THE EQUIVALENCE OF FUNCTIONAL CENTRAL LIMIT THEOREMS FOR COUNTING PROCESSES AND ASSOCIATED PARTIAL SUMS

BY

DONALD L. IGLEHART AND WARD WHITT

TECHNICAL REPORT NO. 121

June 1969

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Gerald J. Lieberman, Project Director

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NONTECHNICAL SUMMARY

It is well known that limit theorems for renewal counting processes are intimately related to corresponding limit theorems for the sequence of partial sums of the times between events. The partial sum process is essentially the inverse of the renewal counting process. In this paper we prove that weak convergence theorems, functional central limit theorems, or invariance principles hold for general counting processes (no independence or common distribution assumptions) if and only if corresponding statements hold for the associated partial sum processes. The entire discussion is in the context of weak convergence of probability measures on the function space $D[0,1]$, cf. [1].

These results should be useful for many applications. We have used these results to prove limit theorems for stochastic processes arising in queues in heavy traffic, cf. [2] and [3]. Heavy traffic means that the traffic intensity in a single queueing system is greater than or equal to $1$ or the sequence of traffic intensities associated with a sequence of queueing systems approaches a limit greater than or equal to $1$. The heavy traffic limit theorems give useful descriptions of unstable queues and, perhaps, useful approximations for stable queues. The limits, usually simple functions of Brownian motion, are much more tractable than the typical queueing expressions.
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by
Donald L. Iglehart and Ward Whitt
Stanford University and Yale University

1. Introduction

Let \{u_n, n \geq 1\} be a sequence of nonnegative random variables, not necessarily independent or identically distributed, with an associated counting process \{N(t), t \geq 0\}, defined by

\[
N(t) = \begin{cases} 
\text{max}\{k: u_1 + \cdots + u_k \leq t\}, & u_1 \leq t \\
0, & u_1 > t.
\end{cases}
\]

We shall show that functional central limit theorems (invariance principles) for \(N(t)\) are equivalent to corresponding statements for the sequence of partial sums of the \(u_n\)'s. This equivalence exists because the counting process and the partial sum process are essentially

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inverses of each other.

Let \( \{X_n, n \geq 1\} \) be the usual sequence of random functions in \( D = D[0,1] \) induced by the sequence of partial sums; that is, let

\[
X_n(t) = (\sigma_n^{-2})^{-1/2} \sum_{i=1}^{[nt]} (u_i - \mu), \quad 0 \leq t \leq 1,
\]

where \( \mu \) and \( \sigma^2 \) are positive constants. Let \( \{N_n, n \geq 1\} \) be the corresponding sequence of random functions induced in \( D \) by \( N(t) \):

\[
N_n(t) = (\sigma_n^{2\mu^{-3}})^{-1/2} [N(nt) - nt/\mu], \quad 0 \leq t \leq 1.
\]

Finally, let \( W \) be the Wiener measure on \( D \). Billingsley [1, Theorem 17.3] has shown that if \( X_n \Rightarrow W \), then \( N_n \Rightarrow W \). (We use \( \Rightarrow \) to denote weak convergence of probability measures. When stochastic processes or ordinary random variables appear in such an expression, we mean the measures induced by these functions.) We shall prove the converse: if \( N_n \Rightarrow W \), then \( X_n \Rightarrow W \). Naturally, this converse is not very useful for showing that \( X_n \Rightarrow W \) when the conditions implying that \( N_n \Rightarrow W \) also directly imply that \( X_n \Rightarrow W \), which is the case when \( \{u_n\} \) is i.i.d., cf. [1, Theorem 16.1]. For nontrivial applications of this converse to queuing problems, see [2, Section 8].

We shall actually prove the necessary and sufficient conditions for weak convergence described above in a more general setting. We shall consider a double sequence of nonnegative random variables and allow the limit measure to be any measure on \( D \) which concentrates on \( C = C[0,1] \) with probability 1. This greater generality is motivated in part by
applications in queueing theory. In particular, the extension of Theorem 17.3 of [1] to double sequences and Theorem 3.1 of Prohorov (1956) are used heavily in [3]. For an application where the limit is not \( \mathcal{W} \), see [2, Theorem 8.4]. Finally, we remark that our result applies to sequences of random variables which are not nonnegative if we consider the maximum cumulative sum process instead of the cumulative sum process itself. Since we have no i.i.d. assumptions, the maximum cumulative sum process can be regarded as a new cumulative sum process in its own right. Our equivalence theorem thus applies to such maximum cumulative sum processes and associated counting processes.

2. The Results

Let \( \{u_i^j; i,j \geq 1\} \) be a double sequence of nonnegative random variables with no independence or common distribution assumptions. For each \( j \geq 1 \), form the counting process \( \{N_j^j(t), t \geq 0\} \), defined by

\[
N_j^j(t) = \begin{cases}
\max\{k: u_1^j + \cdots + u_k^j \leq t\}, & u_1^j \leq t \\
0, & u_1^j > t
\end{cases}
\]

Now construct the (single) sequences of random functions \( \{X_n\} \) and \( \{N_n\} \) in \( D_c \):

\[
X_n(t) = \left(\frac{1}{c_n}\right) \sum_{i=1}^{[a/t]} \left( (u_i^j - b_n^j) \right), \quad 0 \leq t \leq 1,
\]

\[
N_n(t) = \left(\frac{1}{d_n}\right) \sum_{j=1}^{[b/t]} \left( (u_j^j - a_n^j) \right), \quad 0 \leq t \leq 1,
\]

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and

\[ N_n(t) = \left( \frac{b_n}{c_n} \right) [N(a_n b_n t) - a_n t], \quad 0 \leq t \leq 1, \]

where \( \{a_n\}, \{b_n\}, \) and \( \{c_n\} \) are sequences of positive constants.

Our principal result is

**Theorem 1.** If \( a_n b_n/c_n \to \infty \) as \( n \to \infty \) and \( Y \in D \) with \( P(Y \in C) = 1 \), then \( X_n \Rightarrow Y \) if and only if \( N_n \Rightarrow -Y \).

If we have a single sequence \( \{y_n\} \), \( Y = W \), \( a_n = n \), \( b_n = \mu \), and \( c_n = \mu \), then the result mentioned in the introduction is obtained from Theorem 1. To see this, note that

\[
(\mu/\mu^{1/2})[N(\mu t) - \mu t] \equiv (\sigma_{\mu, \eta_m}^2)^{-1/2}[N(\mu t) - \mu t/\mu]
\]

if \( \mu = \eta_m \).

The central idea in Billingsley's proof of Theorem 17.3 of [1] is a random time change [1, p. 144]. Let \( D_0 \) consist of those functions \( \phi \) in \( \mathfrak{M} \) that are nondecreasing and satisfy \( 0 \leq \phi(t) \leq 1 \) for all \( t \), \( 0 \leq t \leq 1 \). Such functions represent transformations of the time interval \([0,1]\). The space \( D_0 \) is a closed subset of \( D \) and a complete separable metric space with Billingsley's version of the Skorohod metric on \( D \), cf. [1, Chapter 3]. Let \( \rho \) represent this metric and \( \rho \) the supremum metric on \( D \) and \( D_0 \).

Let \( \{Z_n\} \) be any sequence of random functions in \( D_0 \) and let \( \{\Phi_n\} \) be any sequence of random functions in \( D_0 \), with \( Z_n \) and \( \Phi_n \) defined.
on a common domain for each \( n \). Assume that the prospective limits \( Z \) and \( \Phi \) are also defined on a common domain. Billingsley [1, p. 145 and Theorem 4.4] has shown

**Lemma 1.** If \( Z_n \Rightarrow Z \) with \( P(Z \in C) = 1 \) and \( d(\Phi_n, \Phi) \Rightarrow 0 \) where \( \Phi \) is a constant function in \( C \cap D_0 \), then

\[
Z_n \circ \Phi_n \Rightarrow Z \circ \Phi,
\]

where

\[
Z_n \circ \Phi_n = Z_n(\Phi_n(t)), \quad 0 \leq t \leq 1,
\]

and

\[
Z \circ \Phi = Z(\Phi(t)), \quad 0 \leq t \leq 1.
\]

We shall also use Lemma 1 to prove Theorem 1. In order to prove the converse of [1, Theorem 17.3] and the corresponding part of Theorem 1, we shall introduce an inverse random time change \( \Phi^{-1} \), defined for any \( \Phi \in D_0 \) by

\[
\Phi^{-1}(\tau) = \begin{cases} 
\inf\{t \geq 0 : \Phi(t) \geq \tau, \quad 0 \leq t \leq 1\}, & \text{if the set is nonempty,} \\
1, & \text{otherwise}
\end{cases}
\]

for \( 0 \leq \tau \leq 1 \). The sample paths of \( \Phi^{-1} \) are left-continuous with limits from the right. For each \( \tau \), \( \Phi^{-1}(\tau) \) is a legitimate random variable, so \( \Phi^{-1} \) is a legitimate random function in \( D_0^L \), cf. [1, p. 128], where \( D_0^L \) is just \( D_0 \) with left-continuity instead of

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right-continuity for all functions. We could define a right-continuous version \( \Phi^{-1} e_{D_0} \) by setting

\[ \Phi^{-1}(\tau) = \lim_{s \uparrow \tau} \Phi^{-1}(s), \quad 0 \leq \tau \leq 1, \]

but it is not necessary; we can use \( \Phi^{-1} e_{D_0}^L \). With this inverse random time change, we shall prove

**Lemma 2.** Let \( Z_n \in D, \Phi_n \in D_0 \), and \( \phi \) be a strictly increasing constant function in \( C \cap D_0 \) with \( \phi(0) = 0 \) and \( \phi(1) = 1 \). If \( Z_n \circ \Phi_n \Rightarrow Z \circ \phi, d(\Phi_n, \phi) \Rightarrow 0, \) and \( P[Z \in C] = 1, \) then \( Z_n \Rightarrow Z \).

For an application of Lemma 2 other than the proof of Theorem 1, see [2, Section 7].

The connection between Theorem 1 and Lemmas 1 and 2 is made apparent by defining the sequence of random functions \( \{Y_n\} \) in \( D_0 \):

\[ Y_n(t) = \frac{1}{c_n} \sum_{i=1}^{N^n(a_n b_n t)} (u_{1,n} - b_n), \quad 0 \leq t \leq 1. \]

Notice that

\[ Y_n(t) \leq [a_n b_n t - b_n N^n(a_n b_n t)]/c_n \leq Y_n(t) + (1/c_n)u_{n}^{N^n(a_n b_n t)+1} \]

from which we shall deduce that \( \rho(Y_n - N_n) \Rightarrow 0 \). Also if we let

\[ \Phi_n(t) = \frac{N^n(a_n b_n t)}{a_n} \wedge 1, \quad 0 \leq t \leq 1, \]
then we shall show that $\rho(Y_n, X_n \circ \Phi_n) \Rightarrow 0$ and $\rho(\Phi_n, I) \Rightarrow 0$ where $I(t) = t$, $0 \leq t \leq 1$. Hence, $\rho(-N_n, X_n \circ \Phi_n) \Rightarrow 0$ and Theorem 1 follows. We now fill in the details.

3. The Proofs

In order to prove Lemma 2 and then Theorem 1, we first prove three technical lemmas.

Since C-tightness is much easier to apply than D-tightness, cf. [1, pp. 55, 125], we would like a condition giving C-tightness based on weak convergence in D. In a sense, we want a converse to Theorem 15.5 of [1]. The property of C-tightness in C or D is expressed in terms of the modulus of continuity $\tilde{w}_x(\delta): D \to R$, defined for any $x \in D$ by

$$\tilde{w}_x(\delta) = \sup_{0 \leq s, t \leq 1, |s-t| < \delta} |x(t) - x(s)| .$$

Lemma 3. Let $Z_n \in D$, $Z \in D$, and $P\{Z \in \mathcal{C}\} = 1$. If $Z_n \Rightarrow Z$, then \{Z_n\} is C-tight: for all positive $\epsilon$ and $\eta$, there exists a $\delta$ $(0 < \delta < 1)$ and an integer $n_0$ such that

$$P\{\tilde{w}_{Z_n}(\delta) > \epsilon\} \leq \eta$$

for $n \geq n_0$.

Proof. Theorem 5.1 of [1] implies that $\tilde{w}_{Z_n}(\delta) \Rightarrow \tilde{w}_Z(\delta)$ for each $\delta$, but $\tilde{w}_Z(\delta) \Rightarrow 0$ as $\delta \downarrow 0$. 

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Lemma 4. Let $\phi_n \in D_0$ and $\phi$ be a strictly increasing constant function in $C \cap D_0$. If $d(\phi_n, \phi) \to 0$, then $\rho(\phi_n^{-1}, \phi^{-1}) \to 0$.

Proof. Since $\phi \in C$, $d(\phi_n, \phi) \to 0$ implies $\rho(\phi_n, \phi) \to 0$. From the graph of $\phi$, it is evident that

$$|\phi_n^{-1}(\tau) - \phi^{-1}(\tau)| \leq |\phi^{-1}([\tau+\varepsilon] \land 1) - \phi^{-1}([\tau-\varepsilon] \lor 0)|$$

if $\rho(\phi_n, \phi) \leq \varepsilon$. Since $\phi^{-1}$ is uniformly continuous, the right side can be made arbitrarily small, uniformly in $\tau$, by choosing an appropriate $\varepsilon$.

Let $J: D \to R$ be the maximum jump functional, defined for any $x \in D$ by

$$J(x) = \sup_{0 \leq t \leq 1} \{|x(t) - x(t^-)|\}$$

For $x \in C$, $J(x) = 0$. (Hint: Use Lemma 3.)

Lemma 5. The function $J$ is measurable and continuous almost everywhere with respect to any measure concentrating on $C$ with probability 1.

Proof. The modulus of continuity $w_x(\delta): D \to R$ is measurable because

$$w_x(\delta) = \sup_{0 \leq s, t \leq 1} \frac{|x(t) - x(s)|}{|s - t| < \delta} = \sup_{0 \leq s, t \leq 1} \frac{|x(t) - x(s)|}{|s - t| < \delta, s, t \in Q}$$
where $Q$ is the set of rational numbers. Obviously, $w_x(\delta) \geq J(x)$ for all $\delta > 0$ and $w_x(\delta)$ decreases as $\delta$ decreases. Applying [1, Lemma 1, p. 110], we have

$$\lim_{\delta \downarrow 0} w_x(\delta) = J(x).$$

If we choose a sequence $\{\delta_n\}$ with $\delta_n \downarrow 0$, then $J$ is the limit of a sequence of measurable functions.

To show continuity, suppose $d(x_n, x) \rightarrow 0$. For $x \in \mathbb{C}$, $\rho(x_n, x) \rightarrow 0$, and

$$|J(x_n) - J(x)| = J(x_n) \leq \sup_{0 \leq t \leq 1} |x_n(t) - x(t)| + \sup_{0 \leq t \leq 1} |x_n(t) - x(t^-)| \leq 2\rho(x_n, x) \rightarrow 0.$$

Proof of Lemma 2. Lemmas 1 and 4 imply that

$$Z_n \circ \phi_n \circ \phi_n^{-1} \Rightarrow Z \circ \phi \circ \phi^{-1} = Z$$

and

$$\phi_n \circ \phi_n^{-1} \Rightarrow \phi \circ \phi^{-1} = I.$$

We complete the proof by showing that $\rho(Z_n \circ \phi_n \circ \phi_n^{-1}, Z_n) \Rightarrow 0$ and applying Theorem 4.1 of [1]. For all $n$, $\delta$, and $\epsilon$,
\[ P\{\rho(Z_n \cdot \Phi_n \cdot \Phi_n^{-1}, Z_n) \geq \varepsilon\} \leq \]

\[ P\{\hat{\mathcal{Z}}_n \cdot \Phi_n \cdot \Phi_n^{-1}(\theta) \geq \varepsilon\} + P\{\rho(\Phi_n \cdot \Phi_n^{-1}, I) > \theta\} . \]

Since \( P(Z \in C) = 1 \) and \( Z_n \cdot \Phi_n \cdot \Phi_n^{-1} \Rightarrow Z \), we have \( C \)-tightness for \( \{Z_n \cdot \Phi_n \cdot \Phi_n^{-1}\} \) by Lemma 3. Since \( I \in C \), \( \rho(\Phi_n \cdot \Phi_n^{-1}, I) \Rightarrow 0 \) and the proof is complete.

**Proof of Theorem 1.** In one direction, the assertion is a consequence of Lemmas 6, 7, and 8 to come and Lemma 1. In the other direction, the assertion is a consequence of Lemmas 9, 10, and 11 to come and Lemma 2. Throughout the following discussion assume \( P(Y \in C) = 1 \), \( a_n b_n / c_n \to \infty \) as \( n \to \infty \), and \( \Phi_n(t) = [N_n(a_n b_n t)/a_n] \wedge 1 \), \( 0 \leq t \leq 1 \).

**Lemma 6.** If \( X_n \Rightarrow Y \), then \( \rho(\Phi_n, I) \Rightarrow 0 \).

**Proof.** Since \( X_n \Rightarrow Y \) and \( a_n b_n / c_n \to \infty \), Theorems 4.4 and 5.1 of [1] imply that

\[ \left| \sum_{i=1}^{[a_n t]} (1/a_n b_n) \sum_{i=1}^{u_n} (u_n - b_n) \right| \to 0 . \]

We now apply the basic relationship: \( N^n(t) > m \) if and only if \( \sum_{i=1}^{m} u_i^n \leq t \). We have \( \inf \{(1/a_n)[N^n(a_n b_n t) - a_n t]\} > -\varepsilon \) if and only if \( 0 \leq t \leq 1 \)

if \( N^n(a_n b_n t) > a_n(t-\varepsilon), 0 \leq t \leq 1 \), or

\[ \sum_{i=1}^{a_n(t-\varepsilon)} u_i^n \leq a_n b_n t, \quad 0 \leq t \leq 1 , \]
or
\[ a_n(t-\epsilon) \leq \sum_{i=1}^{n-\epsilon} (u_i^n - b_i^n) \leq a_n \epsilon, \quad 0 \leq t \leq 1, \]

or
\[ \sup_{0 \leq t \leq 1-\epsilon} \left\{ (1/a_n) \sum_{i=1}^{n} (u_i^n - b_i^n) \right\} \leq \epsilon, \]

but we have just shown that the probability of this event approaches 1 as \( n \to \infty \).

A similar argument shows that for all positive \( \epsilon \)

\[ \lim_{n \to \infty} \sup_{0 \leq t \leq 1} \left\{ (1/a_n) [N_n(a_n b_t, t) - a_n t] \right\} < \epsilon = 1. \]

Thus \( \rho(F_n, I) \to 0 \).

**Lemma 7.** If \( X_n \Rightarrow Y \), then \( \rho(-N_n, Y_n) \Rightarrow 0 \).

**Proof.** By Lemma 5, \( J(X_n) \Rightarrow 0 \). By the proof of Lemma 6, \( N(a_n b_n)/a_n \Rightarrow 1 \). Hence,

\[ \rho(-N_n, Y_n) = \sup_{0 \leq t \leq 1} \left| (1/c_n)u_n^n \left[ N_n(a_n b_t, t) + 1 \right] \right| \to 0. \]

**Lemma 8.** If \( X_n \Rightarrow Y \), then \( \rho(X_n \circ \phi_n, Y_n) \Rightarrow 0 \).

**Proof.** We cannot assert that \( Y_n = X_n \circ \phi_n \) for sufficiently large \( n \) because of the behavior near \( t = 1 \). However, for all \( n, \delta, \) and \( \epsilon \),
\[ P\{\rho(X_n \circ \Phi_n, Y_n) \geq \varepsilon \} \leq P\{\hat{\psi}_{P_n}(\delta) \geq \varepsilon \} + P\{\rho(\Phi_n, I) \geq \delta \} \].

The proof is finished by applying Lemmas 3 and 6.

**Lemma 9.** If \( N_n \Rightarrow -Y \), then \( \rho(\Phi_n, I) \Rightarrow 0 \).

**Proof.** Recall that

\[ \rho(\Phi_n, I) \leq \sup_{0 \leq t \leq 1} |N_n(a_n b_n t)/a_n - t|, \]

but since \( N_n \Rightarrow -Y \) and \( a_n b_n/c_n \to \infty \), Theorems 4.4 and 5.1 of [1] imply that

\[ \sup_{0 \leq t \leq 1} |N_n(a_n b_n t)/a_n - t| \Rightarrow 0. \]

**Lemma 10.** If \( N_n \Rightarrow -Y \), then \( \rho(-N_n, Y_n) \Rightarrow 0 \).

**Proof.** If \( U_n = \sup_{0 \leq t \leq 1} \{(1/c_n)^n u^n_{P_n}(a_n b_n t)/a_n + 1 \} \), then we need to show that \( U_n \Rightarrow 0 \). Since \( P\{-Y \in C\} = 1 \), we have C-tightness for \( \{N_n\} \) by virtue of Lemma 3. We shall show that this C-tightness would be violated if we did not have \( U_n \Rightarrow 0 \). For each \( n \geq 1 \), there are time points \( t_1 \in [0,1] \) and \( t_2 \geq t_1 \) such that

\[ |t_2 - t_1| = \frac{c U_n}{a_n b_n} \]

and \( N_n(a_n b_n t_2) - N_n(a_n b_n t_1) = 0 \). Note that \( t_2 > 1 \) is possible, but it is easy to modify the proof to cover this situation. The statements above imply that
$$\hat{w}_{N_n}(\delta) \geq U_n \land \left( a_n b_n / c_n \right) \delta$$

where $$a_n b_n / c_n \to \infty$$. Hence, $$U_n \Rightarrow 0$$, and the proof is complete.

**Lemma 11.** If $$N_n \Rightarrow -Y$$, then $$\rho(X_n \circ \Phi_n, Y_n) \Rightarrow 0$$.

**Proof.** Use the argument in the proof of Lemma 8 with (c) tightness for $$\{Y_n\}$$ instead of $$\{X_n\}$$.
REFERENCES


The Equivalence of Functional Central Limit Theorems for Counting Processes and Associated Partial Sums

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Let \( \{u^j_i; i, j \geq 1\} \) be a double sequence of nonnegative random variables with no independence or common distribution assumptions. For each \( j \geq 1 \), form the counting process \( \{N_j(t), t \geq 0\} \), defined by \( N_j(t) = \max[k; u^j_1 + \cdots + u^j_k \leq t] \) if \( u^j_1 \leq t; N_j(t) = 0 \), otherwise. Then construct the sequences of random functions \( \{X_n\} \) and \( \{N_n\} \) in \( D[0,1] \):

\[
X_n(t) = (1/c_n) \sum_{i=1}^{a_n} (u^n_i - b^n_i), \quad 0 \leq t \leq 1
\]

and

\[
N_n(t) = \left(b_n/c_n\right)[N^n(a b \cdot t) - a b t], \quad 0 \leq t \leq 1
\]

where \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are sequences of positive constants. We prove that if \( a b_n/c_n \to \infty \) as \( n \to \infty \) and \( Y \in D \) with \( P\{Y \in C\} = 1 \), then \( X_n \Rightarrow Y \) if and only if \( N_n \Rightarrow -Y \).
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