NONPARAMETRIC RANKING PROCEDURES FOR COMPARISON WITH A CONTROL

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M. HASEEB RIZVI, MILTON SOBEL AND GEORGE G. WOODWORTH

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1. Introduction and Summary.

A decision maker is confronted with \( k \) populations, \( \pi_1, \ldots, \pi_k \) (say, \( k \) lots of items available for purchase) and a control population \( \pi_0 \) and must, on the basis of random samples of common size \( n \) from \( \pi_0, \ldots, \pi_k \), select those which are at least as good as \( \pi_0 \). We suppose that items are judged on the basis of a continuously distributed attribute \( X \) and that a known fraction \( \alpha \) \( (0 < \alpha < 1) \) of the items in the control population are deficient (their \( X \)-values are too small). A population is considered to be better than the control if it has a smaller proportion of deficient items; that is, letting \( F_j, j = 0, \ldots, k \), denote the distribution function (df) of \( X \) for population \( \pi_j \) and \( x_\alpha(F_j) \) its \( \alpha \)th quantile, \( \pi_i \) is better than \( \pi_0 \) if \( x_\alpha(F_i) \geq x_\alpha(F_0) \). We also consider the possibility that \( F_0 \) is known in which case \( \pi_0 \) is called a standard and is not sampled. In section 2 we propose a nonparametric procedure \( R \) based on order statistics which guarantees a minimal preassigned probability \( \Pr^* \) that, when each \( F_j \) is stochastically ordered with respect to \( F_0 \), all populations better than the control will be selected; such a selection will be called a correct selection (CS). The corresponding problem of selecting a subset containing the best population (without any control) was treated in [11].
Since the trivial procedure $R_0$ of including all $k$ populations in the selected subset also guarantees the probability requirement it is necessary to investigate the expected number of misclassifications; this is done exactly in section 3 and asymptotically in section 5. Exact results for known standard $F_0$ are given in section 4. Some other aspects of the problem are briefly discussed in section 8.

As a secondary problem we suppose that for some preassigned fraction $\delta^*$ the decision maker considers a population $\pi_i$ to be $\delta^*$-inferior to $\pi_0$ if more than $100(\alpha + \delta^*)$ percent of the items in $\pi_i$ are as bad as at least one of the worst $100(\alpha - \delta^*)$ percent of the items in $\pi_0$; i.e., $\pi_i$ is $\delta^*$-inferior if $X_{\alpha-\delta^*}(F_0) \geq X_{\alpha+\delta^*}(F_1)$.

In section 5 we give asymptotic expressions for the smallest sample size needed to guarantee that the expected proportion of $\delta^*$-inferior populations selected by $R$ will be less than a preassigned number $\beta^*$. An equally reasonable definition of $\pi_i$ to be $\delta^*$-inferior is that more than $100(\alpha + 2\delta^*)$ percent of the items in $\pi_i$ are deficient. Our results with $\alpha$ replaced by $\alpha' = \alpha + \delta^*$ also apply to this problem.

We show in section 6 that for small values of $\delta^*$, a competing nonparametric procedure $S$ based on rank sums and a competing asymptotically nonparametric procedure $M$ based on sample means both require sample sizes proportional to the square of that required by $R$ to achieve the same degree of rejection of $\delta^*$-inferiors. For moderate $\delta^*$-values it is shown that $S$ requires a sample size which has the same order of magnitude as that required by $R$. In section 7 we study a related minimax procedure. We append tables for $\alpha = 1/2$ of (1) the
integer constant $c$ needed to make procedure $R$ explicit, (2) some
required values to make the minimax procedure explicit and (3) efficiency
comparisons of $S$ with respect to $R$.

**A Basic Inequality:**

Let $\mathcal{X} = \{x_{j_1}, 1 \leq j \leq n, 0 \leq i \leq k\}$ denote the combined sample,
thus for each $i$ $x_{j_1}, \ldots, x_{j_n}$ are independent random variables
having the cdf $F_i(x)$. We regard $\omega = (F_0, F_1', \ldots, F_k')$ as the unknown
"parameter" and, for an arbitrary function $\psi$, use the symbol
$E_{\omega} \psi(\mathcal{X})$ to denote the expected value of $\psi(\mathcal{X})$ computed under the
assumption that $\omega$ is the true parameter value. The following lemma
is used extensively in this paper; we state it without proof since it
follows easily from Lemma 2.1 of [1].

**Lemma 1.1.** Let $\psi(\mathcal{X})$ be non-increasing in each $x_{j_0}$, $j=1, \ldots, n$,
and non-decreasing in each $x_{j_1}$, $1 \leq j \leq n$, $1 \leq i \leq k$, and let
$\omega = (F_0, F_1', \ldots, F_k')$ and $\omega' = (F_0', F_1', \ldots, F_k')$ satisfy $F_0(x) \leq F_0'(x)$
and $F_i(x) \geq F_i'(x)$ for $i = 1, \ldots, k$ and all $x$, then

$$E_{\omega} \psi(\mathcal{X}) \leq E_{\omega'} \psi(\mathcal{X}).$$

2. **The Problem and the Proposed Procedure $R$ (Unknown $F_0$).**

Based on a common number $n$ of observations from each of $k+1$
populations $(\pi_0, \pi_1, \ldots, \pi_k)$, all $n(k+1)$ being independent, we want
a procedure $R$ that selects a subset of the $k$ populations which
(with high probability) will contain all populations better than $\pi_0$,
i.e., all $\pi_i$ with $x_{\alpha}(F_i) \geq x_{\alpha}(F_0)$. To make this more precise,
we say \( F_i \) is as good as \( F_0 \) uniformly iff \( F_i(x) \leq F_0(x) \) for all \( x \) and that \( F_i \) is worse than \( F_0 \) uniformly iff \( x_\alpha(F_i) < x_\alpha(F_0) \) and \( F_i(x) \geq F_0(x) \) for all \( x \). Let \( \Omega \) denote the space of all possible \((k+1)\)-tuples \( \omega = (F_0, F_1, \ldots, F_k) \) and let \( \Omega_1 \) denote the subspace of \( \Omega \) consisting of those \( \omega \) such that for each \( i(1, 2, \ldots, k) \) either \( F_i \) is as good as \( F_0 \) uniformly or \( F_i \) is worse than \( F_0 \) uniformly.

For any preassigned \( P^* \) with \( 2^{-k} < P^* < 1 \) we want the procedure \( R \) to be such that

\[
(2.1) \quad P(\mathcal{C}S|R) \geq P^* \quad \text{whenever } \omega \in \Omega_1 .
\]

For any fixed \( \alpha \) with \( 0 < \alpha < 1 \) we assume that

\[
(2.2) \quad 1 \leq (n+1)\alpha \leq n
\]

and define the integer \( r \) by the inequalities

\[
(2.3) \quad r \leq (n+1)\alpha < r+1 .
\]

It follows that \( 1 \leq r \leq n \).

We now define the procedure \( R = R(c) \) in terms of an integer \( c \) and the order statistic \( Y_{ji} \), where \( Y_{ji} \) is the \( j \)th order statistic in a sample of size \( n \) from \( \pi_i \), since the \( F_i \) are unknown we take \( Y_{0i} \) to mean \( \infty \) for each \( i \).

**Procedure R:**

The procedure \( R(c) \) puts \( \pi_i \) in the selected subset for each \( i(i = 1, 2, \ldots, k) \) iff

\[
(2.4) \quad Y_{ri} \geq Y_{r-c, 0} .
\]
The procedure \( R \) will be defined as that \( R(c) \) for which \( c \) is the smallest integer \( 0 \leq c \leq r-1 \) such that \( R(c) \) satisfies (2.1).

In order that the nonrandomized procedure be nondegenerate we limit the \( c \)-values to \( 0 \leq c \leq r-1 \). We shall show that for any \( \alpha \) and \( k \) a value of \( c \leq r-1 \) may not exist for all pairs \( (n, P^*) \) but if \( P^* \) is chosen not greater than some function \( \bar{P}_0 = \bar{P}_0(n,\alpha,k) \), then a value of \( c \leq r-1 \) does exist that satisfies (2.1). \( \bar{P}_C \) will be evaluated by setting \( c = r-1 \) in the \( P(CS) \) and we show that \( \bar{P}_0 \) approaches unity as \( n \) increases. The values of \( P^* \) between \( \bar{P}_0 \) and 1 can be handled by the degenerate procedure \( R_0(c = r) \) or by a randomized combination of the procedures for \( c = r-1 \) and \( c = r \). The expressions for the \( P(CS) \) etc. derived below all hold for \( 0 \leq c \leq r \) unless explicitly stated otherwise.

Letting \( P(CS|R) \) denoted by \( P_0(R) \) we now introduce other functions, some of which were suggested by Lehmann [7]. Some of these functions can be used as alternative criteria for developing new procedures. Let \( k_1 \) denote the number of \( \pi_i \)'s at least as good as \( \pi_0 \), i.e., such that \( x_\alpha(F_i) \geq x_\alpha(F_0) \); we denote the set of subscripts of these \( \pi_i \) by \( I_1 \) and refer to the corresponding set of populations as the superior set. Then \( k_2 = k - k_1 \) is the size of the set \( I_2 \) of subscripts of \( \pi_i \)'s in the interior set.

Let \( P_1(R) \) denote the expected proportion of the \( k_1 \) superior populations that are correctly classified under procedure \( R \). Let \( P_2(R) \) denote the expected proportion of the \( k_2 \) inferior populations that are misclassified. If there are no superior (inferior) populations
then we define \( P_1 = 0 \) (\( P_2 = 0 \)).

If we define a loss function \( L = L(R, F_0, F_1, \ldots, F_k) \) as the total number of misclassifications then we can write the expected loss or risk \( E(L|R) = P_2(R) \) as

\[
(2.5) \quad P_2(R) = k_1[1 - P_1(R)] + k_2 P_2(R).
\]

Obviously we would like \( R \) to be such that \( P_0(R) \) and \( P_1(R) \) are large while \( P_2(R) \) and \( P_3(R) \) are small. We shall therefore be interested in deriving the inf \( P_0(R) \), inf \( P_1(R) \), sup \( P_2(R) \), sup \( P_3(R) \), each taken over \( \Omega_1 \).

3. Exact Expressions for \( P_1(R) \).

Let \( dH_{r1}(y) \) and \( H_{r1}(y) \) denote, respectively, the probability (density) element and the df of the \( r \)th order statistic \( Y_{r1} \) in a sample of size \( n \) from the df \( F_1(y) \). It is well known (and easy to show) that

\[
(3.1) \quad dH_{r1}(y) = r(n) \int F_1^{-1}(y)[1 - F_1(y)]^{n-r-1} dF_1(y),
\]

\[
(3.2) \quad H_{r1}(y) = \sum_{j=r}^{n} \binom{n}{j} F_1^j(y)[1 - F_1(y)]^{n-j} = G_r[F_1(y)],
\]

where \( G_r(p) = I_p[r, n-r+1] \) denotes the standard incomplete beta function

\[
(3.3) \quad G_r(p) = r(n) \int_0^p x^{r-1}(1-x)^{n-r} dx.
\]

Using the above notation, the probability of a correct selection under procedure \( R \) is given by
(3.4) \( P_0(R) = \mathbb{P}\{Y_{ri} \geq Y_{r-c,0}, i \in I_1\} = \int_\infty^{-\infty} \prod_{i \in I_1} [1-H_{r_i}(y)] dH_{r-c,0}(y) \).

Similarly we obtain

(3.5) \( P_1(R) = \frac{1}{k_1} \sum_{i \in I_1} \int_\infty^{-\infty} [1-H_{r_i}(y)] dH_{r-c,0}(y) \),

(3.6) \( P_2(R) = \frac{1}{k_2} \sum_{i \in I_2} \int_\infty^{-\infty} [1-H_{r_i}(y)] dH_{r-c,0}(y) \).

These in turn yield exact expression for \( P_3(R) \).

We now obtain the infimum (or supremum) of these over \( \Omega_1 \). Consider \( P_0(R) \). Since \( G_r(p) \) is strictly increasing in \( p \), it follows as in [11] that the infimum of \( P_0(R) \) over \( \Omega_1 \) is obtained by setting \( P_1(y) = P_0(y) \) for \( i \in I_1 \) and minimizing over \( k_1 \). Thus we obtain

\[
(3.7) \quad \inf_{\Omega_1} P_0(R) = \min_{0 \leq k_1 \leq k} \int_\infty^{-\infty} [1-H_{r_0}(y)]^k dH_{r-c,0}(y)
\]
\[
= \int_0^1 [1-G_r(u)]^k dG_{r-c}(u) = J_c(k) \text{ (say)}.
\]

Since \( G_r(u) \) is decreasing in \( r \) for any \( u \) (see e.g. [11]) it follows that

\[
(3.8) \quad J_c(k) = k \int_0^1 G_{r-c}(u)[1-G_r(u)]^{k-1} dG_r(u)
\]

is an increasing function of \( c \). Since \( J_r(k) = \mathbb{P}(CS|R_0) = 1 \) it follows that our primary \( P^* \)-requirement in (2.1) has a solution for any \( n \). Below we shall consider what values of \( P^* \) allow us to take \( c \leq r-1 \) and avoid the degenerate procedure \( R_0 \) of putting all \( k \) populations in the selected subset. Table 1 gives \((r-c)\)-values for
procedure $R$ for some specified $P^*$ when $\alpha = \frac{1}{2}$.

Similarly, we obtain the supremum of $P_2(R)$ (which is the same as the infimum of $P_1(R)$) by setting $F_1(y) = F_0(y)$ for $y \in I_{n-2}$ and maximizing (3.6) over $k_2$, obtaining

\[
\inf_{\Omega_1} P_1(R) = \sup_{\Omega_1} P_2(R) = \int_{-\infty}^{\infty} [1-H_{r0}(y)]dH_{r-c,0}(y)
\]

\[
= \int_{0}^{1} [1-G_{r}(u)]dG_{r-c}(u) = J_c(1).
\]

To find the supremum of $P_2(R)$ over $\Omega_1$, we first show that $J_c(1) \geq \frac{1}{2}$ for $0 \leq c \leq r$. Integration by parts in (3.9) gives

\[
J_c(1) = \int_{0}^{1} G_{r-c}(u)dG_{r}(u)
\]

and we note that $J_0(1) = \frac{1}{2}$. Since $G_r(x)$ is decreasing in $r$ for any fixed $x$, it follows that $J_c(1) \geq \frac{1}{2}$ for $0 \leq c \leq r$. Hence, taking the supremum for fixed $k_1$ and then the maximum over $k_1$,

\[
\sup_{\Omega_1} P_2(R) = \max_{0 \leq k_1 \leq k} \left\{ k_1 \sup_{\Omega_1} [1-P_1(R)] + k_2 \sup_{\Omega_1} P_2(R) \right\}
\]

\[
= \max_{0 \leq k_1 \leq k} \left\{ k_1 [1-J_c(1)] + (k-k_1)J_c(1) \right\} = kJ_c(1).
\]

In order to use the procedure $R$ with $c \leq r-1$ and avoid the degenerate procedure $R_0$ for $c = r$, it is necessary to specify $P^*$ not greater than $\bar{P}_0$, where $\bar{P}_0$ is the value of $\inf_{\Omega_1} P_0(R)$ for $c = r-1$. From (3.7) we obtain

\[
\bar{P}_0 = n \int_{0}^{1} [G_{n-r+1}(v)]^k v^{n-1} dv = J_{r-1}(k).
\]
An asymptotic expression for (3.7) is derived in section 5.

The value of \( \overline{P}_1 = \inf_{\Omega_1} P_1(R) \) for \( c = r-1 \) (which also holds for \( P_0 \) with \( k = 1 \)) is

\[
(3.13) \quad \overline{P}_1 = n \int_0^1 G_{n-r+1}(v) v^{n-1} dv = \left( \frac{2n}{n} \right)^{r-1} \sum_{i=0}^{r-1} \left( \frac{2n-i-1}{n-1} \right) = 1 - \frac{(2n-r)}{2n}.
\]

This is also the value of \( \overline{P}_2 \), i.e., \( \sup_{\Omega_1} P_2(R) \) for \( c = r-1 \). The smallest value that can be specified for \( P^* \) under \( \Omega_1 \) using procedure \( R \) is easily seen to be \( 1/(k+1) \), obtained by setting \( c = 0 \) in (3.7).

It is also of some interest to investigate the infimum \( P_0 \) of \( P_0(R) \) under the set \( \Omega \) of all possible configurations. The least favorable configuration here will occur for fixed \( P_0 \) when \( r \leq k \leq k \). \( F_1 \) is as large as possible subject to \( \chi_\alpha(F_1) \geq \chi_\alpha(F_0) \). We thus obtain \( k_1 \) binomial distributions with probability \( 1-\alpha \) at \( \chi_\alpha(F_0) \) and the remaining mass at \( -\infty \). Then

\[
(3.14) \quad \inf_{\Omega} P_0(R) = \min_{1 \leq k_1 \leq k} G_{r-c}(\alpha)\left[ \sum_{j=0}^{r-1} \binom{n}{j} (1-\alpha)^{n-j} \right]^{k_1} = G_{r-c}(\alpha)\left[ 1-G_r(\alpha) \right]^{k_1}.
\]

To get an upper bound for (3.14) we first show that \( G_r(\frac{r}{n+1}) \) is decreasing in \( r \). Writing

\[
(3.15) \quad G_{r+1}(\frac{r}{n+1}) = G_{r+1}(\frac{r}{n+1}) + (n-r)\binom{n}{r} \int_{\frac{r}{n+1}}^{\frac{r+1}{n+1}} x^{r}(1-x)^{n-r-1} dx
\]

and integrating \( G_{r+1}(\frac{r}{n+1}) \) by parts gives
(3.16) \[ G_r\left(\frac{x}{n+1}\right) - G_{r+1}\left(\frac{x+1}{n+1}\right) = \binom{n}{r}\left[\frac{x}{n+1}\right]^r\left(\frac{n-r+1}{n+1}\right)^{n-r} - (n-r) \int_{\frac{x}{n+1}}^{\frac{x+1}{n+1}} x^r(1-x)^{n-r-1} dx. \]

Since the maximum of \( x^r(1-x)^{n-r+1} \) is at \( x = \frac{x}{n+1} \), we obtain from (3.16) for any \( r \)

(3.17) \[ G_r\left(\frac{x}{n+1}\right) - G_{r+1}\left(\frac{x+1}{n+1}\right) \geq \binom{n}{r}\left(\frac{x}{n+1}\right)^r\left(\frac{n-r+1}{n+1}\right)^{n-r}[n-r(\frac{n-r+1}{n+1})] \int_{\frac{x}{n+1}}^{\frac{x+1}{n+1}} \frac{dx}{(1-x)^2} = 0. \]

Hence from (3.17) and the fact that \( r/(n+1) \leq \alpha \), we obtain

(3.18) \[ G_r(\alpha) \geq G_r\left(\frac{x}{n+1}\right) \geq G_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n > \frac{1}{e}. \]

Thus from (3.14) we find that for any \( c \)-value

(3.19) \[ P_0 \leq \left(1 - \frac{1}{e}\right)^k \]

which does not depend on \( \alpha, r \) or \( n \). Since this is less than \( (.65)^k \)

for any values of \( r, n, \alpha \) we cannot use the least favorable configuration over \( \Omega \) as a tool for formulating a ranking problem with the usual \( P^* \)-requirement.

4. **Procedure \( R_1 \) for Known Standard.**

In this case we do not sample the known standard and the form of the procedure changes. Let \( x_{\alpha-B}(F_0) \) denote the (\( \alpha-B \))\(^{th} \) quantile of \( F_0 \) where \( \beta \) corresponds to \( c/(n+1) \) in section 2.

Procedure \( R_1 \):

For each \( i(1 = 1, 2, \ldots, k) \) put \( F_i \) in the selected subset if
(4.1) \[ Y_{ri} \geq x_{\alpha-\beta} \]

where \( \beta \) is the smallest number between 0 and \( \alpha \) for which (2.1) holds.

Corresponding to the results in (3.4) through (3.9) we obtain for \( R_1 \)

(4.2) \[ P_0(R_1) = P(CS|R_1) = P(Y_{ri} \geq x_{\alpha-\beta}, i \in I_1) = \prod_{i \in I_1} [1-H_{ri}(x_{\alpha-\beta})] \]

(4.3) \[ P_1(R_1) = \frac{1}{k_1} \sum_{i \in I_1} [1-H_{ri}(x_{\alpha-\beta})], \]

(4.4) \[ P_2(R_1) = \frac{1}{k_2} \sum_{i \in I_2} [1-H_{ri}(x_{\alpha-\beta})], \]

(4.5) \[ \inf_{\Omega_1} P_0(R_1) = [1-H_{r0}(x_{\alpha-\beta})]^k = [1-G_r(\alpha-\beta)]^k = J_0'((k) \text{ (say)}, \]

(4.6) \[ \inf_{\Omega_1} P_1(R_1) = 1-H_{r0}(x_{\alpha-\beta}) = 1-G_r(\alpha-\beta) = J_0'(1), \]

and the last result also holds for \( \sup_{\Omega_1} P_2(R_1) \) over \( \Omega_1 \).

If \( r/(n+1) \geq 1/2 \) then \( \alpha \geq 1/2 \) and \( 1-x \leq x \) for \( x \geq \alpha \). It follows that for \( r/(n+1) \geq 1/2 \)

(4.7) \[ 1-G_r(\alpha) = r^2 \int_\alpha^1 x^{-1}(1-x)^{n-1}dx \geq r \int_\alpha^1 x^{n-1}(1-x)^{r-1}dx = G_r(\alpha), \]

so that \( J_0'(1) = 1-G_r(\alpha) \geq 1/2 \). Since \( J_0'(1) \) is strictly increasing in \( \beta \) for \( 0 \leq \beta \leq \alpha \), it follows that \( J_0'(1) \geq 1/2 \) for \( r/(n+1) \geq 1/2 \) and any \( \beta \) with \( 0 \leq \beta \leq \alpha \). Hence, corresponding to (3.11), we have for \( r/(n+1) \geq 1/2 \),

(4.8) \[ \sup_{\Omega_1} P_2(R_1) = \max_{0 \leq k_1 \leq k} \{ k_1 [1-J_0'(1)] + (k-k_1)J_0'(1) \} = kJ_0'(1). \]
Since \( J_B'(k) \) approaches 1 as \( \beta \to \alpha \), we need not be concerned with the quantities \( \bar{P}_0, \bar{P}_1 \), etc. when \( F_0 \) is known.

If we take the least favorable configuration over the set \( \Omega \) of all possible configurations then we obtain, as in (3.14) through (3.19)

\[
(4.9) \quad \inf_{\Omega} P_0(R_1) = [1 - \alpha_r(\alpha)]^k \leq (1 - \frac{1}{e})^k \leq (.65)^k.
\]

Hence the terminal remark of section 3 also holds for the case of known \( F_0 \).

5. **Asymptotic Properties of Procedure \( R \).**

Procedure \( R \) is constructed so that with high probability it retains those populations at least as good as the standard; it eliminates only those populations which, on the basis of a sample, appear to be definitely inferior. In this section we define a nonparametric measure, \( \delta_{\alpha}(F,F_0) \), of the inferiority of a population with df \( F \) compared to the control population with df \( F_0 \). It will be seen that

\[
0 \leq \delta_{\alpha}(F,F_0) \leq \min(\alpha, \bar{\alpha}) \quad \text{provided } F(x) \geq F_0(x) \text{ for all } x \text{ and where } \bar{\alpha} = 1 - \alpha.
\]

**\( \delta^* \)-Inferior Populations:** For \( \delta^* \), a specified number between 0 and \( \min(\alpha, 1 - \alpha) \), \( F \) is \( \delta^* \)-inferior to \( F_0 \) if \( F(x) \geq F_0(x) \) for all \( x \) and \( \delta_{\alpha}(F,F_0) \geq \delta^* \). Let \( P_2(\delta^*|R) \) denote the expected proportion of \( \delta^* \)-inferiors in the subset selected by \( R \); if there are no \( \delta^* \)-inferiors then we define \( P_2(\delta^*|R(c)) = 0 \).

Recall that \( R(c) \) is the selection procedure defined by (2.14). In this section we obtain asymptotic expressions \( (n \to \infty) \) for

\[
\inf_{\Omega_1} P_0(R(c)) \quad \text{and} \quad \sup_{\Omega_1} P_2(\delta^*|R(c)).
\]

We use these to obtain asymptotic
expressions for the minimum sample size required by procedure R to
guarantee for specified \( P^* \) and \( \beta^* \), \( \inf \frac{P_0(R)}{n_1} \geq P^* \), and
\( \sup \frac{P_2(\delta^* | R)}{n_1} \leq \beta^* \).

A measure of inferiority:

Let \( F(x) \geq F_0(x) \) for all \( x \) and let \( \delta(F, F_0) \) denote an arbitrary
nonparametric measure of the degree of inferiority of \( F \) to \( F_0 \). \( \delta \) is
nonparametric if and only if for continuous \( F \) and \( F_0 \)

\[ \delta(F, F_0) = \delta(F(F_0^{-1}), U), \text{ where } U \text{ is the uniform } (0, 1) \text{df}. \]

Being a measure of inferiority (degree of stochastic smallness) \( \delta \)
should also satisfy

\[ F_0 = F \implies \delta(F, F_0) = 0 \]

and

\[ F'(x) \geq F(x) \geq F_0(x), \text{ for all } x \implies \delta(F', F_0) \geq \delta(F, F_0). \]

Let \( g \) be an arbitrary non-decreasing function of bounded
variation on \((0, 1)\); a general \( \delta \) satisfying (5.1)-(5.3) is

\[ \delta(F, F_0) = \int \frac{F - F_0}{2} g\left(\frac{F + F_0}{2}\right). \]

One example of such a \( \delta \) is already familiar, namely

\[ \delta(F, F_0) = \int (F - F_0) \frac{F + F_0}{2} d \frac{F + F_0}{2} = \int F dF_0 - \frac{1}{2}. \]

The measure \( \delta_\alpha(F, F_0) \) which we propose is obtained by setting
\( g(u) = 0 \) or \( 1 \) according as \( u < \alpha \) or \( u \geq \alpha \). It is easy to see that
under the assumptions \( F(x) \geq F_0(x) \), for all \( x \), \( F \) and \( F_0 \) continuous,
this choice of \( g \) gives

\[
\delta_\alpha(F, F_0) = \inf_x \left\{ \frac{F(x) - F_0(x)}{2} : \frac{F(x) + F_0(x)}{2} = \alpha \right\}.
\]

Notice that if \( F(x) + F_0(x) = 2\alpha \) then \( F(x) = \alpha + \delta_\alpha(F, F_0) \) and \( F_0(x) = \alpha - \delta_\alpha(F, F_0) \) so that \( \delta_\alpha(F, F_0) \leq \min(\alpha, \bar{\alpha}) \). We can also express (5.4) as

\[
\delta_\alpha(F, F_0) = \inf_d \left\{ d : F_0^{-1}(\alpha - d) \geq F^{-1}(\alpha + d) \right\},
\]

provided we define \( F_0^{-1}(u) = \inf_x \{ x : F_0(x) > u \} \) and

\[
F^{-1}(u) = \sup_x \{ x : F(x) < u \}. \quad \text{Thus } \delta_\alpha(F, F_0) \text{ is the smallest non-negative } d \text{ such that } x_{\alpha - d}(F_0) \geq x_{\alpha + d}(F).
\]

**Asymptotic Expressions for} \( \inf_{\Omega_1} P_0(R(c)) \):

It follows from (2.3) that \( r/n \to \alpha \) as, \( n \to \infty \). We shall consider two rates of growth as \( n \to \infty \) for \( c \) in the procedure \( R(c) \); Case i)\( n^{-1/2} c \to (\alpha \bar{\alpha})^{1/2} A \) where \( A \) is an arbitrary non-negative number and \( \bar{\alpha} = 1 - \alpha \) and Case ii) for some \( \epsilon (0 < \epsilon < \alpha/2), \epsilon \leq c/n \leq \alpha - \epsilon \).

Case i) is involved in questions of **Fitman efficiency** and Case ii) in questions of **Bahadur efficiency**.

**Case i):** From (3.7) we conclude that

\[
\inf_{\Omega_1} P_0(R(c)) = P_{Y_{r_1} \geq Y_{r-c,0}, i = 1, \ldots, k},
\]

where \( Y_1 = \ldots = Y_k = F_0 \) are continuous. We can assume any convenient continuous form for this \( F_0 \); in particular, if \( F_0 \) is exponential then \( Y_{r_1} \) and \( Y_{r-c,0} \) are sums of independent random variables, from which it easily follows that (letting \( \Phi \) denote the standard normal df)
\begin{equation}
\lim \inf_{n \to \infty} P_0(R(c)) = \int_{-\infty}^{\infty} \frac{1}{\theta(x + A)^k} \, d\theta(x),
\end{equation}

where $n^{-1/2} \to A(\alpha \frac{1}{\alpha})^{1/2}$. The integral in (5.6) occurs frequently in the literature of selection procedures and is extensively tabulated among others by Milton [10] and Gupta [5].

**Case II:** In this case clearly $\inf_{\Omega_1} P_0(R(c)) \to 1$. Since $\inf_{\Omega_1} P_0(R(c)) = P[Y_{r1} > Y_{r-c,0}, 1 \leq i \leq k]$ when $F_1 = \cdots = F_k = F_0$, it is clear that

$$P(Y_{r1} < Y_{r-c,0}) \leq 1 - \inf_{\Omega_1} P_0(R(c)) \leq kP(Y_{r1} < Y_{r-c,0}),$$

where $F_1 = F_0$.

The event $\{Y_{r1} < Y_{r-c,0}\}$ is the same as the event that at least $r$ observations from population $\pi_1$ are among the $2r-c-1$ smallest observations from $\pi_0$ and $\pi_1$ together. Thus

\begin{equation}
P(Y_{r1} < Y_{r-c,0}) = \sum_{j=0}^{r-c-1} \binom{n}{j} \left( \frac{n}{2n} \right) \binom{r-c-j}{2r-c-1},
\end{equation}

from which it is easy to obtain

\begin{equation}
P(Y_{r1} < Y_{r-c,0}) \leq (r-c) \frac{\binom{n}{r-c-1}(r)}{2n \binom{2r-c-1}{r}};
\end{equation}

and

\begin{equation}
P(Y_{r1} < Y_{r-c,0}) \geq \frac{\binom{n}{r-c-1}(r)}{2n \binom{2r-c-1}{r}}.
\end{equation}

Since $r/n \to \alpha$ and $0 < \epsilon \leq c/n \leq \alpha - \epsilon$, we can apply Stirling's
approximation to (5.8) and (5.9) to obtain:

\[
P(Y_{r_1} < Y_{r-c,0}) \approx K_n \cdot \left( \frac{n - r + \frac{c}{2}}{n-r} \right)^{n-r} \left( \frac{n - r + \frac{c}{2}}{n-r+c} \right)^{n-r+c} \left( \frac{r - \frac{c}{2}}{r-c} \right)^{r-c} \left( \frac{n - r + \frac{c}{2}}{n-r} \right)^{n-r} \]

(5.8) implies that there exists an \( \epsilon' > 0 \) depending only on \( \epsilon \) and \( \alpha \) such that \( K_n \leq n^{1/2}/\epsilon' \) and (5.9) implies that there exists an \( \epsilon'' > 0 \) depending only on \( \epsilon \) and \( \alpha \) such that \( n^{-1/2}\epsilon'' \leq K_n \).

Thus if \( c/n \to \gamma, \ 0 < \gamma < \alpha, \)

\[
(5.10) \quad \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{\Omega_1} \right) = I(\alpha-\gamma, \alpha - \frac{\gamma}{2}) + I(\alpha, \alpha - \frac{\gamma}{2}),
\]

where \( I(x,y) = x \log \left( \frac{x}{y} \right) + (1-x) \log \left( \frac{1-x}{1-y} \right) \) is the Kullback-Leibler information number.

**Asymptotic Expressions for** \( \sup_{\Omega_1} P_2(\delta^* \mid R(c)) \):

Let \( I_2(\delta^*) \) denote the set of subscripts of those \( F_i \) which are \( \delta^* \)-inferior to \( F_0 \) and let \( k_2(\delta^*) \) be the number of subscripts in \( I_2(\delta^*) \). Then \( P_2(\delta^* \mid R(c)) \) is just (3.6) with \( k_2 \) and \( I_2 \) replaced by \( k_2(\delta^*) \) and \( I_2(\delta^*) \) respectively. The supremum of \( 1-H_{r_1}(y) = 1-G_{r_1}(F_1(y)) \) over \( \Omega_1 \) subject to \( \delta(F_1, F_0) \geq \delta^* \) occurs when

\[
(5.11) \quad F_i(x) = \begin{cases} 
F_0(x), & -\infty < x < x_{\alpha-\delta^*}(F_0) \\
\alpha+\delta^*, & x_{\alpha-\delta^*}(F_0) \leq x < x_{\alpha+\delta^*}(F_0) \\
F_0(x), & x_{\alpha+\delta^*}(F_0) \leq x < \infty
\end{cases}
\]

= \text{F}_i(F_0(x)), \text{ say.}


Thus

\[(5.12) \quad \sup_{\Omega_1} P_2(\delta^* | R(c)) = P[Y_{r,1} \geq Y_{r-c,0}^0],\]

where the latter probability is computed under the assumption that 
\(P_0(x)\) is continuous and 
\(P_1(x) = F_1^*(P_0(x)).\)

Analogous to the two cases studied for \(P_0(R(c))\) we consider as 
\(n \to \infty,\)

Case i) \(n^{-1/2} c \to (\alpha / A)^{1/2}\) and \(n^{1/2} \delta^* \to (\alpha / A)^{1/2}\), where \(A\) and \(f\) are arbitrary non-negative constants, and

Case ii) \(c/n \to \gamma, 0 \leq \gamma < \delta^*, \delta^*\) fixed, \(0 < \delta^* < \min(\alpha, A)\).

In case i) an argument similar to that used for \(\inf_{\Omega_1} P_0(R(c))\) yields

\[(5.13) \quad \lim_{n \to \infty} \sup_{\Omega_1} P_2(\delta^* | R(c)) = \int_{f}^{\infty} [1 + \Phi(x+A) - \Phi(x-A)]d\Phi(x)\]

\[+ \Phi(A-f)[2\Phi(f)-1].\]

In case ii), by introducing \(U_{r,1}\) and \(U_{r-c,0}\) where \(U_{r,1}\) and \(U_{r-c,0}\) are the \(r^{th}\) and \((r-c)^{th}\) order statistics from two independent uniform \((0,1)\) samples each of size \(n\), we can write \((5.12)\) as

\[(5.14) \quad \sup_{\Omega_1} P_2(\delta^* | R(c)) = P[U_{r,1} \geq U_{r-c,0}, \ U_{r,1} < \alpha - \delta^*] + P[\alpha - \delta^* \geq U_{r-c,0}, \ \alpha - \delta^* \leq U_{r,1} < \alpha + \delta^*] + P[U_{r,1} \geq U_{r-c,0}, \ \alpha + \delta^* \leq U_{r,1}].

Thus
(5.15) \[ \sup_{\Omega_1} P_2(\delta^*|R(c)) \leq P(U_{r-c}, 0 \leq \alpha-\delta^*) = P(U_{r-1} \geq \alpha+\delta^*) \]

and

(5.16) \[ \sup_{\Omega_1} P_2(\delta^*|R(c)) \geq P(U_{r-c}, 0 \leq \alpha-\delta^*) \cdot P(\alpha-\delta^* \leq U_{r-1} < \alpha+\delta^*) \]

\[ + P(U_{r-1} \geq \alpha+\delta^*) \cdot P(U_{r-c}, 0 \leq \alpha+\delta^*) \]

\[ \approx P(U_{r-c}, 0 \leq \alpha-\delta^*) + P(U_{r-1} \geq \alpha+\delta^*) \quad \text{as } n \to \infty. \]

Letting \( W(p) \) denote the sum of \( n \) Bernoulli random variables with parameter \( p \), the right side of (5.15) which is the same as the second expression in (5.16) can be written as

(5.17) \[ P(W(\alpha-\delta^*) \geq r-c) + P(W(\alpha+\delta^*) \leq r). \]

Then it follows from standard results on large deviations (eg. [4]) Theorem 1) applied to (5.17) that

(5.18) \[ \lim_{n \to \infty} -\frac{1}{n} \log[\sup_{\Omega_1} P_2(\delta^*|R(c))]] \]

\[ = \min[I(\alpha-\gamma, \alpha-\delta^*), I(\alpha, \alpha+\delta^*)], \]

where \( I(x,y) \) is defined after (5.10).

**Approximations to the Sample Size:**

Let \( n(P^*, \beta^*, \delta^*|R) \) be the smallest sample size required by procedure \( R \) to achieve \( \inf_{\Omega_0} P_0(R) \geq P^* \) and \( \sup_{\Omega_1} P_2(\delta^*|R) \leq \beta^* \). We now derive asymptotic expressions for \( n(P^*, \beta^*, \delta^*|R) \) valid in three regions in the domain of the specified quantities \( (P^*, \beta^*, \delta^*) \); the
first two regions correspond to cases i) and ii).

**Region i):** Let $0 < \beta^* < P^* < 1$ be fixed and $\delta^*$ small. Clearly, as $\delta^* \to 0$, $n(P^*, \beta^*, \delta^*| R) \to \infty$. It follows from (5.6) that 
\[ n^{-1/2} c \to A^*(\alpha - \alpha)\frac{1}{2}, \quad \text{where } A^* \text{ is the solution of the right side of} \]
(5.6) equated to $P^*$. Also, it follows from (5.13) that 
\[ n^{1/2} \delta^* \to f^*(\alpha - \alpha)\frac{1}{2} \]
where $f^*$ is the solution of the right side of (5.13) equated to $\beta^*$ with $A$ replaced by $A^*$.

Thus we have

\[ n(P^*, \beta^*, \delta^*| R) \approx \alpha - \alpha \left( f^* \right)^2 / (\delta^*)^2 \]
and

\[ c \approx n^{1/2} A^*(\alpha - \alpha)\frac{1}{2}. \]

**Region ii):** Let $0 < \beta^* < 1$, $\delta^* > 0$ be fixed and $P^*$ be close to 1.

It is easy to prove that $\frac{c}{n} \to 0^*$; for if not then from (5.15) and (5.16) one concludes that

\[ \limsup_{n \to \infty} P_2(\delta^*| R(c)) = \begin{cases} 0, & c/n \leq \delta^* - \epsilon \\ 1, & c/n \geq \delta^* + \epsilon \end{cases} \]

\[ \Omega_1 \]

for any $\epsilon > 0$, but in fact $\sup_{\Omega_1} P_2(\delta^*| R(c)) = \beta^* \neq 0, 1$. Hence $\frac{c}{n} \to 0^*$ and consequently $\gamma$ of case ii) equals $\delta^*$.

Therefore from (5.18) we have

\[ \lim_{n \to \infty} \left\{ - \frac{1}{n} \log(1 - P^*) \right\} = I(\alpha - \delta^*, \alpha - \delta^*/2) + I(\alpha, \alpha - \delta^*/2). \]

Thus we have
\[(5.22)\quad n(P^*, \beta^*, \delta^*| R) \approx -\frac{\log(1-P^*)}{I(\alpha^*, \alpha - \frac{\delta^*}{2}) + I(\alpha, \alpha - \frac{\delta^*}{2})}\]

and, of course,

\[(5.23)\quad c \approx n\delta^* .\]

**Region iii):** Let \(0 < P^* < 1\) and \(0 < \delta^* < \min(\alpha, \bar{\alpha})\) be fixed and \(\beta^*\) small.

As in region i) we have \(n^{-1/2} c \to A^* (\bar{\alpha}^* - 1/2\) so that \(c/n \to 0\).

Since \(\beta^* = \sup_{\Omega_1} E_{\theta_1}(\delta^*| R(c))\) we have from (5.18) (with \(\gamma = 0\))

\[(5.24)\quad n(P^*, \beta^*, \delta^*| R) \approx -\log \beta^*/\min[I(\alpha^*, \alpha - \delta^*), I(\alpha, \alpha + \delta^*)] .\]

6. **Efficiency Comparisons with Competing Procedures.**

A nonparametric competitor \(S\): Let \(R_{ij} (1 \leq i \leq k, 1 \leq j \leq n)\) denote the rank of \(X_{ji}\) among \(X_{10}, \ldots, X_{n0}, X_{11}, \ldots, X_{ni}\) (the smallest has rank 1) and let \(R_i = \sum_{j=1}^{n} R_{ij}\). Procedure \(S(d)\) puts \(x_i\) in the selected subset iff

\[(6.1)\quad R_i \geq d ,\]

where \(d\) is an integer not less than \(n(n+1)/2\). Procedure \(S\) is determined by setting \(d\) equal to its largest value satisfying the condition \(\inf_{P_{0}} P_0(S(d)) \geq P^*\).

\(S\) is intimately related to a simultaneous inference procedure proposed by Steel (see [9] p. 143); in fact the \(d\) value needed to carry out \(S\) can be obtained from tables of the critical values of Steel's procedure with \(1-P^*\) corresponding to the significance level.
To see this, notice that $R_{i_1}$ is nondecreasing in observations from $\pi_1$ and does not depend on observations from $\pi_{i}', i' \neq 0, i$. Then by an obvious application of Lemma 1.1 we conclude that $P_0(S(d))$ is minimized over $\Omega_1$ when $F_1 = F_2 = \cdots = F_k = F_0$. Under this hypothesis the distribution of $(R_{i_1}, \ldots, R_{i_k})$ is the same as that of $(n(2n+1)-R_{i_1}, \ldots, n(2n+1)-R_{i_k})$. This is proved by taking $F_i$, $0 \leq i \leq k$, to be uniform. Thus $Y_{ji} = (1-X_{ji})$, $0 \leq i \leq k$, $1 \leq j \leq n$, are independent uniform random variables and if 

$S_{ji}$ denotes the rank of $Y_{ji}$ among $Y_{10}, \ldots, Y_{n0}, Y_{11}, \ldots, Y_{ni}$, then clearly $S_{ji} = (2n+1)-R_{ji}$. The array $\{S_{ji}; 1 \leq i \leq k, 1 \leq j \leq n\}$ has the same distribution as $\{R_{ji}; 1 \leq i \leq k, 1 \leq j \leq n\}$ so 

$(R_{i_1}, \ldots, R_{i_k})$ has the same distribution as $(\sum_j S_{j1}, \ldots, \sum_j S_{jk}) = (n(2n+1)-R_{i_1}, \ldots, n(2n+1)-R_{i_k})$ and consequently 

$$(6.2) \quad \inf_{\Omega_1} P_0(S(d)) = P\{\min_{1 \leq i \leq k} R_{i1} \geq d\} = 1-P\{\max_{1 \leq i \leq k} R_{i1} > n(2n+1)-d\},$$

where these probabilities are computed under the assumption that $F_1 = F_2 = \cdots = F_k = F_0$. If (6.2) is equated to $P^*$ then $d = n(2n+1)+1-r^*$, where $r^*$ can be obtained by entering table VIII, p. 250 of [9] at significance level $1-P^*$.

For fixed $P^*$ from (57), p. 151 of [9] we obtain 

$$(6.3) \quad \hat{d} = n(2n+1)/2 - A^* n[(2n+1)/24]^{1/2},$$

where $A^*$ is the solution of the right side of (5.6) equated to $P^*$. 
Proportion of Inferiors Selected by $S$:

Let $F^*$ denote an arbitrary (not necessarily continuous) df on the interval $0 < u < 1$ such that $F^*(u) \geq u$. We shall say that a df $F_1$ is $F^*$-inferior to $F_0$ if $F_1(x) \geq F^*(F_0(x))$, for all $x$; $P_2(F^*|S)$ denotes the expected proportion of $F^*$-inferior populations in the subset selected by $S$ and if no populations are $F^*$-inferior then we define $P_2(F^*|S) = 0$. Again applying Lemma 1.1 we conclude that

$$(6.4) \quad \sup_{\Omega_1} P_2(F^*|S) = P(R_1 \geq d),$$

where the latter probability is computed under the assumption that $F_1(x) = F^*(F_0(x))$.

From Lemma 3.2 of [2] we conclude that $W = n^{1/2}((R_1 - n(2n+1)/2)/n^2 - (1/2 - \int F^*du))$ has the same limiting distribution ($n \to \infty$) as

$$Y = n^{-1/2} \sum_{j=1}^{n} [F_0(X_{j1}) + 1 - F^*(F_0(X_{j0}))] - 2n^{1/2}(1 - \int F^*du).$$

For purposes of analysis suppose that $F^*$ depends on $n$ and as $n \to \infty$ $F^*(u)$ approaches $u$ at such a rate that $n^{1/2}(\int F^*du - 1/2) = o(1)$, then by application of the central limit theorem (as stated in [8], p. 295) we conclude that $Y$ is asymptotically normal with mean zero and variance $1/6$. Hence from (6.3) and (6.4) we obtain

$$(6.5) \quad \sup_{\Omega_1} P_2(F^*|S) \approx \Phi(2^{-1/2}A^* - (6n)^{1/2}(\int F^*du - 1/2)),$$

when $\Phi$ is the standard normal df.

From (5.11) it is clear that $F_1$ is $S^*$-inferior if and only if it is $F^*$-inferior with
(6.6) \[ F^*(x) = F^*_1(x) = \begin{cases} x, & 0 < x < \alpha - \delta^* \text{ or } \alpha + \delta^* \leq x < 1 \\ \alpha + \delta^*, & \alpha - \delta^* \leq x < \alpha + \delta^* \end{cases}. \]

So if \( P_2(\delta^*|S) \) denotes the expected proportion of \( \delta^* \)-inferior populations in the subset selected by \( S \) we have from (6.5),

(6.7) \[ \sup_{\Omega_1} P_2(\delta^*|S) \approx \Phi(2^{-1/2}A^* - (24n)^{1/2}(\delta^*)^2), \]

provided \( n^{1/2}(\delta^*)^2 = o(1) \).

Defining \( n(P^*, \beta^*, \delta^*|S) \) as in section 5 it follows from (6.7) that for fixed \( 0 < \beta^* < P^* < 1 \) and small \( \delta^* \)

(6.8) \[ n(P^*, \beta^*, \delta^*|S) \approx (z^* - 2^{-1/2}A^*)^2/24(\delta^*)^4, \]

where \( \Phi(z^*) = \beta^* \).

**Asymptotic Relative Efficiencies for S Compared to R:**

Comparison of (6.8) with (5.19) shows that for small \( \delta^* \) the sample size required by \( S \) is proportional to the square of the sample size required by \( R \). Thus the Pitman efficiency of \( S \) with respect to \( R \) is zero. It should be noted that if the extremal are taken over a smaller class than \( \Omega_1 \) (such as a location parameter family) then the sample size comparison need not be so unfavorable to \( S \), indeed, \( S \) may even require a smaller sample size than \( R \).

Next we consider an efficiency comparison of the sort urged by Bahadur [3]. Here we hold \( \delta^* \) and \( \beta^* \) fixed and study the behavior of the sample size as \( P^* \) approaches one.
In view of (6.4) and (6.5) and the asymptotic normality of $R_{1}$ the assumption that $\sup_{Q_{1}}P(\delta^{*}|S) = \delta^{*}$ determines $d$ and implies

$$d = \text{ER}_{1} + O(\text{Var } R_{1})^{1/2} = n^{2}\left[\frac{3}{2} - \int P_{1}^{*}du\right] + O(n^{3/2}) \approx n^{2}[1-2(\delta^{*})^{2}].$$

From (6.2) one obtains after some algebra using Bonferroni's inequality

$$-\frac{1}{n} \log[1 - \inf_{Q_{1}}P_{0}(S)] \approx -\frac{1}{n} \log P[R_{1} < d],$$

where the latter probability is computed under the assumption that $F_{1} = F_{0}$. In [12] it is shown that

$$\lim_{n \to \infty} -\frac{1}{n} \log P[R_{1} < d] = 2e_{w}(2(\delta^{*})^{2}),$$

where $e_{w}(\cdot)$ is given by the numerator of (3.4) of [6]. From (6.10) and (6.11) we obtain, for fixed $\beta^{*}$ and $\delta^{*}$ and $P^{*}$ approaching unity,

$$n(P^{*}, \beta^{*}, \delta^{*}|S) \approx -\log(1-P^{*})/2e_{w}(2(\delta^{*})^{2})$$

and comparing (6.12) with (5.22) we have

$$\frac{n(P^{*}, \beta^{*}, \delta^{*}|R)}{n(P^{*}, \beta^{*}, \delta^{*}|S)} \approx \frac{2e_{w}(2(\delta^{*})^{2})}{I(\alpha, \delta^{*}, \alpha - \frac{\delta^{*}}{2}) + I(\alpha, \alpha - \frac{\delta^{*}}{2})}.$$ 

We shall call the right side of (6.13) the Bahadur efficiency of $S$ with respect to $R$; note that it is independent of $\beta^{*}$. Using line 13, p. 1762, of [6], $e_{w}(2(\delta^{*})^{2})$ can be evaluated by entering column 2 of Table I on that page for $\mu = 2^{-1/2}q^{-1}[2(\delta^{*})^{2}+1/2]$, we also do some additional calculations of $e_{w}(\cdot)$ and in Table 3 we tabulate this and the Bahadur efficiency of $S$ with respect to $R$ for $\alpha = \frac{1}{2}$. 

24
An Asymptotically Nonparametric Competitor \( M \):

Let \( \bar{X}_i \) be the sample mean from \( \pi_i, 0 \leq i \leq k \), and let \( \Omega_1(B) \) be the subset of \( \Omega_1 \) on which \( \nu^4(F_0)/\sigma^4(F_0) \leq B < \infty \) where \( \sigma^2(F_0) \) and \( \nu^4(F_0) \) denote the variance and fourth central moment of \( F_0 \).

If \( \sigma^2(F_0) \) is known it is possible to carry out the procedure \( W(d) \) which retains those populations in the selected subset for which

\[
(6.14) \quad n^{1/2} \bar{X}_1 \geq n^{1/2} \bar{X}_0 - \sigma(F_0)d.
\]

It follows from Lemma 1.1 that

\[
(6.15) \quad \inf_{\Omega_1(B)} P_0(W(d)) = \inf_{F_0} \int [1-F_0^{(n)}(x-d)]^k \, dP_0^{(n)}(x),
\]

where \( F_0^{(n)} \) is the cdf of \( n^{1/2}(\bar{X}_0 - \mu(F_0))/\sigma(F_0) \) and the second infimum is taken over those \( F_0 \) for which \( \nu^4(F_0)/\sigma^4(F_0) \leq B \). If \( d \) remains bounded as \( n \to \infty \) then using the Berry-Esseen bound

\[
|F_0^{(n)}(x) - \Phi(x)| \leq \frac{CV_3(F_0)n^{-1/2}}{\sigma^3(F_0)} \leq CB^{3/4} n^{-1/2},
\]

where \( C \) is a constant and \( V_3(F_0) \) is the third absolute central moment of \( F_0 \), one can easily prove that the right side of (6.15) approaches the right side of (5.6) with \( A = d \).

When \( \sigma^2(F_0) \) is unknown, its estimate \( S_0^2 = \frac{1}{n-1} \sum_{j=0}^{n} (X_{j0} - \bar{X}_0)^2/(n-1) \) has the property that \( \sup_{F_0} P(|1-S_0/\sigma(F_0)| \geq \varepsilon) \leq B/\varepsilon^4, \varepsilon > 0 \). Define procedure \( M(d) \) by replacing \( \sigma(F_0) \) by \( S_0 \) in (6.14). Then it is easy to establish that

\[
(6.16) \quad \lim_{n \to \infty} \inf_{\Omega_1(B)} P_0(M(d)) = \int_{-\infty}^{\infty} (\Phi(x+d))^k \, d\Phi(x).
\]
With the further restriction that \( \sigma(F_1) = \ldots = \sigma(F_k) = \sigma(F_0) \) and 
\[ \frac{\nu_4(F_1)}{\nu_4(F_0)} \leq B, \ i = 0, \ldots, k \] one can use the pooled estimate

\[ s^2 = \frac{\sum_{j=1}^{n} \sum_{i=0}^{k} (X_{1j} - \bar{X}_1)^2/(k+1)(n-1)}{s_0^2} \]

in place of \( s_0^2 \) and (6.16) will remain true.

We denote by \( M \) the procedure \( M(d) \) with \( d \) determined so that 
\[ \lim_{n \to \infty} \inf \Omega_1(B) = \Omega_*; \] it follows from (6.16) that \( d \to d^* \), the solution of the right side of (6.16) equated to \( \Omega_* \).

**Proportion of Inferiors Selected by \( M \) for Fixed \( F_0 \):**

We define \( \delta^* \)-inferior populations as usual, thus \( F_1(x) \) is \( \delta^* \)-inferior to \( F_0 \) iff \( F_1(x) \geq F_1(F_0(x)) \), where \( F_1(F_0(x)) \) is the right side of (5.11). Letting \( P_2(\delta^*|M) \) denote the expected proportion of \( \delta^* \)-inferiors selected by \( M \) and \( \Omega_1(F_0) \) denote \( \Omega_1 \) with \( F_0 \) held fixed, it follows from Lemma 1.1 that for any \( \epsilon > 0 \)

\[
(6.17) \quad \sup_{\Omega_1(F_0)} P_2(\delta^*|M) \leq P(\bar{X}_1 \geq \bar{X}_0 - n^{-1/2} \sigma(F_0)(1+\epsilon)d) \\
+ P(\mid 1-s_0/\sigma(F_0) \mid > \epsilon)
\]

where the first probability on the right is computed with \( F_1(x) = F_1(F_0(x)) \) and the second probability depends only on \( F_0 \). It follows easily from (6.17) that

\[
(6.18) \quad \lim_{n \to \infty} \sup_{\Omega_1(F_0)} P_2(\delta^*|M) = \Phi(2^{-1/2}d^* - (n/2)^{1/2} \frac{\mu(F_0) - \mu(F_1(F_0))}{\sigma(F_0)}),
\]

provided \( \delta^* \to 0 \) such that
(6.19) \( n^{1/2}[\mu(F_0) - \mu(F_1^*(F_0))] = n^{1/2} \int_{x_{\alpha-\delta^*}(F_0)}^{x_{\alpha+\delta^*}(F_0)} (x - x_{\alpha-\delta^*}(F_0)) \, dF_0 = o(1) \),

here \( \mu(F) \) denotes the mean of the df \( F \). Assuming \( F_0 \) has a positive derivative at its \( \alpha^{th} \)-quantile and denoting it by \( f_0'(x_\alpha) \), (6.18) becomes

(6.20) \( \lim_{n \to \infty} \sup \Omega_1(F_0) = \phi(2^{-1/2} \delta^* - \frac{(2n)^{1/2} \delta^*^2}{\sigma(F_0)f_0'(x_\alpha)}) \).

Thus for fixed \( F_0, F^* \) and \( \beta^* \) we have (as \( \delta^* \to 0 \))

(6.21) \( n(F^*, \beta^*, \delta^*|M) \approx (z^* - 2^{-1/2} \delta^*)^2 f_0'(x_\alpha) \sigma(F_0)^2 / 2 \delta^*^4 \),

where \( \phi(z^*) = \beta^* \).

**Asymptotic Relative Efficiencies of M Compared to R:**

Comparison of (6.21) and (5.19) shows that \( M \), like \( S \), requires sample sizes proportional to the square of that required by \( R \) for small \( \delta^* \).

In order to obtain Bahadur efficiency comparisons analogous to (6.13) for fixed \( F_0 \) one (essentially) needs a "large deviations" result for the t-statistic computed from a sample drawn from \( F_0 \). To the authors' knowledge such a result is known only when \( F_0 \) is normal. Indeed if \( F_1 = F_0 \) is normal then \( T = (2n)^{1/2} (\bar{X}_1 - \bar{X}_0)/S_0 \) has the t-distribution with \( (n-1) \) degrees of freedom. Klotz [6] shows that for a sequence \( r_n \) approaching a positive constant \( r_0 \),

\[ \lim_{n \to \infty} -n^{-1} \log P(T < -n^{-1/2} r_n) = \log(1+r_0^2)/2. \]
Arguing as in the discussion leading up to (6.12) we conclude that with \( F_0 \) normal, \( \delta^* \) and \( \beta^* \) fixed, as \( P* \to 1 \),

\[
(6.22) \quad n(P^*, \beta^*, \delta^* | M) \approx -2 \frac{\log(1-P^*)/\log(1+r_o^2(\alpha, \delta^*))}{r_o^2(\alpha, \delta^*)},
\]

where (with \( \phi \) denoting the standard normal density function)

\[
(6.23) \quad r_o(\alpha, \delta^*) = 2^{1/2}(\phi(\phi^{-1}(\alpha-\delta^*)) - \phi(\phi^{-1}(\alpha+\delta^*)) - 2\delta^* \phi^{-1}(\alpha-\delta^*)) .
\]

Thus, combining (6.22) and (5.22),

\[
(6.24) \quad \frac{n(P^*, \beta^*, \delta^* | R)}{n(P^*, \beta^*, \delta^* | M)} \approx \frac{\log(1+r_o^2(\alpha, \delta^*))}{2[I(\alpha-\delta^*, \alpha - \frac{\delta^*}{2}) + I(\alpha, \alpha - \frac{\delta^*}{2})]}
\]

We call the right side of (6.24) the Bahadur efficiency of \( M \) with respect to \( R \) when \( F_0 \) is normal; (6.24) is tabulated for \( \alpha = \frac{1}{2} \) in Table 3. Since \( F_0 \) is normal it is not surprising that \( M \) becomes more efficient for larger values of \( \delta^* \).

7. **A Minimax Procedure \( R' \).**

Another problem of interest to us is to define for a given \( b \geq 0 \) the risk function

\[
(7.1) \quad R_\beta(R') = P_\beta(R') + b[1-P_0(R')]
\]

and find the \( c \)-value such that for unknown \( F_0 \) the procedure \( R' = R(c) \) minimizes the maximum of \( P_\beta(R') \) over \( \beta \). This defines a new procedure \( R' \) that does not depend on any specified \( P* \); we refer to it as the minimax procedure and with \( J_c(\cdot) \) as defined by (3.7) obtain in a straightforward manner
\[(7.2) \sup_{\alpha_1} P_4(R') = \max_{0 \leq k_1 \leq k} \left( k_1 [1-J_c(l)] + (k-k_1)J_c(l) + b[1-J_c(k_1)] \right). \]

If \( k \) is not large then we resort to a numerical computation for each value of \( k_1 \) in (7.2) to obtain the maximum, since an analytic maximization is difficult. Then the required \( c \)-value for the minimax procedure \( R' \) is the integer (with \( 0 \leq c \leq r \)) that minimizes these maximal values. Table 2 gives \( c \)-values and the resulting minimax risks for \( \alpha = \frac{1}{2}, b = k, 2k, 3k, k^2 \) and selected values of \( n \) and \( k \).

The trivial procedure \( R'_0 \) that selects one of the \( 2^k \) possible subsets at random without looking at any observations has the constant risk

\[(7.3) \quad P_4(R'_0) = \frac{k}{2} + b(1-2^{-k}), \]

which is an upper bound for the minimax risk for procedure \( R' \).

For the case of known \( F_0 \), the result analogous to (7.2) is obtained by recalling \( J_B'(\cdot) \) of (4.5) and replacing \( J_c(\cdot) \) by \( J_B'(\cdot) \) in (7.2) and the minimax procedure \( R'_1 \) is then defined by taking \( \beta \) equal to the value that minimizes the maximum in the modified version of (7.2). The result (7.3) also holds for the trivial procedure in the case of known \( F_0 \).

It should be noted that the minimax risk of \( R' \) in Table 2 is not necessarily monotonic in \( n \) for fixed \( k \); we believe that this is due to our forcing \( c \) to be an integer. If we use a suitable randomized procedure we can presumably take these "kinks" out of the minimax risk and make it monotonic.

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Related Problems Solved by Procedure R:

The problem of selecting all populations with \( x_{0}(F_{i}) \leq x_{0}(F_{0}) \) so that the probability of a correct selection is no less than \( P^{*} \) is solved by the procedure which selects \( \pi_{1} \) iff

\[
(8.1) \quad Y_{n-r+1,i} \leq Y_{n-r+c+1,0},
\]

where \( r \) is the integer satisfying (2.3) and \( c \) is the solution of (3.7) equated to \( P^{*} \) and may be obtained from a table giving \( c \)-values for procedure \( R \) corresponding to \( 1-\alpha \). This statement is proved simply by noting that if \( X \) has \( df \ F(x) \) then \(-X \) has \( df \ 1-F(-x) \) so that \(-x_{1-\alpha}(F) \) is the \( \alpha \)-quantile of \(-X \).

The procedure defined by (8.1) also solves the classical problem of testing at level \( 1-P^{*} \) that at least one population is better than \( \pi_{0} \). Like Steel's procedure (section 6 and [9], p. 143), it has the property that, with probability at least \( P^{*} \), one may correctly assert that all populations for which (8.1) is not true are better than \( \pi_{0} \).

We remark here that \( R \) has an unbiasedness property: if \( F_{i}(x) \geq F_{j}(x) \) for all \( x \) then \( R \) is more likely to select \( \pi_{j} \) than \( \pi_{1} \).

Scores Procedures:

Procedure \( S \) discussed in section 6 can be generalized by replacing the Wilcoxon statistic \( R_{i} \) in (6.1) by a two-sample scores statistic

\[
(8.2) \quad T_{i} = \sum_{j=1}^{n} J_{n,R_{ij}}
\]
with monotone scores \( J_{n,1} \leq J_{n,2} \leq \cdots \leq J_{n,2n} \); let us call this procedure \( S_J \). It seems clear under the usual assumption (the step function \( J_n(u) = J_{n,j}; \; (j-1)/2n \leq u < j/2n, \; 1 \leq j \leq 2n \), converges in quadratic mean to a function \( J(u) \)) that \( S_J \) will still have zero Pitman efficiency compared to \( R \). Under some additional assumptions on \( J(u) \) (see [12]), there is a function \( I_J(r_o) \) such that, when \( F_1 = F_0 \) and \( r_n \) is a sequence of constants approaching some constant \( r_o \),

\[
(8.3) \quad \lim_{n \to \infty} \left[ -\frac{1}{n} \log P(T_1 \geq nr_n) \right] = I_J(r_o).
\]

In this case the Bahadur efficiency of \( S_J \) with respect to \( R \) will be the right side of (6.13) with the numerator replaced by \( I_J(r^*) \), where \( r^* \) is the probability limit of \( n^{-1}T_1 \) when \( F_1 = F_1^*(F_0) \) (see (5.11)), that is, \( r^* = \int_0^1 J[(F_1^*(u) + u)/2]dF_1^*(u) \).

Acknowledgement.

The authors wish to thank Mr. Richard Freedman of Ohio State University for computing Tables 1 and 2.
Table 1: Largest values $\$ \text{ of } r-c \text{ for which}$

$$\inf P_1(r) \geq P^* \text{ for } \alpha = \frac{1}{2} \text{ and } r = \frac{n+1}{2}.$$ 

$$P^* = .750$$

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$$P^* = .950$$

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32
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\( P^* = .990 \)

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\$\$ Based on the equation \( J_c(k) = P^* \); see (3.7). Other r-c values for \( n > 65 \) can be obtained from Table 3 of [11] by entering that table with the value of \( k \) increased by one. The underlined entries are the only values that differ from the corresponding entries (with \( k \) shifted by one) of Table 3 of [11]; in each case this value is exactly one larger than the value in [11].

\# Degenerate cases in which all the populations go into the selected subset with probability one.
Table 2: Minimax Risk and c-Values for $\alpha = \frac{1}{2}$

(In each cell the risk is followed by the c-value)

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Table 3: Bahadur Efficiency of $S$ with respect to $R$ and of $M$ with respect to $R$ (with $F_0$ normal df) when $\alpha = 1/2$.

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<th>$2e_w (2(\delta^*)^2)$</th>
<th>$\frac{1}{2} \log(1 + r_0^2(\frac{1}{2}, \delta^*))$</th>
<th>Bahadur Efficiency$^#$ of $S$ with respect to $R$</th>
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$^\#$ See (6.13) and (6.24); here $F_0$ is assumed to be the normal df.
REFERENCES


REFERENCES (cont.)

A decision maker is confronted with k populations $\pi_1, \ldots, \pi_k$ (say, k lots of items available for purchase) and a control population $\pi_0$ which, on the basis of random samples of common size from $\pi_1, \ldots, \pi_k$, select those which are at least as good as $\pi_0$. It is assumed that items are judged on the basis of a continuously distributed attribute X and that a known fraction $\alpha$ ($0 < \alpha < 1$) of the items in the control population are deficient (their X-values are too small). Then $\pi_j$ is considered to be better than $\pi_i$ if it has a smaller proportion of deficient items, that is, if $x_\alpha(\pi_j) > x_\alpha(\pi_i)$ where $x_\alpha(\pi)$ denotes the $\alpha$th quantile of $\pi$. A nonparametric procedure $R$ based on order statistics is proposed as a solution for this problem; $R$ guarantees with a preassigned probability $P^*$ that, when each $\pi_j$ is stochastically ordered with respect to $\pi_0$, all populations better than $\pi_0$ will be selected. Further, for a preassigned fraction $\delta^*$, $\pi_j$ is considered to be $\delta^*$-inferior if $x_{1-\delta^*}(\pi_j) > x_{1-\delta^*}(\pi_0)$. Asymptotic expressions for the smallest sample size required to guarantee that the expected proportion of $\delta^*$-instructors selected by $R$ will be less than a specified number $\delta^*$ are obtained. Moreover, it is shown that for small values of $\delta^*$ a competing nonparametric procedure $S$ based on rank sums and a competing asymptotically nonparametric procedure $M$ based on sample means both require sample sizes proportional to the square of that required by $R$ to achieve the same degree of rejection of $\delta^*$-instructors; for moderate values of $\delta^*$, $S$ requires a sample size which has the same order of magnitude as that required by $R$. 
### Key Words

- Nonparametric
- Control population
- Quantile
- Stochastically ordered
- Order statistic
- \( Z^- \)-inferred
- Rank sum
- Pitman efficiency
- Bahadur efficiency

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