APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
CALIFORNIA

SOME NOTES ON MIXED VARIABLES AND
ATTRIBUTES SAMPLING PLANS

By
GEOFFREY GREGORY AND GEORGE J. RESNIKOFF

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Geoffrey Gregory and George J. Reznikoff

1. Introduction.

A variables plan of acceptance inspection is usually based on the assumption of normality. It often utilizes the sample mean and sample standard deviation. Other measures of variability may be used; for example, the sample range is often substituted for the sample standard deviation. However, regardless of the statistics employed in the application of a variables plan some assumption must be made about the distribution of the quality characteristic being tested.

Often the sample items are classified as defective or non-defective with respect to the quality characteristic. For example, the items may be classified as too large to fit or not, even though the actual dimensions could be measured. Such testing of attributes may be compared with variables inspection considering the relative desirability of each. The application of attributes inspection needs no assumption about the form of the distribution of the quality characteristic. On the other hand, in variables inspection fewer items need to be inspected for a given degree of reliability in determining the acceptability of a lot of such items. If the testing is expensive or destructive this consideration of smaller sample size may weigh heavily in favor of variables
sampling. One disadvantage of variables sampling arises in the following way: it may happen that the producer of a lot of manufactured product has performed complete screening of the lot and has removed all items considered defective by an attributes criterion yet the consumer may reject the lot, which although it contains no defectives does contain a sufficient number of individuals close enough to the limit of the quality characteristic to cause it to fail the variables criterion.

Mixed variables and attributes plans have been suggested as possible methods of retaining the advantage of smaller sample size while averting the possibility of rejecting screened lots. Such plans are the extension of the usual variables plans into double sampling plans wherein the first sample is inspected on a variables basis and, if a second sample is drawn, the items in both samples are inspected on an attributes basis.

For the purpose of analysis the mixed procedure can be considered as having two important elements. These are:

(1) The plan of action, i.e., the rules on the basis of which either one sample is drawn and the lot accepted, or two samples are drawn and the lot is either finally accepted or rejected.

(2) The properties of a given plan of action which determine its suitability. These are (a) the amount of inspection required by the plan, i.e., the average number of items which will be inspected when the plan is used for a large number of lots, and (b) the operating characteristics of the plan, i.e., the proportions of submitted lots that will be accepted or rejected when the plan is used for a large number of lots.
One such plan of action for mixed variables and attributes, when sampling from a normal population with unknown mean and variance, is given in the following paragraph. This type of plan is applicable in the one-sided case, that is, when there is a single limit to the quality characteristic being measured such that an item is considered defective if its measurement exceeds this limit in the case of an upper limit, or an item is considered defective if its measurement is exceeded by the limit in the case of a lower limit. (Two-sided cases are those wherein there are two limits specified, an upper and a lower limit, and an item is considered defective if its measurement either exceeds the upper limit or is exceeded by the lower limit.) In this paper only one-sided cases are considered with the discussion being confined to the case of a single upper limit. However all of the subsequent discussion applies with only very slight modification to the one-sided case with a single lower limit.

The plan of action is as follows:

(a) A sample of \( N_1 \) items is selected at random from the lot.

(b) The quality characteristic being inspected is measured and recorded for each item of the sample.

(c) The sample mean \( \bar{x} \) and the sample standard deviation \( s \) are computed from the \( N_1 \) measurements.

(d) The sample standard deviation \( s \) is multiplied by a certain constant \( k \) and added to the mean \( \bar{x} \) to obtain the statistic \( \bar{x} + ks \).

(e) The lot is accepted if \( \bar{x} + ks \) does not exceed the limit of the quality characteristic, say this limit is \( U \).

(f) If \( \bar{x} + ks > U \) a second sample of up to \( N_2 \) items is drawn at
random from the lot, items being drawn only until a decision to accept or reject the lot can be reached.

(g) The items in both samples are inspected by an attributes criterion. This need not involve measuring the items and recording the measurements but perhaps simply comparing the items to a standard or gauge and recording the number which are found to be too large. In practice the first sample would not be reinspected since those items which are found to have measurements exceeding U are classified as defective.

(h) If the total number d of defectives is found to exceed some constant a the sample is rejected, otherwise it is accepted.

This plan of action is characterized by four parameters: the two sample sizes \( N_1 \) and \( N_2 \), the variables criterion \( k \), and the attributes acceptance number \( a \). Any quadruple of numbers, \( N_1, N_2, k, a \) (where the first two are integers and refer to the first and second sample sizes, the third is a number which refers to the constant multiplier of \( s \), and the fourth is an integer) can be considered as a prescription for such a plan of action. Some desiderata which lead to the selection of a given plan are its average sample size and its operating characteristics. In order to carry out the computation required to determine these properties it is necessary, just as in the case of an ordinary variables plan, to assume knowledge of the distribution of the quality characteristic.

It would appear therefore that a mixed variables and attributes plan cannot overcome the handicap imposed by a perhaps unjustified assumption of normality. This is certainly true insofar as the exact determination of the average sample size and operating characteristics is concerned.
However, such a procedure has distinct psychological advantages: whether or not the assumption of normality is justified the lot cannot be rejected by the application of the variables criterion but only on an attributes basis. To the producer the lot seems to have been given a "second chance." On the other hand, if the distribution is very nearly normal, since a variables criterion is used in the first sample the average sample size will be smaller for a mixed plan than the average sample size for a straight attributes plan having the same operating characteristics.

It is true that if the distribution is really normal a straight variables procedure would result in the smaller long run volume of inspection than would either an attributes plan or a mixed plan. However, the application of the attributes procedure in the mixed plan prevents the rejection of a screened lot.

For the remainder of this report the assumption will be made that the quality characteristic has a normal or Gaussian distribution with some unknown mean $\mu$ and a standard deviation $\sigma$ which may or may not be known.

2. Summary.

An account is given of some of the problems involved in obtaining exact or approximate expressions for the average sample size and operating characteristics for some mixed variables and attributes plans.

If the standard deviation is assumed to be known it is shown that the computation may be carried out by a single numerical integration for the case $a = 0$, and by evaluating a double integral by numerical methods for general values of $a$. Another method of computation when $\sigma$ is known is that of using an asymptotic expansion. This latter method is probably much
less tedious than the former and should present little difficulty when performed on automatic machinery. Except for the very smallest first sample sizes it should yield results accurate enough for the practical application of mixed variables and attributes plans. For completeness it is indicated how some tables of V. J. Francis [1] may be used for the special case of \( a = 0, \sigma \) known.

In the case that the population standard deviation is unknown no feasible method of computation for the exact operating characteristic curve has been obtained. For the case \( a = 0, \sigma \) unknown, an exact solution is expressed as a multiple integral of order \( N_1 - 1 \). This serves only to point up the difficulty of obtaining an exact solution for the general case when \( \sigma \) is not known since evaluation of multiple integrals by numerical methods is a very lengthy process even with the most modern high speed machines. (In this connection reference should be made to a scheme suggested by I. R. Savage. In this procedure \( \bar{x} \) and \( s \) are computed only from non-defective items in the first sample. This makes the solution to the problem for \( a = 0 \) a solution for the case of general values of \( a \). If an approximation at least for the case \( a = 0 \) were found to yield practically accurate results this approximation could be used, in Savage's procedure, for all other values of \( a \). For details see [2].)

For the case \( a = 0, \sigma \) unknown, an alternative procedure is described here utilizing the extreme deviation from the mean in place of the sample standard deviation. This procedure would be of value for \( N_1 \) less than twenty since for these sample sizes the efficiency of the extreme deviation from the mean as an estimator of \( \sigma \) is fairly high. With this procedure
the computation of the average sample number curve and operating characteristic curve for a given plan may be carried out by a single numerical integration as in the case of \( a = 0, \sigma \) known.

For the case \( a = 0, \sigma \) unknown, there is given an approximation based on the asymptotic normality of the statistic used in the variables criterion when it is obtained from a truncated population.

For one particular set of parameters an approximation to the operating characteristic curve was obtained by the method of empirical sampling. This method was undertaken to investigate its feasibility in large scale work.

Finally, upper and lower bounds for the operating characteristic curve and average sample size curve are given.


3.1 Independent Plans.

We first consider a type of plan which differs from the one described on page 3 in that the population standard deviation is assumed to be known and only the second sample is subjected to attributes inspection.

Let the measurements \( x_1, x_2, \ldots, x_{N_1} \) be a random sample of size \( N_1 \) from a normal population with unknown mean \( \mu \) and known standard deviation \( \sigma \). An item is considered defective if its measurement exceeds \( U \). The lot of items from which the sample is drawn is accepted if \( \bar{x} \leq U - k \sigma \) where

\[
\bar{x} = \frac{\sum_{i=1}^{N_1} x_i}{N_1}
\]

and \( k \) is some constant. If \( \bar{x} > U - k \sigma \) a second sample of maximum size \( N_2 \) is drawn and this sample is subjected to an attributes test with acceptance number \( a \), i.e., if \( d_2 \), the number of defectives in the second sample, is less than or equal to \( a \) the lot is accepted, otherwise
it is rejected.

Denoting an ordinate on the operating characteristic (OC) curve by \( L_p \), we have

\[
L_p = \Pr \left( \bar{x} \leq U - k\sigma \mid p \right) + \Pr \left( \bar{x} > U - k\sigma \mid p \right) \Pr \left( d_2 \leq a \mid p \right)
\]

\[
= \Pr \left\{ z \leq \sqrt{\frac{N}{1}} (K_p - k) \right\} + \Pr \left\{ z > \sqrt{\frac{N}{1}} (K_p - k) \right\} \sum_{i=0}^{a} \binom{N}{i} p^i q^{N-1-i}
\]

where \( z \) is a standardized normal deviate, \( q = 1 - p \), and \( K_p \) is defined by

\[
\int_{K_p}^{\infty} \frac{-t^2}{2} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \, dt = p.
\]

We consider only the case wherein the first sample is inspected completely and the second sample is curtailed as soon as a decision to accept or reject can be reached. The average sample size (ASN) is given by

\[
ASN = N_1 + N_2: a \Pr \left\{ z > \sqrt{\frac{N}{1}} (K_p - k) \right\}
\]

where

\[
N_2: a = \frac{a+1}{p} \left\{ 1 - \frac{p^{1\prime}}{N_{2\prime}+1} : a+1 \right\} + \frac{N_2-a}{q} \frac{p^{1\prime}}{N_{2\prime}+1} : a 1/
\]

where

\[
p_{N:m}^{1\prime} = \sum_{j=0}^{m} \binom{N}{j} p^j q^{N-j}.
\]

In independent mixed procedures information is lost by failing to take into account the first sample for attributes analysis.

\[
1/ \text{ Equation (43) page 214 of [3].}
\]
3.2 Dependent Plans, $\sigma$ Known.

For the procedure in which both samples are utilized in the attributes analysis the plan of action is as follows: if $\bar{x} \leq U - k\sigma$ accept the lot; if $\bar{x} > U - k\sigma$ draw a second sample of size $N_2$ (sampling will be curtailed as soon as a decision to accept or reject the lot can be made) and accept the lot if $d = d_1 + d_2 \leq a$; if $d > a$ reject the lot. $d_1$ is the number of defectives in the first sample and $d_2$ the number of defectives in the second sample.

Since the OC and ASN curves are plotted as functions of $p$, so that $\frac{U - \mu}{\sigma} = K_p$, we may assume for the purposes of computation that the observations are drawn from a normal population with zero mean and unit standard deviation. Then we have that:

$$L_p = \Pr \left\{ \sqrt{N_1} \bar{x} \leq \sqrt{N_1} (K_p - k) \right\} + \Pr \left\{ \bar{x} > K_p - k, d \leq a \right\} .$$

Since $\sqrt{N_1} \bar{x}$ is a standardized normal deviate evaluating the first term is accomplished by looking up the probability in a table of the normal integral. The second term may be written as:

$$\sum_{i=0}^{a} \Pr \left\{ d_2 \leq a - i \right\} \Pr \left\{ d_1 = i \right\} \Pr \left\{ \bar{x} > K_p - k \mid d_1 = i \right\} = \sum_{i=0}^{a} \binom{N_1}{i} p^i q^{N_1 - i} \Pr \left\{ \bar{x} > K_p - k \mid d_1 = i \right\} \sum_{j=0}^{a-i} \binom{N_2}{j} p^j q^{N_2 - j} .$$

With the exception of the conditional probabilities $\Pr \left\{ \bar{x} > K_p - k \mid d_1 = i; \ i = 0, 1, \ldots, a \right\}$, all the terms in the summand are readily computed.
We also have that

\[ \text{ASN} = N_1 + \sum_{j=0}^{a} N_{2:a-j} \cdot \Pr \left\{ \bar{x} > K_p - k, d_1 = j \right\} \]

where

\[ N_{2:a-j} = \frac{a-j+1}{p} \left\{ 1 - P_{N_{2+1:a-j+1}}^{k-i} \right\} + \frac{N_{2:a-j}}{q} P_{N_{2+1:a-j}}^{k-i} \]

with

\[ P_{N:m} = \sum_{j=0}^{m} \binom{N}{j} p^j q^{N-j} \]

It can be seen that the solution to the problem of computing the OC and ASN curves is essentially that of evaluating either \( \Pr \left\{ \bar{x} > K_p - k \mid d_1 = i \right\} \) or equivalently, evaluating \( \Pr \left\{ \bar{x} > K_p - k, d_1 = i \right\} \). For brevity the following notation will be used:

\( \Pr \left\{ \bar{x} > K_p - k \mid d_1 = i \right\} \) will be denoted by \( P(K_p - k \mid i) \)

\( \Pr \left\{ \bar{x} > K_p - k, d_1 = i \right\} \) will be denoted by \( P(K_p - k, i) \).

3.2.1 **Evaluation of** \( P(K_p - k, 0) \) **by Numerical Integration.**

Let \( x(1), x(2), \ldots, x(N_1) \) be the ordered first sample of \( N_1 \) observations, then

\[ P(K_p - k, 0) = \int_{R} \ldots \int_{R} \int_{\frac{N_1}{N_1}}^{\frac{N_1}{N_1}} e^{-\frac{1}{2} \sum_{j=1}^{N_1} x(j)^2} \prod_{j=1}^{N_1} dx(j) \]
where the region $R$ is given by the following restrictions on the variables

\[
\begin{align*}
K_p &\geq \sum_{j=1}^{N_1} \frac{x^{(j)}}{N_1} > K_p - k, \\
-\infty < x^{(1)} &\leq x^{(2)} \leq \ldots \leq x^{(N_1)} \leq K_p
\end{align*}
\]

Make the orthogonal transformation

\[
\begin{align*}
u_2 &= -x^{(1)} + x^{(2)} \\
u_3 &= -x^{(1)} - x^{(2)} + 2x^{(3)} \\
&\vdots \\
u_{N_1} &= -x^{(1)} - x^{(2)} - \ldots - x^{(N_1-1)} + (N_1 - 1)x^{(N_1)} \\
u_{N_1+1} &= \sqrt{N_1} \bar{x}
\end{align*}
\]

Under this transformation we obtain

\[
P_o(\bar{x}, d) = \int_{R'} \ldots \int_{R'} e^{-\frac{N_1}{2} \bar{x}^2 - \frac{1}{2} \sum_{j=2}^{N_1} u_j^2} \frac{N_1!}{(2\pi)^{N_1/2}} \prod_{j=2}^{N_1} du_j
d\bar{x}
\]

where the region $R'$ is defined by

\[
\begin{align*}
-\infty < K_p - k < \bar{x} &\leq K_p \\
0 &\leq u_2 < \infty \\
u_{j-1} &\leq \sqrt{\frac{1}{j-2}} u_j \\
u_{N_1} &\leq \sqrt{\frac{N_1}{N_1-1}} (K_p - \bar{x})
\end{align*}
\]
If we now make the transformation \( z_k = \sqrt{\frac{k-1}{k}} u_k \), then \( z_{N_1} \) is the extreme deviation from the mean \( (x_{N_1} - \bar{x}) \). Denote by \( F_{N_1}(z) \) the probability that \( z_{N_1} \leq z \) (assuming that we have integrated out the \( N_1 - 2 \) variables \( z_2, z_3, \ldots, z_{N_1-1} \)). We will have

\[
P(K_p - k, 0) = \int_{K_p - k}^{K_p} \frac{\frac{-N_1}{2} \bar{x}^2}{\sqrt{2\pi}} \, F_{N_1}(K_p - \bar{x}) \, d\bar{x}.
\]

This expression can be evaluated by numerical integration provided tables of \( F_{N_1} \) are available. Such tables have been prepared by K. R. Nair [4] for \( N_1 = 2, \ldots, 9 \) and F. E. Grubbs [5] for \( N_1 = 2, 3, \ldots, 25 \). If larger first samples are required the tables \( F_{N_1} \) can be extended by an iterative process of numerical integration since

\[
F_{N+1}(u) = \frac{(N + 1)^2}{2N^2} \int_0^u \frac{\frac{N+1}{2N} x^2}{\sqrt{2\pi}} \, F_N\left(\frac{N+1}{N} x\right) \, dx.
\]

(The above result was first derived by McKay [6].)

3.2.2 Evaluation of \( P(K_p - k, i) \) by Numerical Integration for General Values of \( i \).

\[
P(K_p - k, i) = \int \cdots \int_{R_i} \int_{\mathbb{R}} e^{-\frac{1}{2} \sum_{j=1}^{N_1} x(j)^2} \prod_{j=1}^{N_1} dx(j)
\]
where the region \( R' \) is defined by the following restrictions on the variables

\[
\begin{align*}
K_p - k & < \bar{x} < \infty \\
-\infty & < x(1) \leq x(2) \leq \cdots \leq x(i) \leq K_p < x(i+1) \leq \cdots \leq x(N_1) < \infty
\end{align*}
\]

The integrand can be rewritten as

\[
\begin{pmatrix} N_1 \\ i \end{pmatrix} \left[ \frac{i!}{(2\pi)^{k/2}} e^{-\frac{1}{2} \sum_{j=1}^{i} x(j)^2} \right] \left[ \frac{(N_1 - i)!}{(2\pi)^{k/2}} e^{-\frac{1}{2} \sum_{j=i+1}^{N_1} x(j)^2} \right]
\]

and by transformations similar to those of Section 3.2.1 made separately on the two sets of variables in each bracket we obtain

\[
P(K_p - k, i) = \binom{N_1}{i} \int_{S} \int F_i(K_p - \bar{x}_1)[1 - F_{N_1-i}(\bar{x}_1 - K_p)]
\]

\[
- \frac{1}{2} \left[ i \bar{x}_1^2 + (N_1 - i) \bar{x}_2^2 \right] e^{-\frac{1}{2} \left[ i \bar{x}_1^2 + (N_1 - i) \bar{x}_2^2 \right]}
\]

where \( \bar{x}_1 = \frac{1}{i} \sum_{j=1}^{i} x_j \), \( \bar{x}_2 = \frac{1}{N_1 - i} \sum_{j=i+1}^{N_1} x_j \)

and the functions \( F_k \) are the probability integrals for the extreme deviation from the mean mentioned in the previous section. The region \( S \) is given by the restrictions

\[
\begin{pmatrix} N_1(K_p - k) \\ N_1 - i \end{pmatrix} - \frac{1}{N_1 - i} \leq \bar{x}_1 \leq \bar{x}_2 < \infty
\]

\[
-\infty < \bar{x}_1 \leq K_p
\]
Evaluating this double integral would undoubtedly be very time consuming especially if complete sets of OC curves were required for many values of \( k \) and \( N_1 \). In later sections are given asymptotic approximations which are perhaps more useful.

3.2.3 Evaluation of \( P(K_p - k, 0) \) Using Some Tables of V. J. Francis.

It is evident that \( P(K_p - k, 0) \) can be found from the distribution of the sum of \( N_1 \) sample values from a normal population with zero mean and unit standard deviation truncated from above at \( K_p \). We use some results of V. J. Francis [1].

The probability density of \( x \) truncated from above at \( K_p \) is

\[
f_1(x) = \begin{cases} \frac{-x^2}{2} = \frac{e^{-x^2/2}}{\sqrt{2\pi}q} & \text{for } x \leq K_p \\ 0 & \text{for } x > K_p \end{cases}
\]

Denoting \( \frac{e^{-x^2/2}}{\sqrt{2\pi}} \) by \( \phi(x) \), let

\[
a_n = \int_{-\infty}^{\infty} x^n f_1(x) \, dx = \frac{1}{q} \int_{-\infty}^{K_p} x^n \phi(x) \, dx
\]

then

\[
a_o = 1, \quad a_1 = \frac{-\phi(K_p)}{q}
\]

and, in general

\[
a_n = k_p^{n-1} a_1 + (n-1) a_{n-2}
\]
so that

\[ \sigma^2(x) = a_2 - a_1^2 = 1 + K_p a_1 - a_1^2. \]

Let \( F_{N_1}(z, K_p) \) denote the probability integral of the standardized sum

\[ z = \frac{\sum_{j=1}^{N_1} x_j - N_1 a_1}{\sqrt{N_1} \sigma} \]

where the \( x_j \) are from a normal population with zero mean and unit standard deviation, truncated above at \( K_p \), then

\[ P(K_p - k, 0) = F_{N_1}(z, K_p). \]

This is a two parameter function and is tabulated by V. J. Francis [1] for \( N_1 = 1, 2, 4 \) and for \( K_p = 0, 0.5, 1.0, 1.5 \) and 2. The narrow range of the parameters limit their usefulness for the purpose at hand.

3.2.4 Asymptotic Expansion for \( P(K_p - k \mid 0) \).

Since \( z \) of the previous section is asymptotically normally distributed an Edgeworth series type of asymptotic expansion may be used as an approximation to \( P(K_p - k \mid 0). \) (See [7], Section 17.7.) Let

\[ \mu_n = \int_{-\infty}^{\infty} (y - a_1)^n f_1(y) \, dy \]

and let

\[ \phi(\gamma)(z) = \frac{d^\gamma}{dz^\gamma} \phi(z) = (-1)^\gamma H_{\gamma}(z) \phi(z) \]
where $H_\nu(z)$ is the Hermite polynomial of degree $\nu$. Then the expansion for
\[
\frac{dF_{n_1}}{dz} = f_{n_1},
\]
say, is
\[
f_{n_1}(z) = \phi(z) - \frac{\gamma_1}{6N_1} \phi(3)(z) + \frac{1}{N_1} \left[ \frac{\gamma_2}{24} \phi(4)(z) + \frac{\gamma_1^2}{72} \phi(6)(z) \right] + o\left(\frac{1}{N_1^{\frac{3}{2}}}\right)
\]
where
\[
\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}; \quad \gamma_2 = \frac{\mu_4}{\mu_2} - 3.
\]
Therefore, putting $z_0 = N_1 \left( \frac{K_p - k - a_1}{\sigma_1} \right)$,
\[
P(K_p - k | 0) = F_{n_1}(z_0) \simeq \int_{z_0}^{\infty} \phi(x) \, dx + \frac{\gamma_1}{6N_1} H_2(z_0) + \frac{\phi(z_0)}{N_1} \left[ \frac{\gamma_2}{24} H_3(z_0) + \frac{\gamma_1^2}{72} H_5(z_0) \right].
\]

If additional terms are desired the two following are

---

1/ By $O(r_N)$ is meant a quantity which is of the same order as $r_N$. That is, if $a_N = O(r_N)$, then the ratio of $a_N$ to $r_N$ is bounded as $N$ tends to infinity. There exist constants $M_0$ and $N_0$ such that $\frac{a_N}{r_N} \leq M_0$ for $N \geq N_0$. 
\[
\frac{\phi(z_o)}{N_1^2} \left[ \frac{\gamma_3}{120} H_4(z_o) + \frac{\gamma_1 \gamma_2}{144} H_6(z_o) + \frac{\gamma_3^3}{1296} H_8(z_o) \right] \\
+ \frac{\phi(z)}{N_1^2} \left[ \frac{\gamma_4}{720} H_4(z_o) + \left( \frac{\gamma_2^2}{1152} + \frac{\gamma_1 \gamma_3}{720} \right) H_7(z_o) + \frac{\gamma_1 \gamma_2^2}{1728} H_9(z_o) + \frac{\gamma_4^4}{34312} H_{11}(z_o) \right]
\]

where
\[
\gamma_3 = \frac{\mu_5}{(\mu_2)^2} - 10 \gamma_1
\]
\[
\gamma_4 = \frac{\mu_6}{\mu_2} - 15 \gamma_2 - 10 \gamma_1^2 - 15
\]

3.2.5 Asymptotic Expansion for \( P(K_p - k, l) \).

\[
P(K_p - k | l) = \Pr \left\{ \sum_{i=1}^{N_1} x(i) > N_1(K_p - k) \mid x(N_1 - l) \leq K_p, x(N_1) > K_p \right\}
\]

where \( x(N_1) \) is the largest observation and \( x(N_1 - l) \) the next largest.

The probability density of \( x(N_1) \) truncated from below at \( K_p \) is

\[
\begin{cases} 
\phi(x) \quad \text{for } x > K_p \\
0 \quad \text{for } x \leq K_p 
\end{cases}
\]

therefore, the probability density of \( y \), the sum of the \( N_1 \) \( x \)'s, is the convolution
\[ f_{N_1}^{(1)}(y) = \int_{-\infty}^{\infty} f_{N_1-1}(x) f_1^*(y-x) \, dx \]

\[ = \int_{-\infty}^{\min[(N_1 - 1) K_p, y - K_p]} f_{N_1-1}(x) f_1^*(y-x) \, dx \]

where \( y = \sum_{i=1}^{N_1} x_i \), and \( f_{N_1-1}(x) \) is the probability density of the sum of \( N_1 - 1 \) \( x \)'s truncated from above at \( K_p \) as in the previous section. Let

\[ a_n^* = \int_{-\infty}^{\infty} x^n f_1^*(x) \, dx = \frac{1}{p} \int_{K_p}^{\infty} y^n \phi(y) \, dy. \]

In general, \( a_n^* = k_p^{n-1} a_1^* + (n-1) a_{n-2}^* \). Denoting the central moments and semi-invariants of \( f_1^*(x) \) by \( \mu_n^* \) and \( \kappa_n^* \), and the central moments and semi-invariants of \( f_{N_1}^{(1)}(y) \) by \( \mu_n' \) and \( \kappa_n' \), we have, because of the additivity property for independent variables that

\[ E(y) = (N_1 - 1) a_1 + a_1^* \]

\[ \mu_2' = (N_1 - 1) \mu_2 + \mu_2^* \]

\[ \mu_3' = (N_1 - 1) \mu_3 + \mu_3^* \]

\[ \kappa_4' = (N_1 - 1) \kappa_4 + \kappa_4^* \]

Let

\[ \gamma_1' = \frac{\mu_3'}{(\mu_2')^2}, \quad \gamma_2' = \frac{\kappa_4'}{(\mu_2')^2} \]
and let $z_1 = \frac{1}{\left( N_1 - 1 \right) \mu_2 - \mu_2^* \frac{1}{2}} \left( \frac{N_1}{N_1 - 1} K_p - k \right) - (N_1 - 1) \frac{a_1 - a_1^*}{1}$

then an asymptotic expansion for $P(K_p - k \mid l)$ is given by

$$P(K_p - k \mid l) \sim \int_{N_1(K_p - k)}^{\infty} f_{N_1}^{(1)}(y) \, dy \sim \int_{z_1}^{\infty} \phi(x) \, dx + \frac{\gamma_1'}{6} H_2(z_1) \phi(z_1)$$

$$+ \phi(z_1) \left[ \frac{\gamma_2'}{24} H_3(z) + \frac{(\gamma_1')^2}{72} H_5(z_1) \right].$$

A list of constants to aid in computing OC and ASN curves by this method can be found in Table IV, in the appendix.

To extend this method for general values of $i$, we note that (in the notation of this and the previous sections)

$$P(K_p - k \mid i) = \int_{N_1(K_p - k)}^{\infty} f_{N_1}^{(i)}(y) \, dy$$

where

$$f_{N_1}^{(i)}(y) = \int_{-\infty}^{\infty} f_{N_1 - i}(x) f_{i}^*(y - x) \, dx$$

$$= \int_{-\infty}^{\min \left( (N_1 - i) K_p, y - i K_p \right)} f_{N_1 - i}(x) f_{i}^*(y - x) \, dx$$

and
\[ \chi_n \left( \sum_{i=1}^{N_1} x_i \right) = (N_1 - r) \chi_n^* + r \chi_n^* . \]

4. **Unknown \( \sigma \) Mixed Variables and Attributes Plans.**

Let \( x_1, \ldots, x_{N_1} \) be a random sample of size \( N_1 \) from a normal population with unknown mean \( \mu \) and unknown variance \( \sigma^2 \). An item is considered defective if its measured characteristic lies above \( U \). The lot is accepted on the basis of the first sample of \( N_1 \) items if \( \bar{x} + ks \leq U \) where \( k \) is some constant and

\[
\bar{x} = \frac{1}{N_1} \sum_{j=1}^{N_1} x_j \quad s^2 = \frac{1}{N_1 - 1} \sum_{j=1}^{N_1} (x_j - \bar{x})^2 .
\]

If \( \bar{x} + ks > U \) a second sample of up to \( N_2 \) items is drawn (sampling being curtailed as soon as a decision to accept or reject can be made) and both samples are subjected to an attributes analysis, the lot being rejected if \( d = d_1 + d_2 > a \), accepted if \( d \leq a \).

We then have

\[ L_p = \Pr \left\{ \bar{x} + ks \leq K_p \right\} + \Pr \left\{ \bar{x} + ks > K_p, d \leq a \right\} \]

where the observations can be assumed to have been drawn from a normal population with zero mean and unit standard deviation.

The first term can be written as follows

\[ \Pr \left\{ \bar{x} + ks \leq K_p \right\} = \Pr \left\{ t \geq \sqrt{N_1} k \right\} \]

where \( t \) is a non-central \( t \) variate with \( N_1 - 1 \) degrees of freedom and non-centrality parameter \( \sqrt{N_1} K_p \).

The second term may be written as
\[
\sum_{i=0}^{a} \Pr \left\{ d_2 \leq a - i \right\} \Pr \left\{ \bar{x} + ks > K_p, d_1 = i \right\}.
\]

As in the case of known $\sigma$ plans, our primary concern is with the joint probability, in this case, $\Pr \left\{ \bar{x} + ks > K_p, d_1 = i \right\}$, since the other term of the summand is a simple binomial probability.

Denoting $\Pr \left\{ \bar{x} + ks > K_p, d_1 = i \right\}$ by $P(K_p, k, i)$, and $\Pr \left\{ \bar{x} + ks > K_p \mid d_1 = i \right\}$ by $P(K_p, k \mid i)$ we have for the ASN curve

\[
\text{ASN} = N_1 + \sum_{j=0}^{a} P(K_p, k, i) N_2:a-j
\]

where $N_2:a-j$ is as defined in Section 3.2.

The problem of computing $P(K_p, k \mid i)$ or equivalently $P(K_p, k, i)$ even approximately, is much more difficult than that of computing $P(K_p - k, i)$. Essentially what is required is the conditional probability distribution function for the random variable

\[
t = \sqrt{\frac{K_p - \sum_{i=1}^{N_1} \frac{x(i)}{N_1} \sqrt{\sum_{i=1}^{N_1} (x(i) - \bar{x})^2}}{N_1 - 1}}
\]

given that, for the ordered observations $x(1), x(2), \ldots, x(N_1), x(1) \leq K_p, x(i+1) > K_p; i = 1, 2, \ldots, N_1$.

4.1 $P(K_p, k, 0)$ as a Multiple Integral.

\[
P(K_p, k, 0) = \Pr \left\{ \bar{x} + ks > K_p, x(N_1) \leq K_p \right\}
\]
where \( x_{(N_1)} \) is the largest observation in a sample drawn from a normal population with mean zero and unit variance. This may be written as

\[
\Pr \left\{ \frac{K_p - \bar{x}}{s} < k, \quad \frac{x_{(N_1)} - \bar{x}}{s} \leq \frac{K_p - \bar{x}}{s} \right\}.
\]

It follows from [5], Section 8 that \( \frac{x_{(N_1)} - \bar{x}}{s} \) is independent of \( \frac{K_p - \bar{x}}{s} \) and hence that the above probability is equal to

\[
\int \int \frac{f(\bar{x}) g(s)}{\left( \frac{K_p - \bar{x}}{s} > k \right)} \Pr \left\{ \frac{x_{(N_1)} - \bar{x}}{s} \leq \frac{K_p - \bar{x}}{s} \bigg| \bar{x}, s \right\} d\bar{x} ds
\]

where \( f(\bar{x}) \) and \( g(s) \) are the probability densities for \( \bar{x} \) and \( s \) respectively.

Denoting \( \Pr \left\{ \frac{\sqrt{N_1}}{s} (x_{(N_1)} - \bar{x}) \leq u \bigg| \bar{x}, s \right\} \) by \( G(u) \) and making the transformation \( t = \frac{\sqrt{N_1}}{s} (K_p - \bar{x}) \) we obtain

\[
(1) \quad F(K_p, k, 0) = \int_{\sqrt{N_1} k}^{\infty} h(t) G(t) \, dt.
\]

The function \( h \) is the probability density for a non-central \( t \) variate with \( N_1 - 1 \) degrees of freedom and non-centrality parameter \( \sqrt{N_1} K_p \); and this density itself involves a definite integral which cannot be evaluated in closed form. The function \( G \) is the probability integral for the ratio \( \frac{\sqrt{N_1}}{s} (x_{(N_1)} - \bar{x}) = \frac{\text{(extreme deviation from the mean)}}{\text{sample standard deviation}} \sqrt{N_1} \).

The function \( G \) was derived in the form of an \((N_1 - 2)\) fold multiple.
integral by Smirnov [9] and Grubbs [5]. Chandrasekar and Pearson [9] tabled some of the upper percentage points as did Grubbs and Smirnov. However, neither the density nor the probability integral have been tabled to any extent.

Inasmuch as the expression (1) is given in the form of an $N_1$-fold integral this result is of no practical value for the purpose of this paper but is included here for theoretical interest.

4.2 Plans with $a = 0$, Using the Extreme Deviation from the Mean.

Because of the extreme difficulty of computing the OC and ASN curves for the plans using the statistic $\bar{x} + k\sigma$ in the variables part of the plan, the following scheme is suggested as an alternative for the case $a = 0$, $\sigma$ unknown. This type of plan offers a (relatively) simple computation for values of $N_1$ less than or equal to twenty. It suffers from the handicap that the ease of computation is limited to the case of $a = 0$. However, this case may sometimes be useful and does serve to prevent the acceptance of screened lots.

The plan is as follows: on the basis of a sample of $N_1$ observations compute $\bar{x} + k\sigma$ where $v = x (N_1)^{-1} \bar{x}$, i.e., $v$ is the extreme deviation from the mean, referred to in Section 3.2.1. If $\bar{x} + k\sigma \leq U$ accept the lot; if $\bar{x} + k\sigma > U$ draw a second sample, as for previous plans, and subject the entire sample to an attributes analysis, rejecting the lot as soon as a defective is observed, accepting if no defectives are observed.

Here $v$ is used in place of $s$ as the estimator of $\sigma$, for unknown $\sigma$ plans. The efficiency is fairly high (see [5], page 45) for small values
of \( N_1 \), the variance of \( v \) being smaller than that of the sample range which is often used in place of \( s \) in small samples.

The OC curve is given by

\[
L_p = \Pr \left\{ \frac{\bar{x} + kv}{k} \leq K_p \right\} + \Pr \left\{ \frac{\bar{x} + kv}{k} > K_p, \ x(N_1) \leq K_p \right\} \Pr \left\{ d_2 = 0 \right\}
\]

where \( x(N_1) \) is the largest observation. This may be written as

\[
\Pr \left\{ v \leq \frac{K_p - \bar{x}}{k} \right\} + \Pr \left\{ \frac{K_p - \bar{x}}{k} \leq v \leq \frac{K_p - \bar{x}}{k} \right\} \Pr \left\{ d_2 = 0 \right\}
\]

\( k > 1 \).

Since \( v \) and \( \bar{x} \) are independently distributed and the probability integral of \( v \), for \( N_1 \) from 2 to 25, is adequately tabled, computing the ASN and OC curves involves only computing two single definite integrals by numerical methods, just as in the case of \( a = 0 \), known \( \sigma \) in Section 3.2.1.

4.3 An Asymptotic Approximation for \( P(K_p, k, 0) \)

Let \( \bar{x}, s \) be computed from a sample of size \( N_1 \) from a normal population with zero mean and unit variance truncated from above at \( K_p \). For large \( N_1 \), the statistic \( \bar{x} + ks \) is approximately normally distributed with some mean, say \( \alpha \), and some variance, say \( \beta^2 \). Therefore, if we put

\[
z_0 = \frac{K_p - \alpha}{\beta}
\]

\[
P(K_p, k, 0) = \int_{z_0}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \, dt.
\]

From Cramer ([7], Section 27.7) we have that for any population with variance \( \sigma^2_1 \).
\[ E(s) = \sigma_1 + O\left(\frac{1}{N_1}\right) \]

\[ \text{Var}(s) = \frac{\mu_4 - \sigma_1^4}{4 \sigma_1^2 (N_1 - 1)} + O\left(\frac{1}{N_1^2}\right) \]

and by an argument similar to the one used there to obtain \( E(s) \) and \( \text{Var}(s) \), we have that

\[ \text{Cov}(\bar{x}, s) = \frac{3\mu_3}{4 N_1 \sigma_1} + O\left(\frac{1}{N_1^2}\right) \]

From Section 3.2.4 we recall that

\[ a_1 = \frac{-k_p^2}{e/q} \]

and that

\[ \sigma_1^{-2} = 1 + k_p a_1 - a_1^2 \]

Then it follows that

\[ \alpha = \text{E}(\bar{x} + ks) \approx a_1 + ks_1 - \frac{k(\mu_4 - \sigma_1^4)}{3(N_1 - 1) \sigma_1^2} \]

\[ \beta = \text{Var}(\bar{x} + ks) \approx \frac{\sigma_1^2}{N_1} + \frac{3 k \mu_3}{2 N_1 \sigma_1} + \frac{k^2(\mu_4 - \sigma_1^4)}{4(N_1 - 1) \sigma_1^2} \]

A table of the constants \( k_p, a_1, \sigma_1, \mu_3 \), and \( \mu_4 - \sigma_1^4 \) is given, for various values of \( p \), in Table V, in order to facilitate computation of the large sample approximation to \( F(k_p, k, 0) \).
4.4 **Solution to the Problem by the Method of Empirical Sampling when the Standard Deviation is not Assumed Known.**

It was decided to determine whether or not an OC curve could be drawn from the results of some empirical sampling. One particular plan was selected and the required probabilities were computed in a very simple manner. The reliability of this method can be judged only by the 'smoothness' and 'plausibility' of the curve obtained, although it may be possible to obtain some form of confidence limits for the curve.

The data used were 50,000 individuals drawn at random from a normal population, mean zero, variance one, taken from I.B.M. cards supplied by the Rand Corporation. These individuals were given to three decimal places, that is, they were spaced at intervals of 0.001.

The mixed variables and attributes plan selected was the one with

$$N_1 = 10$$

$$N_2 = 20$$

$$k = 1$$

$$a = 3$$

We had available, therefore, five thousand samples of ten, for each of which the statistic $\bar{x} + s$ was computed. From inspection of the corresponding single variables plan ($N = 10, k = 1$) it was decided to compute ten points on the OC curve at $p = 0.025, 0.05, 0.075, 0.10, 0.15, 0.20, 0.25, 0.30, 0.40, \text{ and } 0.50$.

The data were divided equally between these ten points, allowing 500 samples for each. The same figures were not used for more than one point.
From each of these the joint probability that \( \bar{x} + s \) is greater than the upper limit \( U \), and that the number of defectives is equal to \( i \), where \( i \) runs from zero to \( a \), was estimated.

Here \( U = K_p \), where

\[
1 - p = \int_{-\infty}^{K_p} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} dx.
\]

This probability was estimated simply by finding those samples with \( \bar{x} + s > U \) and counting the number of defectives in each sample.

Suppose for a given point (a given value of \( p \)), the number of samples with \( d_1 = i \) and \( \bar{x} + s > K_p \) is \( n_i \).

Let \( N = \sum_{i=0}^{10} n_i \).

Since \( d_1 \) can only be an integer between 0 and 10,

\[
\sum_{i=0}^{10} \text{Pr}(\bar{x} + s > K_p, d_1 = i) = \text{Pr}(\bar{x} + s > K_p).
\]

The expected value of \( N \), \( E(N) = \sum_{i=0}^{10} E(n_i) \).

\[
E(n_i) = \text{Pr}(\bar{x} + s > K_p, d_1 = i)500, \quad \text{since } n_i \text{ has the binomial distribution}
\]

\[
E(N) = \sum_{i=0}^{10} E(n_i) = 500 \sum_{i} \text{Pr}(\bar{x} + s > K_p, d_1 = i) = 500 \text{Pr}(\bar{x} + s > K_p).
\]
Since the probability of \( \overline{x} + s > K_p \) is known, the expected value of \( N \) is known.

Given the fact that there are \( N \) samples with \( \overline{x} + s > K_p \),

\[
E(n_i | N) = N \frac{\Pr(d_1 = i \mid \overline{x} + s > K_p)}{\Pr(\overline{x} + s > K_p)} = N \frac{\Pr(d_1 = i, \overline{x} + s > K_p)}{\Pr(\overline{x} + s > K_p)}.
\]

This is intended as a 'rough' justification for taking \( u_i = \frac{n_i}{N} \Pr(\overline{x} + s > K_p) \) as an estimate of \( \Pr(d_1 = i, \overline{x} + s > K_p) \) instead of the more usual \( \frac{n_i}{500} \).

Note that \( \sum_{i=0}^{10} u_i = \Pr(\overline{x} + s > K_p) \)

\[
= \sum_{i=0}^{10} \Pr(d_1 = i, \overline{x} + s > K_p).
\]

This property does not hold for the estimate \( \frac{n_i}{500} \).

Also the estimated ordinate of the OC curve,

\[
\hat{L}_p = \Pr(\overline{x} + s < K_p) + \sum_{i=0}^{a} \frac{n_i}{N} \Pr(\overline{x} + s > K_p) \Pr(d_2 \leq a - i)
\]

\[
= \Pr(\overline{x} + s < K_p) + \Pr(\overline{x} + s > K_p) \sum_{i=0}^{a} \frac{n_i}{N} \Pr(d_2 \leq a - i)
\]

\[
\leq \Pr(\overline{x} + s < K_p) + \Pr(\overline{x} + s > K_p) \sum_{i=0}^{a} \frac{n_i}{N}
\]

\[
\leq 1.
\]

In effect we are correcting for the difference between the actual value
of N and its expected value. This was found to be of appreciable significance in the case tried only when p, the percent defective was less than 0.10.

Then

\[ L_p = \Pr(\bar{x} + s < K_p) + \frac{\Pr(\bar{x} + s > K_p)}{N} \sum n_i \Pr(d_2 \leq a - i) \]

and the estimate of the OC curve may be plotted.

Data obtained are shown in Table I and the final OC curve compared with the corresponding single variables plan \((N = 10, k = 1)\) is shown on page 31.

It may be noticed that the largest discrepancy from the curve for the single variables plan occurs around the abscissa \(p = 0.10\). At this stage the number of observations with \(\bar{x} + s > K_p\) was 92 and increased thereafter with \(p\). Although it can not be claimed that a histogram of the number of defectives in these 92 samples is completely satisfactory, the indication is that accuracy sufficient for practical purposes of the Operating Characteristic curve can be achieved with empirical sampling of this order. Little would be gained by attempting to obtain a more accurate estimate of the joint probabilities in the cases of the smaller values of \(p\) by means of considering more samples at these levels at the expense of the other higher levels. As \(p\) increases, then only the first variables sample tends to be effective in the rejection or acceptance of the lot. The incoming quality is then such that if the lot is not accepted on the first variables sample it will nearly always contain sufficient defectives to cause it to be rejected on the second attributes sample. Hence the OC curve for the mixed variables and attributes plan tends to coincide with the corresponding single variables plan as \(p\) increases beyond a certain point.
<table>
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<th>% defective</th>
<th>$d_1$</th>
<th>( n_1 )</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
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<tr>
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<td>0.004</td>
<td>0.0120</td>
<td>0.0240</td>
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</tr>
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</tr>
</tbody>
</table>

**TABLE I.**
4.5 Application of this Method to Further Plans.

Each mixed variables and attributes plan is defined by the four constants \( N_1, N_2, k, a \). The empirical sampling involves only \( N_1 \) and \( k \) and it is, therefore, possible from the results of one set of data to calculate the OC curves for any desired combination of values of \( N_2 \) and \( a \). In all plans with the same \( N_1 \) and \( k \), we are estimating the same joint probabilities, and the use of the same data for such plans is justified. It would also be a great saving if the same data could be used for different values of \( k \), changes which could readily be effected. Here, however, we are estimating different joint probabilities and some doubt arises as to whether or not this step is justified.

So far in this section only the Operating Characteristic curve has been considered. The average sample number is given by

\[
ASN = N_1 + \sum_{i=0}^{a} \Pr(\bar{x} > K_p - k, d_1 = i) N_{2:a-1}
\]

where \( N_{2:a-1} \) is as defined in Section 3.2 and may be computed directly.

Hence the only terms remaining are of the form \( \Pr(\bar{x} > K_p - k, d_1 = i) \), the same terms exactly as were used in obtaining the OC curve. Thus from the same computations an estimate of the ASN curve may also be drawn.

5. Bounds for the OC and ASN Curves.

In this section both upper and lower bounds are obtained for the OC and ASN curves.

5.1 Known Standard Deviation.

The ordinate of the OC curve is given by
\[ L_p = \Pr(\bar{x} + k\sigma \leq U) + \sum_{i=0}^{a} \Pr(\bar{x} + k\sigma > U, d_1 \leq i) \Pr(d_2 = a - i). \]

Also, the greater the number of defectives among the \(x\)'s, the greater is the probability that the mean of the \(x\)'s, \(\bar{x}\) exceeds the fixed quantity \(U - k\sigma\).

This may be expressed as
\[ \Pr(\bar{x} + k\sigma > U \mid d_1 = i) \leq \Pr(\bar{x} + k\sigma > U \mid d_1 = i + 1). \]

Hence
\[ \Pr(\bar{x} + k\sigma > U \mid d_1 \leq a) \leq \Pr(\bar{x} + k\sigma > U \mid d_1 > a) \quad \text{for any } a < N_1. \]

Now
\[
\Pr(\bar{x} + k\sigma > U, d_1 \leq i) = \Pr(\bar{x} + k\sigma > U \mid d_1 \leq i) \Pr(d_1 \leq i)
\]
\[
= \Pr(d_1 \leq i) \left\{ \Pr(\bar{x} + k\sigma > U \mid d_1 \leq i) \left[ \Pr(d_1 \leq i) + \Pr(d_1 > i) \right] \right\}
\]
\[
\leq \Pr(d_1 \leq i) \left\{ \Pr(\bar{x} + k\sigma > U \mid d_1 \leq i) \Pr(d_1 \leq i)
\]
\[+ \Pr(\bar{x} + k\sigma > U \mid d_1 > i) \Pr(d_1 > i) \right\}
\]
\[
= \Pr(d_1 \leq i) \left\{ \Pr(\bar{x} + k\sigma > U, d_1 \leq i) + \Pr(\bar{x} + k\sigma > U, d_1 > i) \right\}
\]
\[
= \Pr(d_1 \leq i) \Pr(\bar{x} + k\sigma > U). \]

Both terms of this product have known distributions.

Hence
\[ L_p \leq \Pr(\bar{x} + k\sigma \leq U) + \sum_{i=0}^{a} \Pr(d_1 \leq i) \Pr(\bar{x} + k\sigma > U) \Pr(d_2 = a - i) \]
\[= \Pr(\bar{x} + k\sigma \leq U) + \Pr(\bar{x} + k\sigma > U) \sum_{i=0}^{a} \Pr(d_1 \leq i) \Pr(d_2 = a - i) \]
\[= \Pr(\bar{x} + k\sigma \leq U) + \Pr(\bar{x} + k\sigma > U) \Pr(d_1 + d_2 \leq a). \]
This expression contains terms whose values are easily computed, and hence an upper bound for the OC curve is obtained.

For a lower bound we use the fact that for any two events \( X, Y \), if \( X \cup Y \) denotes the occurrence of one or the other or both of these events and \( X \cap Y \) denotes the occurrence of both of the events, then

\[
\Pr(X \cup Y) + \Pr(X \cap Y) = \Pr(X) + \Pr(Y)
\]

\[.\cdot 1 \geq \Pr(X \cup Y) = \Pr(X) + \Pr(Y) - \Pr(X \cap Y)\]

i.e., \( \Pr(X \cap Y) \geq \Pr(X) + \Pr(Y) - 1 \).

We use the inequality in the form

\[
\Pr(\bar{x} + k\sigma > U, \ d_1 + d_2 \leq a) \geq \Pr(\bar{x} + k\sigma > U) + \Pr(d_1 + d_2 \leq a) - 1.
\]

The expression on the right hand side may be negative. Hence we write

\[
\Pr(\bar{x} + k\sigma > U, \ d_1 + d_2 \leq a) \geq \max \left[ 0, \Pr(\bar{x} + k\sigma > U) + \Pr(d_1 + d_2 \leq a) - 1 \right]
\]

\[
L_p = \Pr(\bar{x} + k\sigma \leq U) + \Pr(\bar{x} + k\sigma > U, \ d_1 + d_2 \leq a)
\]

\[\geq \max \left[ \Pr(\bar{x} + k\sigma \leq U), \ \Pr(d_1 + d_2 \leq a) \right]
\]

a lower bound for the OC curve.

5.2 Standard Deviation Unknown.

\[
L_p = \Pr(\bar{x} + ks \leq U) + \Pr(\bar{x} + ks > U, \ d_1 + d_2 \leq a).
\]

Now

\[
\Pr(\bar{x} + ks > U, \ d_1 + d_2 \leq a) \leq \Pr(\bar{x} + ks > U, \ d_2 \leq a)
\]

\[
\Pr(\bar{x} + ks > U, \ d_1 + d_2 \leq a) \leq \Pr(\bar{x} + ks > U) \ \Pr(d_2 \leq a)
\]

we also have

\[
\Pr(\bar{x} + ks > U, \ d_1 + d_2 \leq a) \leq \Pr(d_1 + d_2 \leq a)
\]
\[ \Pr(\bar{x} + k_s > U, d_1 + d_2 \leq a) \leq \min \left[ \Pr(\bar{x} + k_s > U) \Pr(d_2 \leq a), \Pr(d_1 + d_2 \leq a) \right] \]

\[ \therefore L_p \leq \min \left[ \Pr(\bar{x} + k_s \leq U) + \Pr(\bar{x} + k_s > U) \Pr(d_2 \leq a), \Pr(\bar{x} + k_s \leq U) + \Pr(d_1 + d_2 \leq a) \right] \]

an upper bound.

For a lower bound we use the same procedure as in the case of known \( \sigma \).

\[ \therefore L_p \geq \max \left[ \Pr(\bar{x} + k_s \leq U), \Pr(d_1 + d_2 \leq a) \right] \]

completing the bounds for the entire OC curve in this case.

6. Conclusion.

For the case that the sample standard deviation is assumed to be known the problem of computing the OC and ASN curves is essentially solved. Either the method of Sections 3.21 and 3.22 or that of Sections 3.24 and 3.25 may be used, the latter method however being much less tedious. A comparison of the two methods (numerical integration and asymptotic expansion) is given in Tables II and III which present the computations for one set of parameters \( (K_p = 1.6, k = 1.6, n = 25) \). The value of \( \Pr \left( \frac{\bar{x}}{\sigma} > K_p - k, d_1 = 0 \right) \) is .06311 computed by asymptotic expansion and .06344 computed by numerical integration. The agreement to three-places of decimals represents probably a greater degree of accuracy than is required for practical use, since the expression is multiplied by two small numbers when computing the OC curve. Also it must be remembered that these results are finally used in the preparation of graphs which seldom require more than two decimal places.

For the case that the sample standard deviation is not known no practical
methods of computation for the OC and ASN curves have been devised. It may be that, since only a small degree of accuracy is required, for practical application a sharpening of the bounds given in Section 5 would suffice.

The experience of one of the authors in obtaining the approximate results through empirical sampling indicate that such methods may be feasible providing high speed automatic computing machinery is available.

Either the method of empirical sampling or improved bounds may in fact be the only possible solutions in the case that one wishes to use the average-range in place of the sample standard deviation in the variables part of the mixed procedures, or in the case that there are two specification limits (an upper and lower) which define quality since the analysis is even more complex for these cases than for the cases considered.

Important questions which also have been left unanswered include finding criteria for relating the second sample size to the first sample size. The effect of increasing the second sample size (for the case of known-σ) is illustrated on the following two pages which give some typical OC curves.

Acknowledgement

A good deal of the material on the asymptotic expansions and the bounds has been gleaned from an unpublished manuscript by S. Vora. The authors are especially grateful for helpful discussions with Albert H. Bowker and Gerald J. Lieberman.
Operating characteristic curves for mixed variables & attribute plan; known sigma
\[ N_1 = 15, \quad k = 1.0, \quad a = 4 \]

Note: Curves for \( N_2 = 60 \) and \( N_2 = 75 \) are approximately the same.
Operating characteristic curves for mixed variables & attribute plan; known sigma

\[ N_1 = 15, \quad k = 1.0, \quad a = 6 \]
References


TABLE II. $P(\bar{x} > k_{\bar{x}, d_1=0})$ Evaluated by an Asymptotic Expansion

($\bar{x}$ is the mean of $N_1$ observations from a normal population with zero mean and unit variance, $N_1 = 25$.)

\[ p = .054799 \]
\[ a_1 = \frac{-1109208}{.945201} = -.11735155 \]
\[ a_2 = .8122375 \]
\[ \sigma_1^2 = .7984661 \]
\[ \sigma_1^4 = .6375481 \]
\[ a_3 = -.5351231 \]
\[ a_4 = 1.95604055 \]
\[ \mu_3 = -.2524033 \]
\[ \mu_4 = 1.7713954 \]
\[ \sigma_1 = -.3537613 \]
\[ \sigma_2 = -.2215502 \]

\[ z = \frac{5(.11735155)}{.8935693} = .6566449 \]

\[ z^2 = .43118252 \]
\[ z^3 = .28313380 \]
\[ z^5 = .2208235 \]

\[ P(\bar{x} > k_{\bar{x}, d_1=0}) = q_1 P(\bar{x} > k_{\bar{x}, d_1=0}) = \left[ \int_{\bar{x}}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2 \pi}} dt + \frac{\sigma_1^2}{6 \sqrt{N_1}} H_2(z) \phi(z) \right. \]
\[ + \left. \frac{\phi(z)}{25} \left( \frac{\gamma_2}{24} H_3(z) + \frac{\gamma_4^2}{72} H_5(z) \right)^{N_1} \right] q_1 \]

\[ z = .6566449 \]
\[ \phi(z) = .3215733 \]
\[ H_2(z) = -.5688175 \]
\[ H_3(z) = -1.6868009 \]
\[ H_5(z) = 7.14041785 \]
\[ q_1 = .2444 \]

\[ P(\bar{x} > k_{\bar{x}, d_1=0}) \approx .06311 \]
TABLE III. \( P(\bar{x} > K_p - k, d_1 = 0) \) Evaluated by Numerical Integration

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* This quantity is obtained by the application of Simpson's rule to the ordinates of column (5). \( P(\bar{x} > K_p - k, d_1 = 0) \approx 0.06311. \)
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