MULTIPRODUCT INVENTORY MODELS WITH SET-UP

BY

ALAN WHEELER

TECHNICAL REPORT NO. 106

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WITH THE OFFICE OF NAVAL RESEARCH

DEPARTMENT OF OPERATIONS RESEARCH
AND
DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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Gerald J. Lieberman, Project Director

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NONTECHNICAL SUMMARY

This report presents a mathematical model of an inventory problem frequently encountered in practice and studies the nature of the best ordering policy to use for it. At the beginning of each of a sequence of periods an inventory manager must decide whether or not to place an order and, if so, how much of which products. His order is filled immediately and then during the period a random demand draws upon his inventory until at the beginning of the next period he makes another ordering decision.

When he orders, a unit cost for each item and a set-up cost for the order are charged. The set-up cost is thus independent of the size of the order and may represent the price of such things as paperwork, tooling up for a production run, or construction of facilities. On the basis of the inventory levels at the end of a period an inventory cost is charged. This represents a holding cost for those products still in stock and a penalty cost for those whose supply is depleted. Since the actual demand is not known but random, the inventory manager does not have full control over the costs he will be charged. It is assumed, however, that he can obtain his expected costs by averaging over the random demands, and his aim then is to use an ordering policy which minimizes his expected costs.

As the mathematical inventory literature is quite large it is important to emphasize four distinctive features of the model in this report: periodic review, random demand, set-up costs, and interdependence of
product costs. The present study is the most general one known by the
author to combine all four attributes. It is shown that the best policy
treating various products simultaneously may be quite different from
that obtained by looking at each product independently. That this would
be true is evident from considering situations where products compete for
the same warehouse or set-up costs may be reduced by ordering different
products at the same time. As the best policies may be quite difficult
to compute and to implement, much of the report is concerned with finding
situations in which the best policies are relatively simple in form.
MULTIPRODUCT INVENTORY MODELS WITH SET-UP COST

Section I. Formulation of the Basic Model

This section presents a formulation of a general multiproduct version of the classical \((s,S)\) stochastic inventory model with periodic review. For almost all situations in this paper the analysis of only two products provides sufficient generality to indicate the proper results for more than two products. The lightening of the notational burden by restricting attention to this case easily offsets any loss of generality due to the limitation.

Consider an inventory of two different items upon which a random demand \(d_{t,l} \geq 0\) is made for product \(l = 1\) or \(2\) in each of a sequence of well-defined periods \(t = 1,2,\ldots\). The demands in separate periods are independent and identically distributed as a random variable \(d\) with cumulative distribution function \(\Phi(d)\) and finite mean \(\mu\).

In order to exploit notationally the dynamic programming formulation which states the solution to the \(n\)-period problem in terms of that to an \((n-1)\)-period problem, periods are numbered from the end of a planning horizon. Thus, period \(t-1\) follows period \(t\). At the beginning of period \(t\) there are \(x_t = (x_{t1},x_{t2})\) units on hand where \(x_{t,l}\) may be negative and therefore indicate a backlog. The problem is to determine an appropriate ordering policy yielding for \(n \geq t \geq 1\)

\[ y_t = (y_{t1},y_{t2}) \in \mathcal{X}(x_t) = \{y | y \geq x_t\} \cap \mathcal{Y}, \]
where \( Y \) is most often \( \mathbb{R}^2 \), euclidean 2-space, but may sometimes be \( \mathbb{I}^2 \), the space of integral two-tuples, and is quite special in Section V. The random \( y_t \) represents the inventory level after immediate delivery of the order \( y_t - x_t \) but before demand has occurred. Since unfulfilled demand is merely backlogged

\[ x_{t-1} = y_t - \xi_t \quad t = n, n-1, \ldots, 1. \]

The criterion for choosing an appropriate policy is that of minimizing the expected total cost over the planning horizon. After a presentation of the costs involved the notion of a policy will be formalized.

The costs associated with the model are directly analogous to those in the single-product situation. If at the end of a period \( z \) units are on hand an inventory cost \( h(z) \) is incurred. This charge may be thought of as a holding cost when \( z \geq 0 \) and a penalty cost when \( z \leq 0 \). It is assumed that \( h(z) \) is a Borel function bounded below so that \( L(y) = E [ h(y - \xi) ] \) is defined. \( L(y) \) represents the expected inventory cost incurred for a single period begun with \( y \) units in inventory. In this section \( L(y) \) is assumed only to be finite, but the subsequent sections will require additional assumptions, including continuity. For convenience only, let \( h(z) \) be charged at the beginning of the period. Costs \( t \) periods in the future are discounted to the present by a factor \( \alpha^t \) where \( 0 \leq \alpha \leq 1 \).

The cost of ordering \( z \geq 0 \) units is equal to a set-up cost plus

\[ cz^T = (c_1, c_2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \]
where \( c_j \geq 0 \). The set-up cost may be written

\[
K(z) = K_1 \delta(z_1)(1-\delta(z_2)) + K_2(1-\delta(z_1))\delta(z_2) + K_{12}\delta(z_1)\delta(z_2),
\]

where

\[
\delta(z_j) = \begin{cases} 1 & z_j > 0 \\ 0 & z_j = 0 \end{cases},
\]

and \( K_1, K_2, \) and \( K_{12} \) are all non-negative. Special models of interest might require \( K_1 = K_2 = K_{12}, \) or \( K_1 = K_2 = \frac{1}{2} K_{12}, \) or \( K_1 + K_2 = K_{12}, \) or \( K_1 = K_2 = 0, K_{12} > 0, \) or simply \( K_1 + K_2 > K_{12}. \) The detailed analysis will cover only the first of these cases although the results for the general case will be indicated at the end of Section II. A model with more than two products has a wide variety of interesting set-up cost assumptions. Again the corresponding results are given without proof at the end of Section II.

The formulation of a policy utilizes the approaches of Blackwell [4] and Veinott [11] with slight alterations. An ordering policy is a sequence of Borel functions \( Y = (\cdots, Y_n, Y_{n-1}, \cdots, Y_1, Y_0) \) such that if at the beginning of period \( t \) in the \( n \)-period problem the history is \( H^n_t = (x_n, y_{n-1}, \cdots, x_t) \) then an order (perhaps zero) is placed to bring the level up to \( y_t = Y_t(H^n_t) \in \mathcal{X}(x_t) \). Because \( Y_t(\cdot) \) takes account of the dimension of the history vector \( Y \) is simultaneously meaningful for problems of all finite horizons. With this convention, of course, two policies may have identical components for the \( n \)-period problem and yet differ because of unequal components among those irrelevant to the
problem. This pathology is unimportant, however, since the concern in this paper is not with the uniqueness of optimal policies but with their existence and the properties of all optimal policies in certain classes. Conditions for uniqueness of optimal policies, in fact, appear well out of reach for the basic model studied here. Next, $Y$ is defined to be a Markov policy by the property that $Y_t(H^n_t) = Y_t(x_t)$ for all $H^n_t$ for all $n \geq t \geq 1$. Finally, by adopting a useful convention of Veinott it is required that $Y_0(H^n_0) = 0$. To effect this any surplus $x_{0^+} > 0$ at the end of the horizon is disposed of at a "cost" of $-c^+_x x_{0^+}$ and any backlog $-x_{0^-} > 0$ is made up at a cost of $-c^-_x x_{0^-}$.

There are two equivalent expressions for the optimal cost of the $n$-period problem: one a direct form, the other a dynamic programming formulation. Again the development of each here depends heavily on the work of Veinott [11], [12]. By convention there is a zero-period problem, and its cost when $x_0$ units are on hand and policy $Y$ is followed is

$$f_0(x_0|Y) = -cx_0^T.$$ 

For $n > 0$ the $n$-period problem has a random total cost whose expectation is really of interest. Hence, the (expected) cost of the $n$-period problem when $x_n$ units are on hand and policy $Y$ is followed is

$$f_n(x_n|Y) = E[\sum_{t=1}^{n} \alpha^{n-t}[k(y_t-x_t) + c(y_t-x_t)^T + h(x_{t-1})] - \alpha^nx_0^T]$$

$$= E[\sum_{t=1}^{n} \alpha^{n-t}[k(y_t-x_t) + h(y_t-x_t) + (1-\alpha)c_y^T + \alpha c_x^T]] - cx_n^T,$$
after selective replacement of \( x_t \) by \( y_{t+1} - \xi_{t+1} \) and a regrouping of terms. Since \( y_t \) is bounded below by the random variable \( x_n - \sum_{i=t+1}^{n} \xi_i \) having finite expectation the order of expectation and summation can be interchanged. Also, for a fixed \( x_n \) \( f_n(x_n | Y) \) is bounded below as a function of \( Y \) but may be infinite. If \( f_n(x_n | Y) \) is finite then so is each term in the sum and

\[
f_n(x_n | Y) = \sum_{t=1}^{n} \alpha^{n-t} E[E[K(y_t - x_t) + h(y_t - \xi_t) + (1-\alpha)cy_t^T + \alpha c_{t+1}^T | y_t] - cx_n^T
\]

\[
= \sum_{t=1}^{n} \alpha^{n-t} E[K(y_t - x_t) + L(y_t) + (1-\alpha)cy_t^T + \alpha c_u^T] - cx_n^T
\]

Note that the conditional expectations exist since the unconditional expectations are finite and each \( y_t(\cdot) \) is a Borel function. By letting

\[G(y) = L(y) + (1-\alpha)cy^T + \alpha c_u^T \]

it is found then that

\[f_n(x_n | Y) = \sum_{t=1}^{n} \alpha^{n-t} E[K(y_t - x_t) + G(y_t)] - cx_n^T\]

Since each \( G(y_t) \) is bounded below by a random variable with finite expectation the last sum is bounded below. If it is finite then each term is finite and a reversal of the above steps shows that \( f_n(x_n | Y) \) is finite. Thus, both sides of the last equality are finite or \( +\infty \) together and equality always holds.

A policy \( Y^* \) is optimal for the n-period problem if for all \( x_n \)

\[\infty > f_n(x_n | Y^*) = \min_Y f_n(x_n | Y) = f_n(x_n)\]
Since a policy with \( Y_t^N (H^N_y) = x_n \) for \( n \geq t \geq 1 \) has finite cost, \( f_n^*(x_n) \) is defined if and only if the minimization is achieved by some \( Y^* \). By the definition of \( f_n^*(x_n | Y) \) only the last \( n+1 \) components of \( Y \) affect the cost so that the other components of \( Y^* \) are arbitrary. Furthermore, \( Y^* \) is called optimal when it is optimal for the \( n \)-period problem for every \( n \geq 1 \). Since \( cx_n^T \) is a constant unaffected by the policy the minimization problem is unchanged if \( f_n^*(x_n | Y) + cx_n^T \) replaces \( f_n^*(x_n | Y) \). Thus, the cost of the \( n \)-period problem following policy \( Y \) commenced with \( x_n \) units on hand is

\[
\begin{cases}
  f_n^*(x_n | Y) = \sum_{t=1}^{n} \alpha^{n-t} E[K(y_t - x_t) + G(y_t)] & n \geq 1, \\
  f_0^*(x_0 | Y) = 0.
\end{cases}
\]

Consequently, the original problem is equivalent to a new one with single-period expected inventory cost function \( G(y) \), proportional ordering cost \( c = (0, 0) \), and unrestricted \( Y_0(\cdot) \). Furthermore, in discussing any period \( n \) it is assumed that \( n \geq 1 \).

The dynamic programming formulation utilizes a recursive relationship between the costs for problems of different horizons. Suppose in the \( n \)-period problem that some policy generates the history \( H^N_t \) for \( n \geq t \) and that a possibly different policy \( Y \) is used from periods \( t \) through \( 1 \). Then at the start of period \( t \) the conditional expected remaining cost given \( H^N_t \) is found as above to be
\[ f_t(H^n_t|Y) = K(y_t-x_t) + G(y_t) + \sum_{i=1}^{t-1} \alpha^{t-i} E[K(y_i-x_i) + G(y_i)] \]

(1.2) \quad = K(y_t-x_t) + G(y_t) + \alpha E[f_{t-1}(H^n_{t-1}|Y)] ,

where \( H^n_{t-1} \) is the resulting history vector. When \( Y \) is a Markov policy it is seen by backwards induction on \( 1 \leq t \) that each \( y_i \) depends on \( H^n_{t} \) only through \( x_t \) so that \( f_t(H^n_{t}|Y) \) may always be written \( f_t(x_t|Y) \). Taking \( t = n-1 \) in (1.2) yields the following expression equivalent to (1.1):

\[
\begin{align*}
\left\{ 
\begin{array}{l}
f_n(x_n|Y) = K(y_n-x_n) + G(y_n) + \alpha E[f_{n-1}(y_n-y_{n-1}|Y)] \quad n \geq 1 , \\
f_0(x_0|Y) = 0 .
\end{array}
\right.
\]

(1.3)

Furthermore, suppose that there exists a Markov policy \( Y^{n*} \) optimal for both the \( n \)- and the \((n-1)\)-period problems. Then if \( Y^n \) denotes an arbitrary Markov policy satisfying \( Y^n_t(x) = Y^{n*}_t(x) \) for \( t \leq n-1 \) it follows by (1.3) that

\[ f_n(x_n) = f_n(x_n|Y^{n*}) \]

\[ = \min_{Y^n} f_n(x_n|Y^n) \]

\[ = \min_{y_n \in Y^n(x_n)} \{ K(y_n-x_n) + G(y_n) + \alpha E[f_{n-1}(y_n-y_{n-1}|Y^{n*})] \} . \]
Thus,

\[
(1.4) \quad f_n(x_n) = \min_{y_n \in Y(x_n)} \left( K(y_n-x_n) + G(y_n) + \alpha \mathbb{E}[f_{n-1}(y_n-\xi)] \right),
\]

which is the finite version of the functional equation of inventory theory. Frequently in the literature (1.4) appears without proof because, first, only Markov policies are considered, and, second, the optimality for the (n-1)-period problem of some optimal n-period policy either is felt to be obvious or follows from an appeal to Bellman's principle of optimality. In the present formulation, however, optimality of Markov policies must be demonstrated since more general policies are allowed, and even for the former a comment regarding the interchange of minimization and expectation operators appears necessary to verify the simultaneous optimality of some optimal Markov policy. The following lemma serves these purposes.

**Lemma 1.1** For a fixed \( n \) suppose that for each \( t \leq n \) there is an optimal policy for the t-period problem. Then there is a Markov policy optimal for the t-period problem simultaneously for all \( t \leq n \).

**Proof** For \( n=1 \) the lemma is trivial so suppose that it is true for \( n-1 \geq 1 \) and consider its truth for \( n \). If \( Y^n \) is now any optimal policy for the n-period problem and \( Y^{(n-1)*} \) is a Markov policy optimal for the t-period problem for all \( t \leq n-1 \), define \( Y^{n*} \) to be a Markov policy with \( Y^{n*}_n(x_n) = Y^n_n(x_n) \) and \( Y^{n*}_t(x_t) = Y^{(n-1)*}_t(x_t) \) for \( t < n \). Then it remains only to show that \( Y^{n*} \) is optimal for the n-period problem. To this end it will be shown by induction on \( t \leq n \) that for \( \mathbb{H}_t^n \) generated by any policy
\[ f_t(x_t|y^{n*}) = \min_{y_t, \ldots, y_1} f_t(H^n_t|Y) . \]

For \( t = 1 \)

\[
\inf f_1(H^n_1|Y) = \inf_{y_1 \in \mathcal{Y}(x_1)} [K(y_1-x_1) + G(y_1)]
= f_1(x_1|y^{n*})
\]

so that the minimization is achieved. Suppose that the desired equality is true for \( t-1 < n \). From (1.2)

\[
\inf f_t(H^n_t|Y) = \inf_{Y_t, \ldots, y_1} [K(Y_t(H^n_t)-x_t) + G(Y_t(H^n_t)) + \alpha E[f_{t-1}(H^n_{t-1}|Y)]]
= \inf_{Y_t, \ldots, y_1} (K(Y_t(H^n_t)-x_t) + G(Y_t(H^n_t)) + \alpha \inf_{Y_{t-1}, \ldots, y_1} E[f_{t-1}(H^n_{t-1}|Y)]) .
\]

By the induction hypothesis \( f_{t-1}(H^n_{t-1}|Y) \) is minimized simultaneously for all \( H^n_{t-1} \) by \( Y^{n*} \) so that the infimum and expectation operators may be interchanged. Thus, for \( t \leq n \)

\[
\inf f_t(H^n_t|Y) = \inf_{y_t \in \mathcal{Y}(x_t)} (K(y_t-x_t) + G(y_t) + \alpha E[f_{t-1}(y_{t-1}^{n*})]) .
\]

For \( t < n \) \( f_{t-1}(y_{t-1}^{n*}|Y^{n*}) = f_{t-1}(y_{t-1}^{n*}) \) and \( f_t(x_t|Y^{n*}) = f_t(x_t) \) by construction. Then since the lemma is true for \( t < n \) it follows by (1.4) that the right side above is \( f_t(x_t|Y^{n*}) \), which implies the desired equality. For \( t=n \) the above becomes by (1.3)
\[ f_n(x_n | y^n) = K(y_n^x(x_n) - x_n) + G(y_n(x_n)) + \alpha E[f_{n-1}(y_{n-1}^x - \xi | y^{n-1})] \]

\[ = f_n(x_n | y_n^x) . \]

Thus, the desired equality again holds and, in fact, \( y_n^x \) is optimal for the n-period problem.

Q.E.D.

By the lemma (1.4) holds and attention may be restricted to Markov policies. Furthermore, it will be assumed that there is an optimal policy for the t-period problem for each \( t < n \) before the n-period problem is considered. With this convention it can be said that \( y_n^x \) is optimal for the n-period problem if

(1.5) \[ f_n(x_n | y_n^x) = \min_{y_n \in \mathcal{Y}(x_n)} \{ K(y_n - x_n) + J_n(y_n) \} , \]

where

\[ J_n(y_n) = G(y_n) + \alpha E[f_{n-1}(y_{n-1} - \xi)] . \]

It should be stressed here that stationarity of the cost and demand parameters over time is assumed only for notational convenience. As in the one-product problem the results of Sections I, II, and III may be extended easily when these parameters vary from one period to the next. In fact, the general nature of Sections I and II eliminates the need for the usual assumption of monotonicity of the discounted set-up costs over time [12]. Furthermore, a constant delivery lag which is the same for all products introduces no difficulties.
Section II. \((0, S(x))\) Policies

Throughout this section \(G(y)\) is quasi-convex and continuous, \(G(y) \to \infty\) as \(\|y\| \to \infty\), and \(\Xi(x_t) = \{y | y \geq x_t\}\). Major attention is confined to the case \(K_1 = K_2 = K_{12} = K\). As is the situation with single-product models, motivation for deriving optimal multiproduct policies is best provided by studying the one-period problem.

By (1.5) it is desired to find a policy \(Y^*\) satisfying

\[
f_1(x) = f_1(x|Y^*) = \min_{y \in \Xi(x)} (K(y-x) + J_1(y))
\]

\[
= \min_{y \in \Xi(x)} (K(y-x) + G(y)) ,
\]

where \(\delta(y-x) = 1\) if either \(y_1 > x_1\) or \(y_2 > x_2\) and is zero otherwise. Note that the period subscript on \(x\) and \(y\) may be suppressed without ambiguity. \(G(y)\) is defined to be quasi-convex \([1]\) if \(\{y | G(y) \leq c\}\) is a convex set for each real number \(c\). Equivalently, \(G(y)\) is quasi-convex if \(G(y) \leq G(y_0)\) implies \(G(\theta y + (1-\theta) y_0) \leq G(y_0)\) for all \(y, y_0, \) and \(\theta \in [0,1]\). Since this eliminates the existence of \(y, y_0, \) and \(\theta\) such that \(G(y) < G(\theta y + (1-\theta) y_0) > G(y_0)\) it is true that \(-G(y)\) is unimodal as a function of \(y\) constrained to any straight line. Thus, in particular, \(G(y)\) is quasi-convex over straight lines. Since \(G(y)\) is continuous and \(\to \infty\) as \(\|y\| \to \infty\) there exists a finite point \(S\) at which the function is minimized:

\[
G(S) = \min_{y} G(y) .
\]

Furthermore, whenever one of the variables is fixed \(G(x)\) can be minimized over the full range of the other:
\[ G(x_1, z_2(x_1)) = \min_{x_2} G(x) \]
\[ G(z_1(x_2), x_2) = \min_{x_1} G(x) \, . \]

Thus, if \( S \) minimizes \( G(y) \) there is a value of \( z_2(S_1) = S_2 \) and a value of \( z_1(S_2) = S_1 \). The set of possible \( z_1(x_j) \) for fixed \( x_j \) is closed by continuity of \( G(x) \). Also, since \( G(x) \) is quasi-convex as a function of \( x_j \) alone these sets are convex and there exist determinations \( z_1(x_j) \) and \( \overline{z}_1(x_j) \) such that \( z_1(x_j) \leq \overline{z}_1(x_j) \) for all \( x_j \).

**Theorem 2.1** (a) \( G(x_1, z_2(x_1)) \) and \( G(z_1(x_2), x_2) \) are continuous quasi-convex functions of \( x_1 \) and \( x_2 \) respectively for any determination of \( z_1(x_j) \).

(b) The set \( U \{(x_1, z_2(x_1)) | G(x_1, z_2(x_1)) = \min_{x_1} G(x) \} \) is connected in the sense that for every \( x_1 \) for all \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) such that for all \( 0 < \delta < \delta(\epsilon) \)

\[ \overline{z}_2(x_1) \geq z_2(x_1 + \delta) - \epsilon \]

and

\[ z_2(x_1) \leq \overline{z}_2(x_1 + \delta) + \epsilon \, . \]

Similarly \( U \{(z_1(x_2), x_2) | G(z_1(x_2), x_2) = \min_{x_1} G(x) \} \) is connected.

(c) For all \( S \) minimizing \( G(x) \)

\( x \geq S \) implies \( x_1 \geq z_1(x_2) \) or \( x_2 \geq \overline{z}_2(x_1) \),

\( x_1 \geq S_1, x_2 \leq S_2 \) implies \( x_1 \geq z_1(x_2) \) or \( x_2 \leq \overline{z}_2(x_1) \),
\( x_1 \leq S_1, x_2 \geq S_2 \) implies \( x_2 \leq \bar{z}_1(x_2) \) or \( x_2 \geq \bar{z}_2(x_1) \),

and

\( x \leq S \) implies \( x_1 \leq \bar{z}_1(x_2) \) or \( x_2 \leq \bar{z}_2(x_1) \).

**Proof** Fix \( x_1 \). Since \( G(y) \) is continuous and \( G(y) \to \infty \) as \( \| y \| \to \infty \) the set

\[
A' = \{ y \mid G(y) \leq G(x_1, z_2(x_1)) + \eta \},
\]

where \( \eta > 0 \), is compact. If \( A \) is a rectangle containing \( A' \) then continuity of \( G(y) \) implies uniform continuity of \( G(y) \) over \( A \). Furthermore, there exists a \( \delta(\eta) > 0 \) such that for all \( 0 < \delta < \delta(\eta) \)

\[
G(x_1, z_2(x_1)) + \eta \geq G(x_1 - \delta, z_2(x_1)) \geq G(x_1 - \delta, z_2(x_1 - \delta)) ,
\]

so that \( (x_1 - \delta, z_2(x_1)) \) and \( (x_1 - \delta, z_2(x_1 - \delta)) \) \( \in A \) and since \( A \) is a rectangle \( (x_1, z_2(x_1 - \delta)) \) \( \in A \) also. Thus, by uniform continuity, for all \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) and so a \( \delta(\epsilon, \eta) = \min\{\delta(\epsilon), \delta(\eta)\} \) such that for all \( 0 < \delta < \delta(\epsilon, \eta) \)

\[
| G(x_1, z_2(x_1)) - G(x_1 - \delta, z_2(x_1)) | < \epsilon
\]

and

\[
| G(x_1 - \delta, z_2(x_1 - \delta)) - G(x_1, z_2(x_1 - \delta)) | < \epsilon .
\]

Therefore,

\[
G(x_1, z_2(x_1)) > G(x_1 - \delta, z_2(x_1)) - \epsilon \geq G(x_1 - \delta, z_2(x_1 - \delta)) - \epsilon
\]

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\[ G(x_1, z_2(x_1 - \delta)) - 2\epsilon \geq G(x_1, z_2(x_1)) - 2\epsilon \]

for all \( 0 < \delta < \delta(\epsilon, \eta) \). Since \( \epsilon \) may be chosen arbitrarily small

\[ G(x_1, z_2(x_1)) = \lim_{\delta \to 0} G(x_1 - \delta, z_2(x_1 - \delta)) . \]

Similar steps yield the fact that

\[ G(x_1, z_2(x_1)) = \lim_{\delta \to 0} G(x_1 + \delta, z_2(x_1 + \delta)) \]

so that \( G(x_1, z_2(x_1)) \) is a continuous function of \( x_1 \). An interchange of components in these steps demonstrates continuity of \( G(z_1(x_2), x_2) \).

Since \( G(y) \) is quasi-convex over any straight line through \( S \) and \( S \) minimizes \( G(y) \) it is true that for \( x_1 < x_1 + \delta \leq S_1 \)

\[ G(x_1, z_2(x_1)) \geq G(x_1 + \delta, z_2(x_1)) + \frac{\delta}{S_1 - x_1}(z_2(x_1) - z_2(x_1 + \delta)) \]

\[ \geq G(x_1 + \delta, z_2(x_1 + \delta)) . \]

Thus, \( G(x_1, z_2(x_1)) \) decreases monotonically as \( x_1 \) approaches \( S_1 \) from below. A similar proof shows the same result as \( x_1 \) approaches \( S_1 \) from above and so \( G(x_1, z_2(x_1)) \) is quasi-convex. The treatment of \( G(z_1(x_2), x_2) \) is directly analogous.

(b) Suppose to the contrary that for some \( x_1 \) there exists an \( \epsilon > 0 \) such that for all \( \delta > 0 \) there is a \( 0 < \delta_0(\epsilon, \delta) = \delta_0 < \delta \) such that

\[ z_2(x_1) < z_2(x_1 + \delta_0) - \epsilon . \]

Let

\[ \eta = G(x_1, \overline{z}_2(x_1) + \epsilon) - G(x_1, \overline{z}_2(x_1)) > 0 . \]
Since

\[ z_2(x_1 + \delta_0) > \bar{z}_2(x_1) + \epsilon > \bar{z}_2(x_1) , \]

it follows that

\[ G(x_1 + \delta_0, \bar{z}_2(x_1) + \epsilon) \leq G(x_1 + \delta_0, \bar{z}_2(x_1)) \]

by quasi-convexity of \( G(y) \) as a function of one component. By choosing \( \delta \) small enough \( \delta_0 \) is ensured of being small enough that

\[ |G(x_1, \bar{z}_2(x_1) + \epsilon) - G(x_1 + \delta_0, \bar{z}_2(x_1) + \epsilon)| < \frac{\eta}{2} . \]

Thus,

\[ \eta < G(x_1 + \delta_0, \bar{z}_2(x_1) + \epsilon) + \frac{\eta}{2} - G(x_1, \bar{z}_2(x_1)) , \]

which implies that

\[ 0 < \frac{\eta}{2} < G(x_1 + \delta_0, \bar{z}_2(x_1)) - G(x_1, \bar{z}_2(x_1)) , \]

for some \( 0 < \delta_0(\epsilon) < \delta(\eta) \) for all \( \delta(\eta) \) small. This contradicts continuity of \( G(x) \), however, and so the conclusion is that for all \( \epsilon > 0 \) there exists \( \delta(\epsilon) > 0 \) such that for all \( 0 < \delta < \delta(\epsilon) \) \( \bar{z}_2(x_1) \geq z_2(x_1 + \delta) - \epsilon \). By following the same steps for \( x_1 - \delta \) and then taking the smaller of the two \( \delta(\epsilon) \) values obtained it is shown that in fact

\[ \bar{z}_2(x_1) \geq z_2(x_1 + \delta) - \epsilon . \]

The remaining statements in (b) have analogous proofs.

Connectedness implies the existence of a continuous curve of points \( z^*(x_1) \) such that \( G(x_1, z^*(x_1)) = \min_{x_2} G(x) \), but \( z^*(x_1) \) may not be a continuous function since a fixed \( x_1 \) may require an interval of values \( z^*(x_1) \) to ensure continuity of \( z^*(x_1) \) as a curve.
(c) Suppose \( x > S \) and \( x_1 < z_1(x_2) \) and \( x_2 < z_2(x_1) \). If \( x_1 = s_1 \) then

\[
S_2 = z_2(s_1) \\
\geq z_2(s_1) \\
= z_2(x_1) \\
> x_2 ,
\]

which is false since \( S \leq x \). Thus, \( x_1 > s_1 \) and similarly \( x_2 > s_2 \). Hence,

\[
\theta = \frac{x_1 - s_1}{x_2 - s_2} \left[ \frac{1}{x_2 - s_2} + \frac{x_1 - s_1}{x_2 - s_2} \frac{z_1(x_2) - x_1}{z_2(x_1) - x_2} \right] \in (0,1) .
\]

Now, since \( G(x) \) is quasi-convex over straight lines the point

\[
x' = \theta(z_1(x_2), x_2) + (1-\theta)(x_1, z_2(x_1))
\]

must satisfy

\[
G(x') \leq \max[G(x_1, z_2(x_1)), G(z_1(x_2), x_2)]
\]

\[
< G(x)
\]

since \( x_1 < z_1(x_2) \) and \( x_2 < z_2(x_1) \). However, it can be verified that \( S < x < x' \) are colinear so that

\[
G(s) \leq G(x) \leq G(x') ,
\]

which is a contradiction. Hence, \( x \geq S \) implies that \( x_1 \geq z_1(x_2) \) or \( x_2 \geq z_2(x_1) \). Proofs of the other relations are similar. Q.E.D.
The conditions on $G(y)$ similarly ensure that for every $x$ there exists a point $S(x) \geq x$ such that

$$G(S(x)) = \min_{y \geq x} G(y).$$

(2.1)

If $S(x)$ is any single-valued determination, i.e., a function, satisfying (2.1) then

$$f_1(x) = \min_{y \geq x} [K^0(y-x) + G(y)]$$

$$= \min [G(x), K + G(S(x))].$$

Let

$$\sigma = \{x | G(x) > K + G(S(x))\}.$$

Then any policy $Y^*$ having

$$Y^*_1(x) = \begin{cases} x & \text{if } x \in \sigma \cup \\ S(x) & \text{if } x \in \sigma \end{cases}$$

is optimal for the one-period problem. It will be useful in what follows to consider a policy derived from one particular class of determinations of $S(x)$.

Since $\{S | G(S) = \min G(y)\}$ is closed and bounded it contains points $S'$ and $S''$ for which $S'_1 = \min S_1$ and $S''_2 = \min S_2$. In particular, the points $(\min S_1, z_2(\min S_1))$ and $(z_1(\min S_2), \min S_2)$ are well-defined and minimize $G(y)$. Let $S$ be used to denote any value of $S \leq (z_1(\min S_2), z_2(\min S_1))$ which minimizes $G(y)$ and satisfies
$S = (z_1(S_2), z_2(S_1))$. Since $(z_1(z_2(\min S_1)), z_2(\min S_1))$ also minimizes $G(y)$ it follows that $z_1(z_2(\min S_1)) \leq \min S_1$ and so equality must hold. Hence, $(\min S_1, z_2(\min S_1))$ and similarly $(z_1(\min S_2), \min S_2)$ are values of $S$. Thus, $\min S_1 \leq \min S_1$, but also $\min S_1 \geq \min S_1$ since every $S$ minimizes $G(y)$. Consequently, $\min S_1 = \min S_1$.

**Lemma 2.2** Consider $x$ such that $\min S_1 < x_1 \leq z_1(\min S_1)$ and $\min S_2 < x_2 \leq z_2(\min S_1)$. If either $x_1 = z_1(x_1)$ then $x$ is a value of $S$.

Suppose $x_1 = z_1(x_1)$.

**Proof** Since $(z_1(\min S_2), \min S_2)$ and $(\min S_1, z_2(\min S_1))$ minimize $G(y)$ and $\min S_1 < x_2 \leq z_2(\min S_1)$ so does $(z_1(x_2), x_2) = x$ by (a) of Theorem 2.1. Thus $x_2 = z_2(x_1)$ and it remains to show equality.

Suppose that $x_2 > z_2(x_1)$. Since $(x_1, z_2(x_1))$ also minimizes $G(y)$ so do all points $y$ on the line from $(x_1, z_2(x_1))$ to $(\min S_1, z_2(\min S_1))$.

In particular, there is a $y$ having $y_2 = x_2$ and $y_1 < x_1$. Since this contradicts the fact that $x_1 = z_1(x_2)$ it is concluded that $x_2 = z_2(x_1)$.

The proof is analogous when it is assumed at first that $x_2 = z_2(x_1)$.

Let the points $S(x)$ be defined as follows:

$$S(x) = \begin{cases} S & \text{if (a): } x \leq S, \\ (x_1, z_2(x_1)) & \text{if (b): } x_1 > z_1(\min S_2), x_2 \leq z_2(x_1), \\ (z_1(x_2), x_2) & \text{if (c): } x_1 \leq z_1(x_2), x_2 > z_2(\min S_1), \\ x & \text{if (d): } x \text{ does not satisfy (a), (b), or (c)}. \end{cases}$$

Although $S(x)$ may not be uniquely defined attention will generally be restricted to single-valued determinations of it, as was the case with $z_1(x_j)$ and $S(x)$. 

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Theorem 2.3 If for each $x$ in (a) $S(x)$ is assigned a single value then $S(x)$ is a function satisfying (2.1).

Proof $S(x)$ is single-valued and, hence, a function, if $S(x)$ is the same for any $x$ satisfying both (b) and (c). For such an $x$,

$$z_1(x_2) > x_1 > z_1(\min S_2) \geq \min S_1$$

and

$$z_2(x_1) > x_2 > z_2(\min S_1) \geq \min S_2.$$ 

Since $x > (z_1(\min S_2), \min S_2)$, which minimizes $G(y)$, it follows from (c) of Theorem 2.1 that $x_1 \geq z_1(x_2)$ or $x_2 \geq z_2(x_1)$ and so $x_1 = z_1(x_2)$ or $x_2 = z_2(x_1)$.

Suppose that $x_1 = z_1(x_2)$ and $x_2 < z_2(x_1)$. Then

$$G(x) > G(x_1, z_2(x_1)) \geq G(z_1(\min S_2), \min S_2).$$

By quasi-convexity then any point $y$ on the line between $(z_1(\min S_2), \min S_2)$ and $(x_1, z_2(x_1))$ satisfies $G(y) < G(x)$. In particular, there is a point on the line with $y_2 = x_2$ so that $G(y_1, x_2) < G(x)$, which contradicts the fact that $x_1 = z_1(x_2)$. Thus, $x_2 \notin z_2(x_1)$ and similarly $x_1 \notin z_1(x_2)$ if $x_2 = z_2(x_1)$.

Thus, $x = (z_1(x_2), z_2(x_1))$ and $S(x) = (x_1, z_2(x_1)) = (z_1(x_2), x_2)$ by (b) or (c). Hence, if $x$ satisfies (b) and (c) only a single value of $S(x)$ is obtained.

To show that $S(x)$ satisfies (2.1) it is useful first to show that $G(y) > G(x)$ for all $y \geq x$ for certain $x$ satisfying (d).
(a) Consider \( x \) satisfying \( \min S_1 < x \leq (\min S_j)^\ast \) and \( x > \bar{z}_1(x_j) \) for \( 1, j = 1 \) and \( 2, j \neq j \).

For all \( \theta \in [0,1] \) let \( S(\theta) = \theta(\bar{z}_1(\min S_2), \min S_2) + (1-\theta)(\min S_1, \bar{z}_2(\min S_1)) \). By quasi-convexity of \( G(y) \), \( S(\theta) \) minimizes \( G(y) \) for all \( \theta \in [0,1] \). Furthermore, there exist \( \theta_1, \theta_2 \in [0,1] \) such that \( (S(\theta_1))_1 = x_1 \) and \( (S(\theta_2))_2 = x_2 \), where \( (a_1, a_2)_1 = a_1 \). By definition of \( \bar{z}_1(x_j) \) then \( \bar{z}_2(x_1) \geq (S(\theta_1))_2 \geq \bar{z}_2(x_1) \) and \( \bar{z}_1(x_2) \geq (S(\theta_2))_1 \geq \bar{z}_1(x_2) \).

If \( x_1 \leq (S(\theta_2))_1 \) or \( x_2 \leq (S(\theta_1))_2 \) then \( x \) minimizes \( G(y) \) and so \( G(y) \geq G(x) \) for all \( y \geq x \).

If \( x_1 > (S(\theta_2))_1 \) and \( x_2 > (S(\theta_1))_2 \) then for all \( y \geq x \) there is a \( \theta_3(y) \) between \( \theta_1 \) and \( \theta_2 \) such that \( S(\theta_3(y)), x \), and \( y \) are colinear. By quasi-convexity \( G(S(\theta(y))) \leq G(x) \leq G(y) \).

(b) Consider \( x_1 > \bar{z}_1(\min S_2), x_1 \geq \bar{z}_2(x_2), x_2 > \bar{z}_2(x_1) \). Let \( y \) be any point \( \geq x \). If \( x_2 \geq \bar{z}_2(y_1) \) then since \( x_2 \geq \bar{z}_2(y_1) \) and \( y_1 \geq x_1 \geq \bar{z}_2(y_1) \):

\[
G(y) \geq G(x) \geq G(y_1, x_2)
\]

If \( x_2 < \bar{z}_2(y_1) \) then \( \bar{z}_2(x_1) < x_2 < \bar{z}_2(y_1) \) so that \( x_1 < y_1 \). By the connectedness in (b) of Theorem 2.1 there exists a \( z \), and a value \( z_2(z_1) \) such that \( x_1 \leq z_1 < y_1 \) and \( z_2(z_1) = x_2 \). Thus,

\[
G(y) \geq G(y_1, \bar{z}_2(y_1)) \geq G(z_1, \bar{z}_2(z_1))
\]

by (a) of Theorem 2.1 since \( y_1 > z_1 \geq x_1 > \bar{z}_1(\min S_2) \) and \( \bar{z}_1(\min S_2), \min S_2 \) minimizes \( G(y) \). Hence,
G(y) \geq G(z_1, x_2) \\
\geq G(x)

since \( z_1 \geq x_1 \geq z_1(x_2) \).

(γ) Consider \( x \) satisfying \( z_1(x_2) > x_1 > z_1(\min S_2) \) and \( z_2(\min S_1) \)
\( \geq x_2 > z_2(x_1) \). The point \( (\min S_1, z_2(\min S_1)) \) minimizes \( G(y) \). Therefore, from the fact that \( x_1 > z_1(\min S_2) \geq \min S_1 \) and \( x_2 \leq z_2(\min S_1) \)

it follows that \( x_2 \leq z_2(x_1) \) by (a) of Theorem 2.1 since \( x_1 < z_1(x_2) \).
Thus, \( z_2(x_1) < x_2 \leq z_2(x_1) \) so that there exists a value \( z_2(x_1) = x_2 \).
Now, for any \( y \geq x \)

\[
G(y) \geq G(y_1, z_2(y_1)) \\
\geq G(x_1, z_2(x_1)) = G(x)
\]

by (a) of Theorem 2.1 since \( y_1 \geq x_1 > z_1(\min S_2) \) and \( (z_1(\min S_2), \min S_2) \)

minimizes \( G(y) \).

Now (α), (β), and (γ) will be used to complete the theorem.

Suppose that \( x \leq (z_1(\min S_2), z_2(\min S_1)) \) so that (b) and (c)
aren't satisfied. If \( x \leq \) some \( S \) then \( S(x) = \) some \( S \) by (a) and so
satisfies (2.1) since all \( S \) minimize \( G(y) \). On the other hand, if \( x \)
\( \notin \) any \( S \) then (a) is not satisfied so that \( S(x) = x \) by (d) and also \( x_1 \)
\( > \) \( \min S_1 \) and \( x_2 \geq \min S_2 \). It can further be argued that each \( x_1 \geq z_1(x_j) \).
Thus, by Lemma 2.2 and (α) (2.1) is satisfied.

Suppose that \( x_1 \geq z_1(\min S_2) \) and \( x_2 \leq z_2(\min S_1) \) so that (a) and
(c) aren't satisfied. If \( x_2 \leq z_2(x_1) \) then by (b) \( S(x) = (x_1, z_2(x_1)) \).
For all \( y \geq x \)

\[
G(y) \geq G(y_1, z_2(y_1)) \\
\geq G(x_1, z_2(x_1)) = G(S(x))
\]

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by (a) of Theorem 2.1 so that (2.1) is satisfied. If, on the other hand, 
\[ x_2 > z_2(x_1) \] then (b) is not satisfied. Thus, \( S(x) = x \) by (d). If \( x_1 \geq z_1(x_2) \) or \( x_1 < z_1(x_2) \) then (2.1) is satisfied by (β) or (γ) respectively,

When \( x_1 \leq z_1(\min \frac{S_2}{2}) \) and \( x_2 > z_2(\min \frac{S_1}{2}) \) the proof is similar.

Suppose that \( x_1 > z_1(\min \frac{S_2}{2}) \) and \( x_2 > z_2(\min \frac{S_1}{2}) \) so that (a) is not satisfied. If \( x_1 \leq z_2(x_1) \) or \( x_1 \leq z_1(x_2) \) then by (b) or (c) \( S(x) = (x_1, z_2(x_1)) \) or \( S(x) = (z_1(x_2), x_2) \). In either case (2.1) is satisfied as above. If, on the other hand, \( x_1 > z_1(x_2) \) and \( x_2 > z_2(x_1) \) then (b) and (c) aren't satisfied so that \( S(x) = x \) by (d). Then, by (β) (2.1) is satisfied.

Q.E.D.

Let \( \sigma \) consist of the following points \( x_1 \):

(a') \( x < \text{some } S, G(x) > K + G(S) \),

(b') \( x_1 > z_1(\min \frac{S_2}{2}), x_2 < z_2(x_1), G(x) > K + G(x_1, z_2(x_1)) \),

or (c') \( x_1 < z_1(x_2), x_2 > z_2(\min \frac{S_1}{2}), G(x) > K + G(z_1(x_2), x_2) \).

By Theorem 2.3 any policy with \( \sigma = \sigma \) and \( S(x) = S(x) \) is optimal for the one-period problem.

In studying the n-period problem much use is made of the following lemma given by Veinott (Lemma 1 in [12]) for the single-product problem. All statements here are special cases of his except that vectors replace his scalar inventory levels and periods are numbered in reverse order.

**Lemma 2.4** For every \( i \) if \( x \leq x' \) then \( f_i(x) \leq f_i(x') + K \) and if \( y \leq y' \) then \( J_i(y') - J_i(y) \geq G(y') - G(y) - K \), provided that \( f_i(\cdot) \) and \( J_i(\cdot) \) are defined.

**Proof** If \( x \leq x' \) then when \( f_i(x) \) exists

\[
f_i(x) = \min_{y \geq x} \left\{ K5(y-x) + J_i(y) \right\}
\]

\[
\leq K + \inf_{y \geq x} J_i(y)
\]

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\[ \leq K + \inf_{y \geq x'} J_1(y) \]
\[ \leq K + \min_{y \geq x'} [Ks(y-x') + J_1(y)] \]
\[ = K + f_i(x') \]

When \( J_1(y) \) exists and if \( y \leq y' \) then

\[ J_1(y') - J_1(y) = G(y') - G(y) + \alpha E[f_{i-1}(y'-\xi) - f_{i-1}(y-\xi)] \]
\[ \geq G(y') - G(y) - \alpha K \]

by the first part of the lemma. \( Q.E.D. \)

Furthermore, the following lemma is needed.

**Lemma 2.5** If \( g(x) \) is a continuous function and \( \to \infty \) as \( \|x\| \to \infty \) then \( g^*(x) = \min_{y \geq x} g(y) \) is continuous.

**Proof** The proof is a modified version of that in (a) of Theorem 2.1.

By the conditions on \( g(y) \) the set

\[ A = \{y | g(y) \leq g^*(x) + \eta\} \]

where \( \eta > 0 \), is compact and \( g(y) \) is uniformly continuous over it.

For every \( x \) there is a \( y(x) \) satisfying \( g^*(x) = g(y(x)) \). Furthermore, there exists a \( \delta(\eta) > 0 \) such that for all \( z \) satisfying \( \max|x'_z - z'_z| \leq \delta(\eta) \)

\[ g(y(x)) + \eta \geq g(z) \]
\[ \geq g(y(z)) \]

so that \( z \) and \( y(z) \in A \). By uniform continuity for all \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) and so \( \delta_0 = \delta(\epsilon, \eta) = \min(\delta(\epsilon), \delta(\eta)) \) such that for all \( z \) satisfying \( \max|x'_z - z'_z| \leq \delta_0 \)

\[ |g(y(x)) - g(z)| < \epsilon \]
and \[ |g(y(z)) - g(x_1 + \delta_0, x_2 + \delta_0)| < \epsilon \]
since \( (x_1 + \delta_0, x_2 + \delta_0) \) is such a \( z \). Therefore,

\[
g^*(x) = g(y(x)) > g(z) - \varepsilon
\geq g(y(z)) - \varepsilon.
> g(x_1 + \delta_0, x_2 + \delta_0) - 2\varepsilon
\geq g(y(x)) - 2\varepsilon
= g^*(x) - 2\varepsilon.
\]

Because \( \varepsilon \) may be chosen arbitrarily small

\[
g^*(x) = \lim_{\delta \to 0} g^*(z)
\]

for all \( z \) satisfying \( \max |x_i - z_i| < \delta \) so that \( g^*(x) \) is continuous.

Q.E.D.

It is now possible to show existence of optimal policies for the \( n \)-period problem and to construct examples.

**Theorem 2.6** For every \( n \): (i) \( J_n(y) \) exists and is continuous.

(ii) For every \( x \) there exists a point \( S_n(x) \geq x \) such that

\[
J_n(S_n(x)) = \min_{y \geq x} J_n(y),
\]

and, in particular, there exists a point \( S_n \) minimizing \( J_n(y) \).

(iii) There exists a policy \( Y^* \) optimal for the \( n \)-period problem.

Specifically, for \( i=1,2,\ldots,n \) for any single-valued determination of \( S_i(x) \) in (ii) for all \( x = x_i \)

\[
Y^*_i(x) = \begin{cases} x & \text{if } x \in \sigma^c_i \\ S_i(x) & \text{if } x \in \sigma_i \end{cases},
\]

where \( \sigma_i = \{x | J_i(x) > K + J_i(S_i(x)) \} \).
yields an optimal policy \( Y^* \).

Furthermore, for every \( x \)

\[
f_n(x) = \begin{cases} 
J_1(x) & x \in \sigma_1 \\
K + J_1(S_1(x)) & x \in \sigma_1
\end{cases}
\]

Proof. The proof is by induction on \( n \). Consider \( n = 1 \).

(i) \( J_1(y) = G(y) \), which exists and is continuous.

(ii) \( S_1(x) = S(x) \).

(iii) This was established after Theorem 2.1 in the solution to the one-period problem.

Suppose (i)-(iii) hold for \( n \). It remains to show that they hold for \( n+1 \).

(i) The proof is similar to that of Veinott [12] for the single-product problem. By (iii) for \( n \) \( f_n(x) \) exists and for all \( x \)

\[
f_n(x) = \min(J_n(x), K + J_n(S_n(x)))
\]

\( J_n(x) \) is continuous by (i) for \( n \) and so \( J_n(S_n(x)) \) is continuous by Lemma 2.5. Since the minimum of two continuous functions is continuous it follows that \( f_n(x) \) is continuous. Now \( J_{n+1}(y) = G(y) + \alpha E[f_n(y-\xi)] \) and exists by boundedness of \( f_n(x) \) for \( x \leq y \). Since \( G(y) \) is continuous it suffices to show that \( E[f_n(y-\xi)] \) is continuous. Consider any finite rectangle \( R(a,b) = \{ y \mid a \leq y \leq b \} \). For any \( y \in R(a,b) \) for all \( \xi > 0 \)

\[
f_n(S_n) \leq f_n(y-\xi) \quad \text{by (ii) for } n
\]

\[
\leq K + f_n(b) \quad \text{by Lemma 2.4.}
\]

Thus, for any \( y \) it is true that for all \( \xi > 0 \) \( f_n(y-\xi) \) is continuous in \( y \) and uniformly bounded in any rectangle around \( y \). Therefore, by the Lebesgue dominated convergence theorem \( E[f_n(y-\xi)] \) is continuous.
(ii) By (ii) for \( n \) there exists \( S_n \) minimizing \( J_n(y) \) so that for all \( x \)

\[
 f_n(x) = \min_{y \geq x} \{ K_\delta(y-x) + J_n(y) \} \geq J_n(S_n).
\]

Thus,

\[
 J_{n+1}(y) = G(y) + \alpha E[f_n(y-\xi)] \geq G(y) + \alpha J_n(S_n).
\]

Hence, \( J_{n+1}(y) \to \infty \) as \( \|y\| \to \infty \). This, together with continuity of \( J_{n+1}(y) \) by (i) implies the existence of \( S_{n+1}(x) \) and \( S_{n+1} \).

(iii) By (iii) for \( n \) there exists a policy \( Y \) which is optimal for the \( n \)-period problem where for \( i=1,2,\ldots,n \), \( Y_i(x) = Y^*_i(x) \) satisfies (2.2) but \( Y_i(x) \) for \( i > n \) may be arbitrary. By (i) for \( n+1 \), \( \sigma_{n+1} \)
and \( S_{n+1}(x) \) exist so that \( Y^*_{n+1}(x) \) may be defined as in the current hypothesis. Consider any policy \( Y \) where \( Y_{n+1}(x) = Y^*_{n+1}(x) \) for \( i=1,2,\ldots,n \) and \( Y^*_{n+1}(x) \) is used for period \( n+1 \). Now for any policy \( Y \)

\[
 f_{n+1}(x|Y) = K_\delta(Y_{n+1}(x)-x) + G(Y_{n+1}(x)) + \alpha E[f_n(Y_{n+1}(x)-\xi|Y)] \geq K_\delta(Y_{n+1}(x)-x) + G(Y_{n+1}(x)) + \alpha E[f_n(Y_{n+1}(x)-\xi)]
\]

(where equality is achieved if \( Y \) is optimal for the \( n \)-period problem, as \( Y^* \) is)

\[
 (2.3) \quad f_{n+1}(x|Y) - f_{n+1}(x|Y^*) \geq K_\delta(Y_{n+1}(x)-x) + J_{n+1}(Y_{n+1}(x)) - K_\delta(Y^*_{n+1}(x)-x) - J_{n+1}(Y^*_{n+1}(x)).
\]

Therefore, for any policy \( Y \)

\[
 f_{n+1}(x|Y) - f_{n+1}(x|Y^*) \geq K_\delta(Y_{n+1}(x)-x) + J_{n+1}(Y_{n+1}(x)) - K_\delta(Y^*_{n+1}(x)-x) - J_{n+1}(Y^*_{n+1}(x)).
\]

If \( x \in \sigma_{n+1} \) and \( Y_{n+1}(x) = x \) this equals \( J_{n+1}(x) - K - J_{n+1}(S_{n+1}(x)) > 0 \).

If \( x \in \sigma_{n+1} \) and \( Y_{n+1}(x) > x \) this equals \( K + J_{n+1}(Y_{n+1}(x)) - K - J_{n+1}(S_{n+1}(x)) > 0 \).
If \( x \in \sigma_{n+1}^c \) and \( Y_{n+1}(x) = x \) this equals \( J_{n+1}(x) - J_{n+1}(x) = 0 \). If \( x \in \sigma_{n+1}^c \) and \( Y_{n+1}(x) > x \) this equals \( K + J_{n+1}(Y_{n+1}(x)) - J_{n+1}(x) \) 
\[ \geq K + J_{n+1}(S_{n+1}(x)) - J_{n+1}(x) \geq 0. \] Thus, \( f_{n+1}(x) - f_{n+1}(x|y^*) \geq 0 \) for all \( Y \) so that \( y^* \) is an optimal policy for the \((n+1)\)-period problem.

Furthermore, by (2.3)

\[
f_{n+1}(x) = f_{n+1}(x|y^*) = \begin{cases} 
J_{n+1}(x) & \text{if } x \in \sigma_{n+1}^c \\
K + J_{n+1}(S_{n+1}(x)) & \text{if } x \in \sigma_{n+1}^c. 
\end{cases}
\]

Q.E.D.

All of the policies studied in this section can be characterized by a sequence of pairs \((\sigma_i, S_i(x))\) composed of a set and a function defined on it. In fact, any Markov policy, optimal or not, can be characterized by such pairs simply by defining

\[
\sigma_i = \{x | Y_i(x) > x\}
\]

\[
S_i(x) = Y_i(x) \text{ if } x \in \sigma_i.
\]

(2.4)

Although no restriction has been made that \( S_i(x) \in \sigma_i \) it is definitely desirable that this be the case for optimal policies. To see this suppose that a policy \( Y \) is such that for some \( i \) there exist \( y \) and \( z \in \sigma_i \) such that \( Y_i(z) = y \). Now, if the beginning inventory is \( z \), policy \( Y \) requires bringing the level up to \( y \) and then awaiting the demand for the period. This is inconsistent, however, with the requirement that if the beginning level is \( y \) it should be brought up to \( Y_i(y) > y \) before demand occurs. In light of this and Theorem 2.7 below the following definition is made by analogy to the single product problem.
Definition A Markov policy \( Y \) is a \((\sigma, S(x))\) policy if for each \( i \) the
\( \sigma_i \) and \( S_i(x) \) defined by (2.4) satisfy \( S_i(x) \in \sigma_i^c \).

In the single-product problem some optimal policy has \( \sigma_i = (-\infty, s_i) \) for
some point \( s_i \) and \( S_i(x) = S_i \), and this policy is called more briefly
an \((s, S)\) policy. Johnson [9] uses the term "\((\sigma, S)\) policies" in ana-
lyzing the infinite-horizon, stationary, multiproduct problem with the
present set-up cost function. Since he studies optimality only in the
equilibrium region, where \( x \leq S \), he can take \( S(x) = S \). A complete
characterization of the optimal policy in the finite-period problem, as
was seen particularly in the case of one period requires specification
of the ordering policy in the transient region, where \( x \notin S \), as well.

Part (iii) of Theorem 2.6 gives existence and a characterization of
some optimal \((\sigma, S(x))\) policies. The following theorem demonstrates
that in searching for an optimal policy attention may be restricted to
\((\sigma, S(x))\) policies satisfying (2.2). Recall that some optimal policy
is Markov.

Theorem 2.7 (a) If \( Y^* \) is optimal and Markov then for all \( i \) there is a single-
valued determination of \( S_i(x) \) in (ii) of Theorem 2.6 such that
\( x \in \sigma_i \) implies \( Y_i^*(x) = S_i(x) \). (b) If \( K > 0 \) then any optimal Markov
policy is a \((\sigma, S(x))\) policy.

Proof (a) Suppose there exists an \( x \in \sigma_i \) for which there does not
exist a point \( S_i(x) = Y_i^*(x) \). Define a policy \( Y' \) by

\[
Y'_i(z) = \begin{cases} 
Y_i^*(z) & \text{if } j \neq i, \\
Y_i^*(z) & \text{if } j = i \text{ and } z \neq x, \\
S_i(x) & \text{if } j = i \text{ and } z = x,
\end{cases}
\]

for some value \( S_i(x) \). Hence, since \( Y_i^*(x) \neq S_i(x) \) and \( Y' \) is optimal for
the (i-1)-period problem,

\[ f_i(x|Y^*) = K + J_i(Y^*_1(x)) \]

\[ > K + J_i(S_i(x)) \]

\[ = f_i(x|Y') . \]

Since this contradicts optimality of \( Y^* \) there must not exist an \( x \) such as was hypothesized.

(b) It suffices to show that \( S_i(x) \in \sigma_i^c \) for all \( x \in \sigma_i \) for all \( i \).

Let \( x \in \sigma_i \) so that \( Y^*_1(x) = S_i(x) = y > x \). Now, if \( y \in \sigma_i \) then

\[ f_i(y|Y^*) = K + J_i(Y^*_1(y)) \leq J_i(y) \] where \( Y^*_1(y) > y \). But

\[ f_i(x|Y^*) = K + J_i(y) \]

\[ \leq K + J_i(Y^*_1(y)) \]

since \( Y^*_1(y) > x \). Thus,

\[ K + J_i(y) \leq J_i(y) , \]

which is impossible since \( K > 0 \). Hence, there does not exist an \( x \in \sigma_i \) for which \( S_i(x) \in \sigma_i \).

Q.E.D.

For theoretical and computational purposes it is desirable to obtain bounds on the sets \( \sigma_i \) and the functions \( S_i(x) \) characterizing optimal policies. To this end for each \( x \) and for any \( S_i(x) \) define the points \( z^*_i(x) \) to be those satisfying

\[ G(z^*_i(x)) = \min_{x \leq y \leq S_i(x)} G(y) . \]
In particular, \( z_1^*(x) = S_1(x) \) is a possibility, and in general one may take \( z_1^*(x) = S_1(x) \) if \( x \leq S_1(x) \leq S_4(x) \).

**Lemma 2.8** \( G(S_1(x)) \leq G(z_1^*(x)) + \alpha K \) for any \( x \) and \( S_1(x) \).

**Proof.** By the definitions of \( S_1(x) \) and \( z_1^*(x) \) and by Lemma 2.4 since \( z_1^*(x) \leq S_4(x) \),

\[
0 \geq J_1(S_1(x)) - J_1(z_1^*(x)) \\
> G(S_1(x)) - G(z_1^*(x)) - \alpha K
\]

Q.E.D.

By an appropriate choice of \( x \) in the lemma the following partial upper bounds on \( S_1 \) are obtained:

\[
G(S_1) \leq \begin{cases} 
G(S_1, z_2(S_1)) + \alpha K & \text{if some } S_1 \geq \text{some } S \\
G(S_1, z_2(S_1)) + \alpha K & \text{if some } S_1 \text{ satisfies } S_1 > z_1(\min S_2) \\
& \text{ and } S_1 \geq z_2(S_1) \\
G(z_1(S_1), S_1) + \alpha K & \text{if some } S_1 \text{ satisfies } S_1 > z_1(S_1) \\
& \text{ and } S_1 > z_2(\min S_1) 
\end{cases}
\]

The next lemma yields an upper bound on \( \sigma^c_1 \).

**Lemma 2.9** If \( G(x) < G(z_1^*(x)) + (1-\alpha)K \) then \( x \in \sigma^c_1 \), and if \( G(x) \leq G(z_1^*(x)) + (1-\alpha)K \) then \( x \in \sigma^c_1 \) for some optimal \( \sigma_1^c \).

**Proof.** Since \( x \leq S_1(x) \) it follows from Lemma 2.4 that

\[
K + J_1(S_1(x)) - J_1(x) \geq G(S_1(x)) - G(x) + (1-\alpha)K \\
\geq G(z_1^*(x)) - G(x) + (1-\alpha)K \\
> 0
\]

By the definition of \( \sigma_1^c \) it is concluded that \( x \in \sigma_1^c \) if the last inequality is strict. If equality holds the expected cost after not
ordering is still the same as that after ordering up to the best level so that \( x \in \sigma_1^c \) for some optimal \( \sigma_1 \). 

Q.E.D.

Define the set \( \overline{\sigma} \) as follows:

\[
\overline{\sigma} = \{ x | S(x) > x \text{ and } G(x) \geq G(S(x)) + (1-\alpha)K \}.
\]

**Corollary 2.10** If \((1-\alpha)K > 0\) then \( \sigma_1 \subseteq \overline{\sigma} \); and if \((1-\alpha)K = 0\) some optimal \( \sigma_1 \subseteq \overline{\sigma} \).

**Proof.** If \((1-\alpha)K > 0\) and \( S(x) = x \) then \( G(x) < G(S(x)) + (1-\alpha)K \) trivially. Hence, for all \( x \in \overline{\sigma}^c \)

\[
G(x) < G(S(x)) + (1-\alpha)K \leq G(z_1^*(x)) + (1-\alpha)K
\]

by definition of \( S(x) \). Thus, by Lemma 2.9 \( x \in \sigma_1^c \) and so \( \sigma_1 \subseteq \overline{\sigma} \). If \((1-\alpha)K = 0\) then for \( x \in \overline{\sigma}^c \) \( S(x) = x = z_1^*(x) \) so that \( x \in \sigma_1^c \) for some optimal \( \sigma_1 \) by Lemma 2.9 and so \( \sigma_1 \subseteq \overline{\sigma} \).

Q.E.D.

The establishment of bounds is completed by a cruder approach leading to less sharp results. Let \( S(x) \) be a specific function now and let \( Y' \) be an appropriate \((\sigma, S(x))\) policy. For theoretical uses it is sufficient to take \( Y'_n(x) = S(x) \) for all \( x \) and \( Y'_i(S(x)-\xi) = S(x) \) for all \( i < n \) and \( \xi > 0 \). For practical computation, however, a policy whose cost is closer to that of the optimal policy \( Y \) is preferable. Eligible candidates are those characterized by \( \sigma'_n = \sigma, S'_n(x) = S(x) \), and \( \sigma'_1 = \sigma_1 \) and \( S'_1(x) = S_1(x) \) for \( i < n \); or \( \sigma'_1 = \sigma_1 \) and \( S'_1(x) = S_1(x) \) for all \( i \); or \( \sigma'_1 = \{ x | S(x) > x, G(x) + E[f_{n-1}(x-\xi | Y')] > K + G(S(x)) + E[f_{n-1}(S(x)-\xi | Y')] \} \) and \( S'_1(x) = S(x) \) for all \( i \). For any of these \( Y' \) for all \( x \)

\[
G(s) \sum_{j=1}^{n} \alpha^{j-1} \leq f_n(S(x) | Y') \leq K \sum_{j=2}^{n} \alpha^{j-1} + G(S(x)) \sum_{j=1}^{n} \alpha^{j-1}.
\]

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Define the set $\sigma_n$ as follows:

$$
\sigma_n = \{ x | S(x) > x, \ G(x) > K + G(S) + f_n(S(x)|Y') - G(S) \sum_{j=1}^{n} \alpha^{j-1} \}.
$$

In particular, $\sigma_1 = \sigma$ and for general $n$, $\sigma_n$ is non-empty since $G(y) \to \infty$ as $\|y\| \to \infty$.

**Lemma 2.11** For all $n$, $\sigma_n \subset \sigma_n$.

**Proof** If $x \in \sigma_n$, then

$$
J_n(x) = G(x) + \alpha \mathbb{E}[f_{n-1}(x-t)]
\geq K + G(S) + f_n(S(x)|Y') - G(S) \sum_{j=1}^{n} \alpha^{j-1} + \alpha G(S) \sum_{j=1}^{n-1} \alpha^{j-1}
= K + f_n(S(x)|Y')
\geq K + f_n(S(x))
\geq K + f_n(S_n(x))
$$

since $S(x) > x$ for $x \in \sigma_n$. But the last expression equals $K + J_n(S_n(x))$ since $S_n(x) \in \sigma_n$ so that it must be that $f_n(x) = K + J_n(S_n(x)) < J_n(x)$ and $x \in \sigma_n$. Thus, $\sigma_n \subset \sigma_n$. Q.E.D.

For any $y = S(x) > x$ for some $x$ define $\Sigma_n(y)$ as follows:

$$
\Sigma_n(y) = \{ z | G(z) \leq G(S) + f_n(y|Y') - G(S) \sum_{j=1}^{n} \alpha^{j-1} \}.
$$

**Lemma 2.12** If $x \in \sigma_n \subset \sigma$ then $S_n(x) \in \Sigma_n(y)$ where $y = S(x) > x$.

**Proof** Consider any $x \in \sigma_n \subset \sigma$ so that $x < y = S(x)$.

Suppose that $S_n(x) \in \Sigma_n(y)$. Then, since $x \in \sigma_n$,

$$
f_n(S_n(x)) = J_n(S_n(x))
\geq G(S) + f_n(y|Y') - G(S) \sum_{j=1}^{n} \alpha^{j-1} + \alpha \mathbb{E}[f_{n-1}(S_n(x)-t)]
\geq f_n(y|Y')
\geq f_n(y)
\geq f_n(S_n(x))
$$


since \( y > x \). Because this is a contradiction it is concluded that 
\[ S_n(x) \in \Sigma_n(y). \]
Q.E.D.

Up to this point it has been assumed that \( K(y-x) = K\delta(y-x) \). It will now be indicated how the preceding analysis may be extended to cover more general set-up cost functions. Recall that

\[ K(y-x) = K_1 \delta(y_1-x_1)\delta(y_2-x_2) + K_2(1-\delta(y_1-x_1))\delta(y_2-x_2) + K_{12} \delta(y_1-x_1)\delta(y_2-x_2), \]

where \( K_1, K_2, \) and \( K_{12} \) are non-negative. It will also be assumed that \( K_{12} \geq \max(K_1, K_2) \). In particular, if the inventory has a continuous measurement violation of this inequality will lead to levels for which there is no optimal order. For example, if the minimizing \( S \) is unique and if \( K_{12} < K_1 \) then for \( x_2 = S_2 \) and small enough \( x_1 \min_{y \geq x}(K(y-x)+G(y)) \) does not exist since \( S(x) = S \). In a discrete problem where there is a smallest positive quantity of a product this indeterminacy does not exist; but it would still appear to be unusual to find a genuine two-product situation in which the set-up charge for two products is less than that for just one.

Denote \( S(x) \), as defined in (2.1), by \( S^{12}(x) \) and define \( S^1(x) \) and \( S^2(x) \) to satisfy \( S^1(x)_2 = x_2, (S^2(x))_1 = x_1 \),

\[ G(S^1(x)) = \min_{y_1 \geq x_1} G(y_1, x_2) \]

\[ G(S^2(x)) = \min_{y_2 \geq x_2} G(x_1, y_2). \]

By (1.5)

\[ f^1(x) = \min_{y \geq x} \{ K(y-x) + G(y) \} = \min(G(x), K_1 + G(S^1(x)), K_2 + G(S^2(x)), K_{12} + G(S^{12}(x))) \].
For $\beta = 1, 2,$ or $12$ define $\sigma^\beta$ by

$$
\sigma^1 = \{ x | f_1^1(x) = K_1 + G(S^1(x)) < G(x) \},
$$

$$
\sigma^2 = \{ x | f_1^2(x) = K_2 + G(S^2(x)) < G(x),\ x \notin \sigma^1 \},
$$

and

$$
\sigma^{12} = \{ x | f_1^{12}(x) = K_{12} + G(S^{12}(x)) < G(x),\ x \notin \sigma^1 \cup \sigma^2 \}.
$$

Then any policy $Y^*_1$ having

$$
Y^*_1(x) = \begin{cases} 
  x & \text{if } x \notin \sigma_c^\beta \\
  \sigma^\beta(x) & \text{if } x \in \sigma^\beta 
\end{cases}
$$

is optimal for the one-period problem. By letting $\sigma = \bigcup \sigma^\beta$ and $S(x) = S^\beta(x)$ if $x \in \sigma^\beta$ one can characterize $Y^*_1(\cdot)$ by the pair $(\sigma, S(x))$. From here it is clear that the relevant results in this section may be extended and the optimality of $(\sigma, S(x))$ policies for a general set-up cost thereby established.

In fact, there is little difficulty in characterizing the solution for $m$ products. Let $\beta$ be any of the $2^m - 1$ non-empty subsets of \{1, 2, ..., m\}. Then one may write

$$
K(y-x) = \sum_{\beta} K_{\beta} (\prod_{i \in \beta} 8(y_i - x_i)) (\prod_{j \notin \beta} (1-8(y_j - x_j))),
$$

where it is assumed that for all $\beta$ and $\beta' \subseteq \beta$ $K_{\beta'} \leq K_{\beta}$. It is convenient to let $y_{\beta}(x)$ denote an $m$-vector whose $j^{\text{th}}$ component is $x_j$ for all $j \notin \beta$ and to impose on the $\beta$'s an arbitrary strict linear order denoted by a strict inequality. As before, any Markov policy may be characterized by a sequence of pairs $(\sigma_1, S_1(x))$ by letting $\sigma_1 = \bigcup_{\beta} \sigma^\beta_1$. 

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where
\[ \sigma^\beta_i = \{ x | (Y_i(x))_j > x_j \text{ if and only if } j \in \beta \}, \]
and \( S_i(x) = Y_i(x) \) if \( x \in \sigma_i \). The general form of a \((\sigma, S(x))\) policy, however, has notational alterations to be explained below:

**Definition** A Markov policy is a \((\sigma, S(x))\) policy if \( x \in \sigma_i^\beta \) implies \( S_i(x) \in (\sigma_i^\beta)^c \).

With this formulation the extension of all the specialized two-product results is apparent and may be encapsulated in the following modification of Theorems 2.6 and 2.7. Since the proof is tedious and adds little enlightenment beyond that found in the already cumbersome proofs for the special model it is omitted. Note first that for every \( i, \beta, \) and \( x \) there exists a point \( S_i^\beta(x) \) such that
\[ J_i(S_i^\beta(x)) = \min_{y = y_\beta(x) \geq x} J_i(y) \]
and so for any single-valued determination of \( S_i^\beta(x) \) one may let
\[ \sigma_i^\beta = \{ x | f_i(x) = K_\beta + J_i(S_i^\beta(x)) < J_i(x), x \in \sigma_i^\beta \text{ for any } \beta' < \beta \} \]

**Theorem 2.13** (i) \( Y^* \) is an optimal policy if for all \( i \)
\[ Y^*_i(x) = \begin{cases} x & \text{if } x \in \bigcap_{\beta} (\sigma_i^\beta)^c \\ S_i^\beta(x) & \text{if } x \in \sigma_i^\beta. \end{cases} \]
(11) If \( Y^* \) is an optimal policy characterized by the sequence \((\sigma_i^*, S_i^*(x))\) then for all \( i \) there is a single-valued determination of \( S_i^\beta(x) \) for all \( \beta \) such that \( x \in \sigma_i^* \) implies \( S_i^*(x) = S_i^\beta(x) \).
(iii) If $K_\beta > 0$ for all $\beta$ then any optimal Markov policy is a $(\sigma, S(x))$ policy.

At first glance it may appear that the discussion preceding the earlier definition would lead to the requirement that $x \in \sigma_1$ imply $S_1(x) \in \sigma_1^c$ for an m-product $(\sigma, S(x))$ policy. If for all $\beta$ for any partition of $\beta$ into two or more non-empty subsets $\beta_1, \beta_2, \ldots, \beta_k$

$$0 < K_\beta < \sum_{j=1}^{k} K_{\beta_j}$$

then for any optimal policy $x \in \sigma_1$ does imply that $S_1(x) \in \sigma_1^c$.

Notice that in the special model of this section this is true since $0 < K < 2K$. On the other hand, if (we drop the subset braces)

$K_{12} = K_1 + K_2$, which is often reasonable, there will be an $x$, $S_1(x), S_2(x)$, and $S_{12}(x) = S_2(S_1(x))$ so that $G(x)$ exceeds

$$K_2 + G(S_2(x)) > K_1 + G(S_1(x))$$

$$= K_1 + K_2 + G(S_2(S_1(x)))$$

$$= K_{12} + G(S_{12}(x)).$$

Here one optimal policy uses $Y_1(x) = S_1(x) \in \sigma_1 \notin \sigma_1^c$. If $K_{12} > K_1 + K_2$ then for such an $x$ any optimal Markov policy has $Y_1(x) = S_1(x)$.

This set-up cost relation might occur if either of two items may be produced on a given machine in a period but production of both items would require purchase of an additional machine.
Section III. Special Characterizations of \((\sigma, S(x))\) Policies

Studies of the present model without a set-up cost have found useful the mathematical concept of substitutable and complimentary products. Two products will be called substitutable if \(z_1(x_j)\) and \(\bar{z}_1(x_j)\) are monotonically non-increasing functions of their arguments; the products are complimentary if these functions are monotonically non-decreasing.

A justification of this terminology is seen by considering the one-period problem with \(K = 0\). If \(x_2 > z_2(\min S_1)\), \(8 > 0\), \(x_1 < z_1(x_2)\), and \(x_1 < z_1(x_2 + 8)\), then it is optimal to order up to \((z_1(x_2), x_2)\) from \(x\) and up to \((z_1(x_2 + 8), x_2 + 8)\) from \((x_1, x_2 + 8)\). In the substitutable case \(z_1(x_2 + 8) \leq z_1(x_2)\) so that a larger initial stock of the second product requires at most as large an order of the first; whereas in the complementary case \(z_1(x_2 + 8) > z_1(x_2)\) so that an order of product one no less than that when starting at \(x\) is called for. As Veinott [11] points out, if \(G(x)\) is convex and twice differentiable then non-negativity and non-positivity of the cross partial derivatives ensure substitutability and complementarity, respectively.

These definitions may be weakened by requiring merely some single-valued determination of each \(z_1(x_j)\) to be monotonically non-increasing or non-decreasing, but the definition given carries the essential idea and has greater tractability. In particular, the results below with slight modifications and somewhat more complicated proofs hold when restrictions are not imposed on \(\bar{z}_1(x_j)\). The price of this greater mathematical generality, however, is a weakening of the relationship between the mathematical concepts and the intuitive notions of substitutability and complementarity. As an example suppose that the products are
substitutable, \( x_2 > z_2 \) (\( \min S_1 \)), \( z_1(x_2) \leq x_1 \leq \bar{z}_1(x_2 + \delta) \), and \( \bar{z}_1(x_2) < x_1 \). For a one-period problem \( x_1 \) is an optimal level for product one when the second is at \( x_2 + \delta \), but it is too much when the second is at \( x_2 \). Since the products are substitutes, however, one would expect an optimal level no lower for the first product when the second starts lower. This difficulty will be avoided and the mathematics will agree with intuition if it is required that \( z_1(x_j) \) and \( \bar{z}_1(x_j) \) be simultaneously monotonically non-increasing for substitutability or non-decreasing for complementarity.

**Theorem 3.1** If the products are substitutable and if for \( i=1,2,\ldots,n, \)

\( (\alpha) \) for every \( x \) for any \( S(x) \) some \( S_1(x) \geq S(x), \)

then (i) \( g \subset g_n; \)

(ii) \( J_n(y') - J_n(y) \leq G(y') - G(y) \leq 0 \)

if \( y \leq y' \leq \text{some } S, \)

or if \( y_1 = y_1' > z_1(\min S_2), y_2 \leq y_2' \leq z_2(y_1), \)

or if \( y_1 \leq y_1' \leq z_1(y_2), y_2 = y_2' > z_2(\min S_1); \)

(iii) \( S_n(x) \neq \text{any } S(x); \)

and (iv) \( f_n(x') - f_n(x) \leq 0 \) for \( x \) and \( x' \) satisfying the same relations as \( y \) and \( y' \) in (ii).

**Proof** For \( n = 1 \) \( (\alpha) \) is superfluous and (i)-(iv) are true as follows.

(i) \( g = g \subset g_1 \) by Lemma 2.11.

(ii) Since \( J_1(y) = G(y) \) it is necessary only to show that \( G(y) \geq G(y') \) for all \( y \) and \( y' \) considered. For the last two cases of \( y \) and \( y' \) this is true by quasi-convexity of \( G(y) \) over straight lines. For the first case if \( y' = S \) the result is true by definition so take \( y' < S \). Consider the line through \( S \) and \( y' \), which
passes through \((y'_1, z'_2)\) and \((z_1, y_2)\) for some \(z_1\) and \(z_2\). Since 
\(y'_1 < \underline{z}\) it follows that 
\[ z_1 \leq y'_1 \leq \underline{z}_1 = z_1(\underline{z}_2) \quad \text{and} \quad z_2 \leq y'_2 \leq \underline{z}_2 = z_2(\underline{z}_1). \]
Hence, by substitutability
\[ z_1 \leq z_1(\underline{z}_2) \leq z_1(\underline{y}_1) \leq \underline{z}_1(\underline{y}_2), \]
and, similarly,
\[ z_2 \leq z_2(\underline{y}_1). \]
If \(z_1 \geq y_1\) then \(y_1 \leq z_1 \leq z_1(\underline{y}_2)\) so that
\[ G(y) \geq G(z_1, y_2) \geq G(y'). \]
by quasi-convexity of \(G(y)\) over straight lines.
If \(z_1 < y_1\) then it is necessary that \(z_2 > y_2\) and
\[ G(y) \geq G(y', z_2) \geq G(y'). \]
(iii) Consider any \(x\) for which \(x \leq S(x) < \underline{S}(x)\), so that \(x\) satisfies (a), (b), or (c) preceding Theorem 2.3. If \(x_1 > \underline{z}_1(\min \underline{z}_2)\) and \(x_2 \leq z_2(x_1)\) then \(S(x) = (x_1, z_2(x_1))\) so that \((S(x))_1 = x_1\) and \((S(x))_2 < z_2(x_1)\). But this is impossible since then \(G(S(x)) = G(S(x)) < G(S(x))\).
Thus, (b) and, similarly, (c) are not satisfied. Consequently, \(x\) satisfies (a) and \(\underline{S}(x)\) is some \(\underline{z}\).
Now, either \((S(x))_1 < \underline{z}_1\) or \((S(x))_2 < \underline{z}_2\). In the first case since \(S(x)\) minimizes \(G(y)\) over \(y \geq x\) and the products are substitutable

\[ z_1(\underline{z}_2) = \underline{z}_1 > (S(x))_1 \geq \underline{z}_1((S(x))_2) \geq z_1(\underline{z}_2), \]
which is a contradiction. Similarly, if \((S(x))_2 < \underline{z}_2\) a contradiction is reached. Hence, \(S(x) \not< \text{any } \underline{S}(x)\).
(iv) For all $x$ $f_1(x) = \min\{G(x), K + G(S(x))\}$. For $x$ and $x'$ under consideration $G(x) \geq G(x')$ by (ii). Now, if $x \leq x' \leq$ some $\underline{S}$ then $G(\underline{S}) = G(S(x')) = G(S(x))$ for these $x$ and $x'$. For the remaining cases notice that if $x_j = x'_j > z_j(min_{S_1})$ and $x'_i \leq x_i \leq z_i(x_j)$ for $i \neq j$ then $S(x) = S(x')$ by (b) and (c) preceding Theorem 2.3. Hence, $G(S(x)) = G(S(x'))$ for all $x$ and $x'$ considered and so $f_1(x) \geq f_1(x')$.

Assume now that the theorem is true for $n$ and consider the statement for $n+1$.

(i) Consider $x \in \underline{S}$ so that (a'), (b'), or (c') following Theorem 2.3 holds.

If (a') holds then $S(x)$ can be chosen to be $S$ and $f_n(x-\xi) - f_n(S(x)-\xi)$ $\geq 0$ for all $\xi \geq 0$ by (iv) for $n$. If (b') holds then $S(x) = (x_1, z_2(x_1))$. If $x_1-\xi_1 > z_1(min_{S_2})$ then for all $\xi_2 \geq 0$ by substitutability

$$x_2-\xi_2 < z_2(x_1) - \xi_2 \leq z_2(x_1-\xi_1) - \xi_2.$$  

Hence, with $x-\xi$ and $S(x)-\xi$ as $x$ and $x'$ (iv) for $n$ applies.

On the other hand, if $x_1-\xi_1 \leq z_1(min_{S_2})$ then for all $\xi_2 \geq 0$ by substitutability

$$x_2-\xi_2 < z_2(x_1) - \xi_2$$

$$\leq z_2(z_1(min_{S_2})) - \xi_2 = min_{S_2} - \xi_2$$

$$\leq min_{S_2}.$$  

Hence, $x-\xi < S(x)-\xi \leq (z_1(min_{S_2}), min_{S_2})$ and (iv) for $n$ applies.

Thus, for all $\xi \geq 0$ $f_n(x-\xi) - f_n(S(x)-\xi) \geq 0$.

If (c') holds the same result is true.
Now, by using the $S(x)$ supplied here and the definition of $S_{n+1}(x)$ and $\sigma$ it is seen that

$$J_{n+1}(x) - K_{n+1}(S_{n+1}(x)) \geq J_{n+1}(x) - K_{n+1}(S(x))$$

$$= G(x) - K - G(S(x)) + \alpha \mathbb{E}[f_n(x) - f_n(S(x))]$$

$$\geq G(x) - K - G(S(x))$$

$$> 0.$$

Hence, $x \in \sigma_n$ and so $\sigma \subseteq \sigma_n$.

(ii) If $y \leq y' \leq S$ then $y - \xi \leq y' - \xi \leq S$ for all $\xi \geq 0$.

If $y_1 = y'_1 > z_1(\min S_2)$ and $y_2 \leq y'_2 \leq z_2(y_1)$ then since $z_2(y_1) \leq z_2(y'_1 - \xi_1)$ by substitutability it is true that either

$$y_1 - \xi_1 = y'_1 - \xi_1 > z_1(\min S_2)$$

and $y_2 - \xi_2 \leq y'_2 - \xi_2 \leq z_2(y_1 - \xi_1)$,

or

$$y_1 - \xi_1 = y'_1 - \xi_1 \leq z_1(\min S_2)$$

and

$$y_2 - \xi_2 \leq y'_2 - \xi_2 \leq z_2(y_1)$$

$$\leq z_2(z_1(\min S_2)) = \min S_2$$

so that $y - \xi \leq y' - \xi \leq (z_1(\min S_2), \min S_2) = (z_1(\min S_2), z_2(z_1(\min S_2)))$.

For the remaining $y$ and $y'$ considered a similar result holds.

Hence, in all cases $f_n(y' - \xi) - f_n(y - \xi) \leq 0$ by (iv) for $n$, and, consequently,

$$J_{n+1}(y') - J_{n+1}(y) = G(y') - G(y) + \alpha \mathbb{E}[f_n(y' - \xi) - f_n(y - \xi)]$$

$$\leq G(y') - G(y)$$

$$\leq 0$$

by (ii) for $n=1$. 

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(iii) Suppose that there are an $x$ and values $S_{n+1}(x)$ and $\underline{S}(x)$ for which $x \leq S_{n+1}(x) < \underline{S}(x)$. It will be shown that (ii) for $n+1$ may be used. Since $x < \underline{S}(x)$ it follows that (a), (b), or (c) preceding Theorem 2.3 holds for $x$ and $\underline{S}(x)$.

If (a) holds then $\underline{S}(x) = \text{some } S$ and (ii) applies. If (b) holds then $x_1 = (S(x))_1$ so that $(S_{n+1}(x))_1 = x_1$ and $(S_{n+1}(x))_2 < (\underline{S}(x))_2 = z(x)$. Thus, (ii) for $n+1$ applies when (b) or, similarly, (c) holds.

Therefore, the fact that $S_{n+1}(x) < \underline{S}(x)$ implies that

$$0 \leq J_{n+1}(\underline{S}(x)) - J_{n+1}(S_{n+1}(x)) \leq G(\underline{S}(x)) - G(S_{n+1}(x)).$$

By (iii) for $n=1$ $G(\underline{S}(x)) - G(S_{n+1}(x)) < 0$, which yields a contradiction. Therefore, it is concluded that there do not exist an $x$ and values $S_{n+1}(x)$ and $\underline{S}(x)$ for which $x \leq S_{n+1}(x) < \underline{S}(x)$.

(iv) For all $x$ $f_{n+1}(x) = \min(J_{n+1}(x), \lambda J_{n+1}(S_{n+1}(x)))$. For $x$ and $x'$ considered $J_{n+1}(x) \geq J_{n+1}(x')$ by (ii) for $n+1$. Furthermore, for these $x$ and $x'$, one may choose $S(x') = \underline{S}(x)$ so that by

(a) some $S_{n+1}(x) \geq \underline{S}(x') \geq x'$. Therefore,

$$J_{n+1}(S_{n+1}(x)) = \min_{y \geq x} J_{n+1}(y)$$

$$= \min_{y \geq x'} J_{n+1}(y)$$

$$= J_{n+1}(S_{n+1}(x')).$$

Hence, $f_{n+1}(x) \geq f_{n+1}(x').$

Q.E.D.

Since (a) played a direct role only in the proof of (iv) the theorem may be restated as follows using conditions only where they are required in the proof.
Theorem 3.1* If the products are substitutable then

(I) (i)-(iv) are true for \( n = 1 \),

(II) if (iv) is true for \( n \) then (i) and (ii) are true for \( n + 1 \),

(III) if (ii) is true for \( n + 1 \) then (iii) is true for \( n + 1 \),

(IV) if (ii) is true for \( n + 1 \) and for every \( x \) for any

\[ S(x) \text{ some } S_{n+1}(x) \geq S(x) \text{ then (iv) is true for } n + 1. \]

Theorem 3.1 shows that under suitable conditions for substitutable products there is a much simpler characterization of optimal policies than is true for the general model. For any single-valued determination of \( S(x) \) if \( S(x) > x \) let \( s_n(x) \leq S(x) \) be a point on the line from \( S(x) \) through \( x \) satisfying

\[ J_n(s_n(x)) = K + J_n(S_n(x)). \]

Since \( J_n(y) \) is continuous and \( \to \infty \) as \( \|y\| \to \infty \) by the proof of (ii) in Theorem 2.6 \( s_n(x) \) exists.

**Definition** A \((c, S(x))\) policy having the following form for some \( s_n(x) \) and \( S_n(x) \) for all \( n \) will be called an \((s(x), S(x))\) policy:

\[
Y_n(x) = \begin{cases} 
  x & \text{if } x > s_n(x) \\
  x \text{ or } S_n(x) & \text{if } x = s_n(x) \\
  S_n(x) & \text{if } x < s_n(x).
\end{cases}
\]

(3.1)

Thus, for substitutable products any \((s(x), S(x))\) using some \( s_n(x) \) and \( S_n(x) \) from above is optimal by (ii) of Theorem 3.1 when \((\alpha)\) holds. If for each \( n \) \( s_n(x) = s_n \) and \( S_n(x) = S_n \) for \( x \leq S_n \) then the policy is called more simply an \((s, S)\) policy.

In the single-product version of the present model this is true. An \((s(x), S(x))\) policy embodies more of the characteristics of a single-
product \((s, s)\) policy than a \((\sigma, S(x))\) policy does since the decision to order for any \(x'\) on the line from \(x\) to \(S(x)\) depends only on the relation of \(x'\) to a single critical number. The fact that no optimal \(S_n(x)\) need lie on the line through \(x\), \(s_n(x)\), and \(S(x)\), however, produces one difference from the single-product case. Alternatively \(s_n(x)\) could have been defined on the line through \(x\) and \(S_n(x)\), but then different lines would be used for different periods.

Since for \((1-\alpha) K > 0 \sigma \subset \sigma_n \subset \sigma\) for all optimal \(\sigma_n\) the definition of \(s_n(x)\) used here implies that \(s(x) \leq s_n(x) \leq \bar{s}(x)\) where \(\bar{s}(x)\) is the smallest point on the same line such that

\[
G(s(x)) = K + G(s(x))
\]

and \(\bar{s}(x)\) is the largest point \(\leq S(x)\) on it such that

\[
G(\bar{s}(x)) = (1-\alpha) K + G(\bar{s}(x)) .
\]

Furthermore, since \(s_n(x) < S_n(x)\) it follows from Lemma 2.4 that

\[
0 = K + J_n(S_n(x)) - J_n(s_n(x)) \\
\geq K + G(S_n(x)) - G(s_n(x)) - \alpha K .
\]

Thus,

\[(3.2)\]

\[
G(s_n(x)) \geq G(S_n(x)) + (1-\alpha)K ,
\]

which yields a possibly tighter upper bound than \(\bar{s}(x)\) since \(G(S_n(x)) \geq G(\bar{s}(x))\). Any optimal Markov policy is a \((\sigma, S(x))\) policy since \(K > 0\) but not all optimal \((\sigma, S(x))\) policies are \((s(x), S(x))\) policies. Some are, however, and all of these use some value of the \(s_n(x)\) defined above. Hence, for every optimal \((s(x), S(x))\) policy \(\underline{s}(x) \leq s_n(x) \leq \bar{s}(x)\).

If \((1-\alpha)K = 0\) then there are optimal \((s(x), S(x))\) policies, for some of which \(s_n(x) \in [\underline{s}(x), \bar{s}(x)]\) and satisfies \((3.2)\).

For any \(S(x) > x\) for some \(x\) the set \(\{y | G(y) \leq G(\bar{s}(x)) + \alpha K\}\) is closed and convex. Let \(\overline{S}(x)\) be any point \(\geq S(x)\) on its boundary.
If $S_1(x) \geq S(x)$ then by Lemma 2.8 $G(S_1(x)) \leq G(S(x))$ and $S_1(x) \leq$ some $S(x)$. Thus, in Theorem 3.1 $S(x)$ provides upper bounds on $S_1(x)$. In particular, if $x \leq S$ then define $\bar{S} = S(x)$.

Since this special characterization of optimal policies for the case of substitutable products required condition (α) it is interesting to inquire when, if ever, the condition holds. One situation for which this is the case arises when $G(y)$ is separable, i.e., $G(y) = G_1(y_1) + G_2(y_2)$. Here the contributions of the products to the one-period expected holding and shortage cost are independent although the demands for the products may be dependent.

For example, let

$$h(z) = \begin{cases} h_1(z_1) + h_2(z_2) & \text{if } z \geq 0 \\ h_1(z_1) + p_2(-z_2) & \text{if } z_1 \geq 0, z_2 \leq 0 \\ p_1(-z_1) + h_2(z_2) & \text{if } z_1 \leq 0, z_2 \geq 0 \\ p_1(-z_1) + p_2(-z_2) & \text{if } z \leq 0 \end{cases}$$

$$= h_1(z_1) + h_2(z_2).$$

Then,

$$G(y) = L(y) + (1-\alpha)cy^T + \alpha c \mu^T$$

$$= \int_0^\infty \int_0^\infty h(y-\xi) \, d\Phi(\xi) + (1-\alpha)cy^T + \alpha c \mu^T$$

$$= \int_0^\infty h_1((y_1-\xi_1)d\Phi_1(\xi_1) + \int_0^\infty h_2((y_2-\xi_2)d\Phi_2(\xi_2) + (1-\alpha)(c_1y_1 + c_2y_2)$$

$$+ \alpha(c_1\mu_1 + c_2\mu_2)$$

$$= L_1(y_1) + (1-\alpha)c_1y_1 + \alpha c_1 \mu_1 + L_2(y_2) + (1-\alpha)c_2y_2 + \alpha c_2 \mu_2$$

$$= G_1(y_1) + G_2(y_2),$$

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where \( \phi_{\ell}(x_\ell) \) is the marginal cumulative distribution function for product \( \ell \). Two products will be called separable if \( G(y) \) is separable and if there are Borel functions \( h_{\ell}(z_{\ell}) \) such that
\[
L_{\ell}(y_{\ell}) = \int_0^\infty h_{\ell}(y_{\ell} - x_{\ell}) \, d\phi_{\ell}(x_{\ell}).
\]
When products are separable it may then be meaningful to discuss the single-product problem associated with product \( \ell \). This point is examined at the end of this section.

Notice that when \( G(y) \) is separable \( G_{\ell}(y_{\ell}) \to \infty \) as \( |y_{\ell}| \to \infty \),
\[
G(x_1,z_2(x_1)) = \min_{x_2} G(x) = G_1(x_1) + \min_{x_2} G_2(x_2)
\]
and
\[
G(z_1(x_2),x_2) = \min_{x_1} G(x) = \min_{x_1} G_1(x_1) + G_2(x_2),
\]
so that the functions \( z_1(x_2) \) are constants \( s_1 \) and \( s_2 \). Hence, the products are both substitutable and complementary, and in (c) of Theorem 2.1 "and" replaces "or" since, for example, if \( x \geq s_1 \) then \( x_1 \geq s_1 = z_1(x_2) \). Furthermore, if \( s(x) > x \) then either \( (s(x))_1 = s_1 \) and \( (s(x))_2 \geq s_2 \), or \( (s(x))_1 \geq s_1 \) and \( (s(x))_2 = s_2 \). Also, \( s_1 \), defined as before to be any point satisfying \( s = (z_1(s_2), z_2(s_1)) \), is now unique.

Let \( s_1 \) be the smallest \( x_1 \) such that \( G(x) = K + G(s_1,x_2) \), i.e., \( G_1(x_1) = K + G_1(s_1) \) independently of \( x_2 \). Similarly define \( \overline{s}_1 \) as the largest \( x_1 \leq s_1 \) for which \( G_1(x_1) = (1-\alpha)K + G_1(s_1) \). Define \( s_2 \) and \( \overline{s}_2 \) analogously. Then with \( s(x) \) and \( \overline{s}(x) \) defined above it is seen that \( s \) contains the following points:
\[
\begin{cases}
x < s(x), \quad x < S \\
x_1 > S_1, \quad x_2 < S_2 \\
x_1 < S_1, \quad x_2 > S_2
\end{cases}
\]

The set \( \bar{\sigma} \) has an analogous characterization.

**Theorem 3.2** Let \( G(y) \) be separable. If \( (1-\alpha)K > 0 \) optimal \((s(x), S(x))\) policies exist, all of which satisfy \( s_n(x) \leq s(x) \leq \bar{s}(x) < S(x) \leq S_n(x) \leq \text{some } \overline{S}(x) \); and, in particular, \( s_{\ell} \leq (s_{n}(x))_{\ell} \leq \text{max } \mathcal{S}_{\ell} \) for all \( x_j < \mathcal{S}_{\ell} \), \( S_n(x)_{\ell} \leq \mathcal{S}_{\ell} \) for all \( x_j < \mathcal{S}_{\ell} \) if \( x_j > S_j \) for \( j \neq \ell \). If \( (1-\alpha)K = 0 \) there is an optimal \((s(x), S(x))\) policy satisfying the inequalities with \( \bar{s}(x) = \overline{S}(x) \) and \( \bar{s}_{\ell} = \mathcal{S}_{\ell} \).

**Proof** The proof essentially consists in showing that condition (a) of Theorem 3.1 holds when \( G(y) \) is separable. Consider statement (IV) of Theorem 3.1*. If \( \mathcal{S}(x) > x \) then

\[
\mathcal{S}(x) = \begin{cases}
\bar{S} & \text{if } x < \underline{S} \\
(x_1, \underline{S}_2) & \text{if } x_1 > \underline{S}_1, \quad x_2 < \underline{S}_2 \\
(\underline{S}_1, \underline{S}_2) & \text{if } x_1 < \underline{S}_1, \quad x_2 > \underline{S}_2
\end{cases}
\]

If \( (s_n(x))_2 < (\mathcal{S}(x))_2 \) then \( (\mathcal{S}(x))_2 = \underline{S}_2 \). Since by (III) \( (s_n(x))_1 \geq (\mathcal{S}(x))_1, \mathcal{S}(s_n(x)) = ((s_n(x))_1, \underline{S}_2) > S_n(x) = \text{some } S_n(s_n(x)). \) Thus, \( (s_n(x))_{\ell} \geq (\mathcal{S}(x))_{\ell} \) for \( \ell = 2 \) and 1. Hence the condition of statement (IV) holds a priori and the induction argument of Theorem 3.1 is valid without imposing (a). Thus, by
Theorem 3.1 and the discussion following it, optimal $(s(x), S(x))$
the inequalities if $(1-\alpha) K > 0$ or $\alpha = 0$, respectively. Furthermore, the
policies exist, for all or none of which $\sigma_n(x)$ satisfies the indicated
upper bounds for $S_n(x)$ hold by Lemma 2.8.

Q.E.D.

From Theorem 3.2 follows immediately the solution to the single-
product model by taking $G_2(y_2)$ to be identically zero. In this case
the superflluous product subscript will be suppressed.

Corollary 3.3 Let $G(y)$ be a continuous, quasi-convex function which
$\to \infty$ as $|y| \to \infty$. If $K > 0$ any optimal policy is an $(s,S)$ policy. If
also $\alpha < 1$, $s \leq s_n \leq \bar{s} < S \leq S_n \leq \bar{S}$. If $(1-\alpha)K = 0$ there is an
optimal $(s,S)$ policy satisfying the inequalities with $\bar{s} = \bar{S}$.

The corollary is close to (a) of Theorem 4 of Veinott and Wagner
[13], for convex $G(y)$ and to Theorem 1 of Veinott [12] for quasi-
convex $G(y)$, although the bounds $\bar{s}$ and $\bar{S}$ here may be less sharp.
The choice of the present bounds was determined by a desire to make them
all-inclusive when $(1-\alpha)K > 0$. A slightly different definition of $\bar{s}(x)$
and $\bar{S}(x)$ together with a weakening of Theorem 3.2 to state that some
optimal policy satisfies the given inequalities would yield as a corol-
lary the results referred to.

A particular situation in which optimal policies have a relatively
simple characterization is that in which a $n$ minimum level of demand is
guaranteed. If the demand exceeds with probability one the difference
between the upper bounds on $S_n(x)$ and the lower bounds on $\sigma_n$ then an
optimal policy consists in taking $\sigma_n = \sigma$ and $S_n(x) = S(x)$ for all $n$.
Although this statement may be verified for the general model the essential elements of the proof are found in analyzing the case of substitut-
able products.
Theorem 3.4  If the products are substitutable and if for all $\bar{S}(x) \phi(\xi) = 0$ for all $0 \leq \xi$ for which $\bar{S}(x) - \xi \in \sigma^c$ then any policy $Y$ characterized by $\sigma_n = \sigma$ and $S_n(x) = \bar{S}(x)$ for each $n$ is optimal.

Proof  For the one-period problem any policy with $(\sigma, S_1(x)) = (\sigma, \bar{S}(x))$ is optimal. Suppose that $Y$ is optimal for the $n$-period problem for an arbitrary $n$ and consider the $(n+1)$-period problem. It suffices to show that $f_{n+1}(x|Y) \leq f_{n+1}(x)$ for all $x$. Note that $(\alpha)$ of Theorem 3.1 is satisfied for $i = 1, 2, \ldots, n$ by the induction hypothesis. By Theorem 3.1* and Corollary 2.10 there is an optimal $(\sigma, S(x))$ policy with $\sigma \subset \sigma_{n+1} \subset \sigma^c$.

Hence,

$$f_{n+1}(x) = \begin{cases} J_{n+1}(x) & \text{if } x \in \sigma^c_{n+1} \\ K + J_{n+1}(S_{n+1}(x)) & \text{if } x \in \sigma_{n+1} \\ G(x) + \alpha \mathbb{E}[f_n(x-\xi|Y)] & \text{if } x \in \sigma^c_{n+1} \\ K + G(S_{n+1}(x)) + \alpha \mathbb{E}[f_n(S_{n+1}(x)-\xi|Y)] & \text{if } x \in \sigma_{n+1}. \end{cases}$$

Consider any $x \in \sigma_{n+1}$. It will be shown that there exist $x'$, $\bar{S}(x')$, and $S(x')$ satisfying

$$x' \leq \bar{S}(x') \leq S(x')$$

and

$$x \leq S(x') \leq S_{n+1}(x).$$

Now if $\bar{S}(x) \leq S_{n+1}(x)$ the existence is immediate so suppose that $(S_{n+1}(x))_2 < (\bar{S}(x))_2$. By (III) of Theorem 3.1* $(S_{n+1}(x))_1 > (\bar{S}(x))_1$.

Since $x_2 < (\bar{S}(x))_2$ either $S(x)$ is some $\bar{S}$ or $(\bar{S}(x))_1 = x_1 > z_1 (\min S_2)$ and $(\bar{S}(x))_2 = z_2(x_1)$. In either case

$$\min S_1 \leq (\bar{S}(x))_1 < (S_{n+1}(x))_1.$$
If $z_2((s_{n+1}(x))_1) > (s_{n+1}(x))_2$ then $S(s_{n+1}(x))$ may be chosen to be

$((s_{n+1}(x))_1, z_2((s_{n+1}(x))_1) > s_{n+1}(x))$, which itself may be chosen to be

$s_{n+1}(S_{n+1}(x))$, and thus (III) is violated. Hence,

$$(3.2) \quad (s_{n+1}(x))_2 \geq z_2((s_{n+1}(x))_1).$$

If $(s_{n+1}(x))_1 < z_1((s_{n+1}(x))_2)$ then if also $(s_{n+1}(x))_2 \geq z_2(\min S_1)$ it follows that $(S(x))_2 \geq z_2(\min S_1)$ and so by substitutability

$$x_2 = (S(x))_2 > (s_{n+1}(x))_2 \geq x_2,$$

which is a contradiction. Thus, $(s_{n+1}(x))_2 < z_2(\min S_1)$ and by (c) of Theorem 2.1 $(s_{n+1}(x))_2 \leq z_2((s_{n+1}(x))_1)$. This together with (3.2) implies that one may take $x_1 = (S(x'))_1 = (S(x'))_1 = (s_{n+1}(x))_1$ and $x_2$ $< z_2(x_1) = (S(x'))_2 \leq (S(x'))_2 = (s_{n+1}(x))_2$.

Alternatively, if $(s_{n+1}(x))_1 \geq z_1((s_{n+1}(x))_2)$ and if also $x_2 \leq z_2((s_{n+1}(x))_1)$ then $x_1 = (s_{n+1}(x))_1$ and $x_2 < z_2(x_1)$ yield appropriate

$S(x') = (x_1, z_2(x_1))$ and $S(x') = (x_1, z_2(x_1))$ or $S_{n+1}(x)$ depending on whether $(s_{n+1}(x))_2 > z_2(x_1)$ or $\leq z_2(x_1)$. If, on the other hand, $x_2 > z_2((s_{n+1}(x))_1)$ then since $x_2 < (S(x))_2 = z_2((S(x))_1)$ and (3.1) holds there is an $x_1 \in ((S(x))_1, (s_{n+1}(x))_1)$ for which some $z_2(x_1) = x_2$ by connectedness of (b) of Theorem 2.1. Then for $x' < S(x') = (x_1, z_2(x_1))$ and $S(x') = (x_1, z_2(x_1))$ the desired result is shown.

An initial assumption that $(s_{n+1}(x))_2 > (S(x))_2$ and will imply that $(s_{n+1}(x))_1 < (S(x))_1$ utilizes a symmetric proof. Finally, if $(s_{n+1}(x))_2 = (S(x))_2$ and $(s_{n+1}(x))_1 < (S(x))_1$ then (III) is violated.

Now since $x \leq S(x') \leq s_{n+1}(x)$ it follows that $S_{n+1}(S(x')) = S_{n+1}(x)$ and by Lemma 2.4.
0 \geq J_{n+1}(S_{n+1}(x)) - J_{n+1}(S(x'))
\geq G(S_{n+1}(x)) - G(S(x')) - \alpha K
= G(S_{n+1}(x)) - G(S(x')) - \alpha K.

Thus, there is an upper bound \( \overline{S}(x') \) on \( S_{n+1}(x) \) and by assumption \( S_{n+1}(x) - \xi \in \mathcal{G} \) with probability one. Thus, for \( x \in \sigma_{n+1} \)

\[ f_{n+1}(x) = K + G(S_{n+1}(x)) + \alpha \mathbb{E}[f_{n}(S_{n+1}(x) - \xi | Y)] \]
\[ \geq K + G(S(x)) + \alpha \mathbb{E}[K + J_{n}(S(S_{n+1}(x) - \xi))] . \]

If \( x < S \) then with probability one by the induction hypothesis

\[ f_{n}(S(x) - \xi | Y) = K + J_{n}(S) \]
\[ \leq K + J_{n}(S(S_{n+1}(x) - \xi)) . \]

If, secondly, \( x_1 > z_{n+1}(\min S_2) \) and \( x_2 < z_2(x_1) \) then \( S(x) = (x_1, z_2(x_1)) \) and \( (S(x) - \xi)_1 \leq (S_{n+1}(x) - \xi)_1 \). For \( (S(x) - \xi)_1 \leq z_{n+1}(\min S_2) \) one may take \( S(S(x) - \xi) = \text{some } S \), while for \( (S(x) - \xi)_1 > z_{n+1}(\min S_2) \) one has

\[ (S(S(x) - \xi))_1 = (S(x) - \xi)_1 \]
\[ \leq (S_{n+1}(x) - \xi)_1 \]
\[ = (S(S_{n+1}(x) - \xi))_1 , \]

and by optimality of \( (g, S(x)) \) for \( n \)

\[ J_{n}(S(S(x) - \xi)) \leq J_{n}(S(S_{n+1}(x) - \xi)) . \]

In both cases

\[ f_{n}(S(x) - \xi | Y) \leq K + J_{n}(S(S_{n+1}(x) - \xi)) . \]
Finally, if \( x_1 < z_1(x_2) \) and \( x_2 > z_2(\min S_1) \) a similar proof holds. Thus, for \( x \in \sigma_{n+1} \)

\[
f_{n+1}(x) \geq K + G(S(x)) + \alpha E[f_{n}(S(x):x \mid Y)]
\]

\[
= f_{n+1}(x \mid Y).
\]

If \( x \in \sigma^c_{n+1} \) then \( x \in \sigma^c \) and so

\[
f_{n+1}(x) = f_{n+1}(x \mid Y),
\]

which completes the induction. Q.E.D.

The implications of the theorem when \( G(y) \) is separable may be stated more simply. (Compare Theorem 3 of Veinott and Wagner [13].)

**Corollary 3.5** If \( G(y) \) is separable and if \( \Phi(\xi) > 0 \) only for \( \xi > (\max S_1 - s_1, \max S_2 - s_2) \) then one optimal policy is \( Y \) having

\[
Y_n(x) = \begin{cases} 
S & \text{if } x \leq S(x), \\
(x_1, S_2) & \text{if } x_1 > S_1, x_2 < S_2, \\
(S_1, x_2) & \text{if } x_1 < S_1, x_2 > S_2, \\
x & \text{otherwise}.
\end{cases}
\]

**Proof** For \( x \leq S \)

\[
G((S(x))_1, S_2) = G_1((S(x))_1) + K + G_2(S_2)
\]

\[
\geq G_1(S_1) + K + G_2(S_2)
\]

\[
= G(S(x))
\]

\[
= G_1((S(x))_1) + G_2((S(x))_2)
\]

so that \( S_2 \leq (S(x))_2 \) and similarly \( S_1 \leq (S(x))_1 \). Thus, by Theorem 3.2 \( S(x) - \xi \in \sigma \) with probability one and the corollary follows from Theorem 3.4. Q.E.D.
By this time one might wonder at the conspicuous absence of a strong characterization of optimal policies for the general \( n \)-period problem. In the single-product model the optimal policy for longer horizons has the same form as that for one period. Aside from more difficult computations the \( n \)-period problem has a solution no different from that of the one-period problem. For the multiproduct model, however, the characterization in Theorem 2.7 is much weaker than the one in Theorem 2.3. For the one-period problem some optimal policies are characterized by pairs \((s(x), S(x))\) whereas for the \( n \)-period problem this is shown only for substitutable products and only when a special condition holds. Examples will now be presented to demonstrate that the condition on substitutable products is not superfluous; that the parameters of optimal policies for complementary products need not satisfy the inequalities which hold in the substitutable case, and finally that optimal policies for complements may not be \((s(x), S(x))\) policies. Furthermore, since \( G(y) \) is convex for all the examples the blame for these shortcomings of the model does not rest with the weaker assumption that \( G(y) \) is merely quasi-convex.

**Example 1** Let \( c = (0,0), K = 4 \), and \( \alpha = 1 \). Let \( \xi \) be discrete with probability mass function \( \varphi(0,0) = \varphi(0,1) = \varphi(1,0) = \varphi(1,1) = 1/4 \) and define \( h(z) \) on part of \( \mathbb{I}^2 \), the space of integral two-tuples as follows:
<table>
<thead>
<tr>
<th>$h(z)$</th>
<th>$z_1 = \cdots$</th>
<th>-6</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\cdots$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$z_2 = \cdots$</td>
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<td>5</td>
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<td>60</td>
<td>70</td>
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<td>90</td>
<td>100</td>
<td>110</td>
<td>120</td>
<td>130</td>
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<td>4</td>
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<td>50</td>
<td>60</td>
<td>70</td>
<td>80</td>
<td>90</td>
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<td>-1</td>
<td>100</td>
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<td>10</td>
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<td>0</td>
<td>10</td>
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<td>70</td>
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<td>-3</td>
<td>120</td>
<td>90</td>
<td>80</td>
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<td>50</td>
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<tr>
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<td>-4</td>
<td>130</td>
<td>120</td>
<td>110</td>
<td>100</td>
<td>90</td>
<td>80</td>
<td>70</td>
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<td>\vdots</td>
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</tbody>
</table>

Define $h(z)$ on the remainder of $\mathbb{R}^2$ to be the lower envelope of the convex hull of the points above extended over the whole plane. It is apparent that these points do lie on the lower surface of the convex hull so that $h(z)$ is a well-defined convex function. Now, since $c = (0,0)$ $G(y) = E[h(y-\xi)]$ and is defined on $\mathbb{I}^2$ as follows where M indicates an entry greater than 20:

<table>
<thead>
<tr>
<th>$G(y)$</th>
<th>$y_1 = \cdots$</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_2 = \cdots$</td>
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<tr>
<td>5</td>
<td>40</td>
<td>40</td>
<td>50</td>
<td>M</td>
<td>M</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>22.5</td>
<td>22.5</td>
<td>30</td>
<td>M</td>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>22.5</td>
<td>8</td>
<td>8.25</td>
<td>12.75</td>
<td>M</td>
<td>M</td>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>M</td>
<td>30</td>
<td>8.5</td>
<td>1.5</td>
<td>.5</td>
<td>7.5</td>
<td>M</td>
<td>M</td>
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<td></td>
</tr>
<tr>
<td>1</td>
<td>M</td>
<td>M</td>
<td>13.5</td>
<td>1.5</td>
<td>.5</td>
<td>2.5</td>
<td>15</td>
<td>M</td>
<td>M</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>M</td>
<td>M</td>
<td>8.25</td>
<td>2.75</td>
<td>0</td>
<td>7.5</td>
<td>30</td>
<td>M</td>
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<tr>
<td>-1</td>
<td>M</td>
<td>M</td>
<td>15</td>
<td>7.5</td>
<td>7.5</td>
<td>7.5</td>
<td>22.5</td>
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</tr>
<tr>
<td>-2</td>
<td>M</td>
<td>M</td>
<td>30</td>
<td>22.5</td>
<td>22.5</td>
<td>40</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>M</td>
<td>M</td>
<td>50</td>
<td>40</td>
<td>40</td>
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</tbody>
</table>

54
Since $h(z)$ is convex so is $G(y)$. It is seen that $z_k(x_j)$ and $\bar{z}_k(x_j)$ are monotonically non-increasing on $\mathbb{R}^2$ and it can be verified that this is true on the rest of $\mathbb{R}^2$ as well. Since $K = 4$ a portion of $f_1(x) = \min\{G(x), K + G(S(x))\}$ is as follows:

| $f_1(x)$ | $x_1$ = | -3 | -2 | -1 | 0 | 1 |...
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$x_2 =$</td>
<td></td>
<td>3</td>
<td>8</td>
<td>8.25</td>
<td>12.75</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td>2</td>
<td>4.5</td>
<td>1.5</td>
<td>.5</td>
<td>7.5</td>
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<td></td>
<td></td>
<td>1</td>
<td>4.5</td>
<td>1.5</td>
<td>.5</td>
<td>2.5</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>2.75</td>
<td>0</td>
<td>7.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>7.5</td>
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<tr>
<td></td>
<td></td>
<td>-2</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>11.5</td>
<td></td>
</tr>
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<td></td>
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</tr>
</tbody>
</table>

Thus, $J_2(y) = G(y) + E[f_1(y-t)]$ takes on the following values:

| $J_2(y)$ | $y_1$ = | -2 | -1 | 0 | 1 |...
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_2 =$</td>
<td></td>
<td>3</td>
<td>13.81</td>
<td>18.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>4.5</td>
<td>1.5</td>
<td>10.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>5</td>
<td>2.69</td>
<td>3.94</td>
<td>21.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>12.25</td>
<td>6.44</td>
<td>2.69</td>
<td>12.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1</td>
<td>19</td>
<td>11.5</td>
<td>14.25</td>
<td></td>
</tr>
</tbody>
</table>

Among points in $\mathbb{R}^2 (-1, 2)$ minimizes $J_2(y)$ and a detailed analysis will show that in fact $(-1, 2)$ minimizes $J_2(y)$ over $\mathbb{R}^2$ and is the unique minimum. Since $S_1 = (0, 0)$ it follows that $(\alpha)$ of Theorem 3.1 is not satisfied for this example. Furthermore, because of backlogging one optimal policy after a translation of the inventory levels is simply the translation of an originally optimal policy. Hence, the above example actually yields many examples and should not be considered pathological on the basis that $(S_2)_1 < 0$. 

55
Example 2 Let $c, K, \alpha$, and $\varphi(t)$ be as in Example 1 and modify $h(z)$ slightly as follows:

<table>
<thead>
<tr>
<th>$h(z)$</th>
<th>$z_1 = \cdots$</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>( \cdots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_2$</td>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>1</td>
<td>100</td>
<td>80</td>
<td>60</td>
<td>40</td>
<td>20</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>80</td>
<td>40</td>
<td>20</td>
<td>0</td>
<td>0</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>60</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>40</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>20</td>
<td>60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>20</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>40</td>
<td>80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>20</td>
<td>20</td>
<td>40</td>
<td>60</td>
<td>80</td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As in Example 1, $h(z)$ may be defined on the remainder of $\mathbb{R}^2$ so as to be convex and achieve these values. Hence,

<table>
<thead>
<tr>
<th>$g(y)$</th>
<th>$y_1 = \cdots$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>( \cdots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_2$</td>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>75</td>
<td>50</td>
<td>30</td>
<td>15</td>
<td>15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>45.5</td>
<td>15.75</td>
<td>5.25</td>
<td>0</td>
<td>15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>26</td>
<td>1.5</td>
<td>.5</td>
<td>5</td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>16</td>
<td>1.5</td>
<td>.5</td>
<td>15</td>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>15.5</td>
<td>15.75</td>
<td>25.25</td>
<td>45</td>
<td>75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In this example it can be verified that the products are complementary when $h(z)$ is extended appropriately in a manner symmetric to that of Example 1. Thus,
\[
\begin{array}{c|cccccc}
    f_1(x) & x_1 = & \cdots & -3 & -2 & -1 & 0 & 1 \cdots \\
    \hline
    x_2 = & \vdots & & & & & & \\
    1 & 19 & 19 & 19 & 15 & 15 & & 19 \\
    0 & 4 & 4 & 4 & 0 & 15 & & 19 \\
    -1 & 4 & 1.5 & .5 & 4 & 19 & & 19 \\
    -2 & 4 & 1.5 & .5 & 4 & 19 & & 19 \\
    -3 & 4 & 4 & 4 & 4 & 19 & & \\
    \vdots & & & & & & & \\
\end{array}
\]

and

\[
\begin{array}{c|cccccc}
    J_2(y) & y_1 = & \cdots & -2 & -1 & 0 & 1 \cdots \\
    \hline
    y_2 = & \vdots & & & & & & \\
    1 & 61.5 & 41.5 & 24.5 & 26.25 & & & \\
    0 & 19.13 & 7.75 & 2.13 & 24.5 & & & \\
    -1 & 4.25 & 1.5 & 7.25 & 41.5 & & & \\
    -2 & 4.88 & 3 & 18.13 & 61.5 & & & \\
    \vdots & & & & & & & \\
\end{array}
\]

Here, \( S_2 = (-1, -1) < (0, 0) = S_1 \) and so not only is \((x)\) of Theorem 3.1 not satisfied but also statement (iii) of the theorem does not hold.

**Example 3** Let \( c, \alpha, \) and \( \varphi(\xi) \) be as in the first two examples but take \( K = 13 \). Let \( h(z) \) take on the values given in Table I. Again, \( h(z) \) may be defined on the rest of \( \mathbb{R}^2 \) so as to be convex, to achieve these values, and to yield a \( G(y) \) for complements. A partial listing of values of \( G(y), f_1(x), \) and \( J_2(y) \) is given in Table I, where all missing numbers are of a larger order. It is seen that \( S_2 = (0, 0) = S_1 = S \) and the only points in \( \mathbb{R}^2 \) less than or equal to \( S_2 \) which are also members of \( \sigma^c_2 \) are those in the set

\[\{(-14, -1), (-13, -1), (-13, 0), (-12, 0), (10, 0), (-9, 0), \ldots, (0, 0)\}\].
| $z_1$ | -16 | -15 | -14 | -13 | -12 | -11 | -10 | -9  | -8  | -7  | -6  | -5  | -4  | -3  | -2  | -1  | 0   | 1   |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $z_2$ |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| $h(z)$: 1 | 900 | 850 | 800 | 750 | 700 | 650 |     |     |     |     |     |     |     |     |     |     |     |
| 0     | 650 | 25  | 23  | 21  | 19  | 17  | 15  | 13  | 11  | 9   | 7   | 5   | 3   | 2   | 1   | 0   | 0   | 100 |
| -1    | 600 | 0   | 0   | 0   | 0   | 0   |     |     |     |     |     |     |     |     |     |     | 0   | 0   | 150 |
| -2    | 550 | 2   | 2   | 2   | 50  | 100 |     |     |     |     |     |     |     |     |     |     |     |
| -3    | 500 | 550 | 600 | 650 |     |     |     |     |     |     |     |     |     |     |     |     |     |
| $a(z)$: 0 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     | 12  | 11  | 10  | 9   | 8   | 7   | 6   | 5   | 4   | 3   | 2   | 1.25 | .75  | .25  | 0    |
| -1    |     | 1   |     | 1   |     |     |     |     |     |     |     |     |     |     |     | 1   |     |     |
| $r_1(z)$: 0 | 13  | 12  | 11  | 10  | 9   | 8   | 7   | 6   | 5   | 4   | 3   | 2   | 1.25 | .75  | .25  | 0   |     |     |     |     |     |     |     |     |     |
| -1    | 13  | 1   | 1   | 1   | 13  |     |     |     |     |     |     |     |     |     |     | 13  |     |     |     |     |     |     |     |     |
| -2    | 13  | 13  | 13  | 13  |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| $J_2(z)$: 0 | 21.75 | 17.25 | 18.75 | 20.25 | 18.75 | 17.25 | 15.75 | 14.25 | 12.75 | 11.25 | 9.75 | 8.56 | 7.75 | 7   | 6.56 |     |     |     |     |     |     |     |     |     |
| -1    | 11  | 8   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
In particular, for $x_2 = 0$ it is optimal to order up to $(0,0)$ if $x_1 \leq -14$ or if $x_1 = -11$ and it is optimal not to order for the other integral values of $x_1$. Hence, for $x_2 = 0$ the optimal policy for $x_1$ is not of the $(s,S)$ form although it is for the one-period problem. Consequently, although for the general model there is an optimal $(s(x), S(x))$ policy in the one-period problem there is not necessarily one for either the general model or the particular complementary model in the $n$-period problem. This indicates that for complements a result comparable to Theorem 3.1 is attainable only with the imposition of an additional condition and that Theorem 2.7 may provide as strong a characterization as is possible for the general model or the unrestricted complementary model. As a result, an alternative approach to the complementary situation is called for. The special model of Section V offers one solution.

At the end of Section II it was pointed out how other forms of the set-up cost function $K(z)$ generally lead to more complicated optimal policies and correspondingly more difficult numerical calculations. An exception to this statement occurs when $K_{12} = K_1 + K_2$ and the products are separable. In this case the two-product problem may be factored into two single-product problems whose independent solutions when considered jointly constitute a solution to the two-product problem. Unlike the situation in Theorem 3.2, here $K(z)$ also is separable and the following theorem similar to the result of Veinott (Section 5 in [11]) is obtained. First, whether $A$ and $B$ are sets or policies the composition $A \boxtimes B$ will be defined to consist of all two-tuples whose first and second components belong to $A$ and $B$ respectively.
Theorem 3.6 If (i) the products are separable,

(ii) \( K_{12} = K_1 + K_2 \),

and

(iii) \( \Xi(x) = \Xi_1(x_1) \otimes \Xi_2(x_2) \) for all \( x \),

then \( Y^* = Y_{1}^* \otimes Y_{2}^* \) is optimal if \( Y_{1}^* \) is optimal for product \( l \).

Proof Condition (iii) ensures that \( Y^* \) is feasible since each \( Y_{1}^* \) must be feasible. Recall that a policy is optimal if it is optimal for the n-period problem for all \( n \). It will be shown by induction on \( n \) that \( Y^* \) is optimal for the t-period problem for all \( t \leq n \). By (1.5) it is sufficient to show that

\[
(3.4) f_{n}(x_{n} | Y^*) = \min_{Y_{n} \in \Xi(x)} \{ K_{1} \delta_{1} + K_{2} \delta_{2} + G_{1}(y_{n1}) + G_{2}(y_{n2}) + \alpha \int_{0}^{\infty} \int_{0}^{\infty} f_{n-1}(y_{n-t} | Y^*) d\phi(t) \},
\]

where \( \delta_{l} \equiv \delta(y_{n,l} - x_{n,l}) \), since

\[
K(y-x) = K_{1} \delta_{1} (1-\delta_{2}) + K_{2} (1-\delta_{1}) \delta_{2} + (K_{1} + K_{2}) \delta_{1} \delta_{2}
= K_{1} \delta_{1} + K_{2} \delta_{2}.
\]

Now, for \( n \geq 1 \), if \( y^*_t \) is the \( y_t \) resulting from \( Y^* \),

\[
f_{n-1}(x_{n-1} | Y^*) = \sum_{t=1}^{n-1} \alpha^{n-1-t} E[K(y^*_t-x_t) + G(y^*_t)]
= \sum_{t=1}^{n-1} \alpha^{n-1-t} E[K_{1} \delta_{1} (y^*_t-x_{t1}) + G_{1}(y^*_t)]
+ \sum_{t=1}^{n-1} \alpha^{n-1-t} E[K_{2} \delta_{2} (y^*_t-x_{t2}) + G_{2}(y^*_t)]
= f_{n-1,1}(x_{n-1,1} | Y^*_1) + f_{n-1,2}(x_{n-1,2} | Y^*_2).
\]
For \( n=1 \), since \( Y^*_{k} \) is optimal for product \( k \) and \( Y^* = Y^*_1 \otimes Y^*_2 \) the right side of (3.4) is

\[
\min_{(y_{11}, y_{12})\in Y_1(x_{11}) \otimes Y_2(x_{12})} \left\{ K_1 \delta_1 + K_2 \delta_2 + G_1(y_{11}) + G_2(y_{12}) \right\}
\]

\[
= \min_{y_{11} \in Y_1(x_{11})} (K_1 \delta_1 + G_1(y_{11})) + \min_{y_{12} \in Y_2(x_{12})} (K_2 \delta_2 + G_2(y_{12}))
\]

\[
= f_{11}(x_{11}|Y^*_1) + f_{12}(x_{12}|Y^*_2)
\]

\[
= f_{1}(x_1|Y^*) .
\]

Assume now that \( Y^* \) is optimal for the \( t \)-period problem for all \( t < n \).

The right side of (3.4) becomes

\[
\min_{(y_{n1}, y_{n2})\in Y_1(x_{n1}) \otimes Y_2(x_{n2})} \left\{ K_1 \delta_1 + K_2 \delta_2 + G_1(y_{n1}) + G_2(y_{n2}) + \alpha \int_{0}^{\infty} \int_{0}^{\infty} \left[ f_{n-1,1}(y_{n1} - \xi_1|Y^*_1) + f_{n-1,2}(y_{n2} - \xi_2|Y^*_2) \right] d\phi(\xi) \right\}
\]

\[
= \sum_{k=1}^{2} \min_{y_{nk} \in Y_k(x_{nk})} (K_k \delta_k + G_k(y_{nk})) + \alpha \int_{0}^{\infty} f_{n-1,k}(y_{nk} - \xi_k|Y^*_k) d\phi_k(\xi_k)
\]

\[
= f_{n1}(x_{n1}|Y^*_1) + f_{n2}(x_{n2}|Y^*_2)
\]

\[
= f_{n}(x_n|Y^*) .
\]

Q.E.D.

Condition (i) may be relaxed so as to require only that \( G(y) \) is separable, for then the problem of determining an optimal policy still reduces to the problem of solving two independent mathematical minimization problems. The catch is that when there does not exist an \( h_1(z_1) \), for example, such that \( L_1(y_1) = \int_{0}^{\infty} h_1(y_1 - \xi_1) d\phi_1(\xi_1) \) the mathematical problem

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is not really an inventory problem associated with product one. This is especially true when there is no cumulative distribution function $\Phi'(z_1)$ for which $L_1(y_1) = \int_0^\infty h_1(y_1 - z_1) d\Phi'(z_1)$ for some $h_1(z_1)$ other than

$$\Phi'(z_1) = \begin{cases} 1 & z_1 > 0 \\ 0 & z_1 \leq 0 \end{cases}.$$

Also, since $G_1(y_1) = L_1(y_1) + (1-\alpha)c_1y_1 + c_1H_1$ in the inventory problem the existence of an appropriate $h_1(z_1)$ and a $\Phi'(z_1)$ non-degenerate but with mean not $\mu_1$ still does not yield an inventory problem. Finally, if there are an $h_1(z_1)$ and a non-degenerate $\Phi'(z_1)$ with mean $\mu_1$ but not identical to $\Phi_1(z_1)$, then an inventory problem is represented but it is still not the problem associated with product one. Consequently, it is preferable to state (i) in its stronger form. Notice that the existence of appropriate $h_1(z_1)$ and $h_2(z_2)$ is still weaker than separability of $h(z)$.

The theorem was proved for more general $Y(x)$ than the specific quadrants used in the past two sections. By taking $Y(x) = \{y | y \geq x\}$ one deduces from the theorem and Corollary 3.3 that (s,S) policies are optimal when (i)-(iii) hold. This represents the simplest solution to the multiproduct problem with set-up cost and the policy used is the extension of the single-product (s,S) policy closest in form to it. One might offer the wish that these policies are optimal more generally. Notice, however, that without conditions (i)-(iii) the concept of solving the single-product problems lacks meaning. Without these factorization conditions the optimal policy for one product depends on the initial level of the other product and the policy used for it.
Section IV  The Infinite-Period Problem

Sections II and III studied the model of Section I for the case of
a finite horizon. This section will be concerned with the infinite-
period problem. First, a few comments about the meaning and applicability
of this problem will be appropriate, and then it will be related mathe-
metrically to the problem in the earlier sections.

The inventory process in the infinite-period problem simply goes
on indefinitely: there is no finite \( n \) such that the problem ends at
the \( n^{th} \) stage. Thus, typically over the entire horizon an unlimited cost
is charged to purchase an unlimited supply of stock to meet an unlimited
demand. However, if \( \alpha < 1 \) a renumbering of the time indices so that
period \( t \) follows period \( t-1 \) will yield a convergent series in (1.1)
for some policies as \( n \to \infty \) and thus a finite (expected) total discounted
cost. This case then represents an immediate extension of the finite-
horizon problem. When \( \alpha = 1 \), on the other hand, the total cost is
typically infinite for all policies and the objective function must be
altered. One meaningful substitute is to minimize the (expected) average
cost per period over the infinite horizon. In the \( n \)-period problem with
\( \alpha = 1 \) minimizing the average cost is equivalent to minimizing the total
cost, so that this infinite-horizon objective is a consistent extension.

In practice, of course, one is never confronted with an infinite
horizon. The applicability of the infinite-period problem derives from
its role as a limit of and therefore an approximation to the finite-period
problem. Since there are optimal Markov policies, there are optimal
inventory paths which form Markov processes. Under certain conditions
these processes will approach a stationary distribution under the influence of a stationary policy, i.e., one in which the ordering rule is the same each period. In some cases, which seem more likely to occur in discrete situations, the stationary distribution is achieved after a moderate number of periods have elapsed. Here eventually the expected undiscounted cost contribution from the \((n+1)\)\textsuperscript{st} period is the same as that of the \(n\)\textsuperscript{th} period, and the same ordering rule, i.e., that for the optimal stationary infinite-period policy, is optimal for both.

When the distribution of the inventory level converges but not in a finite number of periods the ordering rule in the optimal stationary infinite-horizon policy need not be optimal for any period in any finite-horizon problem. Nevertheless, because of the convergence of distributions the minimal costs may converge, and so the optimal infinite-period policy may yield a cost approximately equal to that of the optimal policy in the finite-period problem. If, in addition, the optimal rules themselves converge to the optimal stationary rule then the latter also yields an approximation to the optimal finite-period policy. Since it is frequently computationally easier to solve the stationary problem, finding its solution can be useful for finite-period problems.

Finally, the infinite-horizon problem is often reasonable because the actual length of the horizon is simply unknown. Because of the convergences cited above an infinite-horizon analysis may yield no worse an approximation than the solution for an arbitrarily chosen finite horizon.

To begin the mathematical analysis of the infinite-period problem the earlier numbering of periods is reversed and \(x_1\) denotes the initial inventory vector. The history at the beginning of the \(t\)\textsuperscript{th} period is then \(H_t = (x_1, y_1, x_2, \ldots, y_{t-1}, x_t)\); and an ordering policy \(Y\) when used for
the problem with finite horizon $n \geq 1$ is denoted by $Y = \ldots, y_1^n, y_2^n, \ldots, y_n^n$, where the initial components need no identification. For $t \leq n$

$y_t = Y_t^H(\mathcal{H}_t)$, but the dependence of $y_t$ on $n$ will be suppressed.

Relation (1.1) becomes

$$
\begin{align*}
&f_n(x_1 | Y) = \sum_{t=1}^{n} \alpha^{t-1} \mathbb{E}[K(y_t - x_t) + G(y_t)] \quad n \geq 1 \\
&f_0(x_1 | Y) = 0 \\
&\text{(4.1)}
\end{align*}
$$

while (1.3) yields for Markov policies $Y$

$$
\begin{align*}
&f_n(x_1 | Y) = K(y_1 - x_1) + G(y_1) + \alpha \mathbb{E}[f_{n-1}(y_1 - \xi | Y)] \quad n \geq 1 \\
&f_0(x_1 | Y) = 0 \\
&\text{(4.2)}
\end{align*}
$$

where $Y_{n-1}^n(y_1 - \xi) = y_2$ and is not to be confused with $y_1$.

Furthermore, by Lemma 1.1 and Theorem 2.6 there is an optimal Markov policy $Y^*$, written $(\ldots, y_1^{n*}, y_2^{n*}, \ldots, y_n^{n*})$ for the $n$-period problem.

Finally, (1.5) is now written

$$
\text{(4.3)} \quad f_n(x_1) = \min_{y_1 \in \mathcal{Y}(x_1)} (K(y_1 - x_1) + J_n(y_1)).
$$

An ordering policy for the infinite-horizon problem is also a sequence of Borel functions defined for all history vectors and is written $Y = (Y_1, Y_2, \ldots)$. If $Y_t(\cdot)$ is independent of $t$ the policy is called stationary. By definition the infinite-period problem has no final period, and, therefore, the convention that final inventory
be zero does not apply. Thus, for the infinite-period problem commenced with \( x_1 \) on hand if the expected total cost following policy \( Y \) is defined it is given by

\[
f(x_1|Y) = E[\sum_{t=1}^{\infty} \alpha^{t-1}[K(y_t-x_t) + c(y_t-x_t)^T + h(x_{t+1})]]
\]

Consider this cost when \( \alpha < 1 \) so that \( f(x_1|Y) \) is bounded below as a function of both \( x_1 \) and \( Y \). If \( Y \) is such that \( \sum_{t=1}^{\infty} \alpha^{t-1} c y_t^T \) is defined and finite with probability one when \( x_1 \) is the initial level, then since \( \sum_{t=1}^{\infty} \alpha^{t-1} \xi_t < \infty \) with probability one

\[
\sum_{t=1}^{\infty} \alpha^{t-1} c(y_t-x_t)^T = \sum_{t=1}^{\infty} \alpha^{t-1}(1-\alpha) c y_t^T + \alpha \sum_{t=1}^{\infty} c y_t^T - c x_1^T
\]

with probability one. Thus, as in the finite-horizon problem

\[
f(x_1|Y) = E[\sum_{t=1}^{\infty} \alpha^{t-1}[K(y_t-x_t) + h(y_t-x_t) + (1-\alpha) c y_t^T + \alpha c \xi_t^T]] - c x_1^T.
\]

Since \( y_t \geq x_1 - \sum_{i=1}^{t-1} \xi_i \) and \( \alpha < 1 \), \( \sum_{t=1}^{\infty} \alpha^{t-1} c y_t^T \) is bounded below by a finite random variable so that if \( y_t \) is bounded above for a fixed \( x_1 \) the series is defined and finite. Also, if it is finite then

\[
f(x_1|Y) = \sum_{t=1}^{\infty} \alpha^{t-1} E[E[K(y_t-x_t) + (1-\alpha) c y_t^T + \alpha c \xi_t^T|y_t]] - c x_1^T
\]

\[
= \sum_{t=1}^{\infty} \alpha^{t-1} E[K(y_t-x_t) + G(y_t)] - c x_1^T.
\]
Conversely, if the latter is finite it must be \( f(x_{l}|Y) \) and so equality holds whenever \( \sum_{t=1}^{\infty} \alpha^{t-l} c y_t^T \) is finite. As in the n-period problem the constant \(-cx_1^T\) may be dropped and \( f(x_{l}|Y) \) redefined. Thus, provided that \( \sum_{t=1}^{\infty} \alpha^{t-l} c y_t^T \) is almost surely finite

\[
(4.4) \quad f(x_{l}|Y) = \lim_{n \to \infty} f_n(x_{l}|Y)
\]

One \( Y \) for which this is finite is the policy which orders up to \( S(x_1) \) every period. Thus, it is meaningful to call a policy \( Y^* \) optimal if it minimizes \( f(x_{l}|Y) \) for every \( x_{l} \), although such a \( Y^* \) may not exist. If \( c = (0,0) \) then \( \sum_{t=1}^{\infty} \alpha^{t-l} c y_t^T = 0 \) so that any optimal policy satisfies (4.4), which is usually given as the definition of \( f(x_{l}|Y) \) [13]. When \( c > (0,0) \), however, it can be concluded by some of the above steps only that any optimal policy satisfies

\[
(4.5) \quad f(x_{l}|Y) = \sum_{t=1}^{\infty} \alpha^{t-l} E[K(y_t-x_t) + c(y_t-x_t)^T + L(y_t)] + cx_1^T.
\]

When \( \alpha=1 \) the series representing the random cost may neither converge nor diverge to \( \pm \infty \). Modifying slightly the idea of Veinott and Wagner call \( Y^* \) optimal if it minimizes for all \( x_{l} \)

\[
(4.6) \quad k(x_{l}|Y) = E[\liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} [K(y_t-x_t) + c(y_t-x_t)^T + h(x_{t+1})] - cx_1^T,
\]

which is bounded below. If the expectation and limit operations may be interchanged then
\[ k(x_1|Y) = \lim \inf_{n \to \infty} \frac{1}{n} \left[ f_n(x_1|Y) - cx_1^T + cE[x_{n+1}^T] \right] \]
\[ = \lim \inf_{n \to \infty} \frac{1}{n} \left[ f_n(x_1|Y) + E[c(y_n - \bar{s}_n)^T] \right]. \]

Finally, if \( cE[y_n^T] \) is bounded for all \( n \) when \( x_1 \) is the initial level

\[(4.7) \quad k(x_1|Y) = \lim \inf_{n \to \infty} \frac{1}{n} f_n(x_1|Y). \]

Again, (4.7) is finite for the policy which orders up to \( s(x_1) \) every period. As above, if \( c = (0,0) \) then \( cy_n^T = 0 \) so that any optimal policy satisfies (4.7), which is usually given as the definition of \( k(x_1|Y) \) [13]. Only (4.6) is known to hold for any optimal policy, however, when \( c > (0,0) \).

Practical interest in the infinite-period problem, as was stated above, stems from its role as an approximation to the problem with a finite horizon. Consequently, when all optimal policies satisfy (4.5) or (4.6) but not (4.4) or (4.7) the problem is much less interesting. It may seem reasonable in this case to search for a best policy among those satisfying (4.4) or (4.7), but such a policy may not exist since, for example, when \( \alpha=1 \) it may be that any policy having \( E[y_t] \) bounded is inferior to some other such policy. In the single-product problem it is known that there exist policies minimizing (4.4) or (4.7) ([7], [8], [13]). For the two-product problem, on the other hand, this result has not been obtained, and, consequently, the corresponding theorems below use the existence of an optimal inventory path bounded above as a prior condition.
Because of this limited interest in the infinite-period problem adopt now the usual definition that $Y^*$ is optimal if it minimizes $f(x_1|Y)$ or $k(x_1|Y)$ among policies satisfying (4.4) or (4.7) respectively. Thus, for $\alpha < 1$ let $f(x_1|Y^*) = f(x_1)$ and for $\alpha = 1$ let $k(x_1|Y^*) = k(x_1)$, which will be denoted $k$ when it is independent of $x_1$. Furthermore, since the minimization in (4.4) and (4.7) is unaffected by addition of a constant to $G(y)$ it can now be assumed without loss of generality that $G(s_1) > 0$.

When a Markov policy is followed the inventory levels at the beginning of each period form a Markov process. Under suitable conditions in the single-product infinite-period problem this process has a limiting stationary distribution when a stationary $(s, S)$ policy is followed (Chapters 14 and 15 of [2], [13]). At the same time, optimal finite-horizon costs converge to the optimal infinite-horizon costs, i.e.,

$$f_n(x_1) \to f(x_1) \quad \text{for } \alpha < 1 \quad \text{and} \quad \frac{1}{n} f_n(x_1) \to k \quad \text{for } \alpha = 1;$$

and when $\alpha < 1$ limit points of optimal finite-horizon ordering functions are optimal for the infinite-horizon problem, i.e., an optimal $Y^*_t(x)$ may be taken to be a limit point of $Y^{*n}_t(x)$ ([7], [8]). Of major importance in the proof of these results is the establishment of bounds for optimal values of the policy parameters $s$ and $S$.

For the discrete multiproduct problem Johnson [9] has developed an algorithm for finding an optimal stationary $(s, S)$ policy when $G(y)$ may have a quite general form (it need not be quasi-convex) and $K(y-x) = K_\theta(y-x)$. By its very nature, of course, a stationary $(s, S)$ policy is feasible only for those $x \leq S$. This section extends the single-product results mentioned above, and thus includes treatment of continuous products.
and the transient regions where \( x \notin S \), although \( G(y) \) is assumed to be quasi-convex. For expositional ease, again, \( K(y-x) = K^0(y-x) \) and \( \mathcal{X}(x) = \{ y \mid y \geq x \} \). The detailed analysis will be given first for \( \alpha < 1 \) and later for \( \alpha = 1 \).

When \( \alpha < 1 \) the solution of the infinite-period problem is closely related to the solution of the functional equation (4.3) when both sides are independent of \( n \). Let \( F(x) \) be any finite solution to the following functional equation which is bounded below by some \( C \):

\[
(4.8) \quad F(x) = \min_{y \geq x} \{ K(y-x) + J(y) \},
\]

where \( J(y) = G(y) + \alpha E[F(y-x)] \). If \( Y(x) \) is a specific minimizing value of \( y \) in (4.8) then \( Y \) having \( Y_t(x) = Y(x) \) for all \( x \) and \( t \) is a stationary Markov policy, provided that \( Y(x) \) is a Borel function. Since the latter requirement is not necessary in what follows, however, any such \( Y \) associated with \( F(x) \) will be called a policy even if \( Y(x) \) is not a Borel function.

Although \( J(y) \) is not assumed continuous, statements similar to the rest of Theorem 2.6 can be made regarding \( F(x) \) and \( J(y) \) since \( Y(x) \) must satisfy

\[
J(Y(x)) = \min_{y \geq x} J(y)
\]

if \( x \in \sigma = \{ x \mid Y(x) > x \} \). The existence of a point minimizing \( J(y) \), however, is argued differently. Since \( J(y') \) is finite for some \( y' \) and \( J(y) \geq G(y) + \alpha C \to \infty \) as \( \|y\| \to \infty \) there exists an \( x' < y' \) such that \( J(y) > K + J(y') \) for all \( y \geq x' \). Hence, for all \( x \leq x' \).
\[ F(x) = F(x') = K + J(Y(x')) , \]

where \( Y(x') > x' \) and so \( x' \in \sigma \). Because the minimum is achieved in (4.8) it follows that \( Y(x') \) minimizes \( J(y) \). If \( K > 0 \) the proof of (b) of Theorem 2.7 shows that \( Y \) is a \( (1, S(x)) \) policy, while if \( K = 0 \) and \( x < Y(x) < Y(Y(x)) \) for any \( x \) then \( Y(Y(x)) \) can be redefined to be \( Y(x) \) and the new \( Y \) is a \( (1, S(x)) \) policy.

To determine bounds on \( Y(x) \) let \( z^*(x) \) be a point satisfying

\[ G(z^*(x)) = \min_{x \leq y \leq Y(x)} G(y) . \]

The statement and proof of Lemma 2.4 extend to \( F(x) \) and \( J(y) \) with the condition on their definition unnecessary. Lemma 2.8 then follows for \( x \in \sigma \). If \( x \in \sigma^c \), however, \( Y(x) \) need not minimize \( J(y) \) over \( y \geq x \) unless \( K = 0 \) so that the proof does not hold. Next, Lemma 2.9 and Corollary 2.10 carry over and establish that some or every \( \sigma \in \hat{\sigma} \) depending on whether \( K = 0 \) or \( K > 0 \).

So far the lower bound \( C \) on \( F(x) \) has been used only to establish the inessential existence of a point \( y \) minimizing \( J(y) \), but the existence of \( C \) is necessary in what follows. If

\[ \sigma = \{ x | \hat{s}(x) > x, G(x) > K + F(s(x)) - \alpha C \} , \]

then the proof of Lemma 2.11 can be used to show that \( \sigma \subset \sigma \). Finally, Lemma 2.12 extends where for \( y = \hat{s}(x) > x \)

\[ \Sigma(y) = \{ z | G(z) \leq F(y) - \alpha C \} . \]

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To study the infinite-horizon single-product problem and its relationship to the finite-horizon problem, Iglehart [8] establishes and makes important use of bounds on the optimal policy parameters. Their existence is essential to his proofs of uniform convergence of costs, convergence of policies, and optimality of stationary policies when the horizon is infinite. In the two-product problem the bounds obtained in Section II are inadequate for an extension of his results because the sets \( g_n \) and \( \Sigma_n(y) \) typically become arbitrarily large as \( n \to \infty \). Consequently, the following property will be used as a condition on a policy \( Y \) when \( R \) is a bounded rectangle of starting inventory levels:

\[(\beta) \text{ there exists a finite } B \text{ depending on } Y \text{ and } R^*_t = R
\]

such that if

\[ R^*_t = \{ x | x \leq Y^*_t(z) \text{ for some } z \in R^*_{t-1} \} \]

for \( n \geq t \geq 2 \) then \( Y^*_t(x) \leq B \text{ if } x \in R^*_t \text{ for } n \geq t \geq 1 \).

Furthermore, despite the fact that for an arbitrary solution \( F(x) \to (4.8) \Sigma(y) \) is bounded, it is not apparent that the sequence of inventory levels after ordering generated by the accompanying policy \( Y \) will be bounded. Suppose for a moment that \( (\beta) \) holds for \( Y \) for all \( R \) (\( R^*_t \) is independent of \( n \) for \( Y \)). Since \( F(x) \) is finite there is an upper bound on \( F(S(x)) \) for all \( x < S(x) \leq B \). Therefore, since \( G(x) \to \infty \) as \( \|x\| \to \infty \) there exists an \( A \) depending on \( B \) such that

\[ \{ x | A \not\leq x \leq B \} \subset g \subset \sigma. \]

Thus, it is concluded that \( Y(x) > A \) if \( A \not\leq x \leq B \).

By the use of \( (\beta) \), the following theorem is established. For the one-product problem Iglehart establishes essentially these results without requiring the condition. For any \( Y^* \) (\( \beta \)) holds because of the bounds on the policy parameters (Corollary 3.3), and for any \( Y \) he shows a stronger form to hold.
Theorem 4.1. For $\alpha < 1$, $f_n(x)$ converges to some $g(x)$ pointwise for every $x$. The convergence is uniform in any bounded rectangle $R$ of points $x$ for which some optimal policy satisfies ($\beta$). If some optimal policy $Y^*$ satisfies ($\beta$) for every $R$ then $g(x)$ satisfies (4.8); limit points of $\{Y^*_t(x)\}$ exist for each $t$ and $x$ and provide an optimal stationary policy $\tilde{Y}$ for $g(x)$; if also $P(x)$ is a finite solution to (4.8) which is bounded below and whose policy $Y$ satisfies ($\beta$) for every $R$ then $P(x) = g(x)$; and, finally, if also $f(\cdot|\tilde{Y})$ is finite and $\tilde{Y}$ satisfies ($\beta$) for every $R$ then $f(x) = f(x|\tilde{Y})$.

Proof. For $n \geq 1$

$$f_n(x) = \min_{Y} \left\{ \sum_{t=1}^{n-1} \alpha^{t-1} E[K(y_t - x_t) + G(y_t)] + \alpha^{n-1} E[K(y_n - x_n) + G(y_n)] \right\}$$

$$\geq f_{n-1}(x) + \alpha^{n-1} G(S_{n-1}).$$

Thus, the sequence $\{f_n(x)\}$ is monotone increasing. Since the infinite-horizon cost of the policy which orders up to $S(x)$ every period provides a finite upper bound for the sequence, it is concluded that $f_n(x)$ converges.

The assumed property of $Y^*$ is sufficient to justify a direct application of the proofs of Iglehart's Theorems 1, 2, and 3. The first of these establishes the uniform convergence of $f_n(x)$ to $g(x)$ and the resulting continuity of the limit; while the second demonstrates that $g(x)$ satisfies (4.8). Limit points of $\{Y^*_t(x)\}$ exist since $x \leq Y^*_t(x) \leq B$ and are independent of $t$ since $Y^*_t(x) = Y^*_t(x)_{n=1}$. 

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Iglehart's proof of Theorem 3 is easily adapted to showing that

$$\lim J_n(y) = G(y) + \alpha E[g(y - \xi)]$$

uniformly in every finite rectangle and that any limit point $\tilde{Y}_1(x)$ minimizes the expression in braces in (4.8) over $y \geq x$ when $g(x)$ replaces $F(x)$. Furthermore, $\tilde{Y}_1(x)$, being a limit of a subsequence of Borel functions, is itself a Borel function. Thus, limit points yield an optimal stationary Markov policy $\tilde{Y}$ for $g(x)$. Then, the assumptions on $F(x)$ are sufficient to apply the iterative proof of Iglehart's Theorem 4 to show that $F(x) = g(x)$.

The final statement of the theorem is proved with an argument expanding that of Veinott and Wagner. Since $\tilde{Y}$ is stationary

$$f(x_1 | \tilde{Y}) = \begin{cases} G(x_1) + \sum_{t=2}^{\infty} \alpha^{t-1} E[K(y_t - x_t) + G(y_t)] & x_1 \in \partial^c \\ K + G(y_1) + \sum_{t=2}^{\infty} \alpha^{t-1} E[K(y_t - x_t) + G(y_t)] & x_1 \in \partial \\ \end{cases}$$

But this is precisely (4.8) when $F(x) = g(x)$. Thus, by uniqueness $g(x) = f(x | \tilde{Y})$ since $f(\cdot | \tilde{Y})$ is finite and bounded below and $\tilde{Y}$ satisfies (3) for every $R$. Finally, because $f_n(x) \leq f_n(x | Y)$ for all $Y$ it follows that $g(x) \leq f(x | Y)$ for all $Y$, and, thus, $f(x | \tilde{Y}) = f(x)$. Q.E.D.
In the theorem the convergence is uniform actually over rectangles bounded above. This is because Iglehart's proofs for the single-product problem can be used to show convergence over intervals bounded above. There the determination of a lower bound on every \( s_n^* \) is required only to establish the existence of limit points of the sequence \( \{ s_n^* \} \). Here Theorem 4.1 yields limit points of \( \{ Y_t^*(x) \} \) instead of \( \{ \sigma_n^* \} \). Notice that a sufficient condition for \( f(\cdot | \bar{Y}) \) to be finite when \( \bar{Y} \) satisfies (\( \delta \)) is boundedness of \( \{ x | x \in \delta^c, x \leq B \} \).

Now consider the case \( \alpha = 1 \). The analysis follows the treatment of Iglehart [7] by first solving a stationary problem and then using the solution to form an optimal policy for the infinite-period problem. Consider any stationary \( (\sigma, S(x)) \) policy having the property that for some \( S \) if \( x \in \sigma \) and \( x \leq S \) then \( S(x) = S \). An optimal policy (not necessarily stationary) in Theorem 2.6, for example, may be chosen so that if \( x \in \sigma_n \) and \( x \leq S_n \) then \( S_n(x) = S_n \).

For \( n \geq 1 \) let

\[
\phi^{(n)}(\xi|y) = P\left[ \sum_{j=1}^{n} \xi_j \leq \xi, y - \sum_{j=1}^{1} \xi_j \in \sigma^c \right] ,
\]

This is an improper distribution function for \( n \) large enough when \( \{ x | x \in \sigma^c, x \leq y \} \) is bounded, except in the trivial case that \( \phi(0, 0) = 1 \), which is henceforth excluded. In fact, \( \phi^{(n)}(\xi|y) \to 0 \) as \( n \to \infty \).

Furthermore, for any non-negative function \( H(\xi) \) whose expectation with respect to \( \phi^{(n)}(\xi) \) exists

\[
0 \leq \int_0^\infty \int_0^\infty H(\xi) d\phi^{(n)}(\xi|y) \leq \int_0^\infty \int_0^\infty H(\xi) d\phi^{(n)}(\xi) \leq \infty .
\]
Thus, if \( H(\xi) \) is integrable with respect to \( \phi^{(n)}(\xi) \) for all \( n \geq 1 \) then

\[
0 \leq \int_0^\infty \int_0^\infty H(\xi) \, dM(\xi | y) \leq \int_0^\infty \int_0^\infty H(\xi) \, dM(\xi) \leq \infty ,
\]

where \( M(\xi) = \sum_{n=1}^\infty \phi^{(n)}(\xi) \), the renewal function, and \( M(\xi | y) = \sum_{n=1}^\infty \phi^{(n)}(\xi | y) \).

In particular, \( 0 \leq M(\xi | y) \leq M(\xi) < \infty \) for all \( \xi \). By extending the arguments of Karlin (Chapters 14 and 15 of [2]) it can be shown that under suitable conditions the inventory level before ordering for \( x_1 \leq S \) converges to a random variable having distribution function

\[
F(x) = 1 - [1 + \int_0^\infty \int_0^\infty dM(\xi | S)]^{-1} \int_0^\infty \int_0^{S_2-x_2} dM(\xi | S) \cdot
\]

\[
+ \int_0^{S_1-x_1} \int_0^\infty dM(\xi | S) - \int_0^{S_1-x_1} \int_0^{S_2-x_2} dM(\xi | S) .
\]

A rigorous justification of this statement requires the establishment of properties of \( M(\xi | S) \) analogous to those of the univariate version of \( M(\xi) \). This will be examined at a later date.

The stationary cost of the policy considered here is defined to be

\[
\lambda(\sigma, S) = (K + G(S)) \int_0^\infty dF(x) + \int_0^{\infty} G(x) \, dF(x)
\]

\[
= [1 + \int_0^\infty \int_0^\infty dM(\xi | S)]^{-1} [K + G(S) + \int_0^\infty \int_0^\infty G(S - \xi) \, dM(\xi | S)] .
\]
By the definition of a policy $\sigma$ must be a Borel set, so that the integrals are defined. If $\{x \mid x \in \sigma^c, x \leq S\}$ is bounded then it follows by the Helly-Bray Lemma that $\mathcal{L}(\sigma, S) = k(x \mid \sigma, S)$. Suppose now that $\mathcal{L}(\sigma, S)$ is minimized by choosing $\sigma = \sigma^*$ and $S = S^*$. Since $G(x) \to \infty$ as $\|x\| \to \infty$, it is evident that $\{x \mid x \in \sigma^*, x \leq S^*\}$ may be assumed bounded. The actual composition of $\sigma^*$ not below $S^*$ is unimportant. Assume also that for $x \leq S^*$

$$
\left[1 + \int_0^\infty \int_0^\infty dM^*(\xi | x) \right]^{-1} \left[ G(x) + \int_0^\infty \int_0^\infty G(x-\xi) dM^*(\xi | x) \right] = \begin{cases} 
\leq \mathcal{L}(\sigma^*, S^*) & x \in \sigma^c \\
\geq \mathcal{L}(\sigma^*, S^*) & x \in \sigma^* 
\end{cases}
$$

where $M^*(\xi | x)$ is the version of $M(\xi | x)$ using $\sigma = \sigma^*$. This amounts to assuming that the expected cost per unit of time from when the level is $x$ until an order is placed does not go above or below, in the respective cases, the corresponding cost (including the set-up cost) of the entire cycle. For the discrete problem Johnson asserts the truth of this property by an appeal to Markov programming (Theorem 2.3 of [9]). It will be shown that $k = \mathcal{L}(\sigma^*, S^*)$.

Assume, as Iglehart does for a single product, that there is a finite function $\psi(y)$ satisfying for $y \leq S^*$

$$
\psi(y) = \begin{cases} 
0 & y \in \sigma^c \\
G(y) - \mathcal{L}(\sigma^*, S^*) + \mathbb{E}[\psi(y-\xi)] & y \in \sigma^* 
\end{cases}
$$

(4.9)

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It is desired to show then that \( \psi(y) \) satisfies for \( x \leq S^* \) the functional equation

\[
(4.10) \quad \psi(x) = \min_{y \geq x} \{ K(y-x) + G(y) - z(s^*, S^*) + E[\psi(y-\xi)] \}.
\]

To this end define for \( \alpha = 1 \) \( J(y) = G(y) - z(s^*, S^*) + E[\psi(y-\xi)] \), which is finite. From (4.9)

\[
J(y) = G(y) - z(s^*, S^*) + \int_{\xi_1}^{\xi_2} \int_{\xi_0}^{\xi} \left[ G(y-\xi_1) - z(s^*, S^*) + E[\psi(y-\xi_1-\xi_2)] \right] d\phi(\xi_1) d\phi(\xi_2)
\]

\[
= G(y) - z(s^*, S^*) + \int_{\xi_0}^{\xi} \left[ G(y-\xi) - z(s^*, S^*) + E[\psi(y-\xi-\xi_2)] \right] d\phi(\xi_2) d\phi(\xi_1)
\]

\[
= G(y) - z(s^*, S^*) + \int_{\xi_0}^{\xi} \left[ G(y-\xi) - z(s^*, S^*) + E[\psi(y-\xi)] \right] d\phi(\xi)
\]

\[
+ \int_{\xi_1}^{\xi_2} \int_{\xi_0}^{\xi} \left[ G(y-\xi) - z(s^*, S^*) + E[\psi(y-\xi)] \right] d\phi(\xi_1) d\phi(\xi_2)
\]

\[
= G(y) - z(s^*, S^*) + \int_{0}^{\infty} \int_{0}^{\infty} \left[ G(y-\xi) - z(s^*, S^*) + E[\psi(y-\xi)] \right] d\phi(\xi_1) d\phi(\xi_2)
\]

Further iteration yields for all \( n \geq 1 \)
\[ J(y) - G(y) = \sum_{t=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} G(y-\xi) d\phi^{(t)}(\xi | y) \]

\[ -\mu_{\sigma, S}[1 + \sum_{t=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \phi^{(t)}(\xi | y)] + \int_{0}^{\infty} \int_{0}^{\infty} E[\psi(y-\xi, n+1)] d\phi^{(n)}(\xi | y) . \]

The last term is smaller in absolute value than

\[ \sup_{x: x \in c} |\psi(x)| \int_{0}^{\infty} \int_{0}^{\infty} d\phi^{(n)}(\xi | y) , \]

which approaches zero as \( n \to \infty \) since \( [x: x \in c, x \leq y] \) is bounded and \( \psi(\cdot) \) is finite. Thus,

\[ J(y) = G(y) + \sum_{t=1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} G(y-\xi) d\phi^{(t)}(\xi | y) - \mu_{\sigma, S}[1 + \sum_{t=1}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \phi^{(t)}(\xi | y)] . \]

Finally, by Fubini's theorem

\[ J(y) = G(y) + \int_{0}^{\infty} \int_{0}^{\infty} G(y-\xi) dM^{(\xi | y)}(\xi) - \mu_{\sigma, S}[1 + \int_{0}^{\infty} \int_{0}^{\infty} dM^{*}(\xi | y)] . \]

Now to show that \( \psi(x) \) satisfies (4.10) for \( x \leq S^{*} \) it suffices to show that

\[ \min\{J(x), K + \min_{y \geq x} J(y)\} = \begin{cases} 
0 & x \in c, x \leq S^{*} \\
J(x) & x \in c, x \leq S^{*} 
\end{cases} . \]

For all \( y \)

\[ [J(y) + K][1 + \int_{0}^{\infty} \int_{0}^{\infty} dM^{*}(\xi | y)]^{-1} = \mu_{\sigma, y} - \mu_{\sigma, S^{*}} \geq 0 , \]

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so that \( \min J(y) = J(S^*) = -K. \) Now for \( x \leq S^* \)

\[
J(x)[1 + \int_0^\infty \int_0^\infty \mathbb{d}M^*(\xi | x)]^{-1} = \left[1 + \int_0^\infty \int_0^\infty \mathbb{d}M^*(\xi | x)\right]^{-1} \int_0^\infty \int_0^\infty G(x - \xi) \mathbb{d}M^*(\xi | x)] - \mathcal{I}(\sigma^*, S^*),
\]

which by assumption is non-positive if \( x \in \sigma^* \) and non-negative if \( x \in \sigma^* \). Thus, if \( x \in \sigma^* \) then \( \min\{J(x), 0\} = J(x) \); while if \( x \in \sigma^* \) then \( \min\{J(x), 0\} = 0 \).

For the one-product problem Iglehart is able to start with (4.9) for all \( y \) and to show that (4.10) is satisfied. Here it is assumed that \( \psi(y) \) can be defined for \( y \neq S^* \) as the solution to (4.10) coinciding with (4.9) when \( y \leq S^* \). Thus, a function \( Y(x) \) can be defined to that

\[
\psi(x) = \begin{cases} 
K + G(Y(x)) - \mathcal{I}(\sigma^*, S^*) + E[\psi(Y(x) - \xi)] & x \in \sigma^* \\
G(x) - \mathcal{I}(\sigma^*, S^*) + E[\psi(x - \xi)] & x \in \sigma^* \setminus S^* 
\end{cases}
\]  

(4.11)

where now \( \sigma^* = \{x | Y(x) > x\} \). As for \( \alpha < 1 \) let \( Y \) denote the stationary Markov policy (if \( Y(x) \) is a Borel function) induced by \( Y(x) \), and assume that \( Y \) satisfies \( (\beta) \) for every \( R \).

**Theorem 4.2** If there exists a finite \( \psi(x) \) satisfying (4.10) whose \( Y \) satisfies \( (\beta) \) for a bounded rectangle \( R \) of points \( x \) and if some policy \( Y^* \) optimal for every finite-horizon problem also satisfies \( (\beta) \) for \( R \), then \( \frac{1}{n} f_n(x) \) converges uniformly in \( R \) to \( \mathcal{I}(\sigma^*, S^*) \).
If \( \lim \frac{1}{n} f_n(x) = \mathcal{L}(\sigma^*, S^*) \) for all \( x \) then the stationary \( Y \) characterized by \((\sigma^*, S^*)\) is optimal for the infinite-period problem.

**Proof** By the use of (\( \beta \)) Iglehart's proof is easily extended to show by induction that there exists a \( C \) such that

\[
|f_n(x) - n\mathcal{L}(\sigma^*, S^*) - \psi(x)| \leq C
\]

for all \( n \) and \( x \in \mathbb{R} \), from which uniform convergence of \( \frac{1}{n} f_n(x) \) follows. Then optimality of the stationary policy \( Y \) is obtained by Veinott and Wagner's argument without the use of \( \psi(x) \) or (\( \beta \)):

\[
\mathcal{L}(\sigma^*, S^*) = \lim \frac{1}{n} f_n(x)
\]

\[
\leq \lim \inf \frac{1}{n} f_n(x|y)
\]

\[
= k(x|y)
\]

\[
= \mathcal{L}(\sigma^*, S^*) .
\]

Q.E.D.

Iglehart establishes the above result for the one-product problem without the use of (\( \beta \)) since, as for \( \alpha < 1 \), he exhibits bounds on the policy parameters. As in Theorem 4.1 rectangles bounded only above may be used here. When \( \alpha = 1 \) nothing appears to have been established about convergence of the policy parameters even for the case of a single product.
Theorems 4.1 and 4.2 are unpleasantly encumbered with conditions whose validity appears difficult to verify. One situation in which (β) holds for some Y* is the case of a guaranteed minimum level of demand as in Theorem 3.4. Furthermore, for the Y* given there Y also satisfies (β) and f(x|Y) is finite.

It would still seem that results closer to those obtained for the single-product problem should hold for a practical multiproduct problem and that any indeterminacy might be due to the generality of the mathematical model. This is true. For example, to this point no bounds have been imposed on the inventory levels, the demands, or the costs. In a real problem, however, all of these will be bounded, although tight limits may not be known.

Suppose, therefore, that for some vector M whose components are large numbers it is sufficient to consider only inventory levels x ≤ M. Thus, in Section II Y = {x|x ∈ ℝ^n, x ≤ M} and a policy Y* is optimal for the n-period problem if f_n(x|Y*) ≤ f_n(x|Y) for all Y for all x ≤ M. For this modified problem the results of Sections II and III may be obtained, in some cases more easily than before. More importantly, however, Theorems 4.1 and 4.2 hold without the imposition of (β), and the desired results are obtained. Since M may be chosen arbitrarily large this restriction of the model would not seem to weaken its applicability to real problems.
Section V. Proportional Stockage

With the exception of the situation considered in Theorem 3.6 it has been assumed that \( \mathcal{Y} = \mathbb{R}^2 \) so that the feasible ordering region \( \mathcal{Y}(x) = \{y | y \geq x\} \). Veinott [11] finds that the present model without a set-up cost has a particularly simple optimal policy if the inventory level after ordering is constrained to lie on a straight line having positive slope.

For certain set-up cost functions this is still true. Such a restricted model may be useful by itself since a constraint of this type may apply in a real problem; but it also provides an approach to the case of complementary products, which was found in Section III not even to have an optimal \((s(x), S(x))\) policy in general. The nature of the restricted model, moreover, yields a fairly easy analysis of m-product problems.

Let \( a \) and \( b \) be fixed m-vectors with \( b_\ell > 0 \) for \( \ell = 1, 2, \ldots, m \), and let \( \mathcal{Y} = \{y | y = a + y'b, y' \in \mathbb{R}\} \), a straight line. Hence,

\[
\mathcal{Y}(x) = \{y | y \geq x\} \cap \mathcal{Y} = \{y | y = a + y'b \geq x, y' \in \mathbb{R}\}.
\]

If \( \mathcal{Y}' = \mathbb{R} \) so that for \( x' \in \mathbb{R} \)

\[
\mathcal{Y}'(x') = \{y' | y' \geq x'\} \cap \mathcal{Y}' = \{y' | y' \geq x'\},
\]

then \( y \in \mathcal{Y}(x) \) if and only if \( y' \in \mathcal{Y}'(x') \) where

\[
\begin{align*}
\begin{cases}
y' &= (y_\ell - a_\ell)/b_\ell \text{ for all } \ell \\
x' &= \max\{(x_\ell - a_\ell)/b_\ell\}
\end{cases}
\end{align*}
\]

(In this section maximization and minimization of vector components is understood to be over \( \ell \) such that \( 1 \leq \ell \leq m \).) For \( y' \in \mathbb{R} \) let
\[ G'(y') = G(a + y' \cdot b) \text{ so that } G'(y') = G(y) \text{ if } y' \text{ in } \mathcal{Y}' \text{ and } y \text{ in } \mathcal{Y} \text{ correspond to one another. Since } G(y) \text{ is quasi-convex and therefore quasi-convex over straight lines } G'(y') \text{ is quasi-convex. Also, using (5.1) one may define } S = a + S' \cdot b \text{ and } S(x) = a + S'(x') \cdot b \text{ so that} \]

\[
G(S) = \min_{y \in \mathcal{Y}} G(y) \\
= \min_{y' \in \mathcal{Y}'} G'(y') \\
= G'(S') ,
\]

and

\[
G(S(x)) = \min_{y \in \mathcal{Y}(x)} G(y) \\
= \min_{y' \in \mathcal{Y}'(x')} G'(y') \\
= G'(S'(x')) .
\]

Except for the set-up cost the necessary associations have now been made to reduce the solution of the inventory problem to a one-dimensional mathematical minimization problem. As was pointed out in the analysis of separable products in Section III, further work is required to obtain a single-product inventory problem. The present situation is different, however, in that the reduced mathematical problem is always an inventory problem. To see this let \( \Phi'(*) \) be the cumulative distribution function for \( \xi' = \min(\xi / b_{\xi}) \) and let \( \mu' \) be its mean. For \( y \in \mathcal{Y} \)

\[
G'(y') = G(y) = L(y) + (1-\alpha)cy^T + \alpha cu^T .
\]

Letting \( z = (z_{\xi}) \) note that

\[
L(y) = E[h(y-\xi)] \\
= E[E[h((y-y_{\xi}))/\xi']]
\]
The conditional expectation is a random variable which may be denoted by

\[ C((y \cdot \xi')b) = C(y \cdot \xi' b) \]

\[ = C(a + y' b \cdot \xi' b) \]

\[ = C(a + (y - \xi)' b) \]

since if \( y \in \mathbb{I} \) then

\[ y' - \xi' = \max\{(y \cdot a)/b\} - \min\{\xi/b\} \]

\[ = \max\{(y \cdot a - s)/b\} \]

\[ = (y - \xi)' . \]

Now let

\[ h'(z') = C(a + z' b) + (1 - \alpha)c a^T + \alpha c (\mu^T - \mu' b^T) \]

so that

\[ G'(y') = E[h'(y' - \xi')] + (1 - \alpha)c y' b^T + \alpha c (\mu' b^T) \]

\[ = L'(y') + (1 - \alpha)c b^T y' + \alpha c b^T \mu'. \]

Thus, \( G'(y') \) is the one-period expected cost function for the single-product inventory problem having demand \( \xi' \) with cumulative distribution function \( \Phi'() \), proportional purchase cost \( cb^T \), and conditional holding and penalty cost function \( h'(z') \). Of course, the single product problem is not completely specified without the set-up cost. This factor has purposely been ignored to this point since several set-up cost functions will be considered in what follows.
Theorem 5.1 If $Y = \{ y \mid y = a + y'b, y' \in R \}$ and $K(z) = K \prod_{\ell=1}^{m} \delta(z_{\ell})$ then one optimal policy is $Y^*$ having for each $i$

$$y^*_i(x) = \begin{cases} a + \max((x_i - a_i)/b_i) b & \text{if } x \notin S_i \\ S_i & \text{if } x < S_i, \end{cases}$$

where $s_i = a + s_i' b$ and $S_i = a + S_i' b$ when $s_i'$ and $S_i'$ are optimal parameters for the related single-product problem with $K'(z') = K\delta(z').$

Proof Since the existence of a single-product inventory problem with $G'(y') = G(y)$ for $y \in Y$ was established above consider $K(y-x)$ and $K'(y'-x').$ The quantities $\prod_{\ell=1}^{m} \delta(y_{\ell}-x_{\ell})$ and $\delta(y'-x')$ are 0 or 1. For $y \in Y(x)$ the first is positive if and only if $\delta(y'-(x_i - a_i)/b_i)$ is positive for all $i$, which is equivalent to positivity of $\delta(y'-x')$ for $y' \in Y'(x').$ Hence, $\prod_{\ell=1}^{m} \delta(y_{\ell}-x_{\ell}) = \delta(y'-x')$ if $y \in Y(x)$ and $y' = (y_1 - a_1)/b_1$ or if $y' \in Y'(x')$ and $y = a + y'b.$

Now for any single-product Markov policy $Y'$ there is a related multi-product policy $Y$ having $Y_1(x) = a + Y_1'(x') b.$ The converse, of course, is not true since no $Y'$ corresponds to a $Y$ for which any $Y_i(x)$ depends on another $x_{\ell}$ in addition to the one yielding $x_i$. Hence, it is not immediate that an optimal $Y'$ will yield an optimal $Y$, although this fact can be established by induction as follows. First, by Corollary 3.3 there are optimal parameters $(s_i', S_i')$ characterizing an optimal policy for period $1$ in the single-product problem. Then for the one-period problem
\[
f_1(x) = \inf_{y \in \mathcal{Y}(x)} \left\{ \sum_{\ell=1}^{m} K(y_{\ell} - x_{\ell}) + G(y) \right\}
\]
\[
= \inf_{y' \in \mathcal{Y}(x')} \left\{ K(\delta(y' - x')) + G'(y') \right\}
\]
\[
= f'_1(x')
\]
\[
= \begin{cases} 
G'(x') & \text{if } x' \geq s'_1 \\
K + G'(s'_1) & \text{if } x' < s'_1 
\end{cases}
\]
\[
= \begin{cases} 
G(a + \max((x_{\ell} - a_{\ell})/b_{\ell})b) & \text{if } x \notin s_1 \\
K + G(s_1) & \text{if } x < s_1 
\end{cases}
\]
\[
= f_1(x | Y^*) .
\]

Suppose that \( f'_1(x) = f'_1(x') = f_1(x | Y^*) \) for all \( i < n \). For \( y \in \mathcal{Y} \)
\((y - \xi)' = y' - \xi' \) so that
\[
J_n(y) = G(y) + \alpha E[f_{n-1} (y - \xi)]
\]
\[
= G'(y') + \alpha E[f'_{n-1} (y' - \xi')]
\]
\[
= J'_n(y') .
\]

Thus, since \( Y^*_{n-1} (y - \xi) = a + Y'_{n-1} ((y - \xi)')b = a + Y'_{n-x} (y' - \xi')b \) it then
follows that \( f_n(x) = f'_n(x') = f_n(x | Y^*) \) as for \( n=1 \) .

Q.E.D.

The set-up cost function in Theorem 5.1 results in an optimal \((a, S(x))\)
policy of a special type. By the very nature of \( K(y-x) \), in fact, for
any policy either the minimum feasible amount is ordered or more than that.
In the latter case orders for positive quantities of all products are placed (since \( \mathbf{z} \) is linearly ordered by "<" componentwise) and so \( K \) is charged.

Such a set-up cost function has limited application from a practical point of view, but when the demand \( \mathbf{g} \) has an absolutely continuous cumulative distribution function Theorem 5.1 can be used to obtain an optimal policy for some other set-up cost functions. Let

\[
I(x) = \{ \ell | (x - a_{i \ell})/b_{i \ell} = \max((x_j - a_{ij})/b_{ij}) \}
\]

and let \( N[A] \) be the number of elements in a set \( A \). A cost function with subscript \( t \) is the function used in period \( t \).

**Theorem 5.2** If \( \mathbf{y} = \{y | y = a + yb, y \in \mathbb{R}\} \), \( K(z) = K \sum_{\ell=1}^{m} \delta(z_{\ell}) \), and \( \phi(\cdot, \cdot, \cdot, \cdot) \) is absolutely continuous then one optimal policy is \( y^* \) having for each \( i \)

\[
y^*_i(x) = \begin{cases} 
  a + \max((x_{i\ell} - a_{i\ell})/b_{i\ell})b & \text{if } N[I(x)] = j \text{ and } x < s^j_1 \\
  s^j_1 & \text{if } N[I(x)] = j \text{ and } x \geq s^j_1,
\end{cases}
\]

where \( s^j_1 = a + s^j_1b \) and \( s^j_1 = a + S^j_1b \) when \( s^j_1 \) and \( S^j_1 \) are optimal parameters for the related single-product problem with \( K^j_t(z^i) = K\delta(z^i) \)

for \( t < i \) and \( K^j_t(z^i) = J\mathbf{K}\delta(z^i) \).

**Proof** If \( N[I(x)] = j \) then for \( y \in \mathbf{y}(x) \)

\[
K(y-x) = (m-j)K + K \sum_{j \in I(x)} \delta(y_{j\ell} - x_{j\ell})
\]

which equals \((m-j)K\) or \( mK \) because \( Y \) is linearly ordered by "<" componentwise. As in Theorem 5.1 for the related \( y^i \in \mathbf{y}^i(x^i) \)

\[
K(y-x) + G(y) = (m-j)K + JK\delta(y^i-x^i) + G^i(y^i) \\
= (m-j)K + K^j_1(y^i-x^i) + G^i(y^i).
\]
Consider the one-period problem when \( N[I(x)] = j \) so that

\[
f_1(x) = \inf_{y \in \mathcal{Y}(x)} \left\{ k \sum_{t=1}^m g(s_t) + G(y) \right\}
\]

\[
= \inf_{y' \in \mathcal{Y}'(x')} \{(m-j)K + K_{j}^j(y'-x') + G'(y')\}
\]

\[
= (m-j)K + f_1^j(x')
\]

\[
= \begin{cases} 
(m-j)K + G'(x') & \text{if } x' \geq s_j^j \\
K + G'(s_j) & \text{if } x' < s_j^j 
\end{cases}
\]

\[
= \begin{cases} 
(m-j)K + G(a+\max((x_t-a_t)/b_t)b) & \text{if } x < s_j^j \\
K + G(s_j) & \text{if } x < s_j^j 
\end{cases}
\]

\[
f_1(x|Y^*) = f_1(x|Y^*)
\]

Suppose that \( f_1(x) = (m-j)K + f_1^j(x') = f_1(x|Y^*) \) for all \( i < n \). For \( y \in \mathcal{Y} \)

\[
\mathcal{J}_n(y) = G(y) + \alpha E[f_{n-1}(y-\tilde{y})]
\]

\[
= G(y) + \alpha E[E[f_{n-1}(y-\tilde{y})|N[I(y-\tilde{y})] = j]]
\]

Now if \( j > 1 \) then \( z|N[I(z)] = j \) is an \((m-j+1)\)-dimensional hyperplane and so has \( m \)-dimensional Lebesgue measure zero. Consequently, by absolute continuity of \( \phi(\cdot,\cdot,\cdot,\cdot) \)

\[
P[N[I(y-\tilde{y})] = j] = \begin{cases} 
1 & j = 1 \\
0 & j = 2, 3, \ldots, m
\end{cases}
\]
Hence, since \((y-t)^t = y^t-x^t\) for \(y \in \mathcal{Y}\)

\[
J_n(y) = G^t(y^t) + \alpha E[f_{n-1}(y-t) | N[y-t]] = 1
\]

\[
= G^t(y^t) + \alpha E[f_{n-1}^{y^t}(y^t-x^t)]
\]

\[
= J_n(y^t).
\]

As in Theorem 5.1 if now \(N[y(x)] = j\)

\[
f_n(x) = \inf_{y \in \mathcal{Y}(x)} \left\{ J_n(y) \right\}
\]

\[
= \inf_{y \in \mathcal{Y}(x)} \left\{ (m-j)K + jK\delta(y^t-x^t) + J_n(y^t) \right\}.
\]

If the infimum is attained it represents the minimum cost of the \(n\)-period problem having \(K_t(z^t) = K\delta(x^t)\) for \(t < n\) and \(K_n^d(z^t) = jK\delta(z^t)\).

Because this cost is not stationary Corollary 3.3 doesn't apply directly. However, it is well known (see Veinott [12]) that the corollary may be extended to non-stationary costs when \(K_{t+1} \geq \omega K_t\) for all \(t \geq 1\). Since \(jK \geq K \geq \omega K\) it follows that there exist \(S_n^d\) and \(S_n^i\) characterizing an optimal policy for period \(n\) and so that

\[
f_n(x) = (m-j)K + f_n^d(x)
\]

\[
= f_n(x|Y^t)
\]

as for \(n=1\). Q.E.D.
From the proof of Theorem 5.2 it is apparent that when \( \phi(\ldots, \ldots, \ldots) \) is absolutely continuous even more general set-up costs are easily handled. In the notation at the end of Section II the set-up cost may be written

\[
K(z) = \sum_{\beta} K_\beta (\prod_{i \in \beta} \delta(z_i))(\prod_{j \in \beta^c} (1-\delta(z_j)))
\]

where \( \beta \) is any of the \( 2^{m-1} \) non-empty subsets of \( \{1,2,\ldots,m\} \), which itself is now denoted by \( B \), and the complement of \( \beta \) is taken with respect to \( B \).

**Theorem 5.3** If \( \mathcal{Y} = \{y | y = a + y'b, y' \in R\} \), \( K_B - K_\beta = K'_t \geq 0 \) for all \( \beta \) having \( N[\beta] = m-1 \), and \( \phi(\ldots, \ldots, \ldots) \) is absolutely continuous then one optimal policy is \( \mathcal{Y}^* \) having for each \( i \)

\[
y^*_i(x) = \begin{cases} 
  a + \text{max}(x_i - a'_{i,t}, 0)b & \text{if } I(x) = \beta^c \text{ and } x \notin S^\beta_i \\
  S^\beta_i & \text{if } I(x) = \beta^c \text{ and } x \in S^\beta_i 
\end{cases}
\]

where \( S^\beta_i = a + S^\beta_i' b \) and \( S_i = a + S^t_i b \) when \( S^\beta_i' \) and \( S^t_i \) are optimal parameters for the related single-product problem with \( K_t'(z') = K'_t \delta(z') \) for \( t < i \) and \( K^t_i(z') = (K_B - K_\beta) \delta(z') \).

**Proof** If \( I(x) = \beta^c \) then for \( y \in \mathcal{Y}(x) \) and the related \( y' \in \mathcal{Y}'(x') \)

\[
K(y-x) + G(y) = K_B + (K_B - K_\beta) \delta(y'-x') + G'(y')
\]

The remainder of the proof follows that of Theorem 5.2 since for all \( \beta \) having \( N[\beta] = m-1 \) \( K_B - K_\beta \) is a non-negative constant and \( K_B - K_\beta \geq K_B - K_\beta' \) for all \( \beta' \subset \beta \). Note that if \( I(x) = \beta^c \) and \( N[I(x)] = j \) then \( s^j_i \) in Theorem 5.2 corresponds to \( s^\beta_i \) here although \( N[\beta] = m-j \).

Q.E.D.

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The proof of Theorem 5.3 does not require that $K_{B'} \leq K_B$ for all $B' \subset B$ if $N[\beta] < m-1$ although this constraint was imposed in Section II. In fact, the optimal action for $x$ such that $I(x) = \beta^c$ and $N[\beta] = n-1$ is unchanged at each stage if $K_{B'} - K_B < K_{B'} - K_B$ for any $\beta' \subset \beta$ although the optimal policy need not be of the $(s,S)$ form for other $x$.

The model of this section has practical application, for example, when the products are nuts and bolts and it is reasonable to require that after ordering there be the same number of nuts as bolts so that $a = (0,0)$ and $b = (1,1)$. Since nuts and bolts may be sold separately, however, other choices of $a$ and $b$ might be preferable. In general if $a$ and $b$ are not specified in advance for the problem they can be treated as parameters and the optimal cost expressed as a function of them. A minimization over possible $a$ and $b$ will then yield the optimal cost and an optimal policy for a specified initial level. Although this approach might suggest taking $\gamma$ to be a more general increasing curve the simple policies above need not be obtained since $G(y)$ may not be quasi-convex over $\gamma$.

Complementarity for $m$ products has not yet been defined. To this end let $\beta$ be a proper subset of $B$ and $x_\beta$ a vector whose components are each component $x_j$ of $x$ for which $j \in \beta$. Define $z_{\beta^c}(x_\beta)$ to satisfy

$$G((x_\beta, z_{\beta^c}(x_\beta))^P) = \min_{x_\beta^c} G(x)$$

where $(x_\beta, y_{\beta^c})^P$ is the permutation of $(x_\beta, y_{\beta^c})$ whose components have the same order as $x$. The $m$ products are complementary if for every non-empty $\beta \subset B, \beta \neq B$, the functions $z_{\beta^c}(x_\beta)$ are non-decreasing (in the multivariate sense) in $x_\beta$. Substitutable products are similarly defined.
Optimal policies for complementary products have a natural tendency toward satisfying the restriction of this section. By this statement reference is made to the fact that in the single-period problem with two products for \( x_1 > \frac{1}{2} \min_j z_j(x_1) \) it is optimal to order to \( (x_1, z_j(x_1)) \) or not to order at all, where \( z_j(x_1) \) is non-decreasing in \( x_1 \). Since the curves \( z_j(x_1) \) will coincide and be straight lines only in very special cases, of course, appropriate \( a \) and \( b \) are not obvious. Also, it should be cautioned that complementarity is a property derived from the expected cost function rather than the probability distribution of demand so that nuts and bolts may not be complements in the present sense. When the products are complementary, however, the complexity of unrestricted optimal policies and their tendency toward optimal policies of the restricted model make the analysis of this section appealing.

In the two-product problem with substitutable products it might seem reasonable to attempt the same analysis with \( b_1 < 0 \) and \( b_2 > 0 \). One drawback to such an approach is the fact that no policy is feasible for points above the line \( y = a + y'b \). More important weaknesses arise from the non-existence of a related single-product problem and the retained complexity of optimal policies for problems with horizon greater than one period. Finally, when there are \( m \) products to consider an \( (m-1) \)-dimensional hyperplane seems a more natural constraint set than a line. As an example, suppose that three products are completely interchangeable. With just two products it might be required that \( y = y'(-1,1) \), but with three the analogous constraint is that \( x_1 + x_2 + x_3 = 0 \). In this case \( x \) is not linearly ordered. Thus, the model of this section does not have a natural connection to the more general model for substitutable products; but, of course, the restricted model with each \( b_k > 0 \) may still be used and simpler optimal policies will be obtained.
Section VI. Stationary \((s,S)\) Policies

Theorem 3.6 contains the only conditions obtained here under which there are always optimal \((s,S)\) policies for the multiproduct problem. When these conditions do not hold, however, optimal policies may be sufficiently complex to defy practical calculation or actual implementation. Even for a single-period problem the two-dimensional minimization required to find \((s,S(x))\) may be quite difficult; and, if it is performed, the inventory manager must refer to a set of numbers much larger than \(\{s_1,s_2,S_1,S_2\}\). The seriousness of these drawbacks, of course, is heightened in the general \(m\)-product problem. In the single-product problem, on the other hand, although \((s,S)\) policies themselves are sometimes computationally difficult their application is simple. It is so simple, in fact, that they are in widespread use even when the expected single-period costs are not quasi-convex. From a practical point of view, therefore, it might be preferable to suboptimize by choosing, if possible, an optimal policy in the class of \((s,S)\) policies.

The problem with this objective, however, is that it fails to reduce the issue. In the one-period problem, for example, if the initial inventory level is some \(x \in c^c\), then any \((s,S)\) policy with \(s < x\) is optimal. Similarly, if \(x \in c\) then any \((s,S)\) policy with \(s \not< x\) and \(S = S(x)\) is optimal. Now to require that a single \((s,S)\) policy be optimal for all \(x\) amounts to insisting that for some optimal \((c_1,S_1)\) the set \(\{x \in c_1^c, x \leq S_1\}\) be a rectangle. In general this is impossible. For the \(n\)-period problem an optimal \((c,S(x))\) policy must first be found and then conditional upon the level at the beginning of each period an optimal \((s,S)\) policy is any one which yields the same action as \((c,S(x))\).

A similar problem arises in the one-product problem when \(G(y)\) is not quasi-convex. First, when \(G(y)\) is quasi-convex it is true that for any initial level there are an undetermined number of optimal parameters \((s_1,S_1)\) for the single-period problem. It is also true, however, that at least one
of these parameter combinations provides a policy simultaneously optimal for all initial levels. In fact, such parameters may be found for every period. Without quasi-convexity of $G(y)$, on the other hand, every optimal policy may require ordering if and only if the initial level lies in certain disjoint intervals which vary with $n$. In this case the best $(s,S)$ policy may depend on the initial inventory level, and there is no policy simultaneously optimal for all levels.

To state an attainable objective one might restrict his attention to the class of stationary $(s,S)$ policies even when the horizon is finite. This has the desirable aspect of moving further into the realm of practicality from the standpoints of computation and implementation. A bonus derived from analyzing the problem with this objective is that it provides an answer to the single-product and even the multiproduct problem when $G(y)$ is not quasi-convex.

When $\alpha < 1$ or the horizon is finite the best stationary $(s,S)$ policy may still depend on the initial inventory level although not on any subsequent level. In any particular problem the initial level is known and the optimal stationary $(s,S)$ policy may then be calculated. To obtain a general solution, however, it is useful to specify an initial level by an assumption which is the same for all problems. For example, it might seem natural to take zero as a common initial level since frequently practical problems do start with no items available and no backlog. It might also be argued, however, that the models in this paper in most cases can't be applied until the inventory process has been observed for a few periods, in order that good assumptions regarding costs and demand can be made. From a mathematical point of view an initial level of zero is unsatisfying since it is difficult
to compare the general cost of the best policy for each of the possible inequalities between \( s \) and zero. With one product, for instance, there is a best \((s, S)\) policy with \( s \geq 0 \) and perhaps a different one with \( s \leq 0 \). The costs of each may not be easily compared. In the \( m \)-product problem there are potentially \( 2^m \) policies.

To resolve this difficulty we shall adopt the convention of assuming that the initial level is \( S \) and thus set it to one of the parameters with respect to which the minimization is performed. If \((s^*, S^*)\) is an optimal policy under this assumption, then it is also an optimal stationary \((s, S)\) policy for all initial levels \( x \leq s^* \) for which it is optimal to order in the first period. If \( 0 \leq s^* \), for example, and an optimal stationary \((s, S)\) policy for an initial level of zero is \((s(0), S(0))\) where \( 0 \leq s(0) \), then it is optimal to take \((s(0), S(0)) = (s^*, S^*)\).

Because of the similarity between the \( m \)-product and the single-product problems of finding an optimal stationary \((s, S)\) policy when \( G(y) \) is not quasi-convex, the detailed analysis will be restricted to the case of a single product. Let \( G(y) \) be continuous and approach \( \infty \) as \( |y| \to \infty \). Thus, the function has a lower bound which, by addition of a constant if necessary, will be assumed positive. Although only finite \( S \) are considered, it is possible that \( s = -\infty \) in what follows. Number periods forwards as in the infinite analysis of Section IV, and adapting the notation of Greenberg [6,] to the present needs let

\[
F_t(x|s, S) = P[x_{t+1} \leq x \text{ when } (s, S) \text{ is followed each period}],
\]

\[
r_t(S - s) = F_t(s|s, S),
\]

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for \( s \) finite and \( t \geq 0 \) where \( x_1 = S \) by assumption. Note that, as is indicated, \( F_t(s|s,S) \) depends only on \( S-s \). Define \( F_t(x|\infty,S) \) in the same way so that \( F_t(x|\infty,S) = 1 - \phi(t)(S-x) \), and let \( r_t(S-s) = 0 = F_t(-\infty|-\infty,S) \) when \( s = \infty \). The (expected) cost for the \( n \)-period problem when \((s,S)\) is followed each period is then for \( n \geq 1 \)

\[
(6.1) \quad f_n(s,S) = G(S) + \sum_{t=1}^{n-1} \alpha^t \left[ (G+S) r_t(S-s) + \int_S^S G(x) dF_t(x|s,S) \right].
\]

For the sake of compactness the convention is adopted here that a summation vanishes if its lower limit exceeds its upper limit. Since \( G(x) \) is continuous and non-negative the integrals in (6.1) exist and \( f_n(s,S) \) is defined to be the same as \( f_n(S|s,S) \) would be in Section I. When \( s \) is finite \( f_n(s,S) \) is finite for all \( n \). For the policy \((\infty,S)\), however, some of the integrals may be infinite and then \( f_n(\infty,S) \) will be also.

Note that if an initial set-up cost is included \( f_n(s,S) \) will be increased by only a constant and so the minimization problem is unchanged. Using Greenberg's recursive relationships for \( F_t(s|s,S) \) and \( r_t(S-s) \) one obtains:

\[
F_t(x|s,S) = 1 - \phi(t)(S-x) - \sum_{i=1}^{t-1} r_i(S-s) \phi(t-i)(S-x)
\]

\[
(6.2) \quad - \int_x^S \phi(v-x) d\phi(t-1)(S-v) - \sum_{i=1}^{t-2} r_i(S-s) \int_x^S \phi(v-x) d\phi(t-i-1)(S-v),
\]

\[
r_t(S-s) = 1 - \phi(t)(S-s) - \sum_{i=1}^{t-1} r_i(S-s) \phi(t-i)(S-s),
\]

for \( t \geq 1 \) where \( \phi^{(t)}(\cdot) \) is the \( t \)-fold convolution of \( \phi(\cdot) \) with itself and \( \phi^{(t)}(S-s) = 1 \) when \( s = \infty \).
For convenience it will be assumed in what follows that \( \phi(\cdot) \) does possess a density \( \varphi(\cdot) \) and that \( \varphi^{(t)}(\cdot) \) denotes its \( t \)-fold convolution with itself. Except where noted, analogous results with slightly different proofs hold for an arbitrary \( \phi(\cdot) \), if to avoid a trivial complication we let \( \phi(0) < 1 \). With the present assumptions it has been shown by Karlin (Chapters 14 and 15 in [2]) that for finite \( s \), \( F_t(x|s, s) \) converges in the sense of distributions to a stationary distribution \( F(x|s, s) \) satisfying in Greenberg's notation

\[
F(x|s, s) = \frac{1}{1 + (1+M(S-s))^{-1}}M(S-x) - \int_{x}^{S} m(S-v)\phi(v-x)dv,
\]

where

\[
m(x) = \sum_{t=1}^{\infty} \varphi^{(t)}(x)
\]

and

\[
M(x) = \sum_{t=1}^{\infty} \varphi^{(t)}(x)
\]

are the renewal density and renewal functions respectively. Thus, as \( t \to \infty \)

\[
r_t(S-s) \to r(S-s) = F(s|s, s)
\]

\[
= (1+M(S-s))^{-1}.
\]
As \( n \to \infty \) with \( \alpha < 1 \), \( f_n(s,S) \) converges to

\[
f(s,S) \equiv g(S) + \sum_{t=1}^{\infty} \alpha^t \left[ (K+G(s))r_t(S-s) + \int_S^S g(x) dF_t(x|s,S) \right],
\]

which may be \( \infty \) if \( s = -\infty \). If \( \alpha = 1 \) then because of the convergence in distribution for finite \( s \), \( k_n(s,S) = \frac{1}{n} f_n(s,S) \) converges to

\[
k(s,S) \equiv (K+G(S))r(S-s) + \int_S^S g(x) dF(x|s,S).
\]

Furthermore, that \( k_n(-\infty,S) \) is unbounded as \( n \to \infty \) is seen as follows.

Let \( M \) be an arbitrarily large number and choose \( A \) so that \( \min_{y \leq A} G(y) \geq M \). Since demand is non-negative there exists an \( N = N(M) \) such that for all \( t > N \), \( \phi(t)(S-A) < M^{-1} \).

Hence, since \( G(y) > 0 \)

\[
\lim_{n \to \infty} k_n(-\infty,S) = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{\infty} \int_{-\infty}^{S} g(x) \phi(t)(S-x) dx
\]

\[
\geq \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{\infty} \int_{-\infty}^{A} g(x) \phi(t)(S-x) dx
\]

\[
\geq \lim_{n \to \infty} \frac{n-N}{n} (1 - \frac{1}{M}) M
\]

\[
= M - 1.
\]

Because \( M \) is arbitrary it is concluded that \( k_n(-\infty,S) \to k(-\infty,S) \equiv \infty \).

The following lemma establishes continuity of costs as functions of policies.

**Lemma 6.1** The functions \( f_n(s,S) \), \( f(s,S) \), and \( k(s,S) \) are continuous in \( (s,S) \). For fixed \( S \) as \( s \to -\infty \), \( f_n(s,S) \to f_n(-\infty,S) \) and \( f(s,S) \to f(-\infty,S) \). Furthermore, as \( S \to -\infty \) \( k(s,S) \) is uniformly unbounded for \( S \) in any finite interval.
Proof. For all $t$, $F_t(x|s,S)$ is seen to be absolutely continuous in $x$ and continuous in $(s,S)$ from the recursive expression (6.2). Thus, each term in (6.1) is continuous in $(s,S)$ and so $f_n(s,S)$ is. If $R$ is that part of an arbitrary bounded rectangle having $s \leq S$, then for all $n$

$$\sup_{(s,S) \in R} [f_{n+1}(s,S) - f_n(s,S)] = \sup_{(s,S) \in R} \alpha^n [(K+G(S))r_n(S-s) + \int_s^S G(x) dF_n(x|s,S)]$$

$$\leq \alpha^n (K + \sup_{a \leq x \leq b} G(x)) ,$$

where $a$ and $b$ are the smallest $s$ and largest $S$, respectively, in $R$. Since $G(x)$ is continuous its supremum over $[a,b]$ is finite. Thus, it is concluded that $f_n(s,S)$ converges uniformly to $f(s,S)$ in $R$, and so the latter must also be continuous. Finally, since $F(x|s,S)$ is absolutely continuous in $x$ and continuous in $(s,S)$, $x(s,S)$ is continuous.

Next, for fixed $S$ as $s \to -\infty$, $r_i(S-s) \to 0$ for $i \geq 1$. It will first be shown that $f(s,S) \to f(-\infty,S)$ as $s \to -\infty$. To this end suppose that $f(-\infty,S) < \infty$, recalling that

$$f(-\infty,S) = G(S) + \sum_{t=1}^{\infty} \alpha^t \int_{-\infty}^{S} G(x) dF_t(x| - \infty, S) .$$

For all $s \leq S$

$$f(s,S) = G(S) + \sum_{t=1}^{\infty} \alpha^t [(K+G(S))r_t(S-s) + \int_s^S G(x) dF_t(x|s,S)] .$$

By (6.2) for $x \geq s$
(6.3) \( F_t(x|s,S) = F_t(x|\infty, S) - \sum_{i=1}^{t-1} r_i(S-s)(1-F_{t-i}(x|\infty, S)) \),
so that
\[
f(s,S) = G(S) + \sum_{t=1}^{\infty} \alpha^t[(K+G(S))r_t(S-s) + \int_S G(x)dF_t(x|\infty, S)]
+ \sum_{i=1}^{t-1} r_i(S-s) \int_S G(x)dF_{t-i}(x|\infty, S) \]
\[
\leq f(\infty, S) + \sum_{i=1}^{\infty} \alpha^i r_i(S-s)[K+G(S) + \sum_{t=1+1}^{\infty} \alpha^t \int_S G(x)dF_{t-i}(x|\infty, S)]
\]
\[
\leq f(\infty, S) + (1-\alpha)^{-1}\alpha(K+f(-\infty, S)) .
\]
Thus, by the dominated convergence theorem
\[
\lim_{s \to -\infty} f(s,S) = f(-\infty, S) .
\]
If \( f(-\infty, S) = \infty \) notice first that by (6.3) since each \( F_{t-1}(x|\infty, S) \)
is monotone increasing
\[
\int_S G(x)dF_t(x|s,S) \geq \int_S G(x)dF_t(x|\infty, S) .
\]
Hence, for all \( s \leq S \)
\[
f(s,S) \geq G(S) + \sum_{t=1}^{\infty} \alpha^t \int_S G(x)dF_t(x|\infty, S) ,
\]
and by Fatou's lemma \( f(s,S) \to \infty \) as \( s \to -\infty \). The same proof applies to
\( f_n(s,S) \) for \( n \geq 1 \) if \( n - 1 \) replaces \( \infty \) as the upper limit of the
summations.
It remains to show that for all finite $A$ and $B$ and large $L$ there exists an $s_o$ such that for all $s \leq s_o$ $k(s,S) \geq L$ for all $S \in [A,B]$. First choose $C < A$ so that

$$\min_{y \leq C} G(y) \geq (L-1)^{-1}L^2$$

then choose $s_o \leq C$ so that

$$[(M(A-s_o))^{-1}+1]^{-1} \geq 1 - (2L)^{-1},$$

and

$$M(B-C)(1+M(A-s_o))^{-1} \leq (2L)^{-1}.$$  

For any $s \leq s_o$ since $G(x)$ is positive

$$k(s,S) = (1+M(S-s))^{-1}(K+G(S) + \int_s^S G(x)m(S-x)dx)$$

$$\geq (1+M(S-s))^{-1} \int_s^C G(x)m(S-x)dx$$

$$\geq (1+M(S-s))^{-1}(M(S-s) - M(S-C))(L-1)^{-1}L^2$$

$$\geq (((M(A-s_o))^{-1}+1)^{-1} - (1+M(A-s_o))^{-1}M(B-C))(L-1)^{-1}L^2$$

$$\geq (1-L^{-1})(L-1)^{-1}L^2$$

$$= L.$$  

Q.E.D.

It is useful to establish bounds within which there exist optimal values of $(s,S)$. Since $G(x)$ need not be quasi-convex the bounds in Corollary 3.3 may not hold. Hence, define $S$ and $s$ as before:
\[ S = \min \{ x \mid G(x) = \min_y G(y) \} , \]
\[ \bar{s} = \min \{ x \mid G(x) = K + G(\bar{s}) \} ; \]

but for this section only let \( \bar{s} \) satisfy the following:
\[ \bar{s} = \max \{ x \mid G(x) = K + G(\bar{s}) \} . \]

For the remainder of this section it is also useful to let \( \eta \in (0,1) \) be an otherwise arbitrary fixed constant. Recalling that \( K + G(S) > 0 \) define \( \underline{s} \) and \( \bar{s} \) as follows:
\[ \underline{s} = \min \{ x \mid G(x) = (K + G(S))/\eta \} \]
\[ \bar{s} = \max \{ x \mid G(x) = (K + G(S))/\eta \} . \]

Also, let \( (s_n, S_n) \) indicate an optimal value of \( (s, S) \) for the \( n \)-period problem, \( f_n \equiv f_n(s_n, S_n) \), and \( k_n = f_n/n \). Similarly \( s^* \) and \( S^* \) will be optimal for the infinite-horizon problem, \( f = f(s^*, S^*) \) and \( k = k(s^*, S^*) \). Attention will be restricted to demonstrating bounds for some optimal values of the parameters, although in many situations every optimal value will satisfy them because they are weak. Recall that in this section the parameters \( (s_n, S_n) \) are used in every one of the \( n \) periods, whereas in Section III a sequence of parameters
\[ (s_{n-1}, S_{n-1}), \ldots, (s_1, S_1) \] was used.

\underline{Lemma 6.2} For all \( n \) and every \( (s_n, S_n) \), \( s_n \leq \bar{s} \) and \( S_n \geq \underline{s} \). Also, \( s^* \leq \bar{s} \) and \( S^* \geq \underline{s} \) for every \( (s^*, S^*) \).
Proof If \( s > \bar{s} \) or \( S < \bar{S} \)

\[
k(s, S) = (K + G(S))r(S-s) + \int_{s}^{S} g(x) dF(x|s,S)
\]

\[
> K + G(S)
\]

\[
= k(S, S)
\]

\[
> k.
\]

The proof for a finite horizon or for \( \alpha < 1 \) is clear since each term in the summation in \( f_n(s, S) \) or \( f(s, S) \) is of the above form. Q.E.D.

For \( \alpha < 1 \) let \( \bar{S}(\alpha) \) be the largest \( S \) such that

\[
G(S) = f(S, S).
\]

Still denoting the mean of \( \xi \) by \( \mu \) let \( n_0 \) be an integer satisfying

\[
(\bar{s} - s)^{\mu^{-1}} + \eta < (1 - \eta)n_0
\]

and choose \( S_0 \) so that

\[
\phi^{(n_0)}(S_0 - \bar{s}) > \eta.
\]

By Blackwell's theorem \([10]\) there is an \( S_1 \) so that for all \( S \geq S_1 \)

\[
M(S - s) - M(S - \bar{s}) < (\bar{s} - s)^{\mu^{-1}} + \eta.
\]

Furthermore, for an appropriate \( S_2 \) and any \( S \geq S_2 \)

\[
(M(S - s) - M(S - \bar{s}))(1 + M(S - \bar{s}))^{-1} < 1 - \eta,
\]

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because the numerator is bounded. Now define $\bar{S}(1) = \max(S_0, S_1, S_2)$.

**Lemma 6.3** For all $\alpha$ and $n$ every $S_n \leq \bar{S}(\alpha)$ and every $S^* \leq \bar{S}(\alpha)$.

**Proof** When $\alpha < 1$ for any $S > \bar{S}(\alpha)$ and any $s \geq -\infty$

$$f(s, S) > f_n(s, S) > g(s)$$

$$> r(s, S)$$

$$> f_n(s, S) .$$

Hence, $S_n \leq \bar{S}(\alpha)$ and $S^* \leq \bar{S}(\alpha)$.

When $\alpha = 1$ it is convenient to express $f_n(s, S)$ in terms of the expected cost incurred in each of the inventory cycles begun with an order returning the level to $S$. Let $\{y_t\}$ be the random sequence of levels generated by the policy $(s, S)$ so that

$$f_n(s, S) = \sum_{t=1}^{n} P[y_t = S](K + g(S)) + \sum_{i=1}^{n-t} \int_{S}^{S} G(x) \varphi(i)(S-x)dx - K .$$

Suppose that $S > \bar{S}(1)$ and, by Lemma 6.2, $s \leq \bar{s}$. If $1 \leq n-t \leq n_0$ the term in braces in (6.4) is larger than

$$K + g(S) + \sum_{i=1}^{n-t} (K + g(S)) \eta^{-1} \varphi(i)(S-\bar{s})$$

$$> (n-t+1)(K + g(S)) ,$$

since $\varphi(i)(S-\bar{s}) \geq \varphi(n_0)(S-\bar{s})$ for $i \leq n_0$. On the other hand, if $n-t > n_0$ a lower bound for this term is
\[ K + G(\mathcal{S}) + (K + G(\mathcal{S})) \sum_{i=1}^{n-t} \phi^{(i)}(S-s) - \phi^{(i)}(S-\text{max}(s, \mathcal{S})) \]

\[ + (K + G(\mathcal{S}))\eta^{-1} \sum_{i=1}^{n-t} \phi^{(i)}(S-\bar{\mathcal{S}}) \]

\[ \geq (K + G(\mathcal{S}))[1+ \sum_{i=1}^{n-t} (\phi^{(i)}(S-s) - \phi^{(i)}(S-\text{max}(s, \mathcal{S})) + \phi^{(i)}(S-\bar{\mathcal{S}}))] \]

\[ + (K + G(\mathcal{S}))\eta^{-1}(1-\eta) \sum_{i=1}^{n_0} \phi^{(i)}(S-\bar{\mathcal{S}}). \]

The last term in this expression is larger than

\[ (K + G(\mathcal{S}))(1-\eta)n_0 > (K + G(\mathcal{S}))(M(S-s) - M(S-\bar{\mathcal{S}})) \]

\[ \geq (K + G(\mathcal{S})) \sum_{i=1}^{n-t} (\phi^{(i)}(S-s) - \phi^{(i)}(S-\bar{\mathcal{S}})) \]

\[ \geq (K + G(\mathcal{S})) \sum_{i=1}^{n-t} (\phi^{(i)}(S-\text{max}(s, \mathcal{S})) - \phi^{(i)}(S-\bar{\mathcal{S}})). \]

Hence, the term in braces in (6.4) is strictly greater than

\[ (K + G(\mathcal{S}))[1 + \sum_{i=1}^{n-t} \phi^{(i)}(S-s)], \]

which is the corresponding contribution to \( f_n(\mathcal{S}, \mathcal{S}) \) independent of \( Y_t \) so that \( S_n \leq \mathbb{S}(1). \)
To consider \( k(s, S) \) for \( S > \bar{S}(1) \) and \( s \leq \bar{s} \) note that

\[
F(\bar{s}|s, S) - F(\max(s, \bar{s})|s, S) = [M(S-\max(s, \bar{s})) - M(S-\bar{s})][1+M(S-s)]^{-1}
\]

\[
\leq [M(S-\bar{s}) - M(S-\bar{s})][1+M(S-s)]^{-1}
\]

\[
< 1-\eta.
\]

Thus,

\[
k(s, S) = (K+G(S))r(S-s) + \int_{S}^{\infty} G(x) dF(x|s, S)
\]

\[
> (K+G(S))\eta^{-1}[1-F(\bar{s}|s, S) + F(\max(s, \bar{s})|s, S)]
\]

\[
> K+G(S)
\]

\[
> k(\bar{s}, S).
\]

Hence, \( S^* \leq \bar{S}(1) \).

Q.E.D.

Whenever \( \alpha \) is considered fixed in what follows \( \bar{S}(\alpha) \) will be denoted simply as \( \bar{S} \).

Finding a lower limit for optimal values of \( s_n \) is a more difficult problem. One reason is that if \( \xi \) is a bounded random variable then when \( K \) is large enough it is optimal to take \( s_n = -\infty \) for small \( n \). Another is that even when \( \xi \) is unbounded \( f_n(-\infty, S) < f_n(\bar{s}, S) \) for appropriate \( n, K, \) and \( \phi(\cdot) \). The following lemma, while not giving a lower bound for all \( s_n \), shows that the parameter may be chosen finite.

**Lemma 6.4** For all \( n \geq 2 \) there exists an \( \bar{s}_n \) such that for all \( S \) either \( f_n(-\infty, S) > f_n \) or \( f_n(-\infty, S) = f_n(s, S) \) for all \( s \leq \bar{s}_n \). The latter occurs only if \( \phi^{(n-1)}(S-\bar{s}_n) = 1 \).
Proof: For each $n$ consider any $S$ such that $f_n(-\infty, S) = f_n$, so that $s \leq S \leq S$. 

For $n = 2$ note that 

$$f_2(-\infty, S) = G(S) + \alpha \int_{-\infty}^{S} G(x) \varphi(S-x) dx,$$

$$f_2(s, S) = G(S) + \alpha \int_{s}^{S} G(x) \varphi(S-x) dx + \int_{-\infty}^{S} (K+G(S)) \varphi(S-x) dx,$$

so that 

$$f_2(-\infty, S) - f_2(s, S) = \alpha \int_{-\infty}^{S} (G(x)-K-G(S)) \varphi(S-x) dx.$$

Since $G(x) \to \infty$ as $x \to -\infty$ there exists an $S_2$ such that for all $x \leq S_2$ 

$$G(x) > \max_{S \leq S \leq S_2} (K+G(S)) + \gamma;$$

and so when $s \leq S_2$ 

$$f_2(-\infty, S) \geq f_2(s, S).$$

Thus, equality must hold, and this can happen only if $\Phi(S-S_2) = 1$. 

To consider an arbitrary $n \geq 3$ let $C_t(s, S)$ be the expected cost incurred in period $t$ when $(s, S)$ is followed so that for all $s$

$$f_n(s, S) = \sum_{t=1}^{n} \alpha^{t-1} C_t(s, S).$$

Let period $T_t$ be the last period before period $t + 1$ in which an order is placed; if there has been no ordering let $T_t = 1$. Thus, for $t \geq 2$ and any $s < S$ so that $\Phi(S-s) > 0$. 

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\[ C_t(s, S) = \int_s^S G(x) \varphi(t-1)(S-x)dx + \sum_{j=2}^{t-1} \int_s^S G(x)(\varphi(t-j)(S-s))^{-1} \varphi(t-j)(S-x)dx \]

\[ d_P \left[ \sum_{i=1}^{j-1} \xi_i > S-y, T_t = j \right] + \int_s^S (K + G(S))d_P \left[ \sum_{i=1}^{t-1} \xi_i > S-y, T_t = t \right]. \]

In terms of the same \( T_t \)

\[ C_t(-\infty, S) = \int_{-\infty}^S G(x) \varphi(t-1)(S-x)dx \]

\[ = \int_s^S G(x) \varphi(t-1)(S-x)dx + \sum_{j=2}^{t-1} \int_{-\infty}^S G(x+y-S)(\varphi(t-j)(S-s))^{-1} \varphi(t-j)(S-x)dx \]

\[ d_P \left[ \sum_{i=1}^{j-1} \xi_i > S-y, T_t = j \right] + \int_s^S G(y)d_P \left[ \sum_{i=1}^{t-1} \xi_i > S-y, T_t = t \right]. \]

Hence,

\[ (6.5) \quad C_t(-\infty, S) - C_t(s, S) = \sum_{j=2}^{t-1} \int_{-\infty}^S \int_s^S [G(x+y-S) - G(x)](\varphi(t-j)(S-s))^{-1} \varphi(t-j)(S-x) \]

\[ dx dy \left[ \sum_{i=1}^{j-1} \xi_i > S-y, T_t = j \right] \]

\[ + \int_{-\infty}^s [G(y) - K - G(S)]d_P \left[ \sum_{i=1}^{t-1} \xi_i > S-y, T_t = t \right]. \]

For \( s \leq s_2 \) the last term is non-negative, so taking any \( s_0 \) such that \( s < s_0 < s_2 \) and suppressing the unchanging argument of the second probability measure consider an arbitrary term in the summation:

\[ \int_{-\infty}^s \left[ \int_s^{s_0} [G(x+y-S) - G(x)](\varphi(t-j)(S-s))^{-1} \varphi(t-j)(S-x)dx \right. \]

\[ + \int_{s_0}^S [G(x+y-S) - G(x)](\varphi(t-j)(S-s))^{-1} \varphi(t-j)(S-x)dx \] \[ \left. d_P \right]. \]
The first of the inner integrals is larger than
\[
\int_{s}^{s_0} [G(s) - G(x)](\phi(t-j)(s-x))^{-1} \varphi(t-j)(s-x) dx
\]
\[
\geq - \int_{-\infty}^{s_0} [G(x)-G(s)](\phi(t-j)(s-x))^{-1} \varphi(t-j)(s-x) dx ,
\]
which is finite since \( f_n(-\infty,S) = f_n \). Thus, there exists an \( s_0 = s_0(\eta,t-j) \) such that
\[
\int_{-\infty}^{s_0} [G(x)-G(s)](\phi(t-j)(s-x))^{-1} \varphi(t-j)(s-x) dx < \eta .
\]

Now, for all \( x \in [s_0,S] \)
\[
G(x+y-S) - G(x) \geq G(x+y-S) - \max_{s_0 \leq x \leq S} G(x)
\]
\[
\geq 2\eta
\]
for all \( y \leq \) some \( s(\eta,t-j) \leq s_0(\eta,t-j) \). Hence, for all \( s \leq s(\eta,t-j) \) each term in the summation is bounded below by
\[
\int_{-\infty}^{s} \eta dy^p \geq 0 .
\]

Thus, for \( s \leq \min \{s(\eta,t-j)\} \) when \( t \geq 3 \)
\[
C_t(-\infty,S) \geq C_t(s,S) .
\]

Also, since \( s_0 \leq s_2 \), \( C_2(-\infty,S) \geq C_2(s,S) \). If \( \bar{s}_n \) is the smallest of these minima for \( n \geq t \geq 3 \) then for \( s \leq \bar{s}_n \)

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\[ f_n = f_n(-\,\infty, S) \geq f_n(s, S) \geq f_n. \]

Hence, equality holds throughout these relations. However, by (6.5)

\[ C_n(-\,\infty, S) = C_n(s, S) \text{ only if } \phi^{n-1}(S-s) = 1. \]

Q.E.D.

Veinott and Wagner's renewal approach to the stationary analysis is useful for studying \( f(s, S) \). Their ideas will be employed with slight notational changes. Let \( M_\alpha(x) \) and \( m_\alpha(x) \) be the discount renewal function and its density:

\[ m_\alpha(x) = \sum_{t=1}^{\infty} \alpha t \phi(t)(x) \]

\[ M_\alpha(x) = \sum_{t=1}^{\infty} \alpha^t \phi(t)(x). \]

Let \( T_\ell \) now denote the random period in which the inventory level falls to or below \( s \) for the \( \ell \)th time when \( (s, S) \) is followed in the infinite-horizon problem. The dependence of \( T_\ell \) on the policy need not be expressed here since specific parameters \( s \) and \( S \) will be chosen. Since \( \phi(0) = 0 \) it follows that \( T_\ell \) is a proper random variable for all \( \ell \geq 1 \) if \( s > -\infty \). Also, let \( T_0 = 0 \). The total expected cost then may be written as the sum of the expected discounted costs during each cycle begun with an order and continuing until the next order is placed; i.e.,

\[ f(s, S) = \sum_{\ell=1}^{\infty} E[\alpha^{T_{\ell-1}}c_\ell] - k \]

where
\[ c_\ell = K + G(S) + \sum_{t=1}^{\infty} \alpha^t \int_0^{S-s} G(S-x) \phi(t)(x) \, dx \]

(6.6) \[ = K + G(S) + \int_0^{S-s} G(S-x) m_\ell(x) \, dx. \]

Now \( c_\ell \) is a constant independent of \( \ell \) and for \( \ell \geq 1 \)

\[ E[\alpha^\ell] = E[E[\alpha^\ell | T_{\ell-1}]] \]

\[ = E[\alpha^{\ell-1}E[\alpha^{\ell-T-1} | T_{\ell-1}]] \]

\[ = E[\alpha^{\ell-1}E[\alpha^T]] \]

\[ = E[\alpha]\alpha^{\ell-1} \].

Thus, by induction for \( \ell \geq 0 \)

\[ E[\alpha^\ell] = (E[\alpha^{1}])^\ell. \]

As Veinott and Wagner point out, letting \( \phi(0)(x) = 1 \) for \( x \geq 0 \)

\[ E[\alpha^{1}] = \sum_{t=1}^{\infty} \alpha^t [\phi(t-1)(S-s) - \phi(t)(S-s)] \]

\[ = \alpha - (1-\alpha)M_\alpha(S-s). \]

Therefore,

\[ f(s,S) = c_1 \sum_{\ell=1}^{\infty} (E[\alpha^{1}])^{\ell-1} - K \]

(6.7) \[ = c_1 [(1-\alpha)(1+M_\alpha(S-s))]^{-1} - K. \]
Lemma 6.5 For all \( \alpha \) \( s^* \) is finite. Furthermore, for \( \alpha = 1 \) there exists an \( s^* \) such that all \( s^* > s^* \).

Proof For \( \alpha = 1 \) any \( S^* \in [s, S] \) by Lemmas 6.2 and 6.3. The existence of \( s^* \) is then ensured by Lemma 6.1. Thus, take \( \alpha < 1 \) and consider any \( S \) such that \( f(-\infty, S) = f \). Choose \( s < s^* \) so that

\[
M_\alpha(s-s) > (1-\alpha)^{-1} \eta ,
\]

\[
\min G(y) > (1+\alpha(1-\alpha)^{-1} \eta)^{-1} f .
\]

\( y \leq s \)

In terms of the \( T_\ell \) for the policy \( (s, S) \), \( f(-\infty, S) \) may be expressed as

\[
f(-\infty, S) = G(S) + \sum_{t=1}^{\infty} \alpha^t \int_0^\infty G(S-x)\varphi(t)(x)dx
\]

(6.8)

\[
= G(S) + \int_0^\infty G(S-x)m_\alpha(x)dx
\]

\[
= \sum_{\ell=1}^{\infty} E[\alpha^{T_{\ell-1}}C^\ell]
\]

where

\[
C^\ell = E[G(S-\sum_{i=1}^{T_{\ell-1}} S_i)+ \sum_{t=1}^{\infty} \alpha^t \int_0^{S-s} G(S-\sum_{i=1}^{T_{\ell-1}} S_i-x)\varphi(t)(x)dx|T_{\ell-1}]
\]

\[
= E[G(S-\sum_{i=1}^{T_{\ell-1}} S_i)+ \int_0^{S-s} G(S-\sum_{i=1}^{T_{\ell-1}} S_i-x)m_\alpha(x)dx|T_{\ell-1}] .
\]

Now \( C^\ell \) is really a constant equal to \( C_1 - K \) so that

\[
f(-\infty, S) - f(s, S) = \sum_{\ell=2}^{\infty} E[\alpha^{T_{\ell-1}}(C^\ell-C_1)] .
\]

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For $k \geq 2$ with probability one,

$$C'_k \geq \min_{y \leq s} G(y)[1 + M_\alpha (S-s)]$$

$$\geq \min_{y \leq s} G(y)[1 + M_\alpha (s-s)]$$

$$> f$$

$$> C_1$$

by (6.6) and (6.8). Since all $T_k$ are finite it follows that

$$f(\infty, S) - f(s, S) > 0,$$

which is a contradiction. Hence, $f(\infty, S) > f$ and $s^*$ is finite. A similar proof would hold for Lemma 2.4 although the finiteness of the horizon necessitates truncated versions of the functions and random variables used here.

Q.E.D.

Unfortunately it is not evident that any sequence \( \{s_n\} \) is bounded below. By Lemma 6.4 when demand is bounded, in fact, it may be possible to take $s_n = -\infty$ for some $n$. Also, finiteness of $s^*$ need not imply finiteness of $\inf \{s^*\}$ if the number of different values of $S^*$ is infinite.

**Lemma 6.6** If there exists an $s_0$ for which

$$\min_{y \leq s_0} G(y) = G(s_0) = \max_{s_0 \leq y \leq \bar{S}} G(y) > k + \max_{s \leq y \leq \bar{S}} G(y)$$

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then there exists an $\omega^*$ such that $\omega^* \geq \omega^*$. If also there exists an $\omega_n$ such that for some $1 \leq t \leq n-1$, $\phi(t)(s-s_n) > \phi(t)(S-s_0)$ then $\omega_n \geq \omega_n$.

Proof It will first be shown that $\sum_{t=1}^{n-1} \alpha^t r_t(S-s) \leq \sum_{t=1}^{n-1} \alpha^t r_t(S-s')$ for all $n \geq 2$ if $s \leq s' \leq S$. For an arbitrary $n \geq 2$ let $N$ be the number of reorders when $(s,s)$ is followed in the $n$-period problem.

Let $T_0, T_1, \ldots, T_N$ be the periods prior to the ordering, where $T_0 = 0$. Define $N'$ and $T'_0, T'_1, \ldots, T'_N$, similarly for $(s',s')$. It is evident that with probability one $N \leq N'$ and $T'_\ell \leq T_\ell$ for $0 \leq \ell \leq N$.

Therefore,

$$\sum_{t=1}^{n-1} \alpha^t r_t(S-s) = \sum_{t=1}^{n-1} \alpha^t \mathbb{P}[$some $T_\ell = t$]

= \sum_{t=1}^{n-1} \alpha^t \mathbb{E}\left[ \sum_{\ell=1}^{N} \mathbb{P}[T_\ell = t | N,N'] \right]

= \mathbb{E}\left[ \sum_{\ell=1}^{N} \sum_{t=1}^{n-1} \alpha^t \mathbb{P}[T_\ell = t | N,N'] \right]

= \mathbb{E}\left[ \sum_{\ell=1}^{N'} \mathbb{E}[\alpha^t | N,N'] \right]

= \sum_{\ell=1}^{n-1} \alpha^t r_t(S-s')(S-s')$.
Next, by (6.2) for $n \geq 2$

$$\sum_{t=1}^{n-1} \alpha^t f_t(x|s,S) = \sum_{t=1}^{n-1} \alpha^t - \sum_{t=1}^{n-1} \alpha^t \phi(t)(S-x) + \sum_{i=1}^{t-1} r_i(S-s) \phi(t-i)(S-x)$$

$$= \frac{\alpha^a}{1-\alpha} - \sum_{t=1}^{n-1} \alpha^t \phi(t)(S-x) - \sum_{i=1}^{n-2} \alpha^i r_i(S-s) \sum_{t=1}^{n-1} \alpha^{t-i} \phi(t-i)(S-x)$$

$$= \frac{\alpha^a}{1-\alpha} - \sum_{t=1}^{n-1} \alpha^t \phi(t)(S-x) - \sum_{i=1}^{n-2} \alpha^i r_i(S-s) \sum_{t=1}^{n-1} \alpha^t \phi(t)(S-x)$$

$$= \frac{\alpha^a}{1-\alpha} - \sum_{t=1}^{n-1} \alpha^t \phi(t)(S-x) \left(1 + \sum_{i=1}^{n-1} \alpha^i r_i(S-s)\right).$$

Now, to prove the lemma note that if an $s_n$ exists as hypothesized, then an $s_m$ exists for all $m > n$ and the sequence $\{s_n\}$ may be chosen to be nondecreasing. Let $S \in [s, \bar{S}]$ and $s \leq s_n$. By (6.1)

$$f_n(s,S) - f_n(s_0,S) = \sum_{t=1}^{n-1} \alpha^t [(K+G(S))(r_t(S-s)-r_t(S-s_0))$$

$$+ \int_s^S G(x) d(F_t(x|s,S)-F_t(x|s_0,S)) + \int_{s_0}^S G(x) dF_t(x|s,S)|$$

$$= (K+G(S)) \sum_{t=1}^{n-1} \alpha^t (r_t(S-s)-r_t(S-s_0)) + \int_{s_0}^S G(x) d \left[ \sum_{t=1}^{n-1} \alpha^t (F_t(x|s,S)$$

$$- F_t(x|s_0,S)) \right] + \sum_{t=1}^{n-1} \alpha^t \int_s^S G(x) dF_t(x|s,S)$$

$$= (K+G(S)) \left( \sum_{t=1}^{n-1} \alpha^t r_t(S-s) - \sum_{t=1}^{n-1} \alpha^t r_t(S-s_0) \right)$$

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\[ + \sum_{t=1}^{n-1} \alpha^t \int_{s_0}^{S} g(x) \phi(t)(S-x) \left( \sum_{i=1}^{n-1-t} \alpha^i r_1(s-s) - \sum_{i=1}^{n-1-t} \alpha^i r_1(s-s_0) \right) dx \\
+ \sum_{t=1}^{n-1} \alpha^t \int_{s}^{s_0} g(x) dF_t(x|s,S) \]

\[ (6.9) \geq G(s_0) \left[ \sum_{t=1}^{n-1} \alpha^t (r_t(S-s) - r_t(S-s_0)) \int_{s_0}^{S} \sum_{t=1}^{n-1} \alpha^t \phi(t)(S-x) \sum_{i=1}^{n-1-t} \alpha^i r_1(s-s) \\
- r_1(s-s_0) dx + \sum_{t=1}^{n-1} \alpha^t \int_{s}^{s_0} dF_t(x|s,S) \right] \]

\[ = G(s_0) \sum_{t=1}^{n-1} \alpha^t [r_t(S-s) - r_t(S-s_0) + F_t(S|s,S) - F_t(S|s_0,S) \\
- F_t(s_0|s,S) + F_t(s_0|s_0,S) + F_t(s_0|s,S) - F_t(s|s,S)] \]

\[ = 0. \]

Finally, the condition on \( s_n \) ensures that \( \phi(t)(S-s) > \phi(t)(S-s_0) \) for some \( 1 \leq t \leq n - 1 \). By inspection of the proof of the first inequality here \( \sum_{t=1}^{n-1} r_t(S-s_0) > \sum_{t=1}^{n-1} r_t(S-s) \), so that the inequality in (6.9) is strict. Thus, \( f_n(s,S) > f_n(s_0,S) \) and \( s_n > s_n \).

For the infinite-horizon problem the above argument applies with an appeal to the dominated convergence theorem. A more direct proof, however, employs the discount renewal function. First, there exists an \( s^* \) such that \( M_{\alpha}(S-s^*) > M_{\alpha}(S-s_0) \) since \( \phi(0) = 0 \). Let \( S \in [s,3] \) and \( s \leq s^* \), so that \( M_{\alpha}(S-s) > M_{\alpha}(S-s_0) \). By (6.7) then
\[(1-\alpha)(r(s,s)-r(s_0,s)) = (K+G(s) + \int_0^{S-s} G(s-x)\mu_\alpha(x)dx)(1+M_\alpha(S-s))^{-1} \]

\[- (K+G(s) + \int_0^{S-s_0} G(s-x)\mu_\alpha(x)dx)(1+M_\alpha(S-s_0))^{-1} \]

\[= (K+G(s) + \int_0^{S-s_0} G(s-x)\mu_\alpha(x)dx)(1+M_\alpha(S-s))^{-1} \]

\[- (1+M_\alpha(S-s_0))^{-1} + (1+M_\alpha(S-s))^{-1} \int_{S-s_0}^{S-s} G(s-x)\mu_\alpha(x)dx \]

\[> G(s_0)(1+M_\alpha(S-s_0))[(1+M_\alpha(S-s))^{-1} - (1+M_\alpha(S-s_0))^{-1}] \]

\[+ G(s_0)(1+M_\alpha(S-s))^{-1}(M_\alpha(S-s) - M_\alpha(S-s_0)) \]

\[= G(s_0)(1+M_\alpha(S-s))^{-1}[1+M_\alpha(S-s_0) - 1 - M_\alpha(S-s)] \]

\[+ M_\alpha(S-s) - M_\alpha(S-s_0) \]

\[= 0. \]

Thus, \(f(s,s) > f(s_0,s) \geq f_n\) so that \(s^* > s^*_n\).

Q.E.D.

If an \(s_0\) satisfying the lemma exists then for large enough \(n\) an \(s_n\) will also exist since \(\phi(0) = 0\). Hence, in this case every sequence \(\{s_n\}\) will eventually be bounded below. Also, the inequality on \(G(s_0)\) can be weakened to require only that

\[G(s_0) > K + \max(G(y) | y = s_1, s_2, \ldots, s^*), \]

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but this would be of limited use since the $s_n$ and $s^*$ are unknown at
the beginning of the problem. Finally, it is important to note that an
$s_0$ can be found if $G(x)$ is quasi-convex.

For $\alpha = 1$ it has been indicated only that $k_n(s, S) \rightarrow k(s, S)$ point-
wise. The previously omitted proof is as follows: each summand converges
to the indicated limit by the convergence in distribution and the Helly-
Bray Lemma; then, this convergence implies convergence in Cesàro mean, i.e.,
k_n(s, S) \rightarrow k(s, S). If the convergence of summands were uniform in $(s, S)$
then the convergence of $k_n(s, S)$ would be also. The standard use of the
Helly-Bray Lemma in obtaining uniform convergence of expectations, however,
requires a fixed sequence of distributions and an equicontinuous family of
functions. In the present situation the sequence is parameterized
(by $(s, S)$ ), and the family of functions contains either $G(x)$ alone or
versions of $G(x)$ which vanish outside the interval $(s, S)$ and therefore
are not even continuous. Feller's proof of his version of the Helly-Bray
Lemma ([5], page 244), however, may be modified to show uniform conver-
gence of the integrals in (6.1) if the convergence in distribution is
uniform in the parameters over bounded rectangles. By the latter phrase
it will be meant specifically that $F_t(x|s, S) \rightarrow F(x|s, S)$ uniformly for
every $(s, S)$ such that $s \leq S$ in any bounded rectangle.

**Lemma 6.7** If $F_t(x|s, S) \rightarrow F(x|s, S)$ uniformly in $(s, S)$ over any bounded
rectangle and in $x \in [s, S]$ then $k_n(s, S) \rightarrow k(s, S)$ uniformly over any
bounded rectangle.

**Proof** Consider an arbitrary bounded rectangle and its subset $R$ in which
$s \leq S$. By the hypothesis and continuity of $G(y)$ it is seen that

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\((K + G(S)) r_t(S-s) \to (K + G(S)) r(S-s)\) uniformly in \(R\). For any function \(A(x)\) let \(A(x|s,S) = A(x)\) for \(s \leq x \leq S\) and vanish elsewhere. Since \(G(x)\) is uniformly continuous on the interval bounded by the smallest \(s\) and the largest \(S\) in \(R\), this interval may be partitioned into a finite number of subintervals \([-a_0, a_1], [a_1, a_2], \ldots, [a_{r-1}, a_r]\) — within each of which \(G(x)\) oscillates by less than \(\varepsilon\). Let \(H_\varepsilon(x)\) be defined on each subinterval \([a_{k-1}, a_k]\) as the smallest value \(m_k\) of \(G(x)\) on that subinterval.

Let

\[
E_t[A(x)|s,S] = \int_{-\infty}^{\infty} A(x|s,S) dF_t(x|s,S)
\]

\[
= \int_s^S A(x) dF_t(x|s,S),
\]

if it has meaning, and let \(E[A(x)|s,S]\) be similarly defined. In the standard approach then

\[
|\int_s^S G(x) dF_t(x|s,S) - \int_s^S G(x) dF(x|s,S)| \leq |E_t[G(x) - H_\varepsilon(x)|s,S]|
\]

\[
+ |E_t[H_\varepsilon(x)|s,S] - E[H_\varepsilon(x)|s,S]| + |E[G(x) - H_\varepsilon(x)|s,S]|.
\]

The first and third expressions with absolute value signs on the right are each less than \(\varepsilon\) independent of \((s, S)\). The second expression is bounded above by

\[
\sum m_k[|F_t(a_k|s,S) - F(a_k|s,S)| + |F_t(a_{k-1}|s,S) - F(a_{k-1}|s,S)|]
\]

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where the summation is over those \( k \) such that \( s \leq a_k \leq S \). Since this is less than \( \epsilon \) for large enough \( t \) independent of \((s,S)\) in \( R \) it is concluded that

\[
(K + G(S))r_t(S - s) + \int_s^S G(x) dF_t(x|s,S) \to (K + G(S))r(S - s) + \int_s^S G(x) dF(x|s,S)
\]

uniformly. Thus, \( k_n(s,S) \to k(s,S) \) uniformly in \( R \).

Q.E.D.

**Theorem 6.3** If some \( \{s_n\} \) is bounded below then \( f_n \to f \). If also \( k_n(s,S) \to k(s,S) \) uniformly over any bounded rectangle then \( k_n \to k \).

**Proof** Let \( R \) be the subset of feasible \((s,S)\) in a rectangle large enough to include \( \overline{s}, \overline{s}, \overline{s}, \) and the \( \{s_n\} \) in the hypothesis. Then there exists an \( N(\eta) \) such that for all \( n \geq N(\eta) \) and all \( S_n \) associated with \( s_n \)

\[
f(s_n, S_n) \leq f_n + \eta.
\]

Since \( \eta \) is arbitrary it is concluded that

\[
\lim \inf f(s_n, S_n) \leq \lim \inf f_n.
\]

On the other hand,

\[
\lim \sup f_n \leq \lim \sup f_n(s^*, S^*) = f
\]

\[
\leq f(s_n, S_n)
\]

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for all \( n \). Therefore, \( \lim \sup f_n \leq \lim \inf f_n \), so that equality holds throughout and \( f_n \to f \). If \( k_n(s,S) \) converges uniformly over \( R \) the same proof applies.

Q.E.D.

The condition on \( \{s_n\} \) may be weakened to requiring only that some \( \{s_n\} \) have finite limit points. Since all \( \{S_n\} \) are bounded there are then limit points of \( \{(s_n,S_n)\} \) for all \( \{S_n\} \) and the same proof applies. Furthermore, by continuity of \( f(s,S) \) and \( k(s,S) \) these limit points are optimal for the infinite-horizon problem. These results are analogous to those of Iglehart alluded to in Section IV. When \( G(x) \) is quasi-convex and \( \alpha < 1 \), limit points of the optimal non-stationary parameters for a finite-horizon problem are optimal for the infinite-horizon problem.

Since stationary policies are optimal for the latter, any policy which is optimal for the infinite-horizon problem of this section where \( x_1 = S^* \) will be optimal for the more general one, i.e., \( f = f(S^*) \). Furthermore, although the earlier optimal cost \( f(x_1) \) depends on \( x_1 \) at least one optimal policy will not; and, in fact, any \( (s^*,S^*) \) in this section will be optimal for all \( x_1 \). Finally, \( f_n(s_n,S_n) \geq f_n(S_n) \), but these costs will be close for large \( n \) since they converge to \( f = f(S^*) \). It is in this sense that the optimal policies \( (s_n,S_n) \) of this section approximate the optimal policies \( (s,S) \) of Section III, where \( s \) and \( S \) refer to vectors of infinite dimension.

When \( G(x) \) is quasi-convex and \( \alpha = 1 \) the general optimal costs may converge, but convergence of policies has not been established. Here, uniform convergence of \( k_n(s,S) \) is assumed to prove convergence of optimal costs, but then the optimality of limiting policies is also obtained.
The same assumption would yield convergence of policies in the general problem since Corollary 3.3 provides bounds on the parameters. In a discrete problem eventual boundedness of \((s_n, S_n)\) permits consideration of only a finite number of policies \((s, S)\) for large \(n\). Thus, there exists an \(N(\eta)\) such that for all \((s_n, S_n)\), \(k(s_n, S_n) \leq k + \eta\) without the aid of uniform convergence of \(k_n(s, S)\). The remainder of the proof of the theorem applies and \(k_n \to k\).

In the two-product problem the inventory levels, demand, and the policy parameters become two-tuples. Allow the set-up cost to have its general form as at the end of Section II, but consider stationary \((\sigma, S(x))\) policies having \(S_1^1(x) = (S_1, x_2), S_2^1(x) = (x_1, S_2)\), and \(S_1^{12}(x) = S\). For \(i \neq j\) let \(\sigma^i(x_j) = (x_j | x \in \sigma^i)\). For \(t \geq 0\) and \(\beta = 1, 2, \text{or } 12\) let

\[
\mathbb{P}_t(x | \sigma, S(x)) = \mathbb{P}[x_{t+1} \leq x \text{ when } (\sigma, S(x)) \text{ is followed}],
\]

\[
x^\beta_t(\sigma, S(x)) = \int \int d\mathbb{F}_t(y | \sigma, S(x)),
\]

\[
x^1_t(x_2 | \sigma, S(x)) = \int \int_{y_2 \leq x_2} d\mathbb{F}_t(y | \sigma, S(x)),
\]

\[
x^2_t(x_1 | \sigma, S(x)) = \int \int_{y_1 \leq x_1} d\mathbb{F}_t(y | \sigma, S(x)),
\]

where the initial level is \(S\). The expected cost of the \(n\)-period problem for \(n \geq 1\) is
\[ f_n(\sigma, S(x)) = G(\sigma) + \sum_{t=1}^{n-1} \alpha^t \left[ (K_{12} + G(\sigma)) r_{12}^t(\sigma, S(x)) \right. \]
\[ + \int_{-\infty}^{S_2} (K_{1} + G(s_1, x_2)) dr_{t}^1(x_2 | \sigma, S(x)) \]
\[ + \int_{-\infty}^{S_1} (K_{2} + G(x_1, s_2)) dr_{t}^2(x_1 | \sigma, S(x)) \]
\[ + \int \int_{\sigma} G(x) dF_t(x | \sigma, S(x)) \right] . \]

Since
\[ F_0(x | \sigma, S(x)) = \begin{cases} 1 & \text{if } x \geq S \\ 0 & \text{if } x < S \end{cases} , \]

it follows that for \( t \geq 1 \)
\[ F_t(x | \sigma, S(x)) = F_0(s_1, x_2 | \sigma, S(x)) + F_t(x_1, s_2 | \sigma, S(x)) - 1 \]
\[ + \int \int_{\sigma} \phi(v-x) dF_{t-1}(v | \sigma, S(x)) + \int_{x_2}^{S_2} \phi(s_1 - x_1, v_2 - x_2) dr_{t-1}^1(v_2 | \sigma, S(x)) \]
\[ + \int_{x_1}^{S_1} \phi(v_1 - x_1, s_2 - x_2) dr_{t-1}^2(v_1 | \sigma, S(x)) + \phi(s-x) r_{12}^{t-1}(\sigma, S(x)) , \]

where
\[ F_t(s_1, x_2 | \sigma, S(x)) = 1 - \int \int_{\sigma} \phi(\omega, v_2 - x_2) dF_{t-1}(v | \sigma, S(x)) \]
\[ - \int_{x_2}^{S_2} \phi(\omega, v_2 - x_2) dr_{t-1}^1(v_2 | \sigma, S(x)) - (r_{12}^{t-1}(\sigma, S(x)) + r_{t-1}^{12}(\sigma, S(x))) \phi(\omega, s_2 - x_2) \]
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and \( F_t(x_1, x_2 | s, S(x)) \) is similar. To obtain a significant simplification of these recursions it is necessary to make further assumptions about \( \sigma \).

As only two particular types of policies are of interest here the expressions for the distribution of the inventory level will be simplified only for these. For convenience it is assumed that \( \Phi(\cdot, \cdot) \) is absolutely continuous.

One policy similar to the single-product \((s, S)\) policy is the \((\sigma, S)\) policy with \( \sigma^1 = \{(x_1, s_2) | x_1 \leq s_1\}, \sigma^2 = \{(s_1, x_2) | x_2 \leq s_2\}, \) and \( s^{12} = \{x | x \leq S, x_1 \neq s_1, x_2 \neq s_2\}. \) Notice, however, that this is not an \((s, S)\) policy. Here \( S(x) = S, r^1_t(\sigma, S(x)) = r^2_t(\sigma, S(x)) = 0, \) and \( r^{12}_t(\sigma, S(x)), \) whose superscript will be dropped along with the subscript of \( K^{12}_t \), depends only on \( S - s \). From the corresponding results for the policy studied below it is seen that for \( t \geq 1 \)

\[
F_t(x | \sigma, S) = F_t(s_1, x_2 | \sigma, S) + F_t(x_1, s_2 | \sigma, S) - 1
+ \phi(t)(S - x) - \phi(t-1)(S - x) + \phi(t-1)(s_1 - \max(s_1, x_1), s_2 - \max(s_2, x_2))
+ \sum_{i=1}^{t-1} r_i(S - s)[\phi(t-i)(S - x) - \phi(t-i-1)(S - x) + \phi(t-i-1)(s_1 - \max(s_1, x_1), s_2 - \max(s_2, x_2))],
\]

\[
F_t(s_1, x_2 | \sigma, S) = 1 - \phi(t)(\omega, s_2 - x_2) + \phi(t-1)(\omega, s_2 - x_2) - \phi(t-1)(s_1 - s_1, s_2 - \max(s_2, x_2))
- \sum_{i=1}^{t-1} r_i(S - s)[\phi(t-i)(\omega, s_2 - x_2) - \phi(t-i-1)(\omega, s_2 - x_2) + \phi(t-i-1)(s_1 - s_1, s_2 - \max(s_2, x_2))],
\]

\[
r_t(S - s) = 1 - \phi(t)(S - s) - \sum_{i=1}^{t-1} r_i(S - s)\phi(t-i)(S - s),
\]

where \( \phi(0)(y) = 1 \) for \( y \geq 0 \) and is zero otherwise.
By use of Karlin's argument for the convergence in distribution in the single-product problem it can be shown that $F_t(x|\sigma,S)$ converges to a stationary distribution satisfying

$$F(x|\sigma,S) = F(S_1, x_2|\sigma,S) + F(x_1, S_2|\sigma,S) - 1$$

$$+ (1 + M(S-s))^{-1} [M(S-x) - \int_{s_1}^{S_1} \int_{s_2}^{S_2} m(S-v)\phi(v-x)dv - \int_{s_1}^{S_1} \int_{s_2}^{S_2} m(S-v)\phi(v-x)dv] ,$$

$$F(S_1, x_2|\sigma,S) = 1 - (1 + M(S-s))^{-1} [M(\sigma, S_2 - x_2) - \int_{-\infty}^{S_1} \int_{s_2}^{S_2} m(S-v)\phi(\sigma, v_2 - x_2)dv$$

$$- \int_{-\infty}^{s_1} \int_{s_2}^{S_2} m(S-v)\phi(\sigma, v_2 - x_2)dv] ,$$

with $F(x_1, S_2|\sigma,S)$ similarly expressed. Here $m(x)$ and $M(x)$ are bivariate functions, and the necessary bivariate versions of the elementary renewal theorem and Blackwell's theorem are given in [3] when the norm $||x|| = \max(|x_1|, |x_2|)$ is used. All of the remaining single-product analysis extends with straightforward modifications except for Lemma 6.6.

To use a similar proof it is sufficient to require that there exist an $s_0$ for which if $x_1 \in [s_{01}^1, s_{11}]$ and $x_2 \in [s_{02}, s_{22}]$

$$\min_{y < (s_{01}, s_{02}) or (s_{11}, \infty)} G(y) = G(y) = \max_{s_{02} \leq y < \infty} G(y) ,$$

$$\min_{y_1 < s_{01}, y_2 < s_{22}} G(y_1, y_2) = G(y_1, x_2) = \max_{(s_{01}, x_2) \leq y < \infty} G(y) ,$$

$$\min_{x_1 < y_1 \leq s_{01}, y_2 < s_{02}} G(x_1, s_{02}) = G(x_1, s_{02}) = \max_{(x_1, s_{02}) \leq y < \infty} G(y) ,$$

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\[
\min(G(s_{01}, \bar{s}_2), G(\bar{s}_1, s_{02})) > \kappa + \max_{y \leq \bar{s}} G(y).
\]

This condition thus requires that \(G(s_0) = G(s_{01}, x_2) = G(x_1, s_{02})\) for all \(x_1 \in [s_{01}, \bar{s}_1]\) and \(x_2 \in [s_{02}, \bar{s}_2]\). This means that part of a constant contour has a corner, which is impossible if \(G(y)\) is strictly quasi-convex and smooth. Furthermore, the weakness of the bound \(\bar{s}\) lessens the possibility that the condition holds. Fortunately, similar but more complex sufficient conditions exist so that Lemma 6.6 may be extended. It appears that in discrete problems this difficulty is apt to be less consequential than it is here.

Partly because of Theorem 3.3 the stationary \((s, \bar{s})\) policies in this section for the finite-horizon single-product problem are nearly optimal. The corresponding link for two products is Theorem 3.2, which is unlikely to yield a \((\sigma, \bar{s})\) policy of the form studied here. Again, the possibility improves in a discrete problem. In any event, the appeal of these stationary \((\sigma, \bar{s})\) policies is diminished because of this difficulty, but it is also increased by the sheer complexity of truly optimal policies.

The policies just studied are especially useful if \(K_1 = K_2 = 0\). When each of these costs is positive, however, it is more natural to consider stationary \((s, \bar{s})\) policies. For these \(\sigma_1 = \{x | x_1 \leq s_1, s_2 < x_2 \leq s_2\}\), \(\sigma_2 = \{x | s_1 < x_1 \leq s_1, x_2 \leq s_2\}\), and \(\sigma_{12} = \{x | x \leq s\}\). For \(t \geq 1\)

\[
F_t(x|s, \bar{s}) = F_t(S_1, x_2|s, \bar{s}) + F_t(x_1, S_2) - 1 + \int_{s_1}^{S_1} \int_{s_2}^{S_2} \phi(v-x) dF_{t-1}(v|s, \bar{s})
\]
\[
+ \int_{s_2}^{s_1} \phi(s_1-x_1, v_2-x_2) dr_{t-1}^1(v_2 | s, s) \\
+ \int_{s_1}^{s_2} \phi(v_1-x_1, s_2-x_2) dr_{t-1}^2(v_1 | s, s) + \phi(s-x) r_{t-1}^{12}(s-s) ,
\]

\[
F_t(s_1, x_2 | s, S) = 1 - \int_{s_1}^{s_2} \int_{s_2}^{s_1} \phi(v_2-x_2) df_{t-1}(v_1 | s, s) \\
- \int_{s_2}^{s_1} \phi(v_2-x_2) dr_{t-1}^1(v_2 | s, s) - (r_{t-1}^2(s-s) + r_{t-1}^{12}(s-s)) \phi(s_2, x_2) .
\]

By induction an expression for \( F_t(s_1, x_2 | s, S) + F_t(x_1, s_2 | s, S) - F_t(x | s, S) \) similar to (6.2) can be obtained for \( s \leq x \leq S \), which when substituted into the above will give the following for all \( x \) and \( t \geq 1 \):

\[
F_t(x | s, S) = F_t(s_1, x_2 | s, S) + F_t(x_1, s_2 | s, S) - 1 \\
+ \phi(t)(s-x) - \phi(t-1)(s-x) + \phi(t-1)(s_1 - \max(s_1, x_1), s_2 - \max(s_2, x_2)) \\
+ \sum_{i=1}^{t-1} r_{t}^{12}(s-s)[\phi(t-1)(s-x) - \phi(t-1)(s-x)] \\
+ \phi(t-1)(s_1 - \max(s_1, x_1), s_2 - \max(s_2, x_2)) \\
+ \sum_{i=1}^{t-1} \int_{s_2}^{s_2} [\phi(t-1)(s_1-x_1, v_2-x_2) - \phi(t-1)(s_1-x_1, v_2-x_2)] dr_{t}^1(v_2 | s, S) \\
+ \phi(t-1)(s_1 - \max(s_1, x_1), v_2 - \max(s_2, x_2)) dr_{t}^1(v_2 | s, S) 
\]
\[
+ \sum_{i=1}^{t-1} \int_{s_1} S_1 [\phi(t-1)(v_{1-x_1}, s_2-x_2) - \phi(t-1)(v_{1-x_1}, s_2-x_2)] \mathrm{d}r_1^{1}(v_1|s,s),
\]

\[
+ \phi(t-1)(v_{1-\max(s_1,x_1)}, s_2-\max(s_2,x_2)) \mathrm{d}r_1^{2}(v_1|s,s),
\]

\[
F_t(S_1, x_2 | s, s) = 1 - \phi(t)(\infty, s_2-x_2) + \phi(t-1)(\infty, s_2-x_2) - \phi(t-1)(s_1-s_1, s_2-\max(s_2,x_2))
\]

\[
- \sum_{i=1}^{t-1} r_{i1}^{12}(s-s) [\phi(t-1)(\alpha, s_2-x_2) - \phi(t-1)(\alpha, s_2-x_2)] \mathrm{d}r_1^{1}(v_1|s,s)
\]

\[
+ \phi(t-1)(s_1-s_1, s_2-\max(s_2,x_2)) \mathrm{d}r_1^{2}(v_1|s,s),
\]

\[
- \sum_{i=1}^{t-1} \int_{s_2} S_2 [\phi(t-1)(\alpha, v_2-x_2) - \phi(t-1)(\alpha, v_2-x_2)] \mathrm{d}r_1^{1}(v_2|s,s)
\]

\[
+ \phi(t-1)(s_1-s_1, v_2-\max(s_2,x_2)) \mathrm{d}r_1^{2}(v_2|s,s),
\]

\[
- \sum_{i=1}^{t-1} \int_{s_1} S_1 [\phi(t-1)(v_{1-x_1}, \infty) - \phi(t-1)(v_{1-x_1}, \infty)] \mathrm{d}r_1^{2}(v_1|s,s),
\]

\[
+ \phi(t-1)(s_{1-x_1}, s_2-s_2) \mathrm{d}r_1^{2}(v_1|s,s),
\]

\[
r_{t}^{12}(s-s) = 1 - r_{t}^{1}(s-s) - r_{t}^{2}(s-s) - \phi(t)(s-s)
\]

\[
- \sum_{i=1}^{t-1} [r_{i}^{12}(s-s) \phi(t-1)(s-s) + \int_{s_2} S_2 [\phi(t-1)(s_1-s_1, v_2-s_2)] \mathrm{d}r_1^{1}(v_2|s,s)
\]

\[
+ \int_{s_1} S_1 [\phi(t-1)(v_{1-s_1}, s_2-s_2)] \mathrm{d}r_1^{2}(v_1|s,s),
\]

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\[
\begin{align*}
\text{\underline{r}_t^1(x_2 | s, S)} &= F_t(s_1, x_2 | s, S) - \text{\underline{r}_t^{12}(S-s)} , \\
\text{\underline{r}_t^2(x_1 | s, S)} &= F_t(x_1, s_2 | s, S) - \text{\underline{r}_t^{12}(S-s)} ,
\end{align*}
\]

where \( \text{\underline{r}_t^1(S-s)} = r_t^1(S_2 | s, S) \) and \( \text{\underline{r}_t^2(S-s)} = r_t^2(S_1 | s, S) \).

It is not apparent that \( F_t(x | s, S) \) converges to a stationary distribution since the earlier proofs for \( F_t(x | s, S) \) in the single-product case and for \( F_t(x | \sigma, S) \) do not apply. However, it is conjectured that \( F_t(x | s, S) \) does converge here since \( \phi(\cdot, \cdot) \) is absolutely continuous. When \( \phi(\cdot, \cdot) \) is discrete the resulting Markov chain may be periodic, in which case the inventory level will have merely an average long-run distribution. If indeed \( F_t(x | s, S) \) converges, its limit satisfies

\[
\begin{align*}
F(x | s, S) &= F(S_1, x_2 | s, S) + F(x_1, S_2 | s, S) - 1 \\
& \quad + \text{\underline{r}_t^{12}(S-s)} M(S_1, \text{\underline{max}}(s_1, x_1), S_2, \text{\underline{max}}(s_2, x_2)) \\
& \quad + \int_{S_2}^{S_1} M(S_1, \text{\underline{max}}(s_1, x_1), v_2, \text{\underline{max}}(s_2, x_2)) \text{d} \text{\underline{r}_t^1}(v_2 | s, S) \\
& \quad + \int_{S_1}^{S_2} M(v_1, \text{\underline{max}}(s_1, x_1), S_2, \text{\underline{max}}(s_2, v_2)) \text{d} \text{\underline{r}_t^2}(v_1 | s, S)
\end{align*}
\]

and similar expressions obtained by letting \( t \to \infty \) in the relations above for the finite distribution. When proper modifications are introduced, the single-product results may be extended to the two-product \((s, S)\) policies in a manner similar to that for the \((\sigma, S)\) policies above.
When $g(y)$ is quasi-convex and the problem is separable as in Theorem 3.6, the policies found here are nearly optimal in the same sense as those above for single products. Furthermore, the comments regarding the usefulness of the particular $(\sigma, S)$ policies studied above are also relevant here. In general a stationary $(s, S)$ policy is probably preferable to one of these $(\sigma, S)$ policies because ordering for a single product at a time is allowed.
REFERENCES


Multiproduct Inventory Models With Set-up


Alan Wheeler

January 12, 1968

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Logistics and Mathematical Statistics Branch
Office of Naval Research
Washington, D.C. 20360

This report presents a mathematical model of an inventory problem frequently encountered in practice and studies the nature of the best ordering policy to use for it. At the beginning of each of a sequence of periods an inventory manager must decide whether or not to place an order and, if so, how much of which products. After the order is filled immediately and then during the period a random demand draw upon his inventory until at the beginning of the next period he makes another ordering decision. When he orders, a unit cost for each item and a set-up cost for the order are charged. The set-up cost is thus independent of the size of the order and may represent the price of such things as paperwork, tooling up for a production run, or construction of facilities. On the basis of the inventory levels at the end of a period an inventory cost is charged. This represents a holding cost for those products still in stock and a penalty cost for those whose supply is depleted. Since the actual demand is not known but random, the inventory manager does not have full control over the costs he will be charged. It is assumed, however, that he can obtain his expected costs by averaging over the random demands, and his aim then is to use an ordering policy which minimizes his expected costs. As the mathematical inventory literature is quite large it is important to emphasize four distinctive features of the model in this report: Periodic review, random demand, set-up costs, and interdependence of product costs. The present study is the most general one known by the author to combine all four attributes. It is shown that the best policy treating various products simultaneously may be quite different from that obtained by looking at each product independently. That this would be true is evident from considering situations (cont)
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