: \(u(z-w,B)\) is feasible for (4) and minimizes the right hand side of (4).

Proof: Observe that \(u^j(z-w,B) < B^j\) implies \(u_i^j(z-w,B) = 0\) for all \(i > j\) and clearly \(u(z-w,B)\) is feasible for (4). Let \(j(u) = \min\{n+1, \{i : u_i^j < B_i^j\}\}\). For the minimization problem in (4) pick an optimal feasible \(\hat{u}\) with the largest possible value of \(j(\hat{u})\) and the largest number of 0 elements among \((\hat{u}_1^j)^+1, \ldots, \hat{u}_n^j\) given \(j(\hat{u})\).

Clearly \(\hat{u} = u(z-w,B)\) if \(j(\hat{u}) = n\), so assume \(j(\hat{u}) < n\). Evidently \(j(u(z-w,B)) \geq j(\hat{u})\). Assume \(u(z-w,B) \neq \hat{u}\). Then there exists \(i > j(\hat{u})\) such that \(\hat{u}_i^j > 0\). Let \(\epsilon = \hat{u}_i^j \wedge (B_i^j(\hat{u}) - \hat{u}_i^j) > 0\) and \(\hat{u} = \hat{u} - \epsilon(\delta_i - \delta_j(\hat{u}))\).

Now \(\hat{u}\) is feasible for the minimization problem in (4) and

\[
p_t \cdot \hat{u} + \delta_{t-1}(w,a_t \hat{u}) - p_t \hat{u} = \delta_{t-1}(w,a_t \hat{u}) - \delta_{t-1}(w,a_t \hat{u} - a_t \epsilon(\delta_i - \delta_j(\hat{u})))
\]

\[
\geq 0 \text{ by } p_t^i \geq p_t^j(\hat{u}) \text{ and Lemma 3 ,}
\]

so \(\hat{u}\) is also an optimal feasible solution for the minimization problem in (4). If \(\epsilon = B_i^j(\hat{u}) - \hat{u}_i^j(\hat{u})\) then \(j(\hat{u}) \geq j(\hat{u}) + 1\) and if \(\epsilon = \hat{u}_i^j < B_i^j(\hat{u}) - \hat{u}_i^j(\hat{u})\) then \(j(\hat{u}) = j(\hat{u})\) and \([\hat{u}^j_1(\hat{u})^+1, \ldots, \hat{u}_n^j]\) has one more 0 element than \([\hat{u}^j_1(\hat{u})^+1, \ldots, \hat{u}_n^j]\), so in either case there
is a contradiction. Thus \( u(z-w, B) = \tilde{u} \).

Applying Lemma 4 for that \( \tilde{w} \) for which \( f^*_t(\tilde{w}; z, B) = f^*_t(z, B) \) we have

**Corollary 1:** Given \( z \) and \( B \) in interval \( t \), there exists an optimal feasible \( u \) such that

\[
u^j < B^j \text{ implies } u^i = 0 \text{ for all } i > j.
\]

When there is either complete backlogging or no backlogging in each interval it will be shown that the desirability of satisfying a marginal unit of class \( j \) demand will be independent of the levels of still unsatisfied demand of classes \( 1, \ldots, j \), and hence the optimal rationing policy in a given interval can be completely characterized by \( n \) critical rationing levels. Unfortunately, as is demonstrated in the following example, if \( a_t \epsilon (0, 1) \) this property does not necessarily hold and the characterization of the optimal rationing policy may become more complex.

**Example:**

Suppose \( v_1(\cdot) = v_2(\cdot) = h_1(\cdot) = h_2(\cdot) = 0 \), \( a_2 \epsilon (0, 1), n = 2 \), \( p_1 = p_2 = 1 \), and \( d_1^1 \) and \( d_1^2 \) are each exponentially distributed with mean 1. Let

\[
\frac{p_1^2}{p_2} = 16\left[1 + \frac{1}{16} - (1-a_2) \frac{1}{6}(1+\lambda n 8) + \frac{1}{2}(1-a_2)(\frac{1}{6}(1+\lambda n 8) - \frac{1}{12}(1+\lambda n 12))\right].
\]

Because \( \frac{1}{x}(1+\lambda n x) \) is strictly decreasing in \( x \) for \( x \geq 1 \), it is easily seen that

\[12\]
\[ 16\left[ 1 + \frac{1}{16} - (1-a_2) \frac{1}{12}(1+\ln 12) \right] > p_1^2 \]

\[ > 16\left[ 1 + \frac{1}{16} - (1-a_2) \frac{1}{8}(1+\ln 8) \right] > 1 = p_1^1. \]

Also let \( z = \ln 16 \), \( B_1 = \frac{1}{a_2} (z-\ln 12) \), and \( B_1 = \frac{1}{a_2} (z-\ln 8) \). We will show that for a stock level \( z \) in period 2 if the backlog vector after demand arrives is \( (B_1^1,0) \), then a strictly positive amount of demand should be satisfied; but if the backlog vector is \( (B_1^1,0) \), then no demand should be satisfied. We allow \( p_1^2, B_1^1 \), and \( B_1^1 \) to depend on \( a_2 \) so that this example is valid for any \( a_2 \in (0,1) \). In interval 1 one incurs a cost \( p_1^1 \) for each unit of total demand in excess of the stock level and an additional unit cost \( p_1^2 - p_1^1 \) is added for the excess of class 2 demand over the stock level. Hence, observing that 

\( (d_1^2+d_1^1) \) has a Gamma distribution function with parameters 1 and 2,

for \( a_2 B_1^1 \leq z \)

\[ e_1(z,a_2(B_1^1,0)) = (p_1^2-p_1^1) \frac{\gamma(z,d_1^2-z)}{\gamma(z-a_2 B_1^1)} + p_1^1 \frac{\gamma(z+a_2 B_1^1-z)}{\gamma(z-a_2 B_1^1)} \]

\[ = (p_1^2-p_1^1) \int_z^\infty (\xi-z)e^{-\xi}d\xi + p_1^1 \int_0^{z-a_2 B_1^1} (\xi+a_2 B_1^1-z)e^{-\xi}d\xi \]

\[ = (p_1^2-p_1^1) \int_0^{\infty} \xi e^{-(\xi+z)}d\xi \]

\[ + p_1^1 \int_0^{z-a_2 B_1^1} \xi e^{-(\xi+z-a_2 B_1^1)}d\xi + p_1^1 \int_0^{z-a_2 B_1^1} (z-a_2 B_1^1)e^{-(\xi+z-a_2 B_1^1)}d\xi \]

\[ = (p_1^2-p_1^1)e^{z} + 2p_1^1 e^{-(z-a_2 B_1^1)} + p_1^1(z-a_2 B_1^1)e^{-(z-a_2 B_1^1)}. \]
Then

\[
\frac{d}{d\epsilon} \left[ p_2^1 e + g_1(z+\epsilon,a_2(\bar{B}^1+\epsilon,0)) \right]/\epsilon = 0
\]

\[
= p_2^1 - (p_1^2 - p_1^1) e^{-z} - 2(1-a_2) p_1^1 e^{-z} a_2^1
\]

\[
+ p_1^1 e^{-z} a_2^1 [1-a_2-(1-a_2)(z-a_2^1)]
\]

\[
= -\frac{1}{16} p_1^2 + 1 + \frac{1}{16} - 2(1-a_2) \left( \frac{1}{12} \right) + \frac{1}{12} [1-a_2-(1-a_2)\ln 12]
\]

\[
= -\frac{1}{16} p_1^2 + 1 + \frac{1}{16} - (1-a_2) \left( \frac{1}{12} \right)(1+\ln 12) > 0
\]

and similarly

\[
\frac{d}{d\epsilon} \left[ p_2^1 e + g_1(z+\epsilon,a_2(\bar{B}^1+\epsilon,0)) \right]/\epsilon = 0
\]

\[
= -\frac{1}{16} p_1^2 + 1 + \frac{1}{16} - (1-a_2) \left( \frac{1}{8} \right)(1+\ln 8) < 0
\]

Thus at stock level \( z \) in interval 2 one should satisfy a positive amount of demand when the backlog vector is \((\bar{B}^1, 0)\) and one should not satisfy any demand if the backlog vector is \((\bar{B}^1, 0)\). This counterexample is particularly surprising because one is willing to satisfy demand when facing one demand vector but not when the amount of demand is increased.

By Corollary 1, we know that it is optimal to not satisfy any class \( j \) demand unless there is no remaining unsatisfied demand of classes
\(j+1, \ldots, n\). If, given that no demand of class \(j+1, \ldots, n\) remained unsatisfied, the optimal amount of stock issued to satisfy class \(j\) demand was determined solely by the relationship of the stock level to some critical rationing level for class \(j\) demand as long as the level of class \(j\) demand was strictly positive, then the counterexample given above shows that in general one may not have \(s_t \in (0, 1)\). However, Theorem 1 below asserts that such a policy is optimal if each \(s_t\) equals either 0 or 1.

To find these critical rationing levels, pick \(z_t^j\) such that \(z_t^j = +\infty\) if the convex function \(p_t^j w + h_t^j(w) + g_{t-1}(w, s_t^j)\) is strictly decreasing on \([0, \infty)\) and \(z_t^j\) is the smallest minimum of \(p_t^j w + h_t^j(w) + g_{t-1}(w, s_t^j)\) on \([0, \infty)\) otherwise. If \(z_t^1 > z_t^2 > \cdots > z_t^n\), then \(u = (u^1, \ldots, u^n)^T\) as defined by

\[
(7) \quad u^j = (B^j_z - z + z_t^j)^+ \land B^j
\]

is equivalent to defining \(u^j\) recursively such that given \(u^{j+1}, \ldots, u^n\) (and the new stock level \(z - (B - u)(j+1)\)) \(u^j\) is determined by satisfying as much class \(j\) demand as possible with existing stock without letting the stock level drop below \(z_t^j\). We refer to this as the rationing level policy. Observe that when \(z_t^1 > z_t^2 > \cdots > z_t^n\) the rationing level policy is of the form described in Corollary 1.

The proof of the specific form of the optimal policy takes advantage of the observation that Corollary 1 actually reduces the minimization problem of (1) to a one-dimensional problem.
For \( w \in [(z-B(1))^+, z) \), let \( s(w) \) be that \( j \) for which \( w \in (z-B(j), z-B(j+1)) \). Then for \( w \in [(z-B(1))^+, z) \),

\[
D^+_w f_t(w; z, B) \\
= p^+_t s(w) + D^+_w h_t(w) + D^+_w g_{t-1}(w + \epsilon, a_t(u(z-w, B) + \epsilon s(w))) / \epsilon = 0 .
\]

\[ (8) \]

**Theorem 1:** Suppose \( a_i \) is 0 or 1 for each \( i \leq t \). Then

(a) The rationing level policy is optimal in interval \( t \),

(b) \( z_t^1 \geq z_t^2 \geq \ldots \geq z_t^n \),

and

(c) for \( \epsilon > 0 \), \( f_t(z, B) = f_t(z + \epsilon, B + \epsilon s_j) \) and

\( g_t(z, b) - g_t(z + \epsilon, b + \epsilon s_j) \) are independent of \( B^1, \ldots, B^j \) and \( b^1, \ldots, b^j \), respectively, for each \( j \).

**Proof:** The second part of (c) trivially holds for \( t = 0 \), so assume that the second part of (c) holds for some \( t, 0 \leq t < k \). Then for \( w \in [(z-B(1))^+, z) \)

\[
D^+_w f_{t+1}(w; z, B) = p^+_t s(w) + D^+_w h_{t+1}(w) + D^+_w g_{t-1}(w, a_t w s(w)) .
\]

Hence by the definition of \( z_{t+1}^{s(w)} \)

\[
D^+_w f_{t+1}(w; z, B) \begin{cases} < 0 & \text{if } w < z_{t+1}^{s(w)} \\ \geq 0 & \text{if } w \geq z_{t+1}^{s(w)} . \end{cases}
\]

(9)

By Lemmas 1 and 2 \( f_{t+1}(w; z, B) \) is convex in \( w \) on \([(z-B(1))^+, z)] . \) Now
suppose that for some \( j \) it is optimal to satisfy (at least) all demand of classes \( j+1, \ldots, n \). Then for \( w = z - B(j+1) \), \( D^+_w f_{t+1}(w; z, B) \geq 0 \); and by (9) if \( B^j > 0 \), then \( s(w) = j \) for \( w \in [(z-B^j), z-B(j+1)] \) and by (9) it is optimal to choose \( u^j \) so (7) holds. This along with the convexity of \( f_{t+1}(w; z, B) \) in \( w \) establishes (a) for \( t+1 \).

Because \( p^1_t \leq p^2_t \leq \cdots \leq p^n_t \) it easily follows that (b) holds for \( t+1 \) if \( a_{t+1} = 0 \). Thus suppose \( a_{t+1} = 1 \). By the induction assumption and Lemma 3 we have for \( i > j \) and \( \epsilon > 0 \)

\[
\begin{align*}
&g_t(z+\epsilon, (z+\epsilon)\delta_j^i) - g_t(z, z\delta_j^i) - g_t(z+\epsilon, z\delta_j^i) + g_t(z, z(\delta_j^i) - g_t(z+\epsilon, (z+\epsilon)\delta_j^i) - g_t(z, z\delta_j^i) - g_t(z+\epsilon, (z+\epsilon)\delta_j^i) + g_t(z, 0) \leq 0, \quad \text{so} \quad D^+_z g_t(z, z\delta_j^i) \leq D^+_z g_t(z, z\delta_j^i). \quad \text{Therefore} \\
&d^+_z[p^i_{t+1} z + h_{t+1}(z) + g_t(z, z\delta_j^i)] \leq D^+_z[p^i_{t+1} z + h_{t+1}(z) + g_t(z, z\delta_j^i)] \quad \text{and} \\
&z_{t+1}^i \geq z_{t+1}^j \quad \text{and (b) also holds with} \quad a_{t+1} = 1 \quad \text{for} \quad t+1.
\end{align*}
\]

From what has been shown above (b) implies \( z_{t+1}^1 \geq z_{t+1}^2 \geq \cdots \geq z_{t+1}^n \) and (a) implies that there exist optimal policies \((u, w)\) and \((\overline{u}, \overline{w})\) in interval \( t+1 \) with \( u \) and \( \overline{u} \) as described in (7) for \((z, B)\) and \((z+\epsilon, B+\epsilon\delta_j)\) respectively. If \( B(\delta_j^i) - z + \overline{z}_{t+1}^j \leq 0 \), then \( u^j = u^i = 0 \) for \( i > j \), \( u^i = \overline{u}^i = (B(\delta_j^i) - z + \overline{z}_{t+1}^j)^+ \), and for \( i < j \)

\[
u^i = (B(\delta_j^i) - z + \overline{z}_{t+1}^j)^+ \land B^j = \overline{u}^i. \quad \text{Thus} \quad (u, w) = (\overline{u}, \overline{w}) \quad \text{and hence}
\]

\[
f_{t+1}(z, B) - f_{t+1}(z+\epsilon, B+\epsilon\delta_j) = 0 \quad \text{is independent of} \quad B^1, \ldots, B^j. \quad \text{If}
\]

\[
B(\delta_j^i) - z + \overline{z}_{t+1}^j > 0, \quad \text{then} \quad u^i = u^i = B^i \quad \text{for} \quad i < j \quad \text{and so}
\]

\[
f_{t+1}(z, B) - f_{t+1}(z+\epsilon, B+\epsilon\delta_j)
\]

\[
= p_{t+1}^i (u - \overline{u}) + h_{t+1}(z - l \cdot (B - u)) - h_{t+1}(z+\epsilon - l \cdot (B+\epsilon\delta_j) - \overline{u})
\]

\[
+ g_t(z - l \cdot (B - u), a_{t+1} u) - g_t(z+\epsilon - l \cdot (B+\epsilon\delta_j - u), a_{t+1} u)
\]

(10)
\[
= \sum_{i=j}^{n} p_{t+1}(u^i - \bar{u}^i) + h_{t+1}(z-(B-u)^{(j)}) - h_{t+1}(z-(B-\bar{u})^{(j)})
+ \varepsilon_t(z-(B-u)^{(j)}, a_{t+1}u) - \varepsilon_t(z-(B-u)^{(j)}) + (\bar{u}-u)^{(j)}, a_{t+1}\bar{u}) .
\]

However, \(u^{j+1}, \ldots, u^n\) and \(\bar{u}^{j+1}, \ldots, \bar{u}^n\) are clearly independent of \(B^1, \ldots, B^j\) and by (7) and \(B^{(j+1)} - z + z_{t+1}^j > 0\)

\[
u^j - B^j = (B^{(j)} - z + z_{t+1}^j) \land B^j = (B^{(j+1)} - z + z_{t+1}^j) \land 0
\]

and

\[
\bar{u} - B^j = (B^{(j)} - z + z_{t+1}^j) \land (B^j + \varepsilon) - B^j = (B^{(j+1)} - z + z_{t+1}^j) \land \varepsilon
\]

which are independent of \(B^1, \ldots, B^j\) as is \(u^j - \bar{u}^j = (B^j - \bar{u}^j) - (B^j - u^j)\).

Using this for \(a_{t+1} = 0\) and additionally using the induction hypothesis for \(a_{t+1} = 1\), (10) is not a function of \(B^1, \ldots, B^j\). It has been shown that in each of two possible cases, neither of which are determined by \(B^1, \ldots, B^j\), \(f_{t+1}(z,B) - f_{t+1}(z+\varepsilon, B+\varepsilon S_j)\) is independent of \(B^1, \ldots, B^j\).

Hence by letting \(B = b + d_{t+1}\) and taking the expected value of \(f_{t+1}(z,b+d_{t+1}) - f_{t+1}(z+\varepsilon, b+\varepsilon S_j + d_{t+1})\) with respect to \(d_{t+1}\), the remainder of (c) follows.

As one might suspect, at any stock level the marginal value of additional stock in any interval is a nondecreasing function of the vector of backlogs.

**Lemma 5:** Suppose \(a_t \leq 0\) or \(1\) for each \(t\). If \(b \leq \bar{b}\), then

\[
D^+ \geq g_t(z, \bar{b}) \geq D^+ \geq g_t(z, \bar{b}).
\]

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Proof: It suffices to prove this for \( b = b + \varepsilon \delta_j \) where \( \varepsilon > 0 \).
Then for \( \varepsilon > 0 \) by Lemma 2 and part (c) of Theorem 1
\[
g_t(z + \varepsilon, b) - g_t(z, b) - g_t(z + \varepsilon, b + \varepsilon \delta_j) + g_t(z, b + \varepsilon \delta_j)
\geq g_t(z + \varepsilon, b + \varepsilon \delta_j) - g_t(z, b) - g_t(z + \varepsilon, b + (\varepsilon + \varepsilon) \delta_j) + g_t(z, b + \varepsilon \delta_j)
= 0.
\]

Lemma 6: Suppose \( a_t \) is 0 or 1 for each \( t \). If \( z \leq \bar{z} \)
and \( y \geq 0 \) then
\[
g_t(\bar{z}, b) - g_t(z, b) \leq g_t(\bar{z} + 1 \cdot y, b + y) - g_t(z + 1 \cdot y, b + y).
\]

Proof: It suffices to prove this lemma for \( y = \varepsilon \delta_j, \ varepsilon > 0 \). By
Lemma 2 and part (c) of Theorem 1
\[
g_t(\bar{z} + \varepsilon, b + \varepsilon \delta_j) - g_t(\bar{z}, b)
= g_t(\bar{z} + \varepsilon, b + (\bar{z} - z) \delta_j + \varepsilon \delta_j) - g_t(\bar{z}, b + (\bar{z} - z) \delta_j)
\geq g_t(z + \varepsilon, b + \varepsilon \delta_j) - g_t(z, b).
\]

Next we show that, ignoring the holding cost in interval \( t \), at a
given stock and backlog level the marginal value of additional stock
is no less in interval \( t \) than in interval \( t-1 \) if either \( a_t = 1 \) or

\[1/\text{This simply says that the increased flexibility added by an equal increase in the stock level and the total backlog is of more value for lower initial stock levels.}\]
b = 0. If \( a_t = 0 \) and \( b \neq 0 \), then a counterexample to this statement can easily be constructed.

**Example:**

If \( a_t = \cdots = a_1 = 1 \), \( a_t = 0, \) \( z = 0, \) \( b = \delta, \) \( \pi \), \( p_i > p_t, \) and no demand is expected in intervals \( 1, \ldots, t-1, \) then
\[
D^+_z g_t(z, b) = -p^n_t < -p^n_{t-1} = D^+_z g_{t-1}(z, b).
\]

**Lemma 7:** If \( a_i \) is 0 or 1 for each \( i \) and either \( a_t = 1 \) or \( b = 0, \) then
\[
D^+_h_t(z) + D^+_z g_{t-1}(z, b) \geq D^+_z g_t(z, b).
\]

**Proof:** If \( b = 0, \) we immediately have
\[
D^+_z f_t(z, b) = D^+_h_t(z) + D^+_z g_{t-1}(z, b).
\]

Define \( z_j = (b, \ldots, b_j, 0, \ldots, 0), \) \( z_t = \infty, \) and \( z_t = 0. \)

If \( a_t = 1 \) then from (1) and (7)
\[
D^+_z f_t(z, b) = \begin{cases}
-p^j_t + D^+_z g_{t-1}(z_t, z_t, \ldots, z_t, z_t, z_t, z_t) \\
\frac{-p^j_t}{z_t} + D^+_z g_{t-1}(z_t, z_t, \ldots, z_t, z_t, z_t, z_t) \\
D^+_h_t(z_t, z_t, \ldots, z_t, z_t, z_t, z_t) \\
D^+_h_t(z_t, z_t, \ldots, z_t, z_t, z_t, z_t)
\end{cases}
\]

for \( j = 0, \ldots, n. \)

For \( z \in [z_t, z_t, \ldots, z_t, z_t, z_t, z_t] \) and small enough \( \epsilon > 0, \) we have
\[
z - b(j+1) \geq \frac{z_t}{z_t}, \quad \text{and the}
\]

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definition of $\overline{z}^j_t$ that

$$\frac{1}{\varepsilon} [f_t(z+\varepsilon, b) - f_t(z, b)]$$

$$= -p^j_t + \frac{1}{\varepsilon} [g_{t-1}(\overline{z}^j_t, j-b + (\overline{z}^j_t + b)(j) - z - \varepsilon) s_j$$

$$- g_{t-1}(\overline{z}^j_t, j-b + (\overline{z}^j_t + b)(j) - z) s_j)]$$

$$= -p^j_t + \frac{1}{\varepsilon} [g_{t-1}(z-b(j+1)+\varepsilon, j) - g_{t-1}(z-b(j+1)+\varepsilon, j+b + \varepsilon s_j)]$$

$$= -p^j_t + \frac{1}{\varepsilon} [g_{t-1}(z-b(j+1)+\varepsilon, j) - g_{t-1}(z-b(j+1), j)$$

$$+ g_{t-1}(z-b(j+1), (z-b(j+1)) s_j$$

$$- g_{t-1}(z-b(j+1)+\varepsilon, (z-b(j+1)+\varepsilon) s_j)]$$

$$\leq -p^j_t + \frac{1}{\varepsilon} [g_{t-1}(z-b(j+1)+\varepsilon, j) - g_{t-1}(z-b(j+1), j)$$

$$+ g_{t-1}(\overline{z}^j_t, \overline{z}^j_t s_j - g_{t-1}(\overline{z}^j_t + \varepsilon, (\overline{z}^j_t + \varepsilon) s_j)]$$

$$\leq \frac{1}{\varepsilon} [h_t(\overline{z}^j_t + \varepsilon) - h_t(\overline{z}^j_t) + g_{t-1}(z-b(j+1)+\varepsilon, j)$$

$$- g_{t-1}(z-b(j+1), j)]$$

$$\leq \frac{1}{\varepsilon} [h_t(z+\varepsilon) - h_t(z) + g_{t-1}(z+\varepsilon, b) - g_{t-1}(z, b)] .$$

For $z\varepsilon[\overline{z}^j_t + b(j+1), \overline{z}^j_t + b(j+1)]$ and small enough $\varepsilon > 0,$

$$\frac{1}{\varepsilon} [f_t(z+\varepsilon, b) - f_t(z, b)]$$

$$= \frac{1}{\varepsilon} [h_t(z-b(j+1)+\varepsilon) - h_t(z-b(j+1)) + g_{t-1}(z-b(j+1)+\varepsilon, j)$$

$$- g_{t-1}(z-b(j+1), j)]$$

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\[ \leq \frac{1}{\varepsilon} \left[ h_t(z+\varepsilon) - h_t(z) + g_t(z+\varepsilon,b) - g_t(z,b) \right] \]

by convexity and Lemma 6.

Now observe that by artificially replacing the demand distribution in interval \( t \) with a distribution with all its mass concentrated at 0, Lemma 5 holds (in general) with \( g_t(\cdot,\cdot) \) replaced by \( f_t(\cdot,\cdot) \). Thus if either \( b = 0 \) or \( a_t = 1 \) for small enough \( \varepsilon > 0 \) and any \( d_t \geq 0 \)

\[ \frac{1}{\varepsilon} \left[ f_t(z+\varepsilon,b+d_t) - f_t(z,b+d_t) \right] \leq \frac{1}{\varepsilon} \left[ f_t(z+\varepsilon,b) - f_t(z,b) \right] \]

(12)

\[ \leq \frac{1}{\varepsilon} \left[ h_t(z+\varepsilon) - h_t(z) + g_t(z+\varepsilon,b) - g_t(z,b) \right] . \]

Taking the expected value of (12) with respect to \( d_t \) and letting \( \varepsilon \rightarrow 0 \) yields the desired result.

Lemma 7 is what is needed to establish that if \( a_{t+1} = a_t = 0 \) and \( p_{t+1} = \lim_{z \to -\infty} \frac{1}{z} h_t(z) \leq p_t \) then \( \frac{z^j_{t+1}}{z^j_t} \geq \frac{z^j_{t+1}}{z^j_t} \). The analogous result is not true in the backlog case. The difficulty then arises because the cumulative class \( j \) penalty costs in the last \( t \) intervals may increase with \( t \), and at a given stock level this might make satisfying class \( j \) demand in interval \( t+1 \) more desirable than doing so in interval \( t \).

Example:

Suppose \( n = 2, a_2 = a_3 = 1, p_1^2 = 5, p_1^3 = p_2^1 = p_2^2 = 2 \), there are no holding or salvage costs, exactly 10 units of class 2 demand are expected with certainty in interval 1, and no class 2 demand is expected in interval 2. This clearly implies \( \frac{z_2^2}{z_2^2} = 10 \) (since \( p_1^1 + p_1^2 = 4 < 5 = p_1^3 \)) and \( \frac{z_3^2}{z_3^2} = 0 \) (since \( p_1^1 + p_1^2 + p_1^3 = 6 > 5 = p_1^3 \)).
Theorem 2: If $a_{t+1} = a_t = 0$ and $p_{t+1}^j + \lim_{z \to \infty} D^+ h_{t+1}(z) \leq p_t^j$, then $\frac{z_{t+1}^j}{z_t^j} \geq \frac{z_{t+1}^j}{z_t^j}$.

Proof: This is trivial if $\frac{z_{t+1}^j}{z_t^j} = \infty$, so assume $\frac{z_{t+1}^j}{z_t^j} < \infty$. Then by the definition of $\frac{z_{t+1}^j}{z_t^j}$

$$0 \leq p_{t+1}^j + D^+ h_{t+1}(\bar{z}_{t+1}^j) + D^+ \xi_t(\bar{z}_{t+1}^j, 0)$$

$$\leq p_t^j + D^+ \xi_t(\bar{z}_{t+1}^j, 0) \quad \text{by assumption}$$

$$\leq p_t^j + D^+ h_t(\bar{z}_{t+1}^j) + D^+ \xi_{t-1}(\bar{z}_{t+1}^j, 0) \quad \text{by Lemma 7}$$

Thus by the definition of $\frac{z_{t+1}^j}{z_t^j}$, $\frac{z_t^j}{z_t^j} \leq \frac{z_{t+1}^j}{z_t^j}$.

Once $\xi_{t-1}(\cdot, \cdot)$ has been determined one can find $\{\bar{z}_t^j\}_{j=1}^n$ and from these critical values one can find $\xi_t(\cdot, \cdot)$ by using (7) and (1) and then taking the expected value with respect to $d_t$. Further comments on this computation will be made in the next section.

Suppose no backlogs exist at the beginning of the production period and $z$ units of stock are initially on hand. The optimal change in inventory then is $y(z)$ where $y(z)$ minimizes $c(y) + g_h(z+y, 0)$ over $y \geq 0$. It can be seen that Assumption (D) assures that $y(z)$ exists. If $c(\cdot)$ is convex, this involves minimizing a convex function over $[0, \infty)$. If $c(y) = c \cdot y$ then clearly $y(z) = (y(0)-z)^+$. 

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2. **Computation of the Optimal Dynamic Rationing Policy Following a Single Procurement**

This section will be devoted to exploring computational simplifications for finding the optimal rationing policies characterized in the preceding section.

The proof of the following lemma is straightforward and will be omitted.

**Lemma 8.2**

(a) If $a_t = a_{t-1} = \ldots = a_1 = 0$, $p^j_t + \sum_{m=1}^{t} D^{+} m(0) \geq p^n_i$ for each $i < t$, and $p^j_t + \sum_{i=1}^{t} D^{+} h_i(0) + D^{+} v_1(0) + \lim_{z \to -\infty} D^{+} v_2(z) \geq 0$, then $\frac{z^j_t}{}, \overline{z} = 0$.

(b) If $a_t = a_{t-1} = \ldots = a_1 = 1, D^{+} h_i(0) \geq 0$ for each $i \leq t$, $\sum_{i=1}^{t} p^j_i + D^{+} h_t(0) \geq \sum_{i=1}^{t-1} p^n_i$, and $\sum_{i=1}^{t} p^j_i + \sum_{i=1}^{t} D^{+} h_i(0) + D^{+} v_1(0) \geq 0$, then $\frac{z^j_t}{}, \overline{z} = 0$.

The results of Lemma 8 are what one might anticipate intuitively. It states that one would want to satisfy class $j$ demand regardless of the stock level if the marginal loss from not doing so is always greater than the marginal gain from holding stock to satisfy any future (certain) demand or to get the salvage value. In most reasonable cases one might expect Lemma 8 to immediately indicate that $\frac{z^j_t}{}, \overline{z} = 0$. If

2/ For linear holding costs and appropriately defined salvage costs, having the conditions of (a) or (b) hold for all $j$ and $t$ is equivalent to the set of conditions given by Veinott [8] to ensure that a rationing level policy with $\frac{z^j_t}{}, \overline{z} = 0$ for all $j$ and $t$ is optimal.
\( n = 2 \) and \( z_t^2 = 0 \) for all \( t \), then the problem reduces to one in which only one critical rationing level need be found in each interval. It is this problem which is considered in [6], [3], and [2].

The No-backlog Case

Now attention will be directed toward specific computational schemes for finding \( \bar{z}_t^j \) in the no backlogging case \( (a_1 = a_2 = \ldots = a_k = 0) \).

Here let \( g_t(z) = g_t(z, 0) \).

Suppose \( \{z_t^m \mid i < t, 1 \leq m \leq n \} \) and \( \{z_t^m + m > j \} \) have been found. If Lemma 8 cannot be applied to immediately show that \( \bar{z}_t^j = 0 \), then one would find \( \bar{z}_t^j \) by finding the smallest minimum of \( p_t^j z + h_t(z) + g_{t-1}(z) \) if it exists or noticing that no minimum exists. If \( t > 1 \) and

\[
\lim_{z \to \infty} D^+h_t(z) = \infty
\]

then by Theorem 2 one can restrict one's attention to the interval \( \{z_t^j \leq \bar{z}_t^j \leq \bar{z}_t^j \} \).

Once the function \( D^+g_{t-1}(z) \) is known, it is a relatively easy matter to find \( \bar{z}_t^1, \bar{z}_t^2, \ldots, \bar{z}_t^n \), so the important computational problem is that of finding \( D^+g_t(z) \) given \( \{z_t^m \}_{m=1}^n \) and \( D^+g_{t-1}(z) \). (Recall that \( D^+g_0(z) = D^+v_1(z) + D^+v_2(z) \).

Substituting (7) into (1) and assuming \( a_t = 0 \) we get

\[
D^+_z f_t(z, d) = \begin{cases} 
-p_t^j & \text{if } z \in [z_t^{j+1}, \bar{z}_t^{j+1}, \bar{z}_t^j + d] \\
D^+h_t(z - d(j+1)) + D^+g_{t-1}(z - d(j+1)) & \text{if } z \in [z_t^{j+1}, \bar{z}_t^{j+1}, \bar{z}_t^j + d] 
\end{cases}
\]

for \( j = 1, \ldots, n \).
Now assume $d_t^1, \ldots, d_t^n$ are independent. By (13) and the Monotone Convergence Theorem [5] which here allows us to interchange the order of the expectation and the right-hand derivative,

$$D^+ g_t(z) = E[D^+_z f_t(z, d_t)]$$

$$= - \sum_{j=1}^{n} p_t^j P[d_t^{(j+1)} \leq z - z_t^j < d_t^{(j)}]$$

$$+ \sum_{j=0}^{n-1} \int_{z=z_t^j}^{z=z_t^{j+1}} \left[ D^+_zh_t(z-w)+D^+_zg_{t-1}(z-w) \right] \phi_t^{j+1}(w)$$

$$+ (D^+h_t(z) + D^+g_{t-1}(z)) \delta(z < z_t^n)$$

where $\phi_t^{j+1}(\cdot)$ is the distribution function of $d_t^{(j+1)}$ for $j < n$ and $\delta(A)$ is 1 if the event $A$ occurs and 0 otherwise.

Denote by $\Phi_{t,j}(\cdot)$ the distribution of $d_t^j$. For $j < n$

$$P[d_t^{(j+1)} \leq z - z_t^j < d_t^{(j)}] = \int_{z-z_t^j}^{z-z_t^{j+1}} [1-\Phi_{t,j}(z-z_t^j-w)] d\phi_t^{j+1}(w)$$

$$= \int_{0}^{z-z_t^j} [\Phi_t^{j+1}(z-w)] d\Phi_{t,j}(w).$$

In the event that the demand distributions are stationary ($\Phi_{1,j}(\cdot) = \Phi_{2,j}(\cdot) = \cdots = \Phi_{k,j}(\cdot)$ for each $j$), the work involved in using (14) for each $t$ would be greatly reduced because $P[d_t^{(j+1)} \leq y < d_t^{(j)}]$ and $\phi_t^{j}(\cdot)$ would only have to be found for $t = 1$ (say).

Much effort can also be saved if $\Phi_t^{j}(\cdot)$ can be found without having to explicitly perform any convolutions. This is possible for certain well-known families of distributions which are closed under convolution. For instance, if $\Phi_{t,1}(\cdot)$ is Poisson with parameter $\lambda_i$ for $j \leq i \leq n$,  

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then \( \Phi_t^j(\cdot) \) will be Poisson with parameter \( \sum_{i=1}^{n} \lambda_i^j \). This property also holds for binomial distributions with the same \( p \), negative binomial distributions with the same \( p \), Gamma distributions with the same scale parameters, and normal distributions (here used as approximations with means and variances such that the probability of negative demand is "insignificant").

**Lemma 2:** If \( a_i \) is 0 for each \( i \leq t \), \( h_t(\cdot), \ldots, h_t(\cdot), v_t(\cdot) \) and \( v_2(\cdot) \) are linear between any two adjacent integers in their domains, and \( d_1, \ldots, d_{t-1} \) are almost surely integer valued vectors, then

(a) for any \( j \) \( z_t^j \) is either an integer or \( \infty \)

and

(b) if \( z \) and \( b \) are integer valued then \( g_t(z+b, b) \), \( g_t(z+b, b+\delta_j) \), \( g_t(z+b, b+\delta_j, b+\delta_j) \), and \( g_t(z+b+b+\delta_{j-1} \delta_{j}) \) [for \( b^i \geq 1 \) are linear in \( \epsilon \) on \([0, 1]\).

The proof of Lemma 9 follows in a straightforward manner by induction and will be omitted. If the hypotheses of Lemma 9 hold then in searching for \( z_t^j \) one need only be concerned with \( p_t^j + D^+ h_t(z) + D^+ g_{t-1}(z) \)

for integer \( z \) (in which case this equals \( p_t^j + h_t(z+1) - h_t(z) + g_{t-1}(z+1) - g_{t-1}(z) \)). Letting \( \varphi_t^j(w) = P(d_t^j = w) \) and \( \varphi_t^0 = P(d_t^0) = w) \), then for \( z \) integer (14) becomes

\[
D^+ g_t(z) = - \sum_{j=1}^{n} p_t^j P[d_t^j+1 \leq z - z_t^j < d_t^j]
\]

\[
+ \sum_{j=0}^{n-1} \left( \sum_{w=z_t^j}^{z_t^j+1} \left[ D^+ h_t(z-w) + D^+ g_{t-1}(z-w) \right] \varphi_t^j(w) \right) \delta(z < z_t^j) .
\]

(15)
Clearly $D^+ e_t^z(z)$ need only be computed for integer $z$.

The Backlog Case

When backlogging is allowed, much more computational effort is required. The difficulty is that with backlogging there seems to be no way of avoiding recursive tabulation of functions of several variables. For large $n$ the amount of effort required to do this is prohibitive.

**Theorem 3:** If $a_{t+1} = a_t = \ldots = a_1 = 1$, then for each $j$

(a) neither $D^+_z e_t^z(z, z \delta_j)$ nor $D^+_e e_t^e(w, \epsilon \delta_j - \delta_s + b)/\epsilon = 0$

depend on $\{p^m_i = 1 \leq m \leq j-1, \text{ all } i\}$ or the distribution of

$\{d^m_i = 1 \leq m \leq j, \text{ all } i\}$ where $s > j$ (and $b^s > 0$),

and

(b) $\overline{z}^j_{t+1}$ does not depend on $\{p^m_i = 1 \leq m \leq j-1, \text{ all } i\}$

or the distributions of $\{d^m_i : 1 \leq m \leq j, \text{ all } i\}$.

**Proof:** One need simply observe from (1) and (7) that

\[
\begin{cases} 
 0 & \text{if } z - d^{(j+1)} \geq \overline{z}^j_t \\
 0 & \text{if } z \in (\overline{z}^j_t + d^{(j+1)}, \overline{z}^j_t + d^{(j+1)} + z - d^{(j+1)}) \\
 p^j_t \overline{p}_t^j + D^+_z e_{t-1}^z(\overline{z}^j_t, \overline{z}^j_t + d^{(j+1)}) & \text{if } z \in (\overline{z}^j_t + d^{(j+1)}, \overline{z}^j_t + d^{(j+1)}) \\
 D^+ h_t(z - d^{(j+1)}) + D^+_z e_{t-1}^z(z - d^{(j+1)}, \overline{z}^j_t + d^{(j+1)}) & \text{if } z \in (\overline{z}^j_t + d^{(j+1)}, \overline{z}^j_t + d^{(j+1)}) \\
 & \text{for } k \geq j
\end{cases}
\]
and for $B^S > 0$ and $s > j$

\[(17) \quad D^+_s f_t(z, \epsilon, s_j, s_i, s_k) / \epsilon = 0 = \begin{cases} 
D^+_s f_t(z - B^S(s+1), \epsilon, s_B, s_j) / \epsilon = 0 \\
\text{if } z \geq z^S_t + B^S(s+1) \\
\frac{p^+_t - p^+_t + D^+_s \epsilon \gamma_{t-1} (z - B(i+1), \epsilon, s_j, s_i, s_k) / \epsilon = 0}{z \epsilon \left( \frac{z^i_t - z^i_t + B(i+1)}{z^i_t + B(i+1)} \right)} \\
\text{for } i > s \\
p^+_t - p^+_t + D^+_s \epsilon \gamma_{t-1} (z^i_t, \epsilon, s_j, s_i, s_k) / \epsilon = 0 \\
\text{if } z \epsilon \left( \frac{z^i_t - z^i_t + B(i+1)}{z^i_t + B(i+1)} \right) \text{ for } i > s.
\end{cases}\]

Now similarly to (c) of Theorem 1 it can be shown that for each $t$ if $j < s$ and $B^S > 0$ then $D^+_s \epsilon \gamma_t (z, \epsilon, s_j, s_i, s_k) / \epsilon = 0$ is independent of $B^1, \ldots, B^j$. Part (a) follows from this together with (16), (17), (c) of Theorem 1, and the fact that here the Monotone Convergence Theorem allows us to change the order of the expectation and the right-hand derivative. Part (b) follows immediately from (a).

Theorem 3 may be used to simplify the problem to some extent. For instance, if there is backlogging in each interval then class 1 demand does not influence any of the critical rationing levels (although it may influence the procurement policy), so in determining the critical rationing levels the actual class 1 demand distribution may be replaced with (say) a distribution having all its mass at 0. In fact, by Theorem 3 the critical rationing levels for class $j$ demand in the backlogging case do not even depend on the existence of any lower class demand. Hence if there
are \( n \) demand classes in the problem, the critical rationing levels for class \( j \) demand may be determined by only considering a problem with \( n-j+1 \) demand classes and with degenerate demand distributions for the lowest class. Even if large \( n \) makes computing all the critical rationing levels too much work, one may still use the above observation to calculate the class \( j \) critical rationing levels for \( j \) near enough to \( n \) by only considering a much smaller problem.

If there is backlogging in each interval then the final stock level minus the sum of all backlogs after interval \( l \) would simply be the initial stock level minus the sum of all demands in all subsequent intervals, and so this quantity is independent of the rationing policy used. Thus the level at which the cost function \( v_2(\cdot) \) is incurred is independent of the rationing policy used, and one may ignore \( v_2(\cdot) \) (i.e., artificially set \( v_2(\cdot) = 0 \)) in calculating the critical rationing levels.

Similarly, if there is backlogging in each interval and Lemma 8 indicated that \( \bar{z}^1_1 = \bar{z}^2_1 = \cdots = \bar{z}^n_1 = 0 \) (because \( \bar{v}^1_1 + D^+ h_1(0) + D^+ v_1(0) \geq 0 \)), then the stock level after interval \( l \) would be independent of the rationing policy used before interval \( l \) and so \( v_1(\cdot) \) could be ignored in calculating the critical rationing levels.
3. **Optimal Dynamic Ordering and Rationing Policies**

In this section the model of Section 1 is imbedded in a model having many ordering opportunities. Under modest assumptions the rationing policy within a production period will be of the sort characterized in Section 1. Additional conditions will be given that imply that the optimal ordering and rationing policies in a given period can be determined independently of the costs, demand distributions, and optimal policies in other periods, i.e., that these policies are myopic.

Consider a problem with $N$ ordering periods such that the structure of the rationing problem within each ordering period is as described in Section 1. Within each period the holding and penalty costs in each interval will satisfy Assumptions (A) and (C), respectively. The ordering periods are indexed beginning with period 1 and with period $m$ directly preceding period $m+1$. At the beginning of period $m$ stock may be purchased at a unit cost $c_m$. Any unsatisfied demand remaining at the end of interval 1 of a period must then be immediately satisfied with either existing or newly acquired (at the beginning of the next period) stock. At the end of period $N$, besides being able to purchase more stock at a unit cost $c_{N+1} \geq 0$ one may also dispose of stock for a unit revenue $v \leq c_{N+1}$. The demands will have finite means in each interval, and they will be independent between different intervals and periods. In each interval of each period unsatisfied demand will either be entirely backlogged or entirely lost. If $h_{t,m}(\cdot)$ is the holding cost function in interval $t$ of period $m$, assume

$$(D') \lim_{w \to \infty} \left[ c_{m} + \sum_{j=m}^{N} \left( \sum_{t} h_{t,j}(w) \right) - v \right] > 0 \quad \text{for all } m \leq N.$$
Observe that $(D')$ is analogous to $(D)$. It assures the existence of an optimal ordering policy in each period.

By "net stock" will be meant the actual stock level minus the sum of all still backlogged unsatisfied demands. Let $C_m(z)$ be $c_m z$ plus the infimum over all policies of all future expected costs if the net stock level is $z$ before ordering at the beginning of period $m$.

Assumption $(D')$ assures that $C_m(z)$ is finite. Let $g^m(z) = c_m z + g_k(z,0)$ where $g_k(z,0)$ is calculated in period $m$ with $v_1(\cdot) = 0$ and $v_2(z) = -c_{m+1} z$. Let $\delta^m_k$ be the matrix of demands (indexed by class and interval) in period $m$. Let $\rho^m_k$ denote a rationing policy to be used in period $m$, and let $s_m(z, \delta^m_k, \rho^m_k)$ be the net stock level after interval $l$ of period $m$ given a stock level $z$ after ordering at the beginning of the period, a demand matrix $\delta^m_k$, and a rationing policy $\rho^m_k$.

The Backlog Case

First suppose that there is backlogging in each interval of period $m$. If $\xi^m_m$ is the sum of all demands in period $m$, then $s_m(z, \delta^m_k, \rho^m_k) = z - \xi^m_m$. Thus the rationing policy used in period $m$ does not affect the costs incurred after period $m$, and so the optimal rationing policy is determined solely by the holding and penalty costs within period $m$. Therefore the optimal rationing policy in this period is characterized as in Section 1 and may be calculated as in Section 2 if one sets $v_1(\cdot) = 0$ and used an arbitrary $v_2(\cdot)$.

Now suppose that there is backlogging in every interval of every period. Then
(18) \[ C_m(w) = \inf_{z \geq w} \left[ g_m^*(z) + E C_{m+1}(z - \xi_m) \right], \quad m = 1, \ldots, N \]

where \( C_{N+1}(w) = (c_{N+1} - v)w^+ \).

By induction and the convexity of \( g_m^*(z) \) it is easily seen that \( C_m(w) \) is real-valued and convex, and so it must be continuous on the open set \((-\infty, \infty)\). Because of Assumption (D') the convex continuous function \( g_m^*(z) + E C_{m+1}(z - \xi_m) \) must attain its minimum on \([0, \infty)\). Let \( \bar{x}_m^* \) be any such minimum. Then if the net stock level is \( w \) before ordering in period \( m \), it will be optimal to order up to \( w \sqrt{\bar{x}_m^*} \).

The No-backlog Case

When unsatisfied demand is not always backlogged, the situation is more involved. Let \( C_{N+1}(w) = (c_{N+1} - v)w^+ \). Given \( C_{m+1}(w) \), let \( \tilde{g}_m^*(z) = c_m z + g_k(z, 0) \) where \( g_k(z, 0) \) is calculated in period \( m \) with \( v_1(\cdot) \equiv 0 \) and \( v_2(w) = -c_{m+1}w + C_{m+1}(w) \). Then

(19) \[ C_m(w) = \inf_{z \geq w} \tilde{g}_m^*(z), \quad m = 1, \ldots, N \]

For \( w < 0 \) \( D^+ v_2(w) = -c_{m+1} > -\infty \), and it is easily seen by induction that \( C_m(w) \) and \( \tilde{g}_m^*(w) \) are continuous and convex and the optimal rationing policy in period \( m \) is characterized as in Section 1 with \( v_1(\cdot) \) and \( v_2(\cdot) \) defined as above. By Assumption (D') the convex function \( \tilde{g}_m^*(z) \) attains its minimum on \([0, \infty)\), and let \( \bar{x}_m^* \) be such a minimum. Then if one has a net stock level \( w \) before ordering in period \( m \), it is optimal to order up to \( w \sqrt{\bar{x}_m^*} \). In the case of
backlogging in each interval of a period the rationing and ordering policies in that period were shown to be myopic in the sense that they depend only on the costs within that period, but when some unsatisfied demand is not backlogged the optimal rationing and ordering policies in a period generally must be determined by a "longer" dynamic programming problem which connects that period with the periods following it.

Myopic Policies

Veinott [7] has shown (in somewhat more generality and precision then will be reproduced here) that if in certain inventory problems

(a) $\overline{y}_m$ minimizing the known single period cost function
(appropriately defined) in period $m$ exists for all $m$,

(b) the stock level at the end of period $m$ is a known non-decreasing function of the initial stock level after ordering and the demands in that period,

(c) if period $m$ begins with stock level $\overline{y}_m$ then it will end with a stock level no greater than $\overline{y}_{m+1}$,

and

(d) the single period cost function is convex,

then for each $m$ it is optimal in period $m$ to order up to but not above $\overline{y}_m$. He applied this result [8,9] to a special case which (as here) involved a problem with several demand classes with each period being broken down into a number of intervals during which demand is observed. However, he considered a special case in which (as he pointed out in [8]) it is optimal to set $\overline{z}_t^j = 0$ for all $j$ and $t$ in each period. It will now be demonstrated that under similar assumptions a
myopic ordering policy will be optimal in the more general situation considered here. In addition, when these conditions do apply and the initial stock level is small enough, one will also have a myopic rationing policy in each period just as one generally has in the backlogging case.

Define \( \overline{y}_m \) to be a minimum of the convex continuous function \( g^m(z) \) on \([0, \infty)\) if such a minimum exists and let \( \overline{y}_m = \infty \) otherwise.

**Theorem 4:** Under the assumptions of this section, if \( v = c_{N+1} \) and \( \overline{y}_m \leq \overline{y}_{m+1} \leq \cdots \leq \overline{y}_N \), then

(a) \( \overline{y}_m < \infty \) and the optimal ordering policy in period \( m \) is a critical level policy with critical level \( \overline{y}_m \),

and

(b) if in addition, the stock level at the beginning of period \( m \) \((< N)\) is no greater than \( \overline{y}_{m+1} \), then the optimal rationing policy in period \( m \) is the optimal single period policy calculated with \( v_1(\cdot) \equiv 0 \) and \( v_2(w) = -c_{m+1} w \).

**Proof:** By Assumption (D') for \( m = N \), \( \overline{y}_N < \infty \) and so \( \overline{y}_m < \infty \). Part (a) clearly holds for \( m = N \), so assume that it holds for some \( m, l < m \leq N \). Then

\[
C_m(w) = C_m(\overline{y}_m) \quad \text{if} \quad w \leq \overline{y}_m
\]

and

\[
C_m(w) \geq C_m(\overline{y}_m) \quad \text{if} \quad w \geq \overline{y}_m
\]

and so with \( \overline{y}_{m-1} \leq \overline{y}_m \).
\[ g^{m-1}(z) = g^{m-1}(z) + c_m(y_m) \quad \text{if} \quad z \leq y_m \]

(20) and

\[ g^{m-1}(z) \geq g^{m-1}(z) + c_m(y_m) \quad \text{if} \quad z \geq y_m. \]

If \(\bar{y}_{m-1} < y_m\), then \(D^+ g^{m-1}(z) = D^+ g^{m-1}(z)\) on \([0, y_m) \Rightarrow [0, y_{m-1}]\) and so \(y_{m-1}\) minimizes \(g^{m-1}(z)\) on \([0, \infty)\). If \(y_{m-1} = y_m\) then by (20) and the definition of \(\bar{y}_{m-1}\) \(D^+ g^{m-1}(\bar{y}_{m-1}) = D^+ g^{m-1}(\bar{y}_{m-1}) \geq 0\), and because on \([0, \bar{y}_{m-1}]\) \(D^+ g(z) = D^+ g(z) \leq 0\), it follows that \(y_m\) minimizes \(g^{m}(z)\) on \([0, \infty)\). Thus (a) holds for \(m-1\). It follows immediately from (20) that (b) holds for \(m-1\).

In the case of \(v = c_{N+1}\) and, except possibly for a discount factor, complete stationarity between periods (although not necessarily between different intervals of the same period), the assumptions of Theorem 4 are clearly satisfied. In addition, \(y_1 = y_2 = \cdots = y_N\), so the same ordering policy is optimal in each period and one need only calculate a critical ordering level once (rather than \(N\) times). If it is also true that the stock level before ordering in period 1 is no greater than \(y_1\), then the same rationing policy is optimal in each period and so an optimal rationing policy need only be calculated for one period (rather than a separate calculation for each period).

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simplify and generalize our earlier proofs for the two demand class case [6].
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