A RELIABILITY REFORMULATION WITH APPLICATION TO COMPLEX SYSTEMS

BY

TIMOTHY M. CORCORAN

TECHNICAL REPORT NO. 135
MAY 10, 1971

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DEPARTMENT OF OPERATIONS RESEARCH
AND
DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
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Gerald J. Lieberman, Project Director

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ABSTRACT

This paper concerns the description of stochastic behavior of a class of complex systems. The class of problems considered is that of semi-Markov processes with finite expected lifetimes. A simple reformulation permits the easy derivation of additional results for the model. The reformulation consists of treating the process mathematically as though it restarts from the beginning upon completion of each lifetime. In this way limiting distributions in the modified process may be used to determine relevant characteristics of the original process.

The application examined is a complex system composed of parts connected in series each with independent exponential failure distributions. An additional spare system and a finite number of spare parts are provided. The spare parts are used for repair; the systems are interchangeable so that one may operate while the other is being repaired. Failure occurs when both systems are out of operation simultaneously or when the supply of spare parts is exhausted. The model combines features of the spare parts problem and the repairable item problem, both of which have been previously studied in the theory of reliability. The expected duration of total lifetime, operating lifetime, and time spent in any state is derived for the cases of exponential and general repair distributions. Modifications and extensions considered include the resupply of spare parts to the system. Appended is an extensive bibliography of reliability theory work in the description of stochastic behavior of complex systems.
CHAPTER I
INTRODUCTION

The purpose of this paper is twofold: first, to consider a reliability model which generalizes certain known models; second, in so doing, to illustrate a reformulation of reliability models to aid in the analysis of their stochastic behavior.

The method is applicable, but not necessarily limited to, models which can be formulated as a Semi-Markov process with finitely many states, and where the cost to the user of the system varies with the state. It has two advantages over other methods: (1) more complex systems can be considered, since the method of solution involves limiting probabilities rather than transient ones, and (2) a more precise reliability evaluation may be obtained, since the formulation is oriented toward state probabilities rather than a general evaluation of system reliability.

In Section 1.1 the method of solution is discussed in detail and compared with traditional methods. Section 1.2 describes the model to be considered, together with its relationship to existing results, and summarizes the remainder of the paper.
1.1. **The Solution Method.**

Many reliability models may be formulated as semi-Markov processes. Following the method of Cinlar [8] in our description, we consider a stochastic process \( X(t); t \geq 0 \) taking values in a countable state space \( \{1, 2, \ldots \} \).

**Definition.** A non-decreasing, right continuous nonnegative function \( F(\cdot) \) on \((-\infty, \infty)\) with \( F(\infty) \leq 1 \) is called a mass function. Let \( \mathcal{G} \) be a subset of \( \{1, 2, \ldots \} \).

Let \( \mathcal{A} \) be a matrix of mass functions \( A_{jk} \) \( (j,k \in \mathcal{G}) \) and let \( B_j = \sum_k A_{jk} \). Then \( \mathcal{A} \) is called a semi-Markov matrix over \( \mathcal{G} \) if

\[
B_j(0^-) = 0
\]
\[
B_j(\infty) \leq 1
\]

for all \( j \in \mathcal{G} \).

Suppose \( \mathcal{A} \) is a semi-Markov matrix defined on a subset \( \mathcal{G} \) of \( \{1, 2, \ldots \} \). Let \( (\Omega, \mathcal{F}, P) \) be a probability space and suppose defined on it

1. A function \( L(\omega), \omega \in \Omega \), taking values in \( \{1, 2, \ldots, \infty\} \).
2. Functions \( X_n(\omega), 0 \leq n < L(\omega), \omega \in \Omega \), taking values in \( \mathcal{G} \).
3. Functions \( T_n(\omega), 0 \leq n < L(\omega), \omega \in \Omega \), taking values in \([0, \infty)\),
   with \( 0 = T_0(\omega) \leq T_1(\omega) \leq \cdots \) for all \( \omega \in \Omega \).

Suppose that \( \mathcal{F} \) contains the \( \sigma \)-algebra generated by the set \( \{X_m = k, T_m \leq t, L > n, 0 \leq m \leq n, k \in \mathcal{G}, t \in [0, \infty)\} \) for each \( n \geq 0 \).
Definition. \( \{X_n, T_n; \ L\} \) is called a Markov renewal process induced by \( \mathcal{A} \) if

\[
P(X_{n+1} = j, T_{n+1} \leq t | X_0, \ldots, X_n, T_0, \ldots, T_n)
\]

\[
= P(X_{n+1} = j, T_{n+1} \leq t | X_n, T_n)
\]

\[
= A_X^n j(t - T_n)
\]

almost everywhere on \( \{L > n\} \), for any \( j \in \mathcal{J}, \ n \geq 0, \) and \( t \in [0, \infty) \).

The variables \( T_n \) represent the epoch of the \( n \)th transition, \( X_n \) the state entered, and \( L \) the total number of transitions (counting the first at \( T_0 = 0 \)) before the termination of the process.

For each \( \omega \in \Omega \) define the functions

\[
(1.1) \quad Z(\omega) = \sup_{0 \leq n \leq L(\omega)} T_n(\omega)
\]

\[
(1.2) \quad M(t, \omega) = \sup \{n | T_n(\omega) \leq t\}, \quad 0 \leq t \leq Z(\omega)
\]

\[
(1.3) \quad X(t, \omega) = X_{M(t, \omega)}(\omega), \quad 0 \leq t \leq Z(\omega).
\]

Definition. \( \{X(t), Z\} \) is called the semi-Markov process induced by \( \mathcal{A} \).

The random variable \( Z(\omega) \) represents the lifetime of the process, \( M(t, \omega) \) the total number of transitions in \( (0, t] \), and \( X(t, \omega) \) the state of the process at time \( t \).

Semi-Markov processes were introduced by Pyke ([21], [22]) and have been extensively studied. Applications have been presented by
Barlow [1], Fox [13], Srinivasan [23], and many others. Recent work and an excellent summary of results in the theory of Markov renewal processes and semi-Markov processes is given by Çinlar [8].

Many applications of Markov renewal processes to reliability have considered models allowing the repair of failed equipment. Models with a limited supply of spares, or with a limited supply of spares combined with repair, however, have not in general been analyzed by this method. The purpose of this paper is to indicate how this type of problem may be reformulated to allow the effective use of Markov renewal theory.

Consider a reliability model, which may be formulated as a semi-Markov or Markov renewal process, with the following characteristics:

(i) There is a subset T (of the state space S) of absorbing states.

(ii) The mean time to reach the set T is finite.

Under the assumptions (i) and (ii), every state in \( N = S \setminus T \) is a transient state, and the probability of the set \( N \) approaches 0 as \( t \to \infty \). A worthwhile goal is to determine \( P_i(t) = P(X(t) = i) \) for \( t \geq 0, i \in S \); or its transform \( P_i^*(\lambda) = \int_0^\infty e^{-\lambda t} P_i(dt) \). For models of even moderate complexity, however, inversion is a difficult process for finite values of \( t \). Limiting results can be obtained for more systems than can finite results.

A simple reformulation, however, enables the easy calculation of a useful result—the expected duration of time spent in any state in the interval \([0,2]\). Of course this quantity may be obtained without a reformulation, but it involves finite results. We obtain it as a function of limiting probabilities of an artificial process; thus it can be obtained for more systems than would be possible otherwise.
Assume the existence of a state \( s \in \mathbb{N} \) such that
\[
P(X(0) = s) = P_s(0) = 1.
\]

For \( \omega \in \Omega \) and \( s \in \mathbb{N} = S \setminus T \), define
\[
I_j(t, \omega) = \begin{cases} 1, & X(t, \omega) = j \\ 0, & \text{otherwise} \end{cases}
\]

\[
T_j(t, \omega) = \int_0^t I_j(u, \omega) \, du
\]

\[
T_j(Z, \omega) = \int_0^Z I_j(u, \omega) \, du.
\]

Since \( T_j(Z) \leq Z \) for every \( \omega \in \Omega \) \( (s \in \mathbb{N}) \) and since by assumption (ii) above, \( E(Z) < \infty \), the dominated convergence theorem implies

\[
E(T_j(Z)) < \infty.
\]  

The expected duration of time spent in state \( j \) in \([0, Z]\) is \( E(T_j(Z)) \).

We proceed to define a modified process \( X'(t) \), which will be indicated by primes, such that

\[
P'_j = \lim_{t \to \infty} P'_j(t) = \frac{E(T'_j(Z))}{E(Z)}, \quad j \in \mathbb{N},
\]

where \( P'_j(t) = P(X'(t) = j) \) and the limits \( P'_j \) exist for \( j \in \mathbb{N} \).
The Modified Process. Recall the original state space is \( S = N \cup T \).

The new state space is

\[
S' = N \cup \{\tilde{s}\}
\]

where \( \tilde{s} \) is an artificial starting state. For \( j \in N \), let

\[
A_{js}^i(t) = \sum_{k \in T} A_{jk}(t)
\]

\[
A_{sj}^i(t) = A_{sj}(t)
\]

and for \( j, k \) in \( N \)

\[
A_{jk}^i(t) = A_{jk}(t).
\]

The new semi-Markov matrix \( A_i^i \) induces a Markov renewal process \( \{X'_n, T'_n; L'\} \) and semi-Markov process \( \{X'(t); Z'\} \) with \( P'(L' = \infty) = 1 \). The new process returns to the starting state \( \tilde{s} \) when the original process entered the set \( T \) of absorbing states and terminated. If \( Z'_k \) denotes the kth time the state \( \tilde{s} \) is entered, the process \( \{Z'_k; k = 1, 2, \ldots \} \) is a renewal process. Mean recurrence time to state \( \tilde{s} \) is equal to \( E(Z) < \infty \) and all states are recurrent. Thus, by Theorem 8.1 in Feller ([12], p. 365),

\[
\lim_{t \to \infty} P_i'(t) = P_i'
\]

exists with \( \sum P_i' = 1 \) and \( P_i' \geq 0 \). We now prove
Theorem 1.1.

\begin{equation}
(1.5) \quad p'_i = \frac{E(T_i(Z))}{E(Z)}, \quad i \in N \setminus s
\end{equation}

\begin{equation}
(1.6) \quad p'_s + p'_s = \frac{E(T_s(Z))}{E(Z)}.
\end{equation}

Proof. The theorem is a direct consequence of the basic renewal theorem. For \( i \in S \setminus \{s, \bar{s}\} \), let \( p'_i(t) \) denote the joint probability in the modified process that \( X'(t) = k \) and \( Z'_i > t \). Then \( p'_i(T) = E I_i(t) \), where \( I_i(t) \) was above defined as 1 if \( X(t) = i \) and 0 otherwise, in the original process.

By Theorem 2, p. 349 in Feller [12],

\begin{equation}
(1.7) \quad \lim_{t \to \infty} p'_i(t) = \frac{1}{E(Z'_i)} \int_{0}^{\infty} p'_i(x) \, dx.
\end{equation}

But \( \lim_{t \to \infty} p'_i(t) = p'_i \) and \( E(Z'_i) = E(Z) \). Also

\[ \int_{0}^{\infty} p'_i(x) \, dx = \int_{0}^{\infty} E I_i(x) \, dx \]

\[ = E \int_{0}^{\infty} I_i(x) \, dx \]

\[ = E \int_{0}^{Z} I_i(x) \, dx \]

\[ = E T'_i(Z) \]
where the interchange is justified by Fubini's theorem, since \( T_i \geq 0 \) and \( E(T_i) \leq E(Z) < \infty \). Thus (1.7) becomes

\[
P'_i = \frac{E(T_i(z))}{E(Z)}
\]

which is (1.5). A similar argument holds for (1.6).

Finally, we note that if \( A_{js} = 0 \) for all \( j \in N \setminus \{s\} \), the creation of an artificial state \( \tilde{s} \) is unnecessary. This is because once the process leaves \( s \), it can never return until the next epoch \( Z'_k \). In this case, we may define

\[
A'_{is}(t) = \sum_{j \in T} A_{ij}(t), \quad i \in N
\]

\[
A'_{ij}(t) = A_{ij}(t), \quad i,j \in N
\]

and a result analogous to Theorem 1.1 holds.

1.2. The Reliability Problem.

The problem to be considered in this paper is based upon two well-known reliability models, which we describe briefly.

Model 1. The spare parts problem. A system is composed of \( I \) part types connected in series, each of which is subject to failure with d.f. \( G_i(t) = 1 - e^{-\lambda_i t} \). A number \( N_i - 1 \) \( (N_i \geq 1) \) of spare parts of type \( i \) are provided initially. System failure occurs when the number of failures of some part type \( i \) equals \( N_i \) for the first time.
Model 2. The repairable-item problem. A system consisting of only one part is set in operation. Failure is governed by the d.f. $G(t)$. Upon failure, the system is replaced by an identical spare system. The failed system is sent to a repair facility where repair is completed according to a d.f. $F(t)$. Upon completion of repair, the system now becomes a spare, ready for replacement. Failure occurs whenever both systems are simultaneously at the repair facility.

The model to be studied combines the features of Models 1 and 2 above as follows. A system is composed of $I$ part types, each subject to failure with d.f. $G_i(t) = 1 - e^{-\lambda_i t}$ as in Model 1. Upon failure, the system is sent to a repair facility where repair is completed according to the d.f. $F(t)$; it is replaced in operation by an identical spare system, as in Model 2.

Repair of failed systems uses up a part of the type which failed. There are $(N_i - 2)$ spare parts of type $i$ initially; added to the two parts in the systems gives a total of $N_i$. Failure occurs either (1) when both systems are at the repair facility, or (2) when the total number of failures of some part type $i$ equals $N_i$. The first type of failure is temporary, as operation resumes upon completion of repair of a system. The second type of failure is permanent; we call the time when it occurs the process lifetime and denote it by the r.v. $Z_i$. It is assumed that $N_i \geq 2$ for all $i$.

We make the assumption that the repair d.f. $F(t)$ is absolutely continuous with $F(0) = 0$, $F(\infty) = 1$. Since $F(t)$ is absolutely continuous, a density $f(t)$ exists and equals $F'(t)$ everywhere; we define $\mu(t)dt = f(t)/(1-F(t))$ to be the repair rate function; $\mu(t)dt = P[\text{repair is completed in } (t, t+dt) | \text{repair has not been completed in } [0,t)]$. 
A number of criteria could be used to evaluate this system. For example, the distribution of the length of time until first failure of any system is of interest; but it is deficient in this case since it considers only a part of the process. In fact, if we simplify by setting \( I = 1 \) and \( N = \infty \) (one part type with an infinite supply of spares) we have a single-server queue with finite waiting room of 1, described above as Model 2. If we then allow more than one spare system, we have the problem known variously as the "repairable item" of "spare parts problem with repair." This problem was investigated by Karusch [17], who derived steady state results in the case of exponential failure and a repair distribution represented as a series of exponential stages, using a Markov Process approach. Goodwin and Giese [15] considered the case of exponential failure and constant repair time and attempted to derive the distribution of time to first failure. Srinivasan [23] used Laplace Transforms to solve the case of a general failure and exponential repair distributions, and was able to derive from this method an expression for the first two moments of the distribution. The problem has also been considered by Barlow [1], Gaver [14], Natarajan [20], and others.

Another possible criterion is the distribution of the total process lifetime \( Z \) and the corresponding distribution of \( X \), the operating lifetime. In Chapter 2 we give expressions for \( X \) and \( Z \) in terms of the random sum of independent random variables, and from this derive the means of the two variables. It will be shown there that the problem involving \( X \) is merely the well-known "spare parts" problem, described above as Model 1, which has been considered by many authors. Black
and Proschan [5] originally solved this problem by showing the concavity of $\ln R(t)$, where $R(t)$ is the probability of system survival to time $t$, as a function of the numbers $N_1$ of spare parts. Bhattacharyya [4], considering a slightly different case including a holding cost for unused spares, has derived cost functions directly using failure probabilities given by Wilken and Langford [25]. This type of system has also been considered by Barnett [3], Epstein [11], and others.

In Chapter 2 the process will be analyzed as a function of underlying random variables, and results from renewal theory will be used to evaluate the means of $X_1$ and $Z_1$. Chapter 3 analyzes the process in the case of exponential repair according to the reformulation given in Section 1.1. Chapter 4 considers the case of a general repair time distribution. In Chapter 5, some extensions and modifications are considered. Also included is a comprehensive bibliography of references considering quantitatively the stochastic behavior of complex systems.
CHAPTER 2

THE PROCESS AS A FUNCTION OF UNDERLYING RANDOM VARIABLES

In this chapter we examine the reliability problem in detail. Section 2.1 gives expressions for the variables $X_k$, $Y_k$, and $Z_k$ in terms of random sums of certain underlying independent sequences of random variables. The presentation will be general in that expressions for $X_k$, $Y_k$, $Z_k$ are given, rather than only for $X_1$, $Y_1$, $Z_1$, although only the latter are required to evaluate the means. This is done in Section 2.2.

2.1. The Underlying Process.

Consider the reliability problem as defined in Section 1.2. Let $(\Omega, \mathcal{F}, P)$ be a probability triple. For $\omega \in \Omega$, a function $S(t, \omega)$ is defined representing the state of the process at time $t$, $0 \leq t < \infty$. The state space $S$ consists of nonnegative integer $(I+1)$-vectors $(r, n_1, \ldots, n_I)$ where $r$ denotes the number of systems in the repair shop and $n_i$ ($i = 1, \ldots, I$) denotes the number of spare parts remaining of type $i$. The exact range of the state space $S$ will be indicated in Chapter 3. For every $\omega \in \Omega$ the initial state $S(0, \omega)$ is $(0, N_1, \ldots, N_I)$ since originally $N_i$ parts of type $i$ are provided and no systems are in repair. For $\omega \in \Omega$, let $Z_i(\omega)$ denote the first process lifetime as defined in Section 1.2; $Z_1$ is
the first time such that the total number of failures of some part type \( i \) equals \( N_i \). We consider the modified process; thus at the epoch \( Z_1 \) the process returns to the state \((0, N_1, \ldots, N_i)\) and proceeds as before. For \( k = 2, 3, \ldots \) let \( Z_k(\omega) \) denote the \( k \)th process lifetime; i.e., let

\[
Z_k(\omega) = \text{kth time the process enters state } (0, N_1, \ldots, N_i) \text{ in the interval } (0, \infty).
\]

For any realization \( \omega \) of the process let

\[
I_X(t, \omega) = \begin{cases} 
1 & \text{S(t) an operating state} \\
0 & \text{otherwise} 
\end{cases}
\]

\[
I_Y(t, \omega) = \begin{cases} 
1 & \text{S(t) not an operating state} \\
0 & \text{otherwise} 
\end{cases}
\]

The operating states will be specified in Chapter 3. Now

\[
X_1(\omega) = \int_0^{Z_1} I_X(t, \omega) \, dt \quad \text{and} \quad Y_1(\omega) = \int_0^{Z_1} I_Y(t, \omega) \, dt.
\]

For \( k \geq 2, \)

\[
X_k(\omega) = \int_{Z_{k-1}(\omega)}^{Z_k(\omega)} I_X(t, \omega) \, dt \quad \text{and} \quad Y_k(\omega) = \int_{Z_{k-1}(\omega)}^{Z_k(\omega)} I_Y(t, \omega) \, dt.
\]

Clearly \( X_k + Y_k = Z_k \); \( X_k \) is the \( k \)th operating lifetime.

We proceed to represent \( X_k, Y_k, \) and \( Z_k \) (the \( \omega \) will be dropped from the notation when the meaning is clear) as random functions of underlying random variables. Let \( f_1, f_2, \ldots \) be a sequence of nonnegative, independent and identically distributed (iid) random
variables defined on \((\Omega, \mathcal{F}, \mathbb{P})\), such that \(P(f_1 \leq t) = 1 - e^{-\lambda t}\) where 
\[
\lambda = \lambda_1 + \cdots + \lambda_I \quad \text{for} \quad t \geq 0.
\]
Let \(c_1, c_2, \ldots\) be another sequence of iid random variables, independent of the \(\{f_n\}\), on \((\Omega, \mathcal{F}, \mathbb{P})\), such that 
\(P(c_i = 1) = \tilde{\lambda}_i\) where 
\[
\tilde{\lambda}_i = \lambda_i/(\lambda_1 + \cdots + \lambda_I) \quad \text{for} \quad i = 1, \ldots, I.
\]
Let \(r_1, r_2, \ldots\) be a third sequence of iid random variables on \((\Omega, \mathcal{F}, \mathbb{P})\), independent of the \(\{f_n\}\) and \(\{c_n\}\), such that 
\(P(r_1 \leq t) = F(t)\), where \(F(t)\) is an absolutely continuous distribution function with \(F(0) = 0\) and \(F(\infty) = \lim_{t \to \infty} F(t) = 1\). Assume \(r_1\) has finite expectation \(1/\mu\). Let \(r\) and \(f\) be random variables distributed as \(r_1\) and \(f_1\), respectively.

For intuitive understanding, we note that \(f_n\) is the nth "inter-failure" time, \(c_n\) denotes the part type which fails, and its repair time is \(r_n\). Let \(k_i(n) = 1\) if \(c_n = i\) and zero otherwise and let \(K_i(n) = k_i(1) + \cdots + k_i(n)\). Then \(K_i(n)\) denotes the number of failures of type \(i\) among the first \(n\) failures.

For given values of \((N_1, \ldots, N_I)\), let \(w_{i1} = \min(n|K_i(n) = N_i - 1)\) and \(x_{i1} = \min(n|K_i(n) = N_i)\), and let \(w_1 = \min(w_{i1}|i = 1, \ldots, I)\) and \(x_1 = \min(x_{i1}|i = 1, \ldots, I)\). The value \(x_1\) represents the number of failures required to run out of some part type \(i\). Let 
\[
W_1 = \sum_{n=1}^{w_1} f_n \quad \text{and let}\]
\[
X_1 = \sum_{n=1}^{x_1} f_n.
\]

The variables \(X_1\) and \(W_1\) represent the first time that either zero or one, respectively, part is left of some type \(i\). Each of the variables \(w_1, x_1, W_1, X_1\) are stopping times; i.e., the event \(\{w_1 > n\}\) is a
measurable function of \( \Theta(f_1, c_1, r_1, \ldots, f_n, c_n, r_n) \), the \( \sigma \)-algebra generated by the variables \( (f_1, c_1, r_1, \ldots, f_n, c_n, r_n) \), etc.

For \( k \geq 1 \), let \( w_{ik} = \min\{n | K_i(n) - K_i(x_{k-1}) = N_i-1\} \),
\( x_{ik} = \min\{n | K_i(n) - K_i(x_{k-1}) = N_i\} \), \( w_k = \min\{w_{ik} | i = 1, \ldots, I\} \),
\( x_k = \min\{x_{ik} | i = 1, \ldots, I\} \), \( \bar{W}_k = \sum_{n=x_{k-1}+1}^{x_k} f_n \), and
\[
X_k = \sum_{n=x_{k-1}+1}^{x_k} f_n .
\]

(2.2)

These variables are also stopping times, and by the Strong Markov Property the sequences \( \{X_k; k \geq 1\} \), \( \{\bar{W}_k; k \geq 1\} \), etc. are each sequences of iid random variables. We shall see below that the \( X_k \) here defined is the desired \( X_k \).

Define \( \theta_1 = 0 \) and
\[
\theta_n = (r_{n-1} - f_n)^+, \quad n = 2, 3, \ldots
\]
to be the positive part of the difference between \( r_{n-1} \) and \( f_n \), where \( x^+ = \max(x, 0) \). Also define
\[
I_n = I(r_{n-1} > f_n)
\]

(2.3)
\[
= \begin{cases} 
1 & r_{n-1} > f_n \\
0 & \text{otherwise}
\end{cases}
\]

Let
\begin{equation}
Y_1 = \sum_{n=2}^{w_1-1} O_n + I_{w_1} (r_{w_1-1} - f_{w_1}) + (1 - I_{w_1}) r_{w_1} + \sum_{n=w_1+1}^{x_1-1} r_n
\end{equation}

and for \( k \geq 2 \)

\begin{equation}
Y_k = \sum_{n=x_{k-1}+2}^{w_k-1} O_n + I_{w_k} (r_{w_k-1} - f_{w_k}) + (1 - I_{w_k}) r_{w_k} + \sum_{n=w_k+1}^{x_k-1} r_n
\end{equation}

For \( k = 1, 2, \ldots \) let

\begin{equation}
Z_k = X_k + Y_k
\end{equation}

To see that (2.1) - (2.2) and (2.4) - (2.6) are the appropriate definitions let us consider the following illustrative example of a realization of the process. For any time \( t \geq 0 \), let \( r(t) \) denote the number of systems in the repair shop at time \( t \). The values of \( r(t) \) as a function of the \( f_n \) and \( r_n \) are shown in Figure 2.1 and Figure 2.2. There are two cases depending on whether \( r_{w_1-1}(\omega) < f_{w_1}(\omega) \) or \( r_{w_1-1}(\omega) > f_{w_1}(\omega) \).

In the figures \( r(t, \omega) \) is shown as a function of \( t \) for given \( \omega \in \Omega \). For \( 1 \leq n \leq w_1 \), let \( g_n = f_n + o_n = \max(f_n, r_{n-1}) \) and let \( G_n = g_1 + \cdots + g_n \). Let \( \bar{W}_1 = G_{w_1-1} + f_{w_1} \). Above the graph of \( r(t) \) in each figure is an indication for each \( t \) of whether \( I_X(t) = 1 \) (\( S(t) \) is an operating state) or \( I_Y(t) = 1 - I_X(t) = 1 \) (\( S(t) \) is not an operating state). Above that are shown the values of \( f_n', r_n', \) and \( o_n' \). The \( f_n', r_n', \) and \( o_n' \) determine \( r(t) \) which in turn determines \( I_X(t) \) and \( I_Y(t) \). Two rules apply:
(i) For $0 \leq t \leq \tilde{W}_1 = G_{w_1-1} + f_{w_1}$, $I_Y(t) = 1$ if and only if $r(t) = 2$.

(ii) For $\tilde{W}_1 < t \leq Z_1$, $I_Y(t) = 1$ if and only if $r(t) = 1$.

For $t \leq \tilde{W}_1$, a system operates unless both are in the repair shop ($r(t) = 2$). For $t > \tilde{W}_1$, (ii) holds since $\tilde{W}_1$ is by definition the epoch of the ($w_1$)st failure; i.e., at $\tilde{W}_1$ the number of failures of some part type $i$ reaches $N_i-1$. After this time only one complete system may be formed from the remaining operable parts; when this system is in repair ($r(t) = 1$) no spare system is available and thus $I_Y(t) = 1$. Notice that

$$
\tilde{W}_1(\omega) = \int_0 \mathcal{I}_X(t,\omega) dt.
$$

We now follow through the example. The two figures are identical for $0 \leq t \leq G_{w_1-1}$. At $t = 0$ the system starts and $I_X(t) = 1$; $r(t)$ starts at 0 and remains there until the first failure at $t = f_1 = g_1 = G_1$. At this point the spare system begins operation and the first goes to repair. If the next event is a repair completion as is shown at $t = G_1 + r_1$, $r(t)$ returns to 0 and stays there until another failure. If the next event is another failure, as shown at $t = G_2 + f_2$, $r(t)$ jumps to 2. A non-operating state is thus entered; $I_Y(t)$ becomes 1 and $I_X(t)$ becomes 0. No system can operate until the first repair is completed, shown at $t = G_3$. At this point the repaired system commences operation and repair starts on the other system; thus $I_X(t)$ becomes 1, $I_Y(t)$ becomes 0, and $r(t)$ drops from 2 to 1. The process continues back and forth this way until the epoch $t = G_{w_1-1}$. It may be seen from the figure that

$$
\int_0^{G_{w_1-1}} I_Y(t) dt = \sum_{n=2}^{w_1-1} 0.
$$

(2.7)
After the epoch $G_{w_1-1}$ there are two cases. The first, shown in Figure 2.1, is $r_{w_1-1} < f_{w_1}$; in this case $\bar{W}_1 = G_{w_1}$. When the $(w_1)$th failure occurs at epoch $\bar{W}_1$, the character of the process changes. No longer are there enough spare parts of all types to maintain an extra system and so the process halts at each failure and waits for the system to be repaired. Hence for $G_{w_1} \leq t \leq Z_1$, $r(t)$ is either 0 or 1 and $I_Y(t) = 1 - I_X(t) = r(t)$. The process continues until the $(x_1)$th failure which exhausts the supply of some part type. Thus in the case $r_{w_1-1} < f_{w_1}$,

$$
(2.8) \quad \int_{G_{w_1-1}}^{Z_1} I_Y(t) dt = \sum_{n=w_1}^{x_1-1} r_n.
$$

Since $Y_1 = \int_{0}^{Z_1} I_Y(t) dt$, we have from (2.7) and (2.8),

**Case I:** If $r_{w_1-1} < f_{w_1}$

$$
X_1 = \sum_{n=1}^{x_1} f_n, \quad Y_1 = \sum_{n=2}^{w_1-1} Y_n + \sum_{n=w_1}^{x_1-1} r_n, \quad Z_1 = X_1 + Y_1.
$$

Case II is shown in Figure 2.2 and differs in form from Case I only near the epoch $\bar{W}_1$. In this case the $(w_1)$th failure occurs again at the epoch $G_{w_1-1} + f_{w_1}$, but here $G_{w_1-1} + f_{w_1} = \bar{W}_1 < G_{w_1-1} + R_{w_1-1} = G_{w_1}$.
The term $r_{w_1}$ no longer appears in the expression for $Y_1$; it is replaced by the term $(r_{w_1-1} - f_{w_1})$. The reason for the difference is easily seen. Suppose $i$ denotes the part type causing the $(w_1)$th failure. By definition of $w_1$, immediately after the epoch $\tilde{w}_1$, there remains only one unfailed part of type $i$. But since in the case $r_{w_1-1} > f_{w_1}$ there is already a system in repair at epoch $\tilde{w}_1$, the remaining part of type $i$ must be in the system already in repair and thus repair of the $(w_1)$th failure never takes place. Hence in this case

$$Z_1 = \int_{G_{w_1-1}} I_X(t)dt = (r_{w_1-1} - f_{w_1}) + \sum_{n=w_1+1}^{x_1-1} r_n$$

(2.9)

and we have from (2.7) and (2.9)

**Case II:** If $r_{w_1-1} > f_{w_1}$

$$X_1 = \sum_{n=1}^{x_1} f_n$$

$$Y_1 = \sum_{n=2}^{w_1-1} O_n + (r_{w_1-1} - f_{w_1}) + \sum_{r=w_1+1}^{x_1-1} r_n$$

Recalling the definition of $I_n = I_{(r_{n-1} < f_n)}$ from (2.3), Cases I and II may be combined to give

$$Y_1 = \sum_{n=2}^{w_1-1} O_n + I_{w_1}(r_{w_1-1} - f_{w_1}) + (1 - I_{w_1})r_{w_1} + \sum_{n=w_1+1}^{x_1-1} r_n$$
which is (2.4). Clearly \( X_1 = \sum_{n=1}^{\infty} f_n \) and \( Z_1 = X_1 + Y_1 \) in both cases; the both cases; the definitions of \( X_k, Y_k, \) and \( Z_k \) for \( k \geq 2 \) are similarly verified.

Thus \( \{X_k | k = 1, 2, \ldots \}, \{Y_k | k = 1, 2, \ldots \}, \) and \( \{Z_k | k = 1, 2, \ldots \} \) are sequences of iid random variables defined on \((\Omega, \mathcal{F}, P)\) with the desired distributions, and such that for every \( \omega \in \Omega \),

\[
X_k(\omega) + Y_k(\omega) = Z_k(\omega).
\]

2.2. Expectations of \( X_1, Y_1, \) and \( Z_1 \).

We first present a well-known result from renewal theory.

**Theorem 2.1.** Let \( X_1, X_2, \ldots \) be a sequence of nonnegative iid random variables with finite expectation \( E(X_1) \). Let \( S \) and \( T \) be nonnegative integer-valued variables with finite means \( E(S) \) and \( E(T) \). Let \( \mathcal{F}_k \) be the Borel field generated by \( \{X_1, \ldots, X_k\} \) and assume that for all \( k \geq 1 \), \( (S \leq k) \in \mathcal{F}_k \) and \( (T \leq k) \in \mathcal{F}_k \), and that \( P(T-S \geq 0) = 1 \). Then

\[
E(\sum_{k=1}^{S} X_k) = E(S) E(X_1)
\]

\[
E(\sum_{r=S+1}^{T} X_k) = [E(T) - E(S)] E(X_1).
\]

**Proof:** See Chung [6], p. 128.
We now calculate the expectation of $X_1$. Recall from (2.1) that

$$X_1 = \sum_{n=1}^{x_1} f_n.$$ 

Since $x_1$ is a function only of the variables $c_n$, and the sequence \{c_n\} is independent of \{f_n\}, we have by Theorem 2.1:

$$E(W_1) = E(w_1) \cdot E(f)$$

(2.10)

$$E(X_1) = E(x_1) \cdot E(f)$$

where $f$ is a random variable distributed as $f_1$. It is at this point in the analysis that the assumption of an exponential failure rate is necessary and crucial. If the failure distributions of the parts were not exponential, not only would the \{f_n\} not be iid, but they would also be dependent upon the $c_n$. Here, however, even though the failure rates of the various part types differ, the failure times \{f_n\} and the part types which fail \{c_n\} are independent of each other and of the history of the process. This property, as is of course well known, is unique to the exponential.

Returning to (2.10), the expectation of $f$ is $1/\lambda$ since $P(f \leq t) = 1 - e^{-\lambda t}$. Let us now consider the r.v. $w_1$. It may take on the values $(\min N_{i-2}, \min N_{i-1}, \ldots, \sum_{i=1}^{\frac{J}{2}} (N_i-2) + 1)$, depending on the order of failure of the parts. We have for any positive integer $n$: 22
\[ P(w_1 > n) = P(K_i(n) \leq N_i - 2; i = 1, \ldots, I \mid \sum_{i=1}^I K_i(n) = n) \]

and by elementary probability

\[ P((K_1(n), \ldots, K_I(n)) = (k_1, \ldots, k_I) \mid \sum_{i=1}^I K_i(n) = n) \]

\[ = \left( \begin{array}{c} n \\ k_1 \cdots k_I \end{array} \right)^{k_1 \cdots k_I} \lambda_1^{-k_1} \cdots \lambda_I^{-k_I}, \]

a multinomial distribution, where \( \left( \begin{array}{c} n \\ k_1 \cdots k_I \end{array} \right) \) denotes \( \frac{n!}{k_1! \cdots k_I!} \)

for \( \sum_{i=1}^I k_i = n \) and \( k_i \) nonnegative integers. Thus

\[ P(w_1 > n) = \sum_{(k_i) \in S_n} \left( \begin{array}{c} n \\ k_1 \cdots k_I \end{array} \right)^{k_1 \cdots k_I} \lambda_1^{-k_1} \cdots \lambda_I^{-k_I} \]

where \( S_n = \{(k_1, \ldots, k_I) \mid \sum k_i = n, k_i \leq N_i - 2, i = 1, \ldots, I\} \).

Since for integer-valued variables \( X \) we have \( E(X) = \sum_{n=0}^{\infty} P(X > n) \),

\[ (2.11) \quad E(w_1) = \sum_{n=0}^{\infty} P(w_1 > n) \]

\[ = \sum_{n=1}^{\infty} \sum_{(k_i) \in S_n} \left( \begin{array}{c} n \\ k_1 \cdots k_I \end{array} \right)^{k_1 \cdots k_I} \lambda_1^{-k_1} \cdots \lambda_I^{-k_I} \]

\[ = \sum_{k_1=0}^{N_1-2} \cdots \sum_{k_I=0}^{N_I-2} \left( \sum_{i=1}^I k_i \right)^{k_1 \cdots k_I} \lambda_1^{-k_1} \cdots \lambda_I^{-k_I} \]

by definition of \( S_n \). This implies
Lemma 2.2.

\[
E(W_1) = \frac{1}{\lambda} \sum_{k_1=0}^{N_1-2} \cdots \sum_{k_I=0}^{N_I-2} \left( \sum_{i=1}^{I} k_i \right) \lambda_1^{-k_1} \cdots \lambda_I^{-k_I}
\]

and

\[
E(X_1) = \frac{1}{\lambda} \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_I=0}^{N_I-1} \left( \sum_{i=1}^{I} k_i \right) \lambda_1^{-k_1} \cdots \lambda_I^{-k_I}.
\]

Proof. The proof for \( W_1 \) is immediate by (2.10) and (2.11), and the proof for \( X_1 \) is similar.

Of course the exact distribution of \( X_1 \) may be determined by taking \( P(X_1 > t) \), and thus \( E(X_1) \) evaluated directly. We have used the above derivation, however, as a model for \( Y_1 \). Recall the definition of \( Y_1 \) given above in (2.4):

\[
Y_1 = \sum_{n=2}^{w_1-1} O_n + I_{w_1} (r_{w_1-1} - f_{w_1}) + (1 - I_{w_1}) r_{w_1} + \sum_{n=w_1+1}^{x_1-1} r_n
\]

(2.4)

where \( r_n \) is distributed as \( r(P(r \leq t) = P(t) \text{ and } E(r) = 1/\mu) \), \( f_n \) as \( f(P(f \leq t) = 1 - e^{-\lambda t}) \), \( O_n = (r_n - f_{n+1})^+ \) is distributed as \( (r-f)^+ \), and \( I_n \) is the indicator of the event \( (r_{n-1} - f_n) > 0 \).

Since, again, \( w_1 \) and \( x_1 \) are independent of \( f \) and \( r \), we have by Theorem 2.1:

\[
E(Y_1) = [E(r-f)^+] \cdot [E(w_1)-2] + E[I_{w_1} (r_{w_1+1} - f_{w_1}) + (1 - I_{w_1}) r_{w_1}]
\]

\[
+ E(r) \cdot [E(x_1) - E(w_1) - 1].
\]

(2.12)
Since \( P[w_1 \geq 2] = P[x_1 - w_1 \geq 1] = 1 \), we may extract the integers as shown.

A useful tool in the analysis of Markov processes and nonnegative random variables is the Laplace (Stieltjes) Transform or LST. For a nonnegative random variable \( X \) with distribution function \( F(t) \) \((F(0^-) = 0)\), the LST of \( X \) is defined to be

\[
F^*(s) = e^{-sX} = \int_0^\infty e^{-st} dF(t).
\]

If \( F(t) \) is absolutely continuous, the Riemann integral exists, and the function may be referred to simply as the Laplace Transform and written

\[
F^*(s) = \int_0^\infty e^{-st} f(t) dt.
\]

Several well-known properties are: \( F^*(s) \) exists for all \( s \geq 0 \), \( F^*(0) = 1 \), \( F^*(s) \) is decreasing in \( s \) for \( s \geq 0 \), and \( \lim_{s \to \infty} F^*(s) = 0 \). \( F^*(\cdot) \) and \( F(\cdot) \) uniquely determine each other. (See, e.g., Chung [6].)

To calculate \( E(Y_1) \) from (2.12), we first find \( E(r-f)^+ \).

\[
(2.13) \quad E(r-f)^+ = \int_0^y \int_0^y (y-x) \lambda e^{-\lambda x} f(y) \, dx \, dy
\]

be definition. The inside integral may be written

\[
(2.14) \quad \int_0^y (y-x) \lambda e^{-\lambda x} \, dx = y \int_0^y \lambda e^{-\lambda x} \, dx - \int_0^y x \lambda e^{-\lambda x} \, dx
\]

\[ = y(1-e^{-\lambda y}) - \int_0^y x \lambda e^{-\lambda x} \, dx. \]
Integrating by parts gives for the second integral in (2.14)

\[ \int_0^y x(-\lambda) e^{-\lambda x} \, dx = ye^{-\lambda y} + \frac{1}{\lambda} e^{-\lambda y} - \frac{1}{\lambda} \]

and therefore (2.13) becomes

\[
(2.15) \quad \mathbb{E}(r-f)^+ = \int_0^\infty f(y) \left[ y + \frac{1}{\lambda} e^{-\lambda y} - \frac{1}{\lambda} \right] \, dy \\
= \frac{1}{\mu} + \frac{1}{\lambda} \cdot \mathbb{P^*(\lambda)} - \frac{1}{\lambda} \\
= \frac{1}{\mu} - \frac{1}{\lambda} (1 - \mathbb{F^*(\lambda)}) .
\]

Returning to (2.12), the expectation of \( r \) is \( \frac{1}{\mu} \) by definition, and those of \( x_1 \) and \( w_1 \) were given above in (2.11). All that remains is the middle term in (2.12). Since \( I_{w_1} = 1 \) if \( (r_{w_1} - 1)^- f_{w_1} \geq 0 \), and \( I_{w_1} \) is independent of \( r_{w_1} \), we may write the term as

\[
\mathbb{E}(r-f)^+ + \mathbb{P}(r \leq f) \mathbb{E}(r) .
\]

As in (2.13) we write
\[
P(r \leq f) = \int_0^\infty \int_y^\infty f(y) \lambda e^{-\lambda x} \, dx \, dy
\]

\[
= \int_0^\infty f(y) \left[ \int_y^\infty \lambda e^{-\lambda x} \, dx \right] \, dy
\]

\[
= \int_0^\infty f(y) \, e^{-\lambda y} \, dy
\]

\[
= F^*(\lambda)
\]

Thus the second term in (2.12) is

\[
(2.16) \quad E(r-f)^+ + P(r \leq f) \, E(r) = \left[ \frac{1}{\mu} - \frac{1}{\lambda} \left( 1 - F^*(\lambda) \right) \right] + \frac{1}{\mu} \, F^*(\lambda)
\]

and we have

**Lemma 2.2.**

\[
(2.17) \quad E(Y_1) = E(w_1) \left[ \frac{1}{\mu} - \frac{1}{\lambda} \left( 1 - F^*(\lambda) \right) \right] + \left[ E(x_1) - E(w_1) \right] \frac{1}{\mu}
\]

\[
- \frac{1}{\mu} \left( 1 - F^*(\lambda) \right) - \left[ \frac{1}{\mu} - \frac{1}{\lambda} \left( 1 - F^*(\lambda) \right) \right] \cdot
\]

**Proof:** By (2.12), (2.15), and (2.16),

\[
E(Y) = E(w_1) \left[ \frac{1}{\mu} - \frac{1}{\lambda} \left( 1 - F^*(\lambda) \right) \right] - 2\left[ \frac{1}{\mu} - \frac{1}{\lambda} \left( 1 - F^*(\lambda) \right) \right]
\]

\[
+ \left[ \frac{1}{\mu} - \frac{1}{\lambda} \left( 1 - F^*(\lambda) \right) \right] + \frac{1}{\mu} \, F^*(\lambda) + \frac{1}{\mu} \left[ E(x_1) - E(w_1) \right] - \frac{1}{\mu}
\]

which is (2.17).  

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Define
\[ \rho = \frac{\Delta}{\mu} \]
and let
\[
(2.18) \quad h(N_1, \ldots, N_I) = \sum_{k_1=0}^{N_1} \cdots \sum_{k_I=0}^{N_I} \binom{k_1 + \cdots + k_I}{k_1 \cdots k_I} \lambda_1 \cdots \lambda_I
\]
\[
(2.19) \quad g(N_1, \ldots, N_I) = h(N_1, \ldots, N_I) - h(N_1-1, \ldots, N_I-1)
\]
Then from (2.11), we have \( E(w_1) = h(N_1-2, \ldots, N_I-2) \) and \( E(x_1) = h(N_1-1, \ldots, N_I-1) \).

**Theorem 2.4.** Let \( X_1 \) denote operating time and \( Y_1 \) denote non-operating time, respectively, in the time interval \( [0, Z_1] \), where \( Z_1 \) is the process lifetime. Then if \( F(t) \) is the repair distribution function, \( \frac{1}{\mu} \) the expected repair time, \( F^*(s) \) the Laplace-Stieltjes Transform of \( F(\cdot) \), \( \lambda_i \) the failure rate for part type \( i \) \((i = 1, \ldots, I, I \geq 2) \), and \( \tilde{\lambda}_i = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_I} \),

\[
E(X_1) = \frac{1}{\lambda} h(N_1-1, \ldots, N_I-1)
\]
\[
E(Y_1) = \frac{1}{\lambda} [h(N_1-1, \ldots, N_I-1) (\rho - (1-F^*(\lambda))
\]
\[+ g(N_1-1, \ldots, N_I-1) (1-F^*(\lambda)) - (\rho(2-F^*(\lambda))) - (1-F^*(\lambda))]\]
\[ E(Z_1) = \frac{1}{\lambda} \left[ h(N_1-1, \ldots, N_I-1) (\rho + F^*(\lambda)) \right. \]
\[ + g(N_1-1, \ldots, N_I-1) \left( 1 - F^*(\lambda) \right) - \rho(2 - F^*(\lambda)) + (1 - F^*(\lambda)) \] \]

where \( \lambda = \lambda_1 + \cdots + \lambda_I \), \( \rho = \frac{\lambda}{\mu} \), \( N_1 (\geq 2) \) is the number of parts initially provided,

\[ h(N_1, \ldots, N_I) = \sum_{k_1=0}^{N_1} \cdots \sum_{k_I=0}^{N_I} \left( \begin{array}{c} k_1 + \cdots + k_I \\ k_1 \cdots k_I \end{array} \right) \frac{k_1}{\lambda_1} \cdots \frac{k_I}{\lambda_I}, \]

and \( g(N_1, \ldots, N_I) = h(N_1, \ldots, N_I) - h(N_1-1, \ldots, N_I-1) \). Each term in the above expressions is nonnegative and in the case where the repair distribution is exponential:

\[ E(Y_1) = \frac{1}{\lambda} \left[ h(N_1-1, \ldots, N_I-1) \left( \frac{\rho^2}{1+\rho} \right) + g(N_1-1, \ldots, N_I-1)(\frac{\rho}{1+\rho}) - \frac{2\rho}{1+\rho} \right] \]

\[ E(Z_1) = \frac{1}{\lambda} \left[ h(N_1-1, \ldots, N_I-1) \left( 1 + \frac{\rho^2}{1+\rho} \right) + g(N_1-1, \ldots, N_I-1)(\frac{\rho}{1+\rho}) - \frac{2\rho^2}{1+\rho} \right]. \]

**Proof:** The value of \( E(X_1) \) is given by Lemma 2.2. From Lemma 2.3,

\[ E(Y_1) = \frac{1}{\lambda} \left[ E(w)(\rho - (1 - F^*(\lambda))) + (E(x) - E(w))\rho - (\rho(2 - F^*(\lambda)) - (1 - F^*(\lambda))) \right] \]

\[ = \frac{1}{\lambda} \left[ E(x)(\rho - (1 - F^*(\lambda))) + (E(x) - E(w))(1 - F^*(\lambda)) - (\rho(2 - F^*(\lambda)) - (1 - F^*(\lambda))) \right] \]

\[ = \frac{1}{\lambda} \left[ h(N_1-1, \ldots, N_I-1)(\rho - (1 - F^*(\lambda))) + g(N_1-1, \ldots, N_I-1)(1 - F^*(\lambda)) \right. \]
\[ - (\rho(2 - F^*(\lambda)) - (1 - F^*(\lambda))). \]
The result for $E(Z_1)$ follows immediately. In the exponential case,

$$F(t) = 1 - e^{-\mu t} \quad \text{and} \quad F^*(\lambda) = \frac{\mu}{\mu + \lambda} = \frac{1}{1 + \rho}.$$  

Thus $1 - F^*(\lambda) = \frac{\rho}{1 + \rho}$, and

$$\rho - (1 - F^*(\lambda)) = \frac{\rho^2}{1 + \rho}.$$  

The final term is

$$\rho(2 - F^*(\lambda)) - (1 - F^*(\lambda)) = \rho(1 + \frac{\rho}{1 + \rho}) - \frac{\rho}{1 + \rho} = \frac{2\rho^2}{1 + \rho}.$$

To verify that all terms are nonnegative it suffices to show that

$$F^*(\lambda) \geq 1 - \rho = 1 - \frac{\lambda}{\mu}.$$  

If $\lambda \geq \mu$ the result is immediate since $F^*(\lambda) > 0$ for all $\lambda < \infty$. (Note we do not need to restrict $\lambda < \mu$.) For $\lambda < \mu$ we have

$$\frac{1}{\mu} = \int_0^\infty (1 - F(t)) \cdot 1 \, dt \geq \int_0^\infty (1 - F(t)) \, e^{-\lambda t} \, dt = \frac{1}{\lambda} (1 - F^*(\lambda))$$

since $e^{-\lambda t} \leq 1$. The first integral equals $\frac{1}{\mu}$, and integrating the second by parts gives $\frac{1}{\lambda} (1 - F^*(\lambda))$. Thus $\frac{1}{\mu} \geq \frac{1}{\lambda} (1 - F^*(\lambda))$ or

$$F^*(\lambda) \geq 1 - \rho.$$
CHAPTER 3

STATE PROBABILITIES--EXPONENTIAL CASE

We proceed to follow the method of solution presented in Section 1.1. Since the repair distribution, \( F(t) = 1 - e^{-ut} \ (t \geq 0) \) is exponential, the process is a Markov Process. We choose to use the well-known differential difference method for obtaining limiting state probabilities.

The manner of presentation is as follows. In Section 3.1 we make a detailed examination of the modified process to determine the transition rates among the various states. In Section 3.1.1, for the case \( I = 2 \), the equations are successively developed for the various states. The first is done in full detail, the rest in less detail. Then in Section 3.1.2, limits for each equation are taken, yielding a set of linear relations in the limiting state probabilities. These equations are solved and the solution presented in Section 3.1.3. The solution for general values of \( I \) is given in Section 3.2, and follows a similar presentation. The limiting probabilities in Sections 3.1.3 and 3.2 are given subject to a normalizing constant \( c \). This constant is determined in Section 3.3, along with an alternate derivation of the limiting probability of the set of operating states.
3.1. **State Transitions.**

We examine the states of the process in detail. Two classes of states are distinguished—operating states and non-operating states. For notational clarity we designate a state by an \((I+1)\)-vector \((r, n_1, \ldots, n_I)\), where \(r\) denotes number of systems at the repair facility and may equal \(0, 1,\) or \(2;\) and \(n_i\) designates the number of spare parts remaining of type \(i\) and may equal \(1, 2, \ldots, N_i-1,\) or \(N_i\).

Let \(m = \min\ n_i\). Any state with \(r = 2\) is non-operating since both systems are at the repair facility. Also, any state with \(m = r = 1\) is non-operating since one system is in repair and not enough parts remain for the other system to be operative.

There are certain combinations of \(r\) and the \(n_i\) which cannot be reached. The combination \(m = 1\) and \(r = 2\) is one, since with \(m = 1\) only enough parts remain to compose one system. The combination

\[\Sigma n_i > \Sigma N_i - r\]

is also impossible since the process starts in state \((0, N_1, \ldots, N_I)\) and \(r\) units in repair require a depletion \((\Sigma N_i - \Sigma n_i)\) greater than or equal to \(r\).

The set \(S\) of states with positive probability is therefore given by

\[S = \{(r, n_1, \ldots, n_I) | 0 \leq r \leq 2; 0 \leq n_i \leq N_i; r \leq n_i, \]

\[i = 1, \ldots, I; \text{ and } \Sigma n_i \leq \Sigma N_i - r\} .\]
If $I = 2$, we may illustrate the states graphically, as shown in Figure 3.1. Shaded states are non-operating, and unshaded states are operating states. States marked with an $\times$ are impossible and not included in $S$.

\begin{align*}
&\text{r = 0} \\
&(0,1,1) \quad (0,1,2) \quad \cdots \quad (0,1,N_2) \\
&(0,2,1) \quad (0,2,2) \quad \cdots \quad (0,2,N_2) \\
&\vdots \quad \vdots \quad \ddots \quad \vdots \\
&(0,N_1,1) \quad (0,N_1,2) \quad \cdots \quad (0,N_1,N_2)
\end{align*}

\begin{align*}
&\text{r = 1} \\
&(1,1,1) \quad (1,1,2) \quad \cdots \quad (1,1,N_2) \\
&(1,2,1) \quad (1,2,2) \quad \cdots \quad (1,2,N_2) \\
&(1,3,1) \quad (1,3,2) \quad \cdots \quad (1,3,N_2) \\
&(1,N_1,1) \quad (1,N_1,2) \quad \cdots \quad (1,N_1,N_2-1) \\
&\text{X}
\end{align*}

\begin{align*}
&\text{r = 2} \\
&\text{X} \quad \text{X} \quad \text{X} \quad \cdots \quad \text{X} \\
&\text{X} \quad (2,2,2) \quad (2,2,3) \quad \cdots \quad (2,2,N_2) \\
&\text{X} \quad (2,3,2) \quad (2,3,3) \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
&\text{X} \quad (2,N_1,2) \quad (2,N_1,3) \quad \cdots \quad (2,N_1-1,N_2-1) \\
&\text{X} \quad \text{X}
\end{align*}

\textbf{Figure 3.1}

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The process starts at time 0 in state \((0, N_1, \ldots, N_t)\) and proceeds from state to state until such time as some \(n_i = 0\). At this time, rather than going into some state so labeled, the process returns to state \((0, N_1, \ldots, N_t)\) according to the reformulation. We will now examine these transitions. Consider a general operating state \((r, n_1, n_2)\). (For simplicity, we assume \(I = 2\) in the discussion, although the results hold for general \(I\).) If \(r = 1\), then one of three things may happen -- failure of part type 1, failure of part type 2, or completion of repair. Since each failure distribution is exponential with parameter \(\lambda_i\), the probability of failure in a time interval of length \(\Delta t\) is given by \(\lambda_i \Delta t + o(\Delta t)\), where the "little o" function \(o(x)\) is one such that \(\frac{o(x)}{x} \rightarrow 0\) as \(x \rightarrow 0\) \((x > 0)\). Since in this section we assume the repair time distribution to be exponential also, the probability of a repair completion in a time interval of length \(\Delta t\), given that a system is in repair at the beginning of the interval, is \(\mu \Delta t + o(\Delta t)\), where \(\frac{1}{\mu}\) is the expected repair time.

In a typical state \((1, n_1, n_2)\), if part type 1 fails, the process goes from \((1, n_1, n_2)\) to \((2, n_1-1, n_2)\); a failure of type 2 leads to \((2, n_1, n_2-1)\), and a repair completion leads to \((0, n_1, n_2)\). In a state of the form \((0, n_1, n_2)\), only two things can happen -- failure of part type 1 leading to \((1, n_1-1, n_2)\), and failure of type 2 leading to \((1, n_1, n_2-1)\). And, in a state \((2, n_1, n_2)\) (a non-operating state), only a repair completion can take place, which takes the process to the state \((1, n_1, n_2)\). However there are several special cases where the above rules do not apply. We shall consider each of them individually.
Suppose $r = 1$ and $m = \min n_i = 1$. This is a non-operating state and thus only a repair completion can take place. That is, $(1,1,1)$ goes only to $(0,1,1)$; $(1,1,n_2)$ goes only to $(0,1,n_2)$ ($n_2 \geq 2$), and $(1, n_1, 1)$ goes only to $(0, n_1, 1)$ ($n_1 \geq 2$).

If $r = 0$ and $m = 1$, we have a state of the form $(0, 1, n_2)$ or $(0, n_1, 1)$. Suppose we are in state $(0, 1, n_2)$ ($n_2 \geq 2$) and a failure of part type 1 occurs. Then the process goes to $(0, N_1, N_2)$. If a failure of part type 2 occurs, the new state, of course, is $(1, 1, n_2-1)$. Similarly, transitions from $(0, n_1, 1)$ are to either $(1, n_1-1, 1)$ or $(0, N_1, N_2)$ ($n_1 \geq 2$). From the state $(0, 1, 1)$, a failure of either part type leads to $(0, N_1, N_2)$.

The final exception to the above rules is a state where $r = 1$ and $m = 2$. If the process is in state $(1, 2, n_2)$ ($n_2 \geq 2$) and part type 1 fails, we do not go to the state $(2, 1, n_2)$. This is simply because there is no way to repair the just-failed system, due to unavailability of part type 1. So, the transition is to the state $(1, 1, n_2)$. A similar result holds for the state $(1, n_1, 2)$; failure of part type 2 leads to $(1, n_1, 1)$. Using this information, we can now set up a set of differential-difference equations involving the state probabilities.

3.1.1. Transient Equations in the State Probabilities.

We consider the case $I = 2$, for ease of exhibition in the detailed analysis. The results for general $I$ will be shown in Section 3.2. Define $P_{r,n_1,n_2}(t) = P(X(t) = (r, n_1, n_2))$ for $0 \leq t < \infty$; $r = 0, 1, 2$; $n_1 = 1, 2, \ldots, N_1$, and $n_2 = 1, 2, \ldots, N_2$. 

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As previously mentioned, we consider only the set $S$; for all values of $(r, n_1, n_2)$ outside $S$, let $P_{r, n_1, n_2}(t) = 0$. Let $P_{r, n_1, n_2}$

$= \lim_{t \to \infty} P_{r, n_1, n_2}(t)$. We have shown in Theorem 1.1 that such a limit does exist for all $(r, n_1, n_2)$.

Let $t \geq 0$ and let $\Delta t$ denote a very short interval of time.

What is $P_{1, n_1, n_2}(t + \Delta t)$? We suppose that $n_1 \geq 2$, $n_2 \geq 2$, and

$\sum_{i=1}^{2} n_i \leq \sum_{i=1}^{2} N_i - 1$; thus $(1, n_1, n_2)$ will be an operating state.

There are several ways for the state of the process at $(t + \Delta t)$ to be $(1, n_1, n_2)$, depending on its state at time $t$. One way is if the process is in state $(1, n_1, n_2)$ at time $t$ and nothing happens in the time interval $(t, t + \Delta t]$. Let $S(t)$ denote the state of the process at time $t$. Then

$$P[S(t) = (1, n_1, n_2)] = P_{1, n_1, n_2}(t)$$

by definition. We now derive $P[\text{no transitions in } (t, t + \Delta t) | S(t) = (1, n_1, n_2)]$.

We have

$$(3.1) \quad P[\text{no transitions in } (t, t + \Delta t) | S(t) = (1, n_1, n_2)]$$

$= 1 - P[\text{repair completion in } (t, t + \Delta t) | S(t) = (1, n_1, n_2)]$

$- P[\text{failure of part type } 1 \text{ in } (t, t + \Delta t) | S(t) = (1, n_1, n_2)]$

$- P[\text{failure of part type } 2 \text{ in } (t, t + \Delta t) | S(t) = (1, n_1, n_2)]$

$- P[\text{two or more transitions in } (t, t + \Delta t) | S(t) = (1, n_1, n_2)]$. 36
Since the five events in (3.1) are mutually exclusive and collectively exhaustive, the sum of their probabilities is 1.

Lemma 3.1.

\[ P[\text{no transitions in } (t, t+\Delta t) \mid S(t) = (1, n_1, n_2)] \]

\[ = 1 - \mu \Delta t - \lambda_1 \Delta t - \lambda_2 \Delta t - o(\Delta t). \]

Proof: We consider each term in (3.1) individually. Due to the "lack of memory" of the exponential distribution, the first term on the right hand side,

\[ P[\text{repair completion in } (t, t+\Delta t) \mid S(t) = (1, n_1, n_2)] \]

is independent of the length of time the system has been in repair, and is equal to

\[ \mu \Delta t + o(\Delta t) \]

since the repair d.f. is \( F(t) = 1 - e^{-\mu t} \). Similarly,

\[ P[\text{failure of part type } i \text{ in } (t, t+\Delta t) \mid S(t) = (1, n_1, n_2)] \]

\[ = \lambda_i \Delta t + o(\Delta t); \quad i = 1, 2, \]

since \((1, n_1, n_2)\) is an operating state. We also have,
\[ P[\text{two or more transitions in } (t, t+\Delta t) | S(t) = (1, n_1, n_2)] = o(\Delta t). \]

Since the sum or difference of \( o(\cdot) \) functions is itself a \( o(\cdot) \) function, we have

\[ P[\text{no transitions in } (t, t+\Delta t) | S(t) = (1, n_1, n_2)] = 1 - \mu_\Delta t - \lambda_1 \Delta t - \lambda_2 \Delta t - o(\Delta t) \]

which is (3.2).

We have shown that if \( S(t) = (1, n_1, n_2) \) and if no transitions take place in \( (t, t+\Delta t) \), then \( S(t+\Delta t) = (1, n_1, n_2) \), and that the probability of the above sequence of events is

\[ (3.3) \quad P_{1,n_1,n_2}(t)[1 - \mu_\Delta t - \lambda_1 \Delta t - \lambda_2 \Delta t - o(\Delta t)]. \]

Suppose \( S(t) = (2, n_1, n_2) \) and that only a repair completion occurs in \( (t, t+\Delta t) \). Then \( S(t+\Delta t) = (1, n_1, n_2) \). By a proof identical to that of Theorem 3.2, the probability of this is \( P_{2,n_1,n_2}(t)[\mu_\Delta t + o(\Delta t)]. \)

Similarly, if \( S(t) = (0, n_1 + 1, n_2) \) and a failure of part type 1 occurs, or if \( S(t) = (0, n_1, n_2 + 1) \) and a failure of part type 2 occurs, then \( S(t+\Delta t) = (1, n_1, n_2) \). The probabilities of the latter two events may be written \( P_{0,n_1+1,n_2}(t)[\lambda_1 \Delta t + o(\Delta t)] \) and \( P_{0,n_1,n_2+1}(t)[\lambda_2 \Delta t + o(\Delta t)]. \)
There is one final way in which $S(t+\Delta t)$ can be $(1, n_1, n_2)$. This is for $S(t)$ to be one of the states other than those mentioned above, and for an appropriate sequence of two or more transitions to take place. Let $T$ denote the set $S \setminus \{ (1, n_1, n_2), (0, n_1+1, n_2), (0, n_1, n_2+1), (2, n_1, n_2) \}$. Then

\[ (3.4) \quad P(S(t) \in T \text{ and } S(t+\Delta t) = (1, n_1, n_2)) \]

\[ = P(S(t) \in T) \, P(\text{appropriate sequence of two or more transitions} | S(t)) \]

\[ \leq \sum_{s \in T} P(S(t) = s) \, P(\text{appropriate sequence of two or more transitions} | S(t) = s) \]

\[ \leq \sum_{s \in T} P(S(t) = s) \, P(\text{two or more transitions in} \quad (t, t+\Delta t) | S(t) = s) \]

\[ = o(\Delta t) \sum_{s \in T} P(S(t) = s) \]

\[ = o(\Delta t) . \]

This completes the discussion of this case, and we have

**Theorem 3.2.** For $2 \leq n_1 \leq N_1$, $2 \leq n_2 \leq N_2$, and $n_1 + n_2 \leq N_1 + N_2 - 1$, and any $t \geq 0$ and $\Delta t > 0$,

\[ (3.5) \quad P_{1, n_1, n_2}(t+\Delta t) = P_{1, n_1, n_2}(t) [1 - \mu \Delta t - \lambda_1 \Delta t - \lambda_2 \Delta t - o(\Delta t)] \]

\[ + P_{2, n_1, n_2}(t) [\mu \Delta t + o(\Delta t)] + P_{0, n_1, n_2+1}(t) [\lambda_1 \Delta t + o(\Delta t)] \]

\[ + P_{0, n_1, n_2+1}(t) [\lambda_2 \Delta t + o(\Delta t)] + o(\Delta t) . \]
**Proof.** By the Markov property of the process

\[ P_{1,n_1,n_2}(t+\Delta t) = \sum_{s \in S} P(S(t) = s) P(S(t+\Delta t) = (1,n_1,n_2) | S(t) = s) \]

\[ = P_{1,n_1,n_2}(t) P(S(t+\Delta t) = (1,n_1,n_2) | S(t) = (1,n_1,n_2)) \]

\[ + P_{2,n_1,n_2}(t) P(S(t+\Delta t) = (1,n_1,n_2) | S(t) = (2,n_1,n_2)) \]

\[ + P_{0,n_1+1,n_2}(t) P(S(t+\Delta t) = (1,n_1,n_2) | S(t) = (0,n_1+1,n_2)) \]

\[ + P_{0,n_1,n_2+1}(t) P(S(t+\Delta t) = (1,n_1,n_2) | S(t) = (0,n_1,n_2+1)) \]

\[ + \sum_{s \in T} P(S(t) = s) P(S(t+\Delta t) = (1,n_1,n_2) | S(t) = s) , \]

and thus (3.5) holds by (3.4) and the preceding.

We will not repeat the proof for the other states, as each is similar to that of Theorem 3.2.

\((0, N_1, N_2)\). This state is reached from \((0,n_1,1)\) \((1 \leq n_1 \leq N_1)\) by a failure of part type 1, from \((0,1,n_2)\) \((1 \leq n_2 \leq N_2)\) by a failure of part type 2. Transitions from the state are generated by failures only. Thus

\[ P_{0,N_1,N_2}(t+\Delta t) = P_{0,N_1,N_2}(t)[1 - \lambda_1\Delta t - \lambda_2\Delta t - o(\Delta t)] \]

\[ + \sum_{i=1}^{N_1} P_{0,i,1}(t)[\lambda_1\Delta t + o(\Delta t)] \]

\[ + \sum_{j=1}^{N_2} P_{0,1,j}(t)[\lambda_2\Delta t + o(\Delta t)] + o(\Delta t) . \]
\((0,n_1,n_2)\) \((1 \leq n_1 \leq N_1, 1 \leq n_2 \leq N_2, n_1+n_2 \leq N_1+N_2 - 1)\). This state may be reached only by the completion of a repair. Thus

\[
(3.7) \quad P_{0,n_1,n_2}(t+\Delta t) = P_{0,n_1,n_2}(t)[1 - \lambda_1 \Delta t - \lambda_2 \Delta t - o(\Delta t)]
\]

\[+ P_{1,n_1,n_2}(t)[\mu \Delta t + o(\Delta t)] .\]

\((1,1,1)\). This state is non-operating and thus the only transition from it is a repair completion. It may be entered only from \((0,2,1)\), by a failure of part type 1, or from \((0,1,2)\) by a failure of part type 2. Therefore,

\[
(3.8) \quad P_{1,1,1}(t+\Delta t) = P_{1,1,1}(t)[1 - \mu \Delta t - o(\Delta t)]
\]

\[+ P_{0,2,1}(t)[\lambda_1 \Delta t + o(\Delta t)]
\]

\[+ P_{0,1,2}(t)[\lambda_2 \Delta t + o(\Delta t)] + o(\Delta t) .\]

\((1,n_1,1)\) \((2 \leq n_1 \leq N_1)\). This state is non-operating and thus the only transition from it is a repair completion. It may be entered from the states \((0,n_1+1,1)\) and \((0,n_1,2)\) by failures. It may also be entered from the state \((1,n_1,2)\) by a failure of part type 2. Thus

\[
(3.9) \quad P_{1,n_1,1}(t+\Delta t) = P_{1,n_1,1}(t)[1 - \mu \Delta t - o(\Delta t)]
\]

\[+ P_{0,n_1+1,1}(t)[\lambda_1 \Delta t + o(\Delta t)] + P_{0,n_1,2}(t)[\lambda_2 \Delta t + o(\Delta t)]
\]

\[+ P_{1,n_1,2}(t)[\lambda_2 \Delta t + o(\Delta t)] + o(\Delta t) .\]
It is worthwhile to note that this "extra" entry described by the fourth term on the right hand side occurs only for \( n_1 \geq 2 \). If \( n_1 = 1 \), the state \((1, n_1, 2)\) is non-operating and then, of course, a failure of part type 2 cannot occur.

\[
(1,1,n_2) \quad (2 \leq n_2 \leq N_2).
\]

This case is symmetric to the case \((1,n_1,1)\) noted above. Similar to (3.9), we have

\[
(3.10) \quad P_{1,1,n_2}(t+\Delta t) = P_{1,1,n_2}(t)[1 - \mu \Delta t - o(\Delta t)]
+ P_{0,2,n_2}(t)[\lambda_1 \Delta t + o(\Delta t)]
+ P_{0,1,n_2+1}(t)[\lambda_2 \Delta t + o(\Delta t)]
+ P_{1,2,n_2}(t)[\lambda_1 \Delta t + o(\Delta t)] + o(\Delta t).
\]

\[
(1,n_1,n_2) \quad (2 \leq n_1 \leq N_1, 2 \leq n_2 \leq N_2, n_1+n_2 \leq N_1+N_2 - 1).
\]

This case was fully discussed above and we have from Theorem 3.2:

\[
(3.11) \quad P_{1,n_1,n_2}(t+\Delta t) = P_{1,n_1,n_2}(t)[1 - \lambda_1 \Delta t - \lambda_2 \Delta t - \mu \Delta t - o(\Delta t)]
+ P_{0,n_1+1,n_2}(t)[\lambda_1 \Delta t + o(\Delta t)]
+ P_{0,n_1,n_2+1}(t)[\lambda_2 \Delta t + o(\Delta t)]
+ P_{2,n_1,n_2}(t)[\mu \Delta t + o(\Delta t)] + o(\Delta t).
\]
\((2, n_1, n_2) \ (2 \leq n_1 \leq N_1, \ 2 \leq n_2 \leq N_2, \ n_1 + n_2 \leq N_1 + N_2 - 2)\). This state is non-operating and is exited from by means of a repair completion only. Since we have assumed that the second system in the repair shop must wait until repair on the first is completed, the rate of transition from this state by means of a repair completion is \(u\), as in the cases where \(r = 1\). The state is entered from either the states \((1, n_1 + 1, n_2)\) or \((1, n_1, n_2 + 1)\). Thus we have

\[
(3.12) \quad P_{2,n_1,n_2}(t + \Delta t) = P_{2,n_1,n_2}(t)[1 - u\Delta t - o(\Delta t)] \\
+ P_{1,n_1+1,n_2}(t)[\lambda_1 \Delta t + o(\Delta t)] \\
+ P_{1,n_1,n_2+1}(t)[\lambda_2 \Delta t + o(\Delta t)] + o(\Delta t).
\]

Finally,

\[
(3.13) \quad P_{r,n_1,n_2}(t) = 0 \quad \text{any } (r,n_1,n_2) \text{ not covered by (3.6) - (3.12)}.
\]

### 3.1.2. Limiting Equations in the State Probabilities.

From equations (3.6) - (3.13), we now proceed to derive a set of linear equations in the limiting probabilities \(P_{r,n_1,n_2}\). As above, we will go through the details only for the case \((1,n_1,n_2)\). The procedure for the other cases is the same. From equation (3.11), we subtract the first term on the right hand side to get
\[(3.14)\quad P_{1,n_1,n_2}(t+\Delta t) - P_{1,n_1,n_2}(t) = P_{1,n_1,n_2}(t)[-\lambda_1\Delta t - \lambda_2\Delta t - \mu\Delta t - o(\Delta t)] + P_{0,n_1+1,n_2}(t)[\lambda_1\Delta t + o(\Delta t)] + P_{0,n_1,n_2+1}(t)[\lambda_2\Delta t + o(\Delta t)] + P_{2,n_1,n_2}(t)[\mu\Delta t + o(\Delta t)] + o(\Delta t).
\]

Dividing both sides of \((3.14)\) by \(\Delta t\) and collecting terms involving \(o(\Delta t)\) we have

\[
\frac{P_{1,n_1,n_2}(t+\Delta t) - P_{1,n_1,n_2}(t)}{\Delta t} = P_{1,n_1,n_2}(t)[-\lambda_1 - \lambda_2 - \mu] + P_{0,n_1+1,n_2}(t)[\lambda_1] + P_{0,n_1,n_2+1}(t)[\lambda_2] + P_{2,n_1,n_2}(t)[\mu] + \frac{o(\Delta t)}{\Delta t}.
\]

Taking the limit of both sides of \((3.15)\) as \(\Delta t\) approaches zero, we have by definition on the left, the term \(\frac{\partial}{\partial t} [P_{1,n_1,n_2}(t)]\). Only the final term on the right hand side involves \(\Delta t\), and it approaches zero by definition of \(o(\Delta t)\). Therefore the derivative exists and equals
\[ \frac{\partial}{\partial t} [P_{1,n_1,n_2}(t)] = P_{1,n_1,n_2}(t)[-\lambda_1 - \lambda_2 - \mu] \]
\[ + P_{0,n_1+1,n_2}(t)[\lambda_1] + P_{0,n_1,n_2+1}(t)[\lambda_2] \]
\[ + P_{2,n_1,n_2}(t)[\mu]. \]

The final step is to let \( t \to \infty \) in (3.16). By Lemma 3.3 below, for each \((r,n_1,n_2)\), \( \lim_{t \to \infty} P_{r,n_1,n_2}(t) \) exists. Defining \( P_{r,n_1,n_2} = \lim_{t \to \infty} P_{r,n_1,n_2}(t) \) and taking the limit as \( t \to \infty \) in (3.16), the right hand side remains unchanged except for the deletion of the argument \( (t) \).

For the left hand side, we have

**Lemma 3.3.** For \( r = 0, 1, 2; 1 \leq n_1 \leq N_1; 1 \leq n_2 \leq N_2; n_1 + n_2 \leq N_1 + N_2 - r \), and \( r \leq \min_{i=1,2} n_i \), we have the existence of

\[
P_{r,n_1,n_2} = \lim_{t \to \infty} P_{r,n_1,n_2}(t).
\]

**Proof:** The process is a finite state Markov Process and the result holds as a direct consequence of Theorem 1.1.

**Lemma 3.4.**

\[
\lim_{t \to \infty} \frac{\partial}{\partial t} P_{1,n_1,n_2}(t) = 0.
\]

**Proof:** By Lemma 3.3, each term on the right approaches a limit between zero and one; thus, \( \lim_{t \to \infty} \frac{\partial}{\partial t} [P_{1,n_1,n_2}(t)] \) must exist and must equal
a constant. However, since $P_{1,n_1,n_2}(t) \to P_{1,n_1,n_2}$, the limit of its derivative must equal zero.

Thus from (3.16)

$$0 = P_{1,n_1,n_2}(-\lambda_1 - \lambda_2 - \mu) + P_{0,n_1+1,n_2}(\lambda_1) + P_{0,n_1,n_2+1}(\lambda_2) + P_{2,n_1,n_2}(\mu).$$

Letting $\lambda = \lambda_1 + \lambda_2$ gives

$$\text{(3.17)} \quad (\lambda+\mu)P_{1,n_1,n_2} = \lambda_1 P_{0,n_1+1,n_2} + \lambda_2 P_{0,n_1,n_2+1} + \mu P_{0,n_1,n_2}.$$

A similar procedure is followed for equations (3.6) - (3.13). Subtracting $P_{0,N_1,N_2}(t)$ from both sides, dividing by $\Delta t$, and collecting terms in $o(\Delta t)$ in (3.6) gives

$$\frac{P_{0,N_1,N_2}(t+\Delta t) - P_{0,N_1,N_2}(t)}{\Delta t} = \lambda_1 \sum_{j=1}^{N_2} P_{0,1,j}(t)$$

$$+ \lambda_2 \sum_{i=1}^{N_1} P_{0,i,1}(t) + \frac{o(\Delta t)}{\Delta t}.$$  

Letting $\Delta t \to 0$ and $t \to \infty$ gives, upon rearranging,

$$\text{(3.18)} \quad \lambda P_{0,N_1,N_2} = \lambda_1 \sum_{j=1}^{N_2} P_{0,1,j} + \lambda_2 \sum_{i=1}^{N_1} P_{0,i,1}.$$
In (3.7), we subtract $P_{0,n_1,n_2}(t)$ from both sides, and divide by $\Delta t$:

$$\frac{P_{0,n_1,n_2}(t+\Delta t) - P_{0,n_1,n_2}(t)}{\Delta t} = P_{0,n_1,n_2}(t)(-\lambda_1-\lambda_2) + \mu P_{1,n_1,n_2}(t) + \frac{o(\Delta t)}{\Delta t}.$$  

Taking limits of $t$ and $\Delta t$:

(3.19) \[ \lambda P_{0,n_1,n_2} = \mu P_{1,n_1,n_2}. \]

In (3.8), the same procedure gives

$$\frac{P_{111}(t+\Delta t) - P_{111}(t)}{\Delta t} = P_{111}(t)(-\mu) + \lambda_1 P_{021}(t) + \lambda_2 P_{012}(t) + \frac{o(\Delta t)}{\Delta t}$$

and

(3.20) \[ \mu P_{111} = \lambda_1 P_{021} + \lambda_2 P_{012}. \]

For (3.9) we have

$$\frac{P_{1,n_1,1}(t+\Delta t) - P_{1,n_1,1}(t)}{\Delta t} = P_{1,n_1,1}(t)(-\mu) + P_{0,n_1+1,1}(t)(\lambda_1)$$

$$+ P_{0,n_1,2}(t)(\lambda_2) + P_{1,n_1,2}(t)(\lambda_2) + \frac{o(\Delta t)}{\Delta t}$$

which becomes upon taking limits

(3.21) \[ \mu P_{1,n_1,1} = \lambda_1 P_{0,n_1+1,1} + \lambda_2 P_{0,n_1,2} + \lambda_2 P_{1,n_1,2}. \]
The case (3.10) is symmetric to (3.9) and leads to

\[(3.22) \quad \mu P_{1,1,n_2} = \lambda_1 P_{0,2,n_2} + \lambda_2 P_{0,1,n_2+1} + \lambda_1 P_{1,2,n_2} \cdot \]

The case of (3.11) was done above in (3.17). In (3.12) the same procedure gives

\[
\frac{P_{2,n_1,n_2}(t+\Delta t) - P_{2,n_1,n_2}(t)}{\Delta t} = P_{2,n_1,n_2}(t)(-\mu) + P_{1,n_1+1,n_2}(t)(\lambda_1) \\
+ P_{1,n_1,n_2+1}(t)(\lambda_2) + \frac{o(\Delta t)}{\Delta t}
\]

letting \( \Delta t \to 0 \) and \( t \to \infty \) gives

\[(3.23) \quad \mu P_{2,n_1,n_2} = \lambda_1 P_{1,n_1+1,n_2} + \lambda_2 P_{1,n_1,n_2+1} \cdot \]

Finally, (3.13) becomes, for other values of \((r,n_1,n_2)\) than considered above:

\[(3.24) \quad P_{r,n_1,n_2} = 0 \]

The equations (3.17) - (3.24) together cover all cases of \((r,n_1,n_2)\). This completes the proof of
Theorem 3.5. The following equations related the $p_{r,n_1,n_2}$:

$\lambda p_{0,1,N_1,N_2} = \lambda_2 \sum_{i=1}^{N_1} p_{0,i,1} + \lambda_1 \sum_{j=1}^{N_2} p_{0,1,j}$

$\lambda p_{0,n_1,n_2} = \mu p_{1,n_1,n_2}$

$\mu p_{1,1,1} = \lambda_1 p_{0,2,1} + \lambda_2 p_{0,1,2}$

$\mu p_{1,n_1,1} = \lambda_1 p_{0,n_1+1,1} + \lambda_2 p_{0,n_1,2} + \lambda_2 p_{1,n_1,2}$

$\mu p_{1,1,n_2} = \lambda_1 p_{0,2,n_2} + \lambda_2 p_{0,1,n_2+1} + \lambda_1 p_{1,2,n_2}$

$(\lambda+\mu)p_{1,n_1,n_2} = \lambda_1 p_{0,n_1+1,n_2} + \lambda_2 p_{0,n_1,n_2+1} + \mu p_{2,n_1,n_2}$

$\mu p_{2,n_1,n_2} = \lambda_1 p_{1,n_1+1,n_2} + \lambda_2 p_{1,n_1,n_2+1}$

$p_{r,n_1,n_2} = 0$ otherwise

$\sum_{r} \sum_{n_1} \sum_{n_2} p_{r,n_1,n_2} = 1$. 

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3.1.3. Solution of the Equations.

Equations (3.25.1) - (3.25.9) in Theorem 3.5 are easily solved. Since the equations are homogenous, an arbitrary value may be set for one of the \( P_{r,N_1,N_2} \), and the others solved in terms of it. Thus we set \( P_{0,N_1,N_2} = c \) for some \( c > 0 \). Solving iteratively, from (3.25.6)

\[
(3.26) \quad P_{1,N_1-1,N_2} = \frac{\lambda_1}{\lambda + \mu} P_{0,N_1,N_2} = \frac{\lambda_1}{\lambda + \mu} c = \lambda_1(\frac{\Delta}{\lambda})(\frac{1}{1 + \frac{\lambda}{\mu}}) c
\]

\[
(3.27) \quad P_{1,N_1,N_2-1} = \frac{\lambda_2}{\lambda + \mu} P_{0,N_1,N_2} = \frac{\lambda_2}{\lambda + \mu} c = \lambda_2(\frac{\Delta}{\lambda})(\frac{1}{1 + \frac{\lambda}{\mu}}) c
\]

since \( P_{1,N_1-1,N_2+1} = P_{1,N_1+1,N_2-1} = 0 \) and \( P_{2,N_1-1,N_2} = P_{2,N_1,N_2-1} = 0 \).

Then from (3.25.2),

\[
(3.28) \quad P_{0,N_1-1,N_2} = \frac{\mu}{\lambda} P_{1,N_1-1,N_2} = \lambda_1(\frac{1}{1 + \frac{\lambda}{\mu}}) c
\]

\[
(3.29) \quad P_{0,N_1,N_2-1} = \frac{\mu}{\lambda} P_{1,N_1,N_2-1} = \lambda_2(\frac{1}{1 + \frac{\lambda}{\mu}}) c
\]

where

\[
\lambda_i = \frac{\lambda_i}{\lambda}, \quad i = 1, 2
\]

We next use (3.25.7) to obtain

\[
(3.30) \quad P_{2,N_1-1,N_2-1} = \frac{\lambda_1}{\mu} P_{1,N_1,N_2-1} + \frac{\lambda_2}{\mu} P_{1,N_1-1,N_2}
\]

\[
= \frac{\lambda_1}{\mu} \lambda_2(\frac{\Delta}{\lambda})(\frac{1}{1 + \frac{\lambda}{\mu}}) c + \frac{\lambda_2}{\mu} \lambda_1(\frac{\Delta}{\lambda})(\frac{1}{1 + \frac{\lambda}{\mu}}) c
\]

\[
= 2\lambda_1\lambda_2(\frac{\Delta}{\lambda})^2 \left( \frac{1}{1 + \frac{\lambda}{\mu}} \right) c.
\]

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Continuing the iterative process gives:

**Lemma 3.6.** For \(0 \leq r \leq 2, 2 \leq n_1 \leq N_1, 2 \leq n_2 \leq N_2, n_1 + n_2 \leq N_1 + N_2 - r,\) and \((r, n_1, n_2) \neq (0, N_1, N_2)\):

\[
P_{r, n_1, n_2} = \left(\frac{(N_1-n_1) + (N_2-n_2)}{N_1-n_1}\right)^{\frac{N_1-n_1}{\lambda_1}} \frac{N_2-n_2}{\lambda_2^r} \left(\frac{1}{1 + \lambda_1 \mu}\right) c.
\]

**Proof:** Let \( i = N_1-n_1, j = N_2-n_2. \) The proof is by induction on the sum \( k = i+j. \) For \( k = 1 \) the result is true by (3.26)-(3.30). Suppose the result holds for \((k-1)\). Then \( \forall i \leq N_1-2, j \leq N_2-2, i+j \leq k-1:\)

\[
(3.31) \quad P_{r, N_1-i, N_2-j} = \left(\frac{i+j}{i}\right)^{\frac{i}{\lambda_1}} \left(\frac{j}{\lambda_2}\right)^{\frac{j}{\mu}} \left(\frac{1}{1 + \lambda_1 \mu}\right) c.
\]

There are now three cases depending on the relationship of \( k \) to \( N_1-2 \) and \( N_2-2.\)

**Case 1:** \( k \leq N_1-2, k \leq N_2-2. \) We must show (3.31) holds for \( i+j = k, i = k, k-1, \ldots, 0. \) For \( i = k, \) by (3.25.7):

\[
(3.32) \quad P_{2, N_1-k, N_2} = \frac{\lambda_1}{\mu} P_{1, N_1-k, N_2} + 0
\]

\[
= \frac{\lambda_1}{\mu} \left(\frac{k-1}{k-1}\right)^{\frac{k-1}{\lambda_1}} \left(\frac{0}{\lambda_2}\right)^{\frac{0}{\mu}} \left(\frac{1}{1 + \lambda_1 \mu}\right) c \quad \text{by induction}
\]

\[
= \frac{\lambda_1}{\mu} \frac{1}{\lambda_1} \left(\frac{k-1}{\lambda_2}\right)^{\frac{k-1}{\mu}} \left(\frac{1}{1 + \lambda_1 \mu}\right) c
\]

\[
= \frac{\lambda_1}{\lambda_2} \left(\frac{k-1}{k}\right)^{\frac{k-1}{\mu}} \left(\frac{1}{1 + \lambda_1 \mu}\right) c.
\]
From (3.25.6):

\[ P_{1,N_1-k,N_2} = \frac{\lambda_1}{\lambda+\mu} P_{0,N_1-k+1,N_2} + o + \frac{\mu}{\lambda+\mu} P_{2,N_1-k,N_2} \]

\[ = \frac{\lambda_1}{\lambda+\mu} \binom{k-1+0}{k-1} \lambda_1^{k-1} \lambda_2^0 \left( \frac{1}{1 + \lambda/\mu} \right)^c \]

\[ + \frac{\mu}{\lambda+\mu} \frac{\lambda}{\lambda_1 \lambda_2^0 (k)} \binom{0}{k} \left( \frac{\lambda}{\mu} \right)^2 \left( \frac{1}{1 + \lambda/\mu} \right)^c \quad \text{by induction and (3.32)} \]

\[ = \binom{k-0}{k} \lambda_1^k \lambda_2^0 \left( \frac{1}{1 + \lambda/\mu} \right)^c \left[ \frac{\lambda}{\lambda_1 + \mu} \left( \frac{\lambda}{\mu} \right)^2 \right] \]

and from (3.25.2)

\[ P_{0,N_1-k,N_2} = \binom{k-0}{k} \lambda_1^k \lambda_2^0 \left( \frac{1}{1 + \lambda/\mu} \right)^c \]

which together are (3.31). For \( i = 0 \), (3.32) - (3.34) hold again with the indices switched. Finally, for \( 1 \leq i \leq k-1 (j-k-i) \), we have by (3.25.7):

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\[(3.35)\quad p_{2, N_1-1, N_2-j} = \frac{\lambda_1}{\mu} p_{1, N_1-1, N_2-j} + \frac{\lambda_2}{\mu} p_{1, N_1-1, N_2-j+1} \]

\[= \frac{\lambda_1}{\mu} \binom{i+j-1}{i-1} \frac{z_1^{i-1}}{\lambda_1} \frac{z_2^j}{\lambda_2} \frac{\lambda_2}{\mu} \left( \frac{1}{1 + \lambda/\mu} \right)^c + \frac{\lambda_2}{\mu} \binom{i+j-1}{i} \frac{z_1^i}{\lambda_1} \frac{z_2^{j-1}}{\lambda_2} \frac{\lambda_2}{\mu} \left( \frac{1}{1 + \lambda/\mu} \right)^c \text{ by induction} \]

\[= \lambda_1^{i} \lambda_2^{j} \left( \frac{\lambda_2}{\mu} \right)^2 \left( \frac{1}{1 + \lambda/\mu} \right)^c \left[ \binom{i+j-1}{i-1} + \binom{i+j-1}{i} \right] \]

\[= \binom{i+j}{i} \lambda_1^{i} \lambda_2^{j} \left( \frac{\lambda_2}{\mu} \right)^2 \left( \frac{1}{1 + \lambda/\mu} \right)^c . \]

From (3.25.6):

\[(3.36)\quad p_{1, N_1-1, N_2-j} = \frac{\lambda_1}{\lambda+\mu} p_{0, N_1-1, N_2-j} + \frac{\lambda_2}{\lambda+\mu} p_{0, N_1-1, N_2-j+1} + \frac{\mu}{\lambda+\mu} \lambda_2 p_{2, N_1-1, N_2-j} \]

\[= \left[ \frac{\lambda_1}{\lambda+\mu} \binom{i+j-1}{i-1} \frac{z_1^{i-1}}{\lambda_1} \frac{z_2^j}{\lambda_2} + \frac{\lambda_2}{\lambda+\mu} \binom{i+j-1}{i} \frac{z_1^i}{\lambda_1} \frac{z_2^{j-1}}{\lambda_2} \right] \left( \frac{1}{1 + \lambda/\mu} \right)^c \text{ by induction} \]

\[= \left[ \frac{\lambda_1}{\lambda+\mu} \binom{i+j}{i} \frac{z_1^i}{\lambda_1} \frac{z_2^j}{\lambda_2} \left( \frac{\lambda_2}{\mu} \right)^2 + \frac{\mu}{\lambda+\mu} \left( \frac{\lambda}{\mu} \right)^2 \right] \left( \frac{1}{1 + \lambda/\mu} \right)^c \]

\[= \binom{i+j}{i} \lambda_1^i \lambda_2^j \left( \frac{\lambda_2}{\mu} \right) \left( \frac{1}{1 + \lambda/\mu} \right)^c . \]

and from (3.25.2) and (3.36),

\[(3.37)\quad p_{0, N_1-1, N_2-j} = \left( \frac{\lambda_1}{\mu} \right)^{-1} \quad p_{1, N_1-1, N_2-j} = \binom{i+j}{i} \lambda_1^i \lambda_2^j \left( \frac{1}{1 + \lambda/\mu} \right)^c . \]
Case 2: \( \mathbf{N}_1 - 2 < k \leq \mathbf{N}_2 - 2 \). We assume without loss of generality that \( \mathbf{N}_1 \leq \mathbf{N}_2 \). If \( \mathbf{N}_1 = \mathbf{N}_2 \) this case is vacuous. We need only show that (3.31) holds for \( i + j = k; i = \mathbf{N}_1 - 2, \ldots , 0 \). For \( i = \mathbf{N}_1 - 2, \ldots , 1 \), (3.35) - (3.37) hold. For \( i = 0 \), (3.34) - (3.36) hold and the proof of this case is complete.

Case 3: \( k > \mathbf{N}_2 - 2 \). It must be shown that (3.31) holds only for \( i = \mathbf{N}_1 - 2, \ldots , k-(\mathbf{N}_2 - 2) \). But in all of these cases, (3.35) - (3.37) hold. This completes the proof of Lemma 3.6.

It remains to calculate \( P_{0,n_1,n_2} \) and \( P_{1,n_1,n_2} \) for

\[
\begin{align*}
m &= \min_{i=1,2} n_i = 1. \text{ From (3.25,5) for } n_1 = 1, n_2 = N_2:

(3.38) \quad P_{1,1,N_2} &= \frac{\lambda_1}{\mu} P_{0,2,N_2} + 0 + \frac{\lambda_1}{\mu} P_{1,2,N_2} \\
&= \left[ \frac{\lambda_1}{\mu} \left( \begin{array}{c} N_1 - 2 + 0 \\ N_1 - 2 \end{array} \right) \right] \frac{N_1 - 2}{\lambda_1} \frac{-0}{\lambda_2}

&\quad + \frac{\lambda_1}{\mu} \left( \frac{N_1 - 2}{N_1 - 2} \right) \frac{N_1 - 2}{\lambda_1} \frac{0}{\lambda_2} \left( \frac{1}{1 + \frac{\lambda_1}{\mu}} \right) c \quad \text{by Lemma 3.6}

&= \frac{\lambda_1}{\lambda_2} \left( \begin{array}{c} N_1 - 1 + 0 \\ N_1 - 1 \end{array} \right) \left[ \frac{\lambda}{\mu} \left( 1 + \frac{\lambda}{\mu} \right) \left( \frac{1}{1 + \frac{\lambda_1}{\mu}} \right) c \right]

&= \left( \begin{array}{c} N_1 - 1 + 0 \\ N_1 - 1 \end{array} \right) \frac{\lambda_1}{\lambda_2} \frac{-0(\lambda)}{\mu} c

\end{align*}
\]

and by (3.25.2):

(3.39) \quad P_{0,1,N_2} = \frac{\lambda_1}{\mu} P_{1,1,N_2} = \left( \begin{array}{c} N_1 - 1 + 0 \\ N_1 - 1 \end{array} \right) \frac{N_1 - 1}{\lambda_1} \frac{-0(\lambda)}{\mu} c \quad .

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Lemma 3.7. For \( 1 \leq n_1 \leq N_1, 1 \leq n_2 \leq N_2, \min_{i=1,2} n_i = 1, \) and \( r = 0, 1 \)

\[
Pr, n_1, n_2 = \binom{(N_1-n_1) + (N_2-n_2)}{N_1-n_1} \frac{N_1-n_1}{\lambda_1} \frac{N_2-n_2}{\lambda_2} \frac{1}{(\Delta_{\mu})^r} c.
\]

Proof: The proof is by reverse induction on \( n_2 \). It suffices to prove (3.40) only for \( P_{1,1,n_2} (n_2 = N_2, N_2-1, \ldots, 1) \), since the cases are symmetric.

Case 1: \( n_2 = N_2 \). This was shown above in (3.38).

Case 2: \( 2 \leq n_2 \leq N_2-1 \). Suppose (3.40) holds for \( n_2+1 \). Thus

\[
P_{r,1,n_2+1} = \binom{(N_1-1) + (N_2-(n_2+1))}{N_1-1} \frac{N_1-1}{\lambda_1} \frac{N_2-n_2-1}{\lambda_2} \frac{1}{(\Delta_{\mu})^r} c, \quad r = 0, 1.
\]

We have for \( n_2 \), by (3.25.5)

\[
P_{1,1,n_2} = \frac{\lambda_1}{\mu} P_{0,2,n_2} + \frac{\lambda_2}{\mu} P_{0,1,n_2+1} + \frac{\lambda_1}{\mu} P_{1,2,n_2}
\]

\[
= \frac{\lambda_1}{\mu} \binom{(N_1-2) + (N_2-n_2)}{(N_1-2)} \frac{N_1-2}{\lambda_1} \frac{N_2-n_2}{\lambda_2} \frac{1}{1 + \lambda/\mu} c
\]

\[
+ \frac{\lambda_2}{\mu} \binom{(N_1-1) + (N_2-n_2-1)}{(N_1-1)} \frac{N_1-1}{\lambda_1} \frac{N_2-n_2-1}{\lambda_2} c
\]

\[
+ \frac{\lambda_1}{\mu} \binom{(N_1-2) + (N_2-n_2)}{(N_1-2)} \frac{N_1-2}{\lambda_1} \frac{N_2-n_2}{\lambda_2} \frac{1}{1 + \lambda/\mu} \frac{1}{(\Delta_{\mu})^r} c
\]

by induction and Lemma 3.6
\[
\begin{align*}
&= \lambda_1^{N_1-1} \lambda_2^{N_2-n_2} \frac{1}{1 + \lambda/\mu} c \left[ \lambda \left( \begin{array}{c} N_1-2+N_2-n_2 \\ N_1-2 \end{array} \right) + \frac{\lambda}{\mu} \left( \begin{array}{c} N_1-1+N_2-n_2+1 \\ N_1-1 \end{array} \right) \right] \\
&\quad + \left( \frac{\lambda}{\mu} \right)^2 \left( \begin{array}{c} N_1-2+N_2-n_2 \\ N_1-2 \end{array} \right) \\
&= \lambda_1^{N_1-1} \lambda_2^{N_2-n_2} \frac{1}{1 + \lambda/\mu} c \left[ (\Delta) \left( \begin{array}{c} N_1-1+N_2-n_2 \\ N_1-1 \end{array} \right) + (\Delta)^2 \left( \begin{array}{c} N_1-1+N_2-n_2 \\ N_1-1 \end{array} \right) \right] \\
&= \left( \begin{array}{c} N_1-1 \\
\frac{N_1-1}{\lambda_1} \lambda_2^{N_2-n_2} (\Delta) \\
\lambda_2^{N_2-n_2} (\Delta) \\
\lambda_2^{N_2-n_2} (\Delta) c \\
\end{array} \right) \\
\end{align*}
\]

and since \( P_{0,1,n_2} = (\Delta)^{-1} P_{1,1,n_2} \), (3.40) holds for this case.

Case 3: \( n_2 = 1 \). We have by (3.25.3)

(3.42) \[ P_{111} = \frac{\lambda_1}{\mu} P_{021} + \frac{\lambda_2}{\mu} P_{012} \]

\[ = \left( \begin{array}{c} N_1-1 \\
\frac{N_1-1}{\lambda_1} \lambda_2^{-1} (\Delta) \\
\end{array} \right) \lambda_2^{-1} \lambda_2 \lambda_2^{-1} (\Delta) c \quad \text{by induction} \]

and by (3.42):

\[ P_{011} = (\Delta)^{-1} P_{111} = \left( \begin{array}{c} N_1-1 \\
\frac{N_1-1}{\lambda_1} \lambda_2^{-1} (\Delta) \\
\end{array} \right) \lambda_2^{-1} \lambda_2 \lambda_2^{-1} c . \]

This completes the proof of the lemma.

Finally, we must verify the validity of equation (3.25.1).

Rewriting we have
\[ \lambda_c = \lambda_2^c \sum_{n_1=1}^{N_1} \left( \frac{(N_1-n_1)(N_2-1)}{(N_1-n_1)} \right) \frac{N_1-n_1}{\lambda_1} \frac{N_2-1}{\lambda_2} \\
+ \lambda_1^c \sum_{n_2=1}^{N_2} \left( \frac{(N_1-1)(N_2-n_2)}{(N_1-1)} \right) \frac{N_1-1}{\lambda_1} \frac{N_2-n_2}{\lambda_2}. \]

Dividing through by \( \lambda_c \) and letting \( i = N_1-n_1; j = N_2-n_2 \), it must be verified that

\[ 1 = \sum_{i=0}^{N_1-1} \left( \binom{N_2-1+i}{i} \right) \frac{N_1-i}{\lambda_1} \frac{N_2-1}{\lambda_2} + \sum_{j=0}^{N_2-1} \left( \binom{N_1-1+j}{j} \right) \frac{N_1-1-j}{\lambda_1} \frac{N_2-j}{\lambda_2}. \]

The following combinatorial lemma proves the result.

**Lemma 3.8.** For any integers \( m \geq 0 \) and \( n \geq 0 \), and \( 0 < \lambda_1 < 1 \), let \( \lambda_2 = 1 - \lambda_1 \). Then

\[ \sum_{i=0}^{m} \binom{n+i}{n} \frac{i}{\lambda_1} \frac{n+1}{\lambda_2} + \sum_{j=0}^{n} \binom{m+j}{m} \frac{m+1}{\lambda_1} \frac{j}{\lambda_2} = 1. \]  

**Proof:** The proof is by induction on \( m \) and \( n \).

**m = n = 0.** Immediate since (3.43) reduces to \( \lambda_1 + \lambda_2 = 1 \).

**m = 0, n > 0.** Suppose the result holds for \( (0,n) \). Then

\[
\binom{n+1+0}{n+1} \lambda_1^0 \lambda_2^{n+2} + \sum_{j=0}^{n+1} \binom{0+j}{0} \lambda_1^1 \lambda_2^j \\
= \left[ \binom{n+0}{n} \lambda_1^0 \lambda_2^{n+1} \right] \lambda_2 + \sum_{j=0}^{n} \binom{0+j}{0} \lambda_1^1 \lambda_2^j + \binom{0+n+1}{0} \lambda_1^1 \lambda_2^{n+1}
\]

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\[\begin{align*}
&= \left(\binom{n+0}{n} \lambda_1^0 \lambda_2^{n+1}\right) \lambda_2 + \sum_{j=0}^{n} \binom{0+j}{0} \lambda_1^1 \lambda_2^j + \left[\binom{n+0}{0} \lambda_1^0 \lambda_2^{n+1}\right] \lambda_1 \\
&= \left(\binom{n+0}{n} \lambda_1^0 \lambda_2^{n+1}\right) + \sum_{j=0}^{n} \binom{0+j}{0} \lambda_1^1 \lambda_2^j = 1
\end{align*}\]

by the induction hypothesis. A similar proof holds for \((m,0)\).

For \(m > 0, n > 0\). Now suppose the result is known to be true for \((m,n-1)\) and \((m-1,n)\). We will show that this implies its validity for \((m,n)\) and thus the Lemma. For \((m,n)\)

\[
\begin{align*}
\lambda_2^{n+1} & \sum_{i=0}^{m} \binom{n+i}{n} \lambda_1^i + \lambda_1^{m+1} \sum_{j=0}^{n} \binom{m+j}{m} \lambda_2^j \\
&= \lambda_2^{n+1}(1 + \sum_{i=1}^{m} \binom{n+i}{n} \lambda_1^i) + \lambda_1^{m+1}(1 + \sum_{j=1}^{n} \binom{m+j}{m} \lambda_2^j) \\
&= \lambda_2^{n+1}(1 + \sum_{i=1}^{m} \binom{n-1+i}{n-1} \lambda_1^i) + \sum_{i=1}^{m} \binom{n+i}{n} \lambda_1^i \\
&\quad + \lambda_1^{m+1}(1 + \sum_{j=1}^{n} \binom{m-1+j}{m-1} \lambda_2^j) + \sum_{j=1}^{n} \binom{m-1+j}{m} \lambda_2^j \text{ since } \binom{n+1}{m+1} = \binom{n}{m} + \binom{n}{m+1} \\
&= \lambda_2^{n+1} \left( \sum_{i=0}^{m} \binom{n-1+i}{n-1} \lambda_1^i \right) + \lambda_1^{m+1} \left( \sum_{j=0}^{n} \binom{m-1+j}{m-1} \lambda_2^j \right) \\
&\quad + \lambda_1^1 \left( \sum_{j=0}^{n} \binom{m-1+j}{m} \lambda_2^j \right) \\
&= \lambda_2^{n+1} \left( \sum_{i=0}^{m} \binom{n-1+i}{n-1} \lambda_1^i \right) + \lambda_1^{m+1} \left( \sum_{j=0}^{n-1} \binom{m+j}{m} \lambda_2^j \right) \\
&\quad + \lambda_1^1 \left( \sum_{j=0}^{n} \binom{m+j}{m} \lambda_2^j \right) \\
&= \lambda_2^{n+1} \left( \sum_{i=0}^{m} \binom{n-1+i}{n-1} \lambda_1^i \right) + \lambda_1^{m+1} \left( \sum_{j=0}^{n-1} \binom{m+j}{m} \lambda_2^j \right) \\
&\quad + \lambda_1^1 \left( \sum_{j=0}^{n-1} \binom{m+j}{m} \lambda_2^j \right) \lambda_2^{n+1} \\
&= \lambda_2^{n+1} \left( 1 + \sum_{i=0}^{m} \binom{n+i}{n} \lambda_1^i \right) + \lambda_1^{m+1} \left( 1 + \sum_{j=0}^{n-1} \binom{m+j}{m} \lambda_2^j \right) \\
&= \lambda_2^{n+1} \lambda_2^{n+1} + \lambda_1^{m+1} \lambda_1^{m+1} \lambda_2^{n+1} = 1.
\end{align*}\]
It is now possible to write explicitly the limiting probabilities of the $P_{r,n_1,n_2}$.

**Theorem 3.9.** For the case of exponential repair time and $I = 2$, the limiting probability $P_{r,n_1,n_2} = \lim_{t \to \infty} P\{S(t) = (r,n_1,n_2)\}$ is given by:

$$P_{0,1,n_2} = c_{1,n_2}, \quad 1 \leq n_2 \leq N_2$$

$$P_{0,n_1,1} = c_{n_1,1}, \quad 1 \leq n_1 \leq N_1$$

$$P_{0,n_1,n_2} = c_{n_1,n_2}(\frac{1}{1 + \lambda/\mu}), \quad 2 \leq n_1 \leq N_1, \quad 2 \leq n_2 \leq N_2, \quad n_1 + n_2 \leq N_1 + N_2 - 1$$

$$P_{0,N_1,N_2} = c_{N_1,N_2}$$

$$P_{1,1,n_2} = c_{1,n_2}(\lambda/\mu), \quad 1 \leq n_2 \leq N_2$$

$$P_{1,n_1,1} = c_{n_1,1}(\lambda/\mu), \quad 1 \leq n_1 \leq N_1$$

$$P_{1,n_1,n_2} = c_{n_1,n_2}(\lambda/\mu)(\frac{1}{1 + \lambda/\mu}), \quad 2 \leq n_1 \leq N_1, \quad 2 \leq n_2 \leq N_2, \quad n_1 + n_2 \leq N_1 + N_2 - 1$$

$$P_{2,n_1,n_2} = c_{n_1,n_2}(\lambda/\mu)^2(\frac{1}{1 + \lambda/\mu}), \quad 2 \leq n_1 \leq N_1, \quad 2 \leq n_2 \leq N_2, \quad n_1 + n_2 \leq N_1 + N_2 - 2$$

$$P_{r,n_1,n_2} = 0 \quad \text{otherwise}$$

where
\[ c_{n_1, n_2} = \frac{(N_1 - n_1) + (N_2 - n_2)}{(N_1 - n_1)} \cdot \frac{N_1 - n_1}{\lambda_1} \cdot \frac{N_2 - n_2}{\lambda_2} \cdot c, \]

and \( c \) is defined such that

\[ \sum_{r=0}^{2} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} p_{r, n_1, n_2} = 1. \]

**Proof:** The proof is immediate by Lemmas 3.6 - 3.8.

Corresponding to Figure 3.1 above, we may indicate the probabilities graphically. Non-operating states are marked by dashes, and states with zero probability are so marked.

---

**Figure 3.2**

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3.2. **General Values of I.**

We have proven the result for the case \( I = 2 \). Generalization to arbitrary values of \( I \) is straightforward. The equations corresponding to Theorem 3.5 will be given without comment, as their derivation is similar to the case \( I = 2 \). From these equations the analogue to Theorem 3.9 will be derived.

Additional notation for the case of arbitrary values of \( I \) is necessary. Let \( \lambda_i \) again denote the failure rate of part type \( i \) when a system is operating, and let \( \mu \) denote the failure rate (i.e., \( F(t) = 1 - e^{-\mu t} \)). Define \( \lambda = \sum_{i=1}^{I} \lambda_i \) and \( \lambda_i^* = \lambda_i / \lambda \), \( i = 1, \ldots, I \).

Let \( P_{r, n_1, \ldots, n_I}(t) = P[S(t) = (r, n_1, \ldots, n_I)] \) and

\[
P_{r, n_1, \ldots, n_I} = \lim_{t \to \infty} P_{r, n_1, \ldots, n_I}(t).
\]

The limit \( P_{r, n_1, \ldots, n_I} \) exists, analogously to Theorem 1.1 and Lemma 3.3, since the state space again is finite. Let \( m = \min_{i=1,\ldots,I} n_i \) as before, and let \( N_m \) be the number of indices \( i \) such that \( n_i = m \). Let \( e_j \) denote a \( 1 \times (I+1) \) row vector, with an \( e \) in the \((j+1)\)st position and zeros elsewhere \((e = 1, 2; j = 1, \ldots, I)\). We will write \( P_{1, n_1, \ldots, n_I+e_j} \) to denote the expression \( P_{1, n_1, \ldots, n_{j-1}, n_j+e_j, n_{j+1}, \ldots, n_I} \). Another shorthand in notation will be the use of \( P_{1, n_1, \ldots, n_I(1)} \) to denote

\[
P_{1, n_1, \ldots, n_{j-1}, 1, n_{j+1}, \ldots, n_I}
\]

and \( P_{1, n_1, \ldots, n_I(2)} \) to denote

\[
P_{1, n_1, \ldots, n_{j-1}, 2, n_{j+1}, \ldots, n_I}.
\]

Finally, we assume for this case without loss of generality that \( \mu = 1 \). This can be accomplished in the process merely by a change in the time scale. The final results, given in Theorem 3.12 below, can
be converted back to the case \( \mu \neq 1 \) simply by replacing \( \sqrt[\lambda]{\mu} \) for each \( \lambda \) occurring in Theorem 3.12. All other quantities are unchanged.

The equations governing the \( P_{r,n_1,\ldots,n_I} \), analagous to Theorem 3.5, are as follows:

(3.44.1) \( \lambda P_{0,N_1,\ldots,N_I} = \lambda_1 \sum_{1}^{N_2} \sum_{1}^{N_I} P_{0,1,n_2,\ldots,n_I} \)
\( + \lambda_2 \sum_{1}^{N_1} \sum_{1}^{N_3} \sum_{1}^{N_I} P_{0,n_1,1,\ldots,n_I} + \cdots + \lambda_I \sum_{1}^{N_1} \sum_{1}^{N_I-1} P_{0,n_1,\ldots,n_{I-1},1} \)

(3.44.2) \( \lambda P_{0n_1,\ldots,n_I} = P_{1n_1,\ldots,n_I} \)
\( \begin{cases} m = \min_{i=1,\ldots,I} n_i \geq 1 \\ \Sigma n_i \leq \Sigma N_i - 1 \end{cases} \)

(3.44.3) \( P_{1,1,\ldots,1} = \lambda_1 P_{01 \ldots 1+1} + \cdots + \lambda_I P_{01 \ldots 1+1} \)

(3.44.4) \( P_{1,n_1,\ldots,n_I(1_k)} = \lambda_k P_{1,n_1,\ldots,n_I(2_k)} \)
\( + \sum_{j=1}^{I} \lambda_j P_{0,n_1,\ldots,n_I(1_k)+1} \)
\( \begin{cases} 2 \leq n_j \leq N_j, j \neq k \\ \Sigma n_i \leq \Sigma N_i - 1 \end{cases} \)

(3.44.5) \( P_{1,n_1,\ldots,n_I} = \sum_{j=1}^{I} \lambda_j P_{0,n_1,\ldots,n_I+1} \)
\( m = 1 \) and \( N_m > 1 \)

(3.44.6) \( P_{1,n_1,\ldots,n_I(1+\lambda)} = \sum_{j=1}^{I} \lambda_j P_{0,n_1,\ldots,n_I+1} \)
\( + P_{2,n_1,\ldots,n_I} \)
\( \begin{cases} 2 \leq n_i \leq N_i \\ \Sigma n_i \leq \Sigma N_i - 1 \end{cases} \)
(3.44.7) \[ P_{2,n_1,\ldots,n_I} = \sum_{j=1}^I \lambda_j P_{1,n_1,\ldots,n_{I+1}} \quad 2 \leq n_1 \leq N_i \]
\[ \sum_{r=1}^{N_1} \sum_{l=1}^{N_I} \sum_{r_{I+1}}^{N_{I+1}} P_{r,n_1,\ldots,n_I} = 1. \]

These equations may be solved as above. Two lemmas are first presented.

**Lemma 3.10.** Let \( N = \sum_{i=1}^I n_i \). Then

(3.45) \[ \binom{N}{n_1 \ldots n_I} = \binom{N-1}{n_1-1 \ldots n_{I-1}} + \binom{N-1}{n_1(n_2-1) \ldots n_{I-1}} + \cdots + \binom{N-1}{n_1 n_2 \ldots (N_{I-1})} \]

**Proof:**

\[ \binom{N}{n_1 \ldots n_I} = \frac{N!}{n_1! \cdots n_I!} = \frac{(N-1)!}{n_1! \cdots n_I!} \]
\[ = \frac{(N-1)!}{(n_1-1)! \cdots n_{I-1}!} + \frac{(N-1)!}{n_1(n_2-1)! \cdots n_{I-1}!} \]
\[ + \cdots + \frac{(N-1)!}{n_1 n_2 \cdots (N_{I-1})!} \]

which is (3.45).
Lemma 3.11. For a fixed set \((N_1, \ldots, N_I)\) define

\[
\begin{align*}
\tilde{c}_{n_1, \ldots, n_I} &= \left( \frac{\Sigma(N_i - n_i)}{(N_i - n_i) \cdots (N_1 - n_1)} \right) \prod_{i=1}^{I} \tilde{\lambda}_i^{N_i - n_i}, \\
0 \leq n_i \leq N_i. \text{ Then}
\end{align*}
\]

\[
c_{n_1, \ldots, n_I} = \sum_{j} \hat{\lambda}_j c_{n_1, \ldots, n_I + 1_j}.
\]

Proof: Immediate by Lemma 3.10.

Theorem 3.12. For the case of exponential repair time and for any integer \(I\), the limiting probability

\[
P_{r, n_1, \ldots, n_I} = \lim_{t \to \infty} P[S(t) = (r, n_1, \ldots, n_I)],
\]

for \(r = 0, 1, 2\) and \(n_i = 1, \ldots, N_i\) \((i = 1, \ldots, I)\) is given by:

\[
\begin{align*}
(3.46.1) & \quad P_{0, n_1, \ldots, n_I} = c_{n_1, \ldots, n_I} \quad \min n_i = 1; \sum_{i} n_i \leq \sum_{i} N_i - 1 \\
(3.46.2) & \quad P_{0, n_1, \ldots, n_I} = c_{n_1, \ldots, n_I} \frac{1}{1+\lambda} \quad \min n_i = 2; \sum_{i} n_i \leq \sum_{i} N_i - 1 \\
(3.46.3) & \quad P_{0, N_1, \ldots, N_I} = c_{N_1, \ldots, N_I} \\
(3.46.4) & \quad P_{1, n_1, \ldots, n_I} = c_{n_1, \ldots, n_I}(\lambda) \quad \min n_i = 1; \sum_{i} n_i \leq \sum_{i} N_i - 1 \\
(3.46.5) & \quad P_{1, n_1, \ldots, n_I} = c_{n_1, \ldots, n_I}(\lambda) \frac{1}{1+\lambda} \quad \min n_i = 2; \sum_{i} n_i \leq \sum_{i} N_i - 1
\end{align*}
\]
(3.46.6) \[ p_{2,n_1,\ldots,n_I} = c_{n_1,\ldots,n_I} (\lambda)^2 \frac{1}{1+\lambda} \min_i n_i \geq 2; \sum_i n_i \leq \sum_i n_{i-1} \]

(3.46.7) \[ p_{r,n_1,\ldots,n_I} = 0 \quad \text{otherwise} \]

where

\[ c_{n_1,\ldots,n_I} = \frac{[\Sigma(N_i-n_i)]!}{(N_1-n_1)! (N_2-n_2)! \cdots (N_I-n_I)!} \cdot \frac{N_1-n_1}{\lambda_1} \frac{N_2-n_2}{\lambda_2} \cdots \frac{N_I-n_I}{\lambda_I} c \]

and \( c > 0 \) is defined so that \( \sum \sum \cdots \sum p_{r,n_1,\ldots,n_I} = 1 \).

**Proof:** We show (3.46.1) - (3.46.7) to be the unique solution to (3.44.1) - (3.44.9). Verifying the equations in turn, by Lemma 3.11, (3.44.7) is

\[ p_{2,n_1,\ldots,n_I} = c_{n_1,\ldots,n_I} \left( \frac{\lambda}{1+\lambda} \right) = \sum_{j=1}^{I} \lambda_{j} c_{n_1,\ldots,n_{I-1}+1} \left( \frac{\lambda}{1+\lambda} \right) \]

Since \( p_{1,n_1,\ldots,n_I} = \frac{\lambda}{1+\lambda} \),

\[ p_{2,n_1,\ldots,n_I} = \sum_{j=1}^{I} \lambda_{j} p_{1,n_1,\ldots,n_{I-1}+1} = \sum_{j=1}^{I} \lambda_{j} p_{1,n_1,\ldots,n_I} \]

Equation (3.44.3) holds also by Lemma 3.11, and (3.44.2) is trivial.

To solve (3.44.6)
\[ \sum_{j=1}^{1} \lambda_j p_{0,n_1,\ldots,n_{I+1,j}} + p_{2,n_1,\ldots,n_{I}} \]

\[ = \sum_{j=1}^{1} \lambda_j c_{n_1,\ldots,n_{I+1,j}} \left( \frac{\lambda}{1+\lambda} \right) + c_{n_1,\ldots,n_{I}} \left( \frac{\lambda^2}{1+\lambda} \right) \]

\[ = c_{n_1,\ldots,n_{I}} \left( \frac{\lambda}{1+\lambda} \right) + c_{n_1,\ldots,n_{I}} \left( \frac{\lambda^2}{1+\lambda} \right) \]

\[ = c_{n_1,\ldots,n_{I}} \lambda = p_{1,n_1,\ldots,n_{I}} (1+\lambda). \]

To solve (3.44.4):

\[ \sum_{j=1}^{1} \lambda_j p_{0,n_1,\ldots,n_{I}(1_k)+1_j} + \lambda_k p_{1,n_1,\ldots,n_{I}(2_k)} \]

\[ = \lambda_k c_{n_1,\ldots,n_{I}(2_k)} \left( \frac{\lambda}{1+\lambda} \right) + \sum_{j=1, j \neq k}^{1} \lambda_j c_{n_1,\ldots,n_{I}(1_k)+1_j} \left( \frac{\lambda^2}{1+\lambda} \right) \]

\[ + \lambda_k c_{n_1,\ldots,n_{I}(2_k)} \left( \frac{\lambda}{1+\lambda} \right) \]

\[ = \lambda_k c_{n_1,\ldots,n_{I}(2_k)} (\lambda) + \sum_{j=1, j \neq k}^{1} \lambda_j c_{n_1,\ldots,n_{I}(1_k)+1_j} (\lambda) \]

\[ = c_{n_1,\ldots,n_{I}(1_k)} (\lambda) = p_{1,n_1,\ldots,n_{I}(1_k)}. \]

Equation (3.44.5) is similar to (3.44.3) and (3.44.7), and (3.44.1) holds by an extension of Lemma 3.12. Since (3.44.8) and (3.44.9) are also guaranteed by (3.46.1)-(3.46.7), the proof of Theorem 3.16 is complete.
3.3. Normalizing Constant and Limiting Operation Probability.

In this section we derive the value of the constant $c$, in terms of which the $P_{r_1, n_1, \ldots, n_I}$ were given, to be chosen so that

$$\sum_{S} P_{r_1, n_1, \ldots, n_I} = 1.$$  

In so doing an alternate derivation of $F(n_1, \ldots, n_I)$, the limiting probability of system operation, is provided. From Theorem 1.1

$$F(n_1, \ldots, n_I) = \frac{E(X_1)}{E(Z_1)}$$

where the values for $E(X_1)$ and $E(Z_1)$ were given in Chapter 2.

Alternatively we may write

$$F(n_1, \ldots, n_I) = \frac{\sum \text{operating states } S P_{r_1, n_1, \ldots, n_I}}{\sum_S P_{r_1, n_1, \ldots, n_I}}.$$  

(3.47)

Since each $P_{r_1, n_1, \ldots, n_I}$ is written in terms of $c$, let

$$P_{r_1, n_1, \ldots, n_I}^c = \frac{1}{c} P_{r_1, n_1, \ldots, n_I}.$$  

Then

$$c^{-1} = \sum_S P_{r_1, n_1, \ldots, n_I}^c$$  

(3.48)

and

$$F(n_1, \ldots, n_I) = \frac{\sum \text{operating states } S P_{r_1, n_1, \ldots, n_I}^c}{\sum_S P_{r_1, n_1, \ldots, n_I}^c}.$$  

(3.49)
We first consider the case $I = 2$. Divide the set $S$ into five mutually exclusive and collectively exhaustive subsets which we shall label $A$, $B$, $C$, $D$, and $E$. Let

$$A = \{(r, n_1, n_2) \in S | r = 0 \text{ and } \min_{i=1,2} n_i = 1\}$$

$$B = \{(r, n_1, n_2) \in S | r = 0 \text{ and } \min_{i=1,2} n_i \geq 2\}$$

$$C = \{(r, n_1, n_2) \in S | r = 1 \text{ and } \min_{i=1,2} n_i = 1\}$$

$$D = \{(r, n_1, n_2) \in S | r = 1 \text{ and } \min_{i=1,2} n_i \geq 2\}$$

$$E = \{(r, n_1, n_2) \in S | r = 2 \text{ and } \min_{i=1,2} n_i \geq 2\}$$

Graphically the sets are as in Figure 3.3. The sets $A$, $B$, and $D$ include only operating states, and the sets $C$ and $E$ include only non-operating states.

We have from (3.48) and (3.49), then

$$F(N_1, N_2) = \frac{P'(A) + P'(B) + P'(D)}{P'(A) + P'(B) + P'(C) + P'(D) + P'(E)}$$

and

$$C^{-1} = P'(A) + P'(B) + P'(C) + P'(D) + P'(E).$$
By Theorem 3.9, for each \((n_1, n_2)\) such that \((0, n_1, n_2) \in A\)

\[(3.50)\]
\[
\frac{\lambda}{\mu} P^i_{0, n_1, n_2} = P^i_{1, n_1, n_2}
\]

and for each \((n_1, n_2)\) such that \((0, n_1, n_2) \in B\) and \(\sum_{i=1}^{2} n_i \leq \sum_{i=1}^{2} \lambda_i - 2\),

\[
\frac{(\lambda)}{(\mu^i)} P^i_{0, n_1, n_2} = \frac{(\lambda)}{(\mu^i)} P^i_{1, n_1, n_2} = P^i_{2, n_1, n_2}.
\]

By (3.50), we have

\[(3.51)\]
\[
P^i(C) = \frac{\lambda}{\mu} P^i(A)
\]
A simple relation such as (3.51), however, does not hold for the sets B, D, and E, since certain pairs of $(n_1, n_2)$ have differing probabilities or do not appear in all sets.

Define, for $0 \leq n_1 \leq N_1$ and $0 \leq n_2 \leq N_2$,

$$\tilde{c}_{n_1, n_2} = \frac{1}{c} c_{n_1, n_2} = \left( \frac{(N_1 - n_1) + (N_2 - n_2)}{(N_1 - n_1)} \right) \frac{N_1 - n_1}{\lambda_1} \frac{N_2 - n_2}{\lambda_2}$$

and

$$h(N_1, N_2) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \tilde{c}_{N_1 - i, N_2 - j} = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (i + j) \frac{\lambda_i}{\lambda_2} \frac{\lambda_i}{\lambda_1}$$

In addition, define

(3.53) \[ \rho = \frac{\lambda}{\mu} \]

Then we have

**Lemma 3.13.**

$$P'(B) = h(N_1 - 2, N_2 - 2) \left( \frac{1}{1 + \rho} \right) + \frac{\rho}{1 + \rho}$$

**Proof:**

$$P'(B) = \sum_{n_1=2}^{N_1} \sum_{n_2=2}^{N_2} P'_{0, n_1, n_2} = \sum_{i=0}^{N_1 - 2} \sum_{j=0}^{N_2 - 2} P'_{0, N_1 - i, N_2 - j}$$

$$= \sum_{i=0}^{N_1 - 2} \sum_{j=0}^{N_2 - 2} \tilde{c}_{N_1 - i, N_2 - j} \left( \frac{1}{1 + \rho} \right) + \tilde{c}_{N_1, N_2}$$

(i, j) \neq (0, 0)
\[
\begin{align*}
\sum_{i=0}^{N_1-2} \sum_{j=0}^{N_2-2} \tilde{c}_{N_1-i, N_2-j} \left( \frac{1}{1+\rho} \right) + \tilde{c}_{N_1, N_2} - \tilde{c}_{N_1, N_2} \left( \frac{1}{1+\rho} \right) \\
= h(N_1-2, N_2-2) \left( \frac{1}{1+\rho} \right) + \left( \frac{\phi}{1+\rho} \right)
\end{align*}
\]

since \( \tilde{c}_{N_1, N_2} = 1. \)

**Lemma 3.14.**

\[
P'(D) = h(N_1-2, N_2-2) \left( \frac{\phi}{1+\rho} \right) - \left( \frac{\phi}{1+\rho} \right)
\]

**Proof:**

\[
P'(D) = \sum_{n_1=2}^{N_1-2} \sum_{n_2=2}^{N_2-2} P^i_{n_1, n_1, n_2}
\]

\[
= \sum_{i=0}^{N_1-2} \sum_{j=0}^{N_2-2} \tilde{c}_{N_1-i, N_2-j} \left( \frac{\phi}{1+\rho} \right) \\
= h(N_1-2, N_2-2) \left( \frac{\phi}{1+\rho} \right) - \left( \frac{\phi}{1+\rho} \right)
\]

**Lemma 3.15.**

\[
P'(E) = h(N_1-2, N_2-2) \left( \frac{2}{1+\rho} \right) - \left( \frac{2\phi}{1+\rho} \right)
\]

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Proof:

\[
P'(E) = \sum_{n_1=2}^{N_1} \sum_{n_2=2}^{N_2} \sum_{i=0}^{N_1-2} \sum_{j=0}^{N_2-2} P_{2,n_1,n_2}^{i,j} \cdot P^i_{2,N_1-i,N_2-j} \cdot (\frac{\rho}{1+\rho})^2 \]

\[
= \sum_{i=0}^{N_1-2} \sum_{j=0}^{N_2-2} \tilde{c}_{N_1-i,N_2-j} \cdot (\frac{\rho}{1+\rho})^2 - \tilde{c}_{N_1,N_2} \cdot (\frac{\rho}{1+\rho})^2 \\
- \tilde{c}_{N_1-1,N_2} \cdot (\frac{\rho}{1+\rho})^2 - \tilde{c}_{N_1,N_2-1} \cdot (\frac{\rho}{1+\rho})^2 \\
= h(N_1-2,N_2-2) \cdot (\frac{\rho}{1+\rho})^2 \cdot (1 + \tilde{\lambda}_1 + \tilde{\lambda}_2) \cdot (\frac{\sigma^2}{1+\rho}) \\
= h(N_1-2,N_2-2) \cdot (\frac{\rho^2}{1+\rho}) - (\frac{\rho^2}{1+\rho}) 
\]

Thus by Lemmas 3.13 - 3.15,

(3.54) \hspace{1cm} P'(B) \cdot (1+\rho) - \rho = P'(D) \cdot (\frac{1+\rho}{\rho}) + 1 = P'(E) \cdot (\frac{1+\rho}{\rho^2}) + 2 = h(N_1-2,N_2-2).

To evaluate \( P'(A) \) and \( P'(C) \), we define, for \( N_1 \geq 0 \) and \( N_2 \geq 0 \),

(3.55) \hspace{1cm} g(N_1,N_2) = \sum_{j=0}^{N_2} \tilde{c}_{0,j} + \sum_{i=0}^{N_1} \tilde{c}_{i,0} - \tilde{c}_{0,0} \\
= \sum_{j=0}^{N_2} \sum_{i=0}^{N_1+j} \tilde{c}_{i,j} \cdot \tilde{\lambda}_1^{i} \tilde{\lambda}_2^{j} + \sum_{i=0}^{N_1} \sum_{j=0}^{i+N_2} \tilde{c}_{i,j} \cdot \tilde{\lambda}_1^{i} \tilde{\lambda}_2^{j} - \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \tilde{c}_{i,j} \cdot \tilde{\lambda}_1^{i} \tilde{\lambda}_2^{j}.

Then
(3.56) \[ P'(A) = g(N_1-1, N_2-1) \]

and

(3.57) \[ P'(C) = \rho g(N_1-1, N_2-1) \]

and finally

**Theorem 3.16.** For the process with exponential repair time and \( I = 2 \), where \( N_i \) parts of type \( i \) are provided, \( i = 1, 2 \); the limiting probability that a system will be operating is given by

\[
F(N_1, N_2) = \frac{h(N_1^{-1}, N_2^{-1})}{h(N_1^{-1}, N_2^{-1}) \left( \frac{1+\rho+\omega^2}{1+\rho} \right) + g(N_1^{-1}, N_2^{-1}) \left( \frac{\rho}{1+\rho} \right) - \frac{2\omega^2}{1+\rho}}
\]

where

\[
h(N_1, N_2) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (i+j) \frac{\lambda_1^i \lambda_2^j}{(1+\rho) + \omega^2} \]

\[
g(N_1, N_2) = \sum_{j=0}^{N_2} \left( \begin{array}{c} N_1+j \cr j \end{array} \right) \frac{\lambda_1^{N_1} \lambda_2^j}{(1+\rho) + \omega^2} + \sum_{i=0}^{N_1} \left( \begin{array}{c} N_2+i \cr i \end{array} \right) \frac{\lambda_1^i \lambda_2^{N_2}}{(1+\rho) + \omega^2} - \left( \begin{array}{c} N_1+N_2 \cr N_1 \end{array} \right) \frac{\lambda_1^{N_1} \lambda_2^{N_2}}{(1+\rho) + \omega^2}
\]

\( \lambda_i \) is the failure rate of part type \( i \), \( \lambda = \sum_{i=1}^2 \lambda_i \), \( \bar{\lambda}_1 = \lambda_1/\lambda \), and \( \omega = \lambda/\mu \).
**Proof:** By (3.50), (3.54), (3.56), and (3.57),

\[
\begin{align*}
F(N_1, N_2) &= \frac{P'(A) + P'(B) + P'(D)}{P'(A) + P'(B) + P'(C) + P'(D) + P'(E)} \\
&= \frac{g(N_1-1, N_2-1) + h(N_1-2, N_2-2)(\frac{1}{1+\rho} + \frac{\rho}{1+\rho}) + \frac{\rho}{1+\rho} - \frac{\rho^2}{1+\rho}}{g(N_1-1, N_2-1)(1+\rho) + h(N_1-2, N_2-2)(\frac{1}{1+\rho} + \frac{\rho^2}{1+\rho}) + \frac{\rho}{1+\rho} + \frac{\rho}{1+\rho} - \frac{2\rho^2}{1+\rho}} \\
&= \frac{g(N_1-1, N_2-1) + h(N_1-2, N_2-2)}{g(N_1-1, N_2-1)(1+\rho) + h(N_1-2, N_2-2)(\frac{1+\rho^2}{1+\rho}) - \frac{2\rho^2}{1+\rho}}.
\end{align*}
\]

We now show that \(g(N_1-1, N_2-1) + h(N_1-2, N_2-2) = h(N_1-1, N_2-1)\):

\[
\begin{align*}
h(N_1-1, N_2-1) &= \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (i+j) \lambda_1^i \lambda_2^j \\
&= \sum_{i=0}^{N_1-2} \sum_{j=0}^{N_2-1} (i+j) \lambda_1^i \lambda_2^j + \sum_{j=0}^{N_2-1} \left( \sum_{i=0}^{N_1-1} \lambda_1^i \right) \lambda_2^j \\
&= \sum_{i=0}^{N_1-2} \sum_{j=0}^{N_2-1} (i+j) \lambda_1^i \lambda_2^j + \sum_{j=0}^{N_2-1} \left( \sum_{i=0}^{N_1-1} \lambda_1^i \right) \lambda_2^j \\
&\quad + \sum_{i=0}^{N_1-2} \left( \sum_{j=0}^{N_2-1} \lambda_2^j \right) \lambda_1^i \\
&= h(N_1-2, N_2-2) + \sum_{j=0}^{N_2-1} \left( \sum_{i=0}^{N_1-1} \lambda_1^i \right) \lambda_2^j \\
&\quad + \sum_{i=0}^{N_1-2} \left( \sum_{j=0}^{N_2-1} \lambda_2^j \right) \lambda_1^i \\
&= h(N_1-2, N_2-2) + g(N_1-1, N_2-1).
\end{align*}
\]
Thus (3.58) becomes

\[
P(N_1, N_2) = \frac{h(N_1 - 1, N_2 - 1)}{g(N_1 - 1, N_2 - 1) (\frac{1 + \rho + \rho^2}{1 + \rho}) + h(N_1 - 2, N_2 - 2) (\frac{\rho}{1 + \rho}) - \frac{2 \rho^2}{1 + \rho}}
\]

\[
= \frac{h(N_1 - 1, N_2 - 1)}{h(N_1 - 1, N_2 - 1) (\frac{1 + \rho + \rho^2}{1 + \rho}) + g(N_1 - 1, N_2 - 1) (\frac{\rho}{1 + \rho}) - \frac{2 \rho^2}{1 + \rho}}
\]

which proves the Theorem.

**Corollary 3.17.** The value of the normalizing constant \( c \) is

\[
[h(N_1 - 1, N_2 - 1) (\frac{1 + \rho + \rho^2}{1 + \rho}) + g(N_1 - 1, N_2 - 1) (\frac{\rho}{1 + \rho}) - \frac{2 \rho^2}{1 + \rho}]^{-1}
\]

**Proof:** Immediate since \( c^{-1} = P'(A) + P'(B) + P'(C) + P'(D) + P'(E) \).

For general values of \( N_i \geq 1 \), define the corresponding functions \( h(N_1, \ldots, N_i) \):

\[
h(N_1, \ldots, n_i) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_i=0}^{N_i} \left( \frac{i_1 + \cdots + i_i}{\lambda_1 \cdots \lambda_i} \right)^{i_1 \cdots i_i}
\]

\[
g(N_1, \ldots, N_i) = h(N_1, \ldots, N_i) - h(N_1 - 1, \ldots, N_i - 1)
\]

\[
= \sum_{i_1=0}^{N_1} \cdots \sum_{i_i=0}^{N_i} \left( \frac{i_1 + \cdots + i_i}{\lambda_1 \cdots \lambda_i} \right)^{i_1 \cdots i_i}
\]

\[
- \sum_{i_1=0}^{N_1 - 1} \cdots \sum_{i_i=0}^{N_i - 1} \left( \frac{i_1 + \cdots + i_i}{\lambda_1 \cdots \lambda_i} \right)^{i_1 \cdots i_i}
\]

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and sets $A$, $B$, $C$, $D$, and $E$ as above, with $I$ replacing $2$. Following a similar line of reasoning, we have $(N_i \geq 2, \ i = 1, \ldots, I)$:

\[(3.59.1) \quad P'(A) = g(N_{i-1}, \ldots, N_{i-1})\]

\[(3.59.2) \quad P'(B) = h(N_{i-2}, \ldots, N_{i-2}) \left(\frac{1}{1+\rho}\right) + \left(\frac{\rho}{1+\rho}\right)\]

\[(3.59.3) \quad P'(C) = \rho \ g(N_{i-1}, \ldots, N_{i-1})\]

\[(3.59.4) \quad P'(D) = h(N_{i-2}, \ldots, N_{i-2}) \left(\frac{\rho}{1+\rho}\right) - \left(\frac{\rho}{1+\rho}\right)\]

\[(3.59.5) \quad P'(E) = h(N_{i-2}, \ldots, N_{i-2}) \left(\frac{\rho}{1+\rho}\right) - \left(\frac{\rho}{1+\rho}\right)\]

and the analogue to Theorem 3.16 carries over to

**Corollary 3.16.** For the process with exponential repair and arbitrary \( I \geq 2 \), with all else as in Theorem 3.16, we have

\[F(N_1, \ldots, N_I) = \frac{h(N_{i-1}, \ldots, N_{i-1})}{[h(N_{i-1}, \ldots, N_{i-1})\left(\frac{1+\rho+\rho^2}{1+\rho}\right) + g(N_{i-1}, \ldots, N_{i-1})\left(\frac{\rho}{1+\rho}\right) - \frac{2\rho^2}{1+\rho}]}\]

\[c = [h(N_{i-1}, \ldots, N_{i-1})\left(\frac{1+\rho+\rho^2}{1+\rho}\right) + g(N_{i-1}, \ldots, N_{i-1})\left(\frac{\rho}{1+\rho}\right) - \frac{2\rho^2}{1+\rho}]^{-1} \]
Proof: Immediate by (3.59.1) - (3.59.5) and the proofs to Theorem 3.16 and Corollary 3.17.
CHAPTER 4

STATE PROBABILITIES -- GENERAL CASE

In this chapter we allow the repair distribution function $F(t)$ to be arbitrary, with the restriction that it be absolutely continuous with $F(0) = 0$. The process no longer is a Markov Process, but the procedure is basically similar to that of the previous chapter. Certain results concerning existence, limits, etc. from Chapter 3 carry over to this case and will not be quoted unless substantial differences exist. Section 4.1 presents the derivation of equations in the limiting state probabilities which are solved and compared with the exponential case in Section 4.2, except for determination of a normalizing constant. This constant is found in Section 4.3, along with the limiting probability of the set of operating states. We consider here only the case $I = 2$; generalization to arbitrary $I$ is straightforward.

4.1. Derivation of the Equations.

In order to consider the case of a general repair time distribution, we use a slightly different method of attack. The method is sometimes called the method of "inclusion of supplementary variables" and has been used in queueing problems (see, e.g., Syski [24]).
We again consider the case where \( I = 2 \). It is necessary to add an additional state variable \( s \), denoting the length of time the item currently in repair (if any) has been in repair.

Define

\[
\begin{align*}
    r(t) &= \text{number of systems out of operation at time } t \\
    u(t) &= \text{length of time the item in repair at time } t \\
    n_1(t) &= \text{number of unfailed parts of type } i \text{ available at time } t \\
    r(t) &= 1, 2 \\
    n_1(t) &= n_2(t) = n_2, \quad u(t) \in (s, s+ds) \quad i = 1, 2
\end{align*}
\]

and let

\[
\begin{align*}
P_{r, n_1, n_2, s}(t) \, ds &= P(r(t) = r, n_1(t) = n_1, n_2(t) = n_2, u(t) \in (s, s+ds)) \\
P_{r, n_1, n_2}(t) &= \int_0^\infty P_{r, n_1, n_2, s}(t) ds \quad r = 1, 2.
\end{align*}
\]

Then \( P_{r, n_1, n_2}(t) \) is the probability that \( S(t) = (r, n_1, n_2); r = 1, 2 \).

As before, let \( P_{0, n_1, n_2}(t) = P(r(t) = 0, n_1(t) = n_1, n_2(t) = n_2) \). For \( 0 \leq t < \infty \), let

\[
\mu(s) \, ds = P(\text{repair completion takes place in } (t, t+ds)|u(t) = s) .
\]

It is well known (see, for example, Barlow and Proschan [2] or Lloyd and Lipow [18]) that

\[
\mu(s) = \frac{f(s)}{1 - F(s)} \quad s > 0
\]

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where \( F(s) \) is the repair time distribution function.

We now proceed to set up equations in the \( P_r, n_1, n_2 (\cdot) \).

Seven classes of states are considered:

**Case I:** \( r = 0, n_1 = N_1, n_2 = N_2 \). We have

\[
P_{0, N_1, N_2} (t+\Delta t) = P_{0, N_1, N_2} (t) \ P\{\text{no events in } (t, t+\Delta t]\}
\]

\[
+ \sum_{j=1}^{N_2} P_{0, 1, j} (t) \ P\{\text{failure of part 1 in } (t, t+\Delta t]\}
\]

\[
+ \sum_{i=1}^{N_1} P_{0, i, 1} (t) \ P\{\text{failure of part 2 in } (t, t+\Delta t]\}
\]

\[
+ P\{\text{two or more events in } (t, t+\Delta t]\}
\]

from which follows

\[
P_{0, N_1, N_2} (t+\Delta t) = P_{0, N_1, N_2} (t) [1 - \lambda_1 \Delta t - \lambda_2 \Delta t - o(\Delta t)]
\]

\[
+ \sum_{i=1}^{N_1} P_{0, i, 1} (t) [\lambda_2 \Delta t + o(\Delta t)]
\]

\[
+ \sum_{j=1}^{N_2} P_{0, i, j} (t) [\lambda_1 \Delta t + o(\Delta t)] + o(\Delta t)
\]

As in Chapter 3, we subtract the first term on the right from both sides of the equation; divide by \( \Delta t \); let \( \Delta t \to 0 \), and let \( t \to \infty \), thereby dropping \( t \) from the equations, to get:
\[ \lim_{t \to \infty} \lim_{\Delta t \to 0} \frac{P_{0,N_1,N_2}(t + \Delta t) - P_{0,N_1,N_2}(t)}{\Delta t} = - (\lambda_1 + \lambda_2) P_{0,N_1,N_2} + \lambda_2 \sum_{i=1}^{N_1} P_{0,i,1} + \lambda_1 \sum_{j=1}^{N_2} P_{0,1,j}. \]

Since the left-hand side is zero, we have

\[ (4.1) \quad (\lambda_1 + \lambda_2) P_{0,N_1,N_2} = \lambda_2 \sum_{i=1}^{N_1} P_{0,i,1} + \lambda_1 \sum_{j=1}^{N_2} P_{0,1,j}. \]

Case II: \( r = 0, n_1 \in [1,N_1], n_2 \in [1,N_2], n_1 + n_2 \leq N_1 + N_2. \) When in this state, failures of part 1 or part 2 are possible. We may enter it from any of the states \([1,n_1,n_2,s]|s \in [0,\infty)\) by means of a repair completion. Thus, by Lemma 4.1:

\[ P_{0,n_1,n_2}(t + \Delta t) = P_{0,n_1,n_2}(t)[1 - \lambda_1 \Delta t - \lambda_2 \Delta t - o(\Delta t)] \]

\[ + \int_0^\infty P_{1,n_1,n_2,s}(t)[\mu(s) \Delta t + o(\Delta t)] \, ds. \]

Subtracting \( P_{0,n_1,n_2}(t) \) from both sides and dividing by \( \Delta t \) gives

\[ \frac{P_{0,n_1,n_2}(t + \Delta t) - P_{0,n_1,n_2}(t)}{\Delta t} = - (\lambda_1 + \lambda_2) P_{0,n_1,n_2}(t) - \frac{o(\Delta t)}{\Delta t} P_{0,n_1,n_2}(t) \]

\[ + \int_0^\infty P_{1,n_1,n_2,s}(t) \mu(s) \, ds + \int_0^\infty P_{1,n_1,n_2,s}(t) \frac{o(\Delta t)}{\Delta t} \, ds. \]
Dividing by $\Delta t$, the second and fourth terms on the right hand side go to zero. Letting $t \to \infty$, the left hand side goes to zero, and we have

$$(l, 2) \quad (\lambda_1 + \lambda_2) P_{0, n_1, n_2} = \int_0^\infty \int P_{1, n_1, n_2, s} \mu(s) \, ds.$$ 

Case III: $r = 1$, $n_1 = 1$, $n_2 = 1$. In this case (recall $m = \min(n_1)$), $r = m$, and the system is inoperative. Thus only a repair can take place. We distinguish two cases depending on whether elapsed repair time $s$ is positive or zero.

Case IIIa: $s \geq 0$. For $s \geq 0$, the state $[1, 1, 1, s + \Delta t]$ cannot be entered by means of a failure. This is so because the item in repair has been there for $s$ units of time. Thus the appropriate equation is:

$$P_{1, 1, 1, s + \Delta t} = P_{1, 1, 1, s}(t) [1 - \mu(s) \Delta t - o(\Delta t)].$$

This equation is of a slightly different form than the differential difference equations above. Rather than making the left hand side equal to zero, we will instead derive the derivative of $P_{1, 1, 1, s}$ with respect to $s$. We first subtract $P_{1, 1, 1, s}(t)$ from both sides to get:

$$P_{1, 1, 1, s + \Delta t} - P_{1, 1, 1, s}(t) = -\mu(s) \Delta t P_{1, 1, 1, s}(t) + o(\Delta t).$$

Dividing by $\Delta t$ yields
\[
\frac{P_{1,1,1,s+\Delta t}(t+\Delta t) - P_{1,1,1,s}(t)}{\Delta t} = -\mu(s) P_{1,1,1,s}(t) + \frac{o(\Delta t)}{\Delta t}.
\]

Now we let \( t \) approach infinity and drop the \( t \) from the notation:

\[
\frac{P_{1,1,1,s+\Delta t} - P_{1,1,1,s}}{\Delta t} = -\mu(s) P_{1,1,1,s} + \frac{o(\Delta t)}{\Delta t}.
\]

Finally let \( \Delta t \to 0 \) and we have

\[
(4.3) \quad \frac{d}{ds} P_{1,1,1,s} = -\mu(s) P_{1,1,1,s}.
\]

**Case IIIb: \( s = 0 \).** We would like a boundary condition on \((4.3)\). To do this, we will seek an equation involving \( P_{1,1,1,0} \). Consider

\[
\int_{0}^{\Delta t} P_{1,1,1,s}(t+\Delta t) ds = P(\text{the event } [1,1,1,0] \text{ occurs during } [t,t+\Delta t])
\]

\[
= P_{0,2,1}(t)[\lambda_1 \Delta t + o(\Delta t)]
\]

\[
+ P_{0,1,2}(t)[\lambda_2 \Delta t + o(\Delta t)].
\]

To solve here we first let \( t \to \infty \) to eliminate \( t \) from the equation:

\[
\int_{0}^{\Delta t} P_{1,1,1,s} ds = P_{0,2,1}[\lambda_1 \Delta t + o(\Delta t)] + P_{0,1,2}[\lambda_2 \Delta t + o(\Delta t)].
\]

Dividing by \( \Delta t \) gives

\[
\frac{1}{\Delta t} \int_{0}^{\Delta t} P_{1,1,1,s} ds = \lambda_1 P_{0,2,1} + \lambda_2 P_{0,1,2} + \frac{o(\Delta t)}{\Delta t}.
\]

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By the continuity of $P_{1,1,1,s}$, the limit of

$$\lim_{\Delta t \to 0} \frac{\int_0^{\Delta t} P_{1,1,1,s}(t) dt}{\Delta t}$$

is just $P_{1,1,1,0}$. Thus

$$(4.4) \quad P_{1,1,1,0} = \lambda_1 P_{0,2,1} + \lambda_2 P_{0,1,2}.$$ 

Case IV: $r = 1$, $n_1 = 1$, $2 \leq n_1 \leq N_1$. The analysis here is similar to that in Case III. For simplicity, we merely give the following table:

<table>
<thead>
<tr>
<th>State at time $t$</th>
<th>Event - Probability</th>
<th>State at time $t+\Delta t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1,n_1,1,s$</td>
<td>nothing $\longrightarrow 1 - \mu(s)\Delta t - o(\Delta t)$</td>
<td>$1,n_1,1,s+\Delta t$</td>
</tr>
<tr>
<td>$1,n_1,2,s$</td>
<td>failure part 2 $\longrightarrow -\lambda_2\Delta t + o(\Delta t)$</td>
<td>$1,n_1,1,s+\Delta t$</td>
</tr>
<tr>
<td>any other</td>
<td>more than one event $\longrightarrow o(\Delta t)$</td>
<td>$1,n_1,1,s+\Delta t$</td>
</tr>
</tbody>
</table>

This gives the equation

$$P_{1,n_1,n,s+\Delta t}(t+\Delta t) = P_{1,n_1,1,s}(t)[1 - \mu(s)\Delta t - o(\Delta t)]$$

$$+ P_{1,n_1,2,s}(t)[\lambda_2\Delta t + o(\Delta t)] + o(\Delta t).$$

Subtracting $P_{1,n_1,1,s}(t)$ from both sides, dividing by $\Delta t$, letting $\Delta t \to 0$, and letting $t \to \infty$, gives
\[ (4.5) \quad \frac{\partial}{\partial s} P_{1,n_1,1,s} = -\mu(s) P_{1,n_1,1,s} + \lambda_2 P_{1,n_1,2,s}. \]

For the boundary condition:

\[ \int_0^{\Delta t} P_{1,n_1,1,s}(t+\Delta t) ds = P_{0,n_1+1,1}(t) \left[ \lambda_1 \Delta t \right] + P_{0,n_1,2}(t) \left[ \lambda_2 \Delta t \right] + o(\Delta t) \]

which gives

\[ (4.6) \quad P_{1,n_1,1,0} = \lambda_1 P_{0,n_1+1,1} + \lambda_2 P_{0,n_1,2}. \]

**Case V:** \( r = 1, n_1 = 1, 2 \leq n_2 \leq N_2 \). This is identical to Case IV with the labels reversed. We give the equations directly:

\[ (4.7) \quad \frac{\partial}{\partial s} P_{1,1,n_2,s} = -\mu(s) P_{1,1,n_2,s} + \lambda_1 P_{1,2,n_2,s} \]

\[ (4.8) \quad P_{1,1,n_2,0} = \lambda_2 P_{0,1,n_2+1} + \lambda_1 P_{0,2,n_2}. \]

**Case VI:** \( r = 1, 2 \leq n_1 \leq N_1, 2 \leq n_2 \leq N_2 \). This case is the most general of the system, i.e., the one free of "exceptions". We have, for \( s \geq 0 \),

\[ P_{1,n_1,n_2,s+\Delta t}(t+\Delta t) = P_{1,n_1,n_2,s}(t) \left[ 1 - \lambda_1 \Delta t - \lambda_2 \Delta t - \mu(s) \Delta t - o(\Delta t) \right] \]

\[ + o(\Delta t). \]

Performing the usual operations gives
\[ (4.9) \quad \frac{\partial}{\partial s} P_{1,n_1,n_2,s} = (-\lambda_1 - \lambda_2 - \mu(s)) P_{1,n_1,n_2,s} \quad . \]

For the boundary equation, consider the following. The state may be entered from state \([0, n_1+1, n_2]\) by a failure of part 1, from \([0, n_1, n_2+1]\) by a failure of part 2, or from any of the states \([2,n_1,n_2,s] : s \in [0,\infty)\) by a repair completion. Thus

\[
\Delta t \int_0^\infty P_{1,n_1,n_2,s}(t) ds = P_{0,n_1+1,n_2}(t)[\lambda_1 \Delta t + o(\Delta t)] \\
+ P_{0,n_1,n_2+1}(t)[\lambda_2 \Delta t + o(\Delta t)] \\
+ \int_0^\infty P_{2,n_1,n_2,s}(t)[\mu(\Delta t) + o(\Delta t)] ds \quad .
\]

Letting \( t \to \infty \), dividing by \( \Delta t \), and letting \( \Delta t \to 0 \) gives

\[ (4.10) \quad P_{1,n_1,n_2,0} = \lambda_1 P_{0,n_1+1,n_2} + \lambda_2 P_{0,n_1,n_2+1} \\
+ \int_0^\infty P_{2,n_1,n_2,s} \mu(s) ds \quad . \]

Case VII: \( r = 2 \), \( 2 \leq n_1 \leq N_1 \), \( 2 \leq n_2 \leq N_2 \), \( n_1+n_2 \leq N_1+N_2 - 2 \). When \( r = 2 \), both systems are in the repair shop. However, since under our assumption, only one system can be undergoing repair at a time, the other failed system is awaiting repair, and thus only one "repair time" state variable is needed. For the first equation, we give the table:
<table>
<thead>
<tr>
<th>State of time ( t )</th>
<th>Event - Probability</th>
<th>State at time ( (t+\Delta t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([2, n_1, n_2, s])</td>
<td>nothing -- ( 1 - \mu(s)\Delta t - o(\Delta t) )</td>
<td>([2, n_1, n_2, s+\Delta s])</td>
</tr>
<tr>
<td>([1, n_1+1, n_2, s])</td>
<td>failure part 1 -- ( \lambda_1\Delta t + o(\Delta t) )</td>
<td>([2, n_1, n_2, s+\Delta s])</td>
</tr>
<tr>
<td>([1, n_1+1, n_2, s])</td>
<td>failure part 2 -- ( \lambda_2\Delta t + o(\Delta t) )</td>
<td>([2, n_1, n_2, s+\Delta s])</td>
</tr>
<tr>
<td>any</td>
<td>more than 1 event -- ( o(\Delta t) )</td>
<td>([2, n_1, n_2, s+\Delta s])</td>
</tr>
</tbody>
</table>

Therefore, we have the equation

\[
P_{2,n_1,n_2,s+\Delta s}(t+\Delta t) = P_{2,n_1,n_2,s}(t)[1 - \mu(s)\Delta t - o(\Delta t)]
\]

\[
+ P_{1,n_1+1,n_2,s}(t)[\lambda_1\Delta t + o(\Delta t)]
\]

\[
+ P_{1,n_1,n_2+1,s}(t)[\lambda_2\Delta t + o(\Delta t)] + o(\Delta t)
\]

which becomes

\[
(4.11) \quad \frac{\partial}{\partial s} P_{2,n_1,n_2,s} = -\mu(s) P_{2,n_1,n_2,s} + \lambda_1 P_{1,n_1+1,n_2,s} + \lambda_2 P_{1,n_1,n_2+1,s}.
\]

Finally, for the boundary equation, we reach a result which is at first surprising, but obvious at a second glance. The only way to reach the state \([2, n_1, n_2, 0]\) in an interval \([t, t+\Delta t]\), is for two or more events to occur in an interval of length \( \Delta t \). Thus

\[
\int_{0}^{\Delta t} P_{2,n_1,n_2,s}(t+\Delta t)ds = o(\Delta t)
\]

and consequently

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\textbf{Case VIII: Otherwise.} For any other values of } r, n_1, n_2, \text{ or } s, \text{ we have }

\begin{equation}
P_{r, n_1, n_2, s} = 0.
\end{equation}

Equations (4.1) - (4.13) together give

\textbf{Theorem 4.1.} The integral-differential equations describing the system with a general repair-time distribution } F(t) \text{ are: } (0 \leq s < \infty):

\begin{align}
(4.14.1) \quad (\lambda_1 + \lambda_2) P_{0, N_1, N_2} &= \lambda_2 \sum_{i=1}^{N_1} P_{0, i, 1} + \lambda_1 \sum_{j=1}^{N_2} P_{0, 1, j} \\
(4.14.2) \quad (\lambda_1 + \lambda_2) P_{0, n_1, n_2} &= \int_0^\infty P_{1, n_1, n_2, s} \mu(s) \, ds \\
&\quad \left\{ \begin{array}{l}
1 \leq n_1 \leq N_1; 1 \leq n_2 \leq N_2 \\
n_1 + n_2 \leq N_1 + N_2 - 1
\end{array} \right.
\\
(4.14.3) \quad \frac{\partial}{\partial s} P_{1, 1, 1, s} &= -\mu(s) P_{1, 1, 1, s} \\
(4.14.4) \quad P_{1, 1, 1, 0} &= \lambda_1 P_{0, 2, 1} + \lambda_2 P_{0, 1, 2} \\
(4.14.5) \quad \frac{\partial}{\partial s} P_{1, n_1, 1, s} &= -\mu(s) P_{1, n_1, 1, s} + \lambda_2 P_{1, n_1, 2, s} \quad 2 \leq n_1 \leq N_1 \\
(4.14.6) \quad P_{1, n_1, 1, 0} &= \lambda_1 P_{0, n_1 + 1, 1} + \lambda_2 P_{0, n_1, 2} \\
(4.14.7) \quad \frac{\partial}{\partial s} P_{1, 1, n_2, s} &= -\mu(s) P_{1, 1, n_2, s} + \lambda_1 P_{1, 2, n_2, s} \quad 2 \leq n_2 \leq N_2
\end{align}
(4.14.8) \[ P_{1, n_1, n_2, 0} = \lambda_1 P_{0, n_1, n_2} + \lambda_2 P_{0, n_1, n_2 + 1} \]

(4.14.9) \[ \frac{\partial}{\partial s} P_{1, n_1, n_2, s} = -\left(\lambda_1 + \lambda_2 + \mu(s)\right) P_{1, n_1, n_2, s} + \left\{ \begin{array}{ll}
2 \leq n_1 \leq N_1 & 2 \leq n_2 \leq N_2 \\
n_1 + n_2 \leq N_1 - N_2 - 1 & 
\end{array} \right. 
+ \int_0^s P_{2, n_1, n_2, s} \mu(s) \, ds 
\]

(4.14.10) \[ P_{1, n_1, n_2, 0} = \lambda_1 P_{0, n_1 + 1, n_2} + \lambda_2 P_{0, n_1, n_2 + 1} + \int_0^s P_{2, n_1, n_2, s} \mu(s) \, ds 
\]

(4.14.11) \[ \frac{\partial}{\partial s} P_{2, n_1, n_2, s} = -\mu(s) P_{2, n_1, n_2, s} + \lambda_1 P_{1, n_1 + 1, n_2, s} + \lambda_2 P_{1, n_1, n_2 + 1, s} 
\]

(4.14.12) \[ P_{2, n_1, n_2, 0} = 0 
\]

(4.14.13) \[ P_{r, n_1, n_2} = P_{r, n_1, n_2, s} = 0 \text{ otherwise} 
\]

4.2. Solution of the Equations.

Four stages will be used in the solution of the equation (4.14) given in Theorem 4.1. They are: (I) Eliminate the differentials. (II) Eliminate the Integrals. (III) Eliminate the fourth state variable s. (IV) Solve the remaining set of linear equations directly for the \[ P_{r, n_1, n_2} \]
Stage (1): Eliminate the Differentials.

Differentials occur in equations (4.14.3), (4.14.9), (4.14.11), (4.14.5), and (4.14.7). We consider the equations in that order.

\[ \frac{\partial}{\partial s} P_{1,1,1,s} = -\mu(s) P_{1,1,1,s}. \] Since \( \mu(s) \) is continuous, we know from Coddington [9] that every solution to (4.14.3) is of the form

\[ P_{1,1,1,s} = C_{11} e^{-\tilde{\mu}(s)} \quad C_{11} > 0 \]

where \( \tilde{\mu}(s) = \int_0^s \mu(t) \, dt \). In addition, we have from (4.14.4)

\[ C_{11}' = P_{1,1,1,0} = \lambda_1 P_{0,2,1} + \lambda_2 P_{0,1,2}. \]

The function \( \tilde{\mu}(s) \) may be written

\[ \tilde{\mu}(s) = -\ln(1 - F(s)) \]

and is closely related to the failure rate function \( \mu(s) \). It is also discussed in Lloyd and Lipow [18] and Barlow and Proschan [2].

\[ \frac{\partial}{\partial s} P_{1,n_1,n_2,s} = -\left(\lambda_1 + \lambda_2 + \mu(s)\right) P_{1,n_1,n_2,s} \quad 2 \leq n_1 \leq N_1; 2 \leq n_2 \leq N_2 \]

Similarly to above, the solution is

\[ P_{1,n_1,n_2,s} = C_{n_1,n_2} e^{-\left(\lambda_1 + \lambda_2\right)s - \tilde{\mu}(s)} \]
\[ (4.19) \quad P_{n_1, n_2}^{1, n_1, n_2, 0} = \binom{n_1}{n_2} C_{n_1, n_2} \quad C_{n_1, n_2} > 0 \]

\[ (4.14.11) \quad \frac{\partial}{\partial s} P_{2, n_1, n_2, s} = -\mu(s) P_{2, n_1, n_2, s} + \lambda_1 P_{1, n_1+1, n_2, s} + \lambda_2 P_{1, n_1, n_2+1, s} \]

\[ 2 \leq n_1 \leq N_1; \quad 2 \leq n_2 \leq N_2; \quad n_1+n_2 \leq N_1+N_2 - 2. \] Again since \( \mu(s) \) is continuous, from Coddington [9], we know that the solution is of the form

\[ (4.20) \quad P_{2, n_1, n_2, s} = C e^{-(\lambda_1+\lambda_2)s-\tilde{\mu}(s)} + g(s) \]

From \((4.18)\), \((4.14.11)\) may be written

\[ (4.21) \quad \frac{d}{ds} P_{2, n_1, n_2, s} = -\mu(s) P_{2, n_1, n_2, s} \]

\[ + [\lambda_1 C_{n_1+1, n_2} + \lambda_2 C_{n_1, n_2+1}] e^{-(\lambda_1+\lambda_2)s-\tilde{\mu}(s)}. \]

If we assume, for some \( g(s) \):

\[ (4.22) \quad P_{2, n_1, n_2, s} = C_{n_1, n_2} e^{-(\lambda_1+\lambda_2)s-\tilde{\mu}(s)} + g(s) \]

then from \((4.22)\):

\[ (4.23) \quad \frac{\partial}{\partial s} P_{2, n_1, n_2, s} = -\mu(s) C_{n_1, n_2} e^{-(\lambda_1+\lambda_2)s-\tilde{\mu}(s)} \]

\[ - (\lambda_1+\lambda_2) C_{n_1, n_2} e^{-(\lambda_1+\lambda_2)s-\tilde{\mu}(s)} + C_{n_1, n_2}^g g'(s) \]

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Equating (4.23) and (4.21) gives

\begin{equation}
(4.24) \quad -\mu(s) \, c'_{n_1, n_2} \left. e^{-(\lambda_1 + \lambda_2)s} \right|_{n_1, n_2} + c''_{n_1, n_2} \, g'(s) = -\mu(s) \, F_{2, n_1, n_2, s} + c'_{n_1, n_2} \left. e^{-(\lambda_1 + \lambda_2)s} \right|_{n_1, n_2}
\end{equation}

\begin{equation}
= [\lambda_1 c'_{n_1+1, n_2} + \lambda_2 c''_{n_1, n_2+1}] \left. e^{-(\lambda_1 + \lambda_2)s} \right|_{n_1, n_2+1}
\end{equation}

The \( g(s) \) which satisfies (4.24) is

\begin{equation}
g(s) = e^{-\tilde{\mu}(s)}
\end{equation}

and solving for \( c'_{n_1, n_2} \) and \( c''_{n_1, n_2} \) we get

\begin{equation}
c'_{n_1, n_2} = -\frac{1}{\lambda_1 + \lambda_2} \left( \lambda_1 \, c'_{n_1+1, n_2} + \lambda_2 \, c'_{n_1, n_2+1} \right)
\end{equation}

\begin{equation}
c''_{n_1, n_2} = \frac{1}{\lambda_1 + \lambda_2} \left( \lambda_1 \, c'_{n_1+1, n_2} + \lambda_2 \, c''_{n_1, n_2+1} \right)
\end{equation}

Thus the solution to (4.14.11) is

\begin{equation}
(4.25) \quad P_{2, n_1, n_2, s} = \frac{\lambda_1 c'_{n_1+1, n_2} + \lambda_2 c''_{n_1, n_2+1}}{\lambda_1 + \lambda_2} \left. e^{-\mu(s)} \right|_{1 - e^{-(\lambda_1 + \lambda_2)s}}
\end{equation}

\begin{equation}
(4.26) \quad P_{2, n_1, n_2, 0} = 0.
\end{equation}
\[ \frac{\partial}{\partial s} P_{l, n_1, 1, s} = -\mu(s) P_{l, n_1, 1, s} + \lambda_2 P_{l, n_1, 2, s} \quad \text{for} \quad 2 \leq n_1 \leq N_1 - 1. \]

This equation is solved in the same way as was (4.14.11). If we assume

\[ P_{l, n_1, 1, s} = C'_{n_1, 1} e^{-(\lambda_1 + \lambda_2)s-\tilde{\mu}(s)} + C''_{n_1, 1} g(s). \]

Then

\[ \frac{\partial}{\partial s} P_{l, n_1, 1, s} = -\mu(s) C'_{n_1, 1} e^{-(\lambda_1 + \lambda_2)s-\tilde{\mu}(s)} - (\lambda_1 + \lambda_2) C'_{n_1, 1} e^{-(\lambda_1 + \lambda_2)s-\tilde{\mu}(s)} + C''_{n_1, 1} g'(s), \]

\[ -C''_{n_1, 1} g(s)\mu(s) = C''_{n_1, 1} g'(s), \]

\[ g(s) = e^{-\tilde{\mu}(s)}. \]

and

\[ P_{l, n_1, 1, s} = C'_{n_1, 1} e^{-(\lambda_1 + \lambda_2)s-\tilde{\mu}(s)} + C''_{n_1, 1} e^{-\tilde{\mu}(s)}. \]

Substituting (4.27) into (4.14.5) gives for the value \( C'_{n_1, 1} \)

\[ (4.28) \quad C'_{n_1, 1} = -\frac{\lambda_2}{\lambda_1 + \lambda_2} C_{n_1, 2}. \]

and the boundary condition (4.14.6) gives for \( C''_{n_1, 1} \)

\[ (4.29) \quad C''_{n_1, 1} = \lambda_1 P_{0, n_1+1, 1} + \lambda_2 P_{0, n_1, 2} - C'_{n_1, 1}. \]

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(4.14.7) \[ \frac{\partial}{\partial s} P_{1,1,n_2,s} = -\mu(s) P_{1,1,n_2,s} + \lambda_1 P_{1,2,n_2,s} \] for \( 2 \leq n_2 \leq N_2 \).

As in (4.14.5) we have

(4.30) \[ P_{1,1,n_2,s} = C'_{1,n_2} e^{-(\lambda_1 + \lambda_2)s - \bar{\mu}(s)} + C''_{1,n_2} e^{-\bar{\mu}(s)} \]

(4.31) \[ C'_{1,n_2} = -\frac{\lambda_1}{\lambda_1 + \lambda_2} C_{2,n_2} \]

(4.32) \[ C''_{1,n_2} = \lambda_1 P_{0,2,n_2} + \lambda_2 P_{0,1,n_2+1} - C'_{1,n_2} \]

**Stage II. Eliminate the Integrals.**

We now have a form for \( P_{i,n_1,n_2,s} \) all \( i, n_1, n_2 \); and can evaluate the integrals in the equations. Let \( \lambda = \lambda_1 + \lambda_2 \) and \( \tilde{\lambda}_i = \lambda_i / \lambda, \ i = 1, 2 \), as before. Let \( F^*(s) = \int_0^\infty e^{-st} dF(t) \) be the Laplace-Stieltjes Transform of the distribution \( F(\cdot) \). We now give the following Lemma evaluating certain integrals.

**Lemma 4.2.** If \( F(t) \) is a distribution function with \( F(0) = 0 \) and having a continuous density \( f(t) \); \( \mu(t) = \frac{f(t)}{1-F(t)} \); \( F^*(s) = \int_0^\infty e^{-st} dF(t) \); \( \tilde{\mu}(t) = \int_0^t \mu(s) ds = -\ln(1-F(t)) \); and \( \frac{1}{\mu} = \int_0^\infty t dF(t) \), then

(4.33.1) \[ \int_0^\infty \mu(t) e^{-\tilde{\mu}(t)} dt = 1 \]

(4.33.2) \[ \int_0^\infty e^{-\tilde{\mu}(t)} dt = \frac{1}{\mu} \]
\[ (4.33.3) \quad \int_0^\infty \mu(t) e^{-\mu(t)} \lambda t \, dt = F^*(\lambda) \]

\[ (4.33.4) \quad \int_0^\infty \mu(t) e^{-\mu(t)} (1 - e^{-\lambda t}) \, dt = 1 - F^*(\lambda) \]

\[ (4.33.5) \quad \int_0^\infty e^{-\mu(t)} - \lambda t \, dt = \frac{1}{\lambda} (1 - F^*(\lambda)) \]

\[ (4.33.6) \quad \int_0^\infty e^{-\mu(t)} (1 - e^{-\lambda t}) \, dt = \frac{1}{\mu} - \frac{1}{\lambda} (1 - F^*(\lambda)) \]

**Proof:** \[ \int_0^\infty e^{-\mu(t)} \, dt = \int_0^\infty e^{\ln(1-F(t))} \, dt = \int_0^\infty (1-F(t)) \, dt = \frac{1}{\mu}, \text{ which proves (4.33.2). To prove (4.33.1),} \]

\[ \int_0^\infty \mu(t) e^{-\mu(t)} \, dt = \int_0^\infty \frac{f(t)}{1-F(t)} (1-F(t)) \, dt = 1. \]

For \((4.33.3),\)

\[ \int_0^\infty \mu(t) e^{-\mu(t)} - \lambda t \, dt = \int_0^\infty \frac{f(t)}{1-F(t)} (1-F(t)) e^{-\lambda t} \, dt \]

\[ = \int_0^\infty e^{-\lambda t} f(t) \, dt = F^*(\lambda) \]

by definition. Equation \((4.33.4)\) follows immediately from \((4.33.1)\) and \((4.33.3),\) as does \((4.33.6)\) from \((4.33.2)\) and \((4.33.5).\) Thus it remains only to prove \((4.33.5).\) Integrating by parts

\[ \int_0^\infty e^{-\mu(t)} - \lambda t \, dt = \int_0^\infty e^{-\lambda t} (1-F(t)) \, dt = \frac{1}{\lambda} - \int_0^\infty \frac{1}{\lambda} e^{-\lambda t} f(t) \, dt \]

\[ = \frac{1}{\lambda} - \frac{1}{\lambda} F^*(\lambda) \]

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which completes the proof of Lemma 4.2.

The equations of Theorem 4.1 now read (from (4.15) - (4.16), (4.18) - (4.19), and (4.25) - (4.32))

\[
\Lambda_{0, N_1, N_2} = \lambda_2 \sum_{i=1}^{N_1} P_{0, i, 1} + \lambda_1 \sum_{j=1}^{N_2} P_{0, 1, j}
\]

(4.34.1)

\[
\Lambda_{0, n_1, n_2} = \int_{0}^{\infty} P_{1, n_1, n_2, s} \mu(s) ds \quad 1 \leq n_1 \leq N_1; \quad 1 \leq n_2 \leq N_2
\]

(4.34.2)

\[
P_{1, 1, 1, s} = C_{11} e^{-\tilde{\mu}(s)}
\]

(4.34.3)

\[
P_{1, 1, 1, 0} = C_{11} = \lambda_1 P_{0, 2, 1} + \lambda_2 P_{0, 1, 2}
\]

(4.34.4)

\[
P_{1, n_1, 1, s} = C_{n_1, 1} e^{-\lambda s - \tilde{\mu}(s)} + C_{n_1, 1} e^{-\tilde{\mu}(s)} \quad 2 \leq n_1 \leq N_1
\]

(4.34.5)

\[
C_{n_1, 1} = -\tilde{\lambda}_2 C_{n_1, 2}
\]

(4.34.6)

\[
C_{n_1, 1} e^{-\lambda s} \tilde{\mu}(s) + C_{n_1, 2} e^{-\tilde{\mu}(s)} \quad 2 \leq n_2 \leq N_2
\]

(4.34.7)

\[
P_{1, n_2, s} = C_{1, n_2} e^{-\lambda s} \tilde{\mu}(s) + C_{1, n_2} e^{-\tilde{\mu}(s)} \quad 2 \leq n_2 \leq N_2
\]

(4.34.8)

\[
C_{1, n_2} = -\tilde{\lambda}_1 C_{2, n_2}
\]

(4.34.9)

\[
C_{1, n_2} = \lambda_1 P_{0, 2, n_2} + \lambda_2 P_{0, 1, n_2+1} - C_{1, n_2}
\]

(4.34.10)
\[ P_{1,n_1,n_2,s} = \sum_{n_1,n_2} C_{n_1,n_2} e^{-\lambda s-m(s)} \]

\[ P_{1,n_1,n_2,0} = \sum_{n_1,n_2} C_{n_1,n_2} = \lambda_1 P_{0,n_1+1,n_2} + \lambda_2 P_{0,n_1,n_2+1} + \sum_{0}^{\infty} P_{2,n_1,n_2,s} \mu(s) \, ds \]

\[ P_{2,n_1,n_2,s} = (\tilde{\lambda}_1 C_{n_1+1,n_2} + \tilde{\lambda}_2 C_{n_1,n_2+1}) \cdot e^{-\bar{\mu}(s)} \cdot (1-e^{-\lambda s}) \]

\[ P_{2,n_1,n_2,0} = 0 \]

\[ P_{r,n_1,n_2,0} = P_{r,n_1,n_2,s} = 0 \quad \text{otherwise} \]

Integrals remain in equations (4.34.12) and (4.34.2). The integral in (4.34.12) is

\[ \int_{0}^{\infty} P_{2,n_1,n_2,s} \mu(s) \, ds = \sum_{n_1,n_2} C_{n_1+1,n_2} + \sum_{n_1,n_2} C_{n_1,n_2+1} \cdot (e^{-\bar{\mu}(s)} - e^{- \bar{\mu}(s)-\lambda s}) \mu(s) \, ds \]

\[ = (\tilde{\lambda}_1 C_{n_1+1,n_2} + \tilde{\lambda}_2 C_{n_1,n_2+1}) (1-F*(\lambda)) \]

by (4.33.1) and (4.33.3). The integral in (4.34.2) is of four forms, depending on the values of \( n_1 \) and \( n_2 \). For \( n_1 = n_2 = 1 \),
\[ \int_0^\infty P_{1,1,1,s} \mu(s) \, ds = \int_0^\infty c_{11} e^{-\tilde{\mu}(s)} \, ds = c_{11}. \]

For \( n_1 \geq 2 \) and \( n_2 = 1 \),

\[ \int_0^\infty P_{1,n_1,1,s} \mu(s) \, ds = \int_0^\infty \left( c_{n_1,1} \mu(s) e^{-\lambda s - \tilde{\mu}(s)} + c_{n_1,1} \mu(s) e^{-\tilde{\mu}(s)} \right) \, ds \]

\[ = c_{n_1,1} F^*(\lambda) + c_{n_1,1}. \]

Similarly, for \( n_1 = 1 \) and \( n_2 \geq 2 \),

\[ \int_0^\infty P_{1,1,n_2,s} \mu(s) \, ds = c_{1,n_2} F^*(\lambda) + c_{1,n_2}. \]

by (4.33.1) and (4.33.3). Finally, for \( n_1 \geq 2 \) and \( n_2 \geq 2 \),

\[ \int_0^\infty P_{1,n_1,n_2,s} \mu(s) \, ds = \int_0^\infty c_{n_1,n_2} e^{-\lambda s - \tilde{\mu}(s)} \mu(s) \, ds = c_{n_1,n_2} F^*(\lambda), \]

again by (4.33.3). This gives us the following which replace (4.33.2) and (4.33.11) - (4.33.12):

Replacing (4.33.2):

(4.35.1) \( \lambda P_{0,1,1} = c_{11} \)

(4.35.2) \( \lambda P_{0,n_1,1} = c_{n_1,1} F^*(\lambda) + c_{n_1,1} \) \quad \( n_1 \geq 2 \)

(4.35.3) \( \lambda P_{0,1,n_2} = c_{1,n_2} F^*(\lambda) + c_{1,n_2} \) \quad \( n_2 \geq 2 \)
\[(4.35.4) \quad \lambda^{0,n_1,n_2} = C_{n_1,n_2}^{n_1,n_2} F^0(\lambda) \quad 2 \leq n_1 \leq N_1; \quad 2 \leq n_2 \leq N_2, \quad n_1 + n_2 \leq N_1 + N_2 - 1\]

Replacing \((4.33.11) - (4.33.12)\):

\[(4.35.5) \quad P_{1,n_1,n_2,s} = C_{n_1,n_2}^{n_1,n_2} e^{-\lambda s - \tilde{\mu}(s)} \quad n_1 \geq 2, \quad n_2 \geq 2\]

\[(4.35.6) \quad P_{1,n_1,n_2,0} = C_{n_1,n_2}^{n_1,n_2} = \lambda_1^{n_1,n_1+1,n_2} + \lambda_2^{n_1,n_1+1,n_2} + n_1 + n_2 \leq N_1 + N_2 - 2\]

\[+ (\tilde{\lambda}_1 C_{n_1,n_2}^{n_1+1,n_2} + \tilde{\lambda}_2 C_{n_1,n_2+1}^{n_1+1,n_2+1})(1 - F^0(\lambda))\]

\[(4.35.7) \quad P_{1,n_1,n_2,0} = C_{n_1,n_2}^{n_1,n_2} = \lambda_1^{n_1,n_1+1,n_2} + \lambda_2^{n_1,n_1+1,n_2} + n_1 + n_2 \leq N_1 + N_2 - 1\]

**Stage III: Eliminate \( s \).**

In this stage we eliminate all equations involving densities with the state variable \( s \) and replace them with corresponding probabilities. The equations involved are \((4.34.3), (4.34.5), (4.34.8), (4.34.13), \) and \((4.35.5)\). In \((4.34.3)\):

\[P_{111} = \int_0^\infty P_{111s} ds = \int_0^\infty C_{11}^{11} e^{-\tilde{\mu}(s)} ds = C_{11}^{11} \cdot \frac{1}{\mu}\]

by \((4.33.2)\). For equations \((4.34.5)\) and \((4.34.8)\) we use \((4.33.2)\) and \((4.33.5)\) in Lemma 4.2 to get
\[ P_{1,n_1,1} = \int_0^\infty P_{1,n_1,1,s} \, ds = C'_{n_1,1} \frac{1}{\lambda} (1 - F^*(\lambda)) + C''_{n_1,1} \cdot \frac{1}{\mu}, \]

\[ P_{1,1,n_2} = \int_0^\infty P_{1,1,n_2,s} \, ds = C'_{1,n_2} \frac{1}{\lambda} (1 - F^*(\lambda)) + C''_{1,n_2} \cdot \frac{1}{\mu}. \]

To do (4.35.5), (4.33.5) gives

\[ P_{1,n_1,n_2} = \int_0^\infty P_{1,n_1,n_2,s} \, ds = C_{n_1,n_2} \frac{1}{\lambda} (1 - F^*(\lambda)). \]

Finally, in (4.34.13),

\[ P_{2,n_1,n_2} = \int_0^\infty P_{2,n_1,n_2,s} \, ds = (\tilde{\lambda}_1 C_{n_1+1,n_2} + \tilde{\lambda}_2 C_{n_1,n_2+1}) \left( \frac{1}{\mu} - \frac{1}{\lambda} (1 - F^*(\lambda)) \right). \]

(We note from Lemma 4.2, equation (4.33.6), that the term \( \frac{1}{\mu} - \frac{1}{\lambda} (1 - F^*(\lambda)) \) is always nonnegative). Equations are now reduced to the variables \( P_{r,n_1,n_2} \) and \( C_{n_1,n_2} \).

(4.36.1) \[ \lambda P_{0,N_1,N_2} = \tilde{\lambda}_2 \sum_{i=1}^{N_1} P_{0,i,1} + \tilde{\lambda}_1 \sum_{j=1}^{N_2} P_{0,1,j} \]

(4.36.2) \[ \lambda P_{0,1,1} = C'_{1,1} \]

(4.36.3) \[ \lambda P_{0,n_1,1} = C'_{n_1,1} F^*(\lambda) + C''_{n_1,1} \quad 2 \leq n_1 \leq N_1 \]

(4.36.4) \[ \lambda P_{0,1,n_2} = C'_{1,n_2} F^*(\lambda) + C''_{1,n_2} \quad 2 \leq n_2 \leq N_2 \]

(4.36.5) \[ \lambda P_{0,n_1,n_2} = C_{n_1,n_2} F^*(\lambda) \quad 2 \leq n_1 \leq N_1; \ 2 \leq n_2 \leq N_2; \ n_1 + n_2 \leq N_1 + N_2 - 1 \]
(4.36.1) \( p_{111} = c_{11}^{'} \frac{1}{\mu} \)

(4.36.7) \( c_{11}^{''} = \lambda_1 p_{021} + \lambda_2 p_{012} \)

(4.36.8) \( p_{1,n_1,1} = c_{n_1,1}^{'} \frac{1}{\lambda} (1-F^{*}(\lambda)) + c_{n_1,1}^{''} \frac{1}{\mu} \quad 2 \leq n_1 \leq N_1 \)

(4.36.9) \( c_{n_1,1}^{'} = -\hat{\lambda}_2 c_{n_1,2} \)

(4.36.10) \( c_{n_1,1}^{''} = \lambda_1 p_{0,n_1+1,1} + \lambda_2 p_{0,n_1,2} - c_{n_1,1}^{'} \)

(4.36.11) \( p_{1,n_2,1} = c_{1,n_2}^{'} \frac{1}{\lambda} (1-F^{*}(\lambda)) + c_{1,n_2}^{''} \frac{1}{\mu} \quad 2 \leq n_2 \leq N_2 \)

(4.36.12) \( c_{1,n_2}^{'} = -\hat{\lambda}_1 c_{2,n_2} \)

(4.36.13) \( c_{n_1,1}^{''} = \lambda_1 p_{0,2,n_2} + \lambda_2 p_{0,1,n_2+1} - c_{1,n_2}^{'} \)

(4.36.14) \( p_{n_1,n_2} = c_{n_1,n_2} \frac{1}{\lambda} (1-F^{*}(\lambda)) \quad n_1+n_2 \leq N_1+N_2-1 \quad 2 \leq n_1 \leq N_1 \)

(4.36.15) \( c_{n_1,n_2} = \lambda_1 p_{0,n_1+1,n_2} + \lambda_2 p_{0,n_1,n_2+1} \)

\[ + (\hat{\lambda}_1 c_{n_1+1,n_2} + \hat{\lambda}_2 c_{n_1,n_2+1})(1-F^{*}(\lambda)) \]

(4.36.16) \( c_{n_1,n_2} = \lambda_1 p_{0,n_1+1,n_2} + \lambda_2 p_{0,n_1,n_2+1} \quad n_1+n_2 = N_1+N_2-1 \quad 2 \leq n_1 \leq N_1 \)

(4.36.17) \( p_{n_1,n_2} = (\hat{\lambda}_1 c_{n_1+1,n_2} + \hat{\lambda}_2 c_{n_1,n_2+1}) \)

\[ . (\frac{1}{\mu} - \frac{1}{\lambda} (1-F^{*}(\lambda))) \quad 2 \leq n_1 \leq N_1 \quad 2 \leq n_2 \leq N_2 \quad n_1+n_2 \leq N_1+N_2-2 \]

(4.36.18) \( p_{r,n_1,n_2} = 0 \) otherwise
Stage IV: Solution for $P_{r,n_1,n_2}$

We can now solve the set of linear equations (4.36.1) - (4.36.18) for the limiting probabilities $P_{r,n_1,n_2}$. First, solving for the

$c_{n_1,n_2}$, by (4.36.5) and (4.36.15), we have for $2 \leq n_1 \leq N_1$, $2 \leq n_2 \leq N_2$;

\[
\sum_{i=1}^{2} n_1 \leq \sum_{i=1}^{2} N_i - 2;
\]

(4.37) \[c_{n_1,n_2} = \lambda_1 P_{0,n_1+1,n_2} + \lambda_2 P_{0,n_1,n_2+1} + (\tilde{\lambda}_1 c_{n_1+1,n_2} + \tilde{\lambda}_2 c_{n_1,n_2+1})(1 - F^*(\lambda)) \]

\[= (\tilde{\lambda}_1 c_{n_1+1,n_2} + \tilde{\lambda}_2 c_{n_1,n_2+1}) (F^*(\lambda) + (1-F^*(\lambda))) \]

\[= \tilde{\lambda}_1 c_{n_1+1,n_2} + \tilde{\lambda}_2 c_{n_1,n_2+1} \]

where $c_{n_1,n_2} = 0$ if $n_1 > N_1$ or $n_2 > N_2$. We still must solve for $c_{N_1-1,N_2}$, $c_{N_1,N_2-1}$, and $c_{N_1,N_2}$. By (4.36.16)

(4.38) \[c_{N_1-1,N_2} = \lambda_1 P_{0,N_1,N_2} \]

(4.39) \[c_{N_1,N_2-1} = \lambda_2 P_{0,N_1,N_2} \cdot \]

Since the equations (4.36.1) - (4.36.18) determine the $P_{r,n_1,n_2}$ up to a normalizing constant, we may set an arbitrary value for one of the $c_{n_1,n_2}$, solve for the others in terms of it, and later normalize so that $\sum_{S} P_{r,n_1,n_2} = 1$. Thus, we assume

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\begin{align*}
\tag{4.40} & \quad P_{0, N_1, N_2} = c_{N_1, N_2} = \frac{1}{\lambda} c \quad \text{some } c > 0.
\end{align*}

Then we may solve \((4.37) - (4.40)\) to get

\begin{align*}
\tag{4.41} & \quad c_{n_1, n_2} = \\
& \begin{cases}
(N_1 - n_1) + (N_2 - n_2) \frac{N_1 - n_1}{\lambda_1} \frac{N_2 - n_2}{\lambda_2} c & 2 \leq n_1 \leq N_1; \\
0 & \text{otherwise}
\end{cases}
\begin{cases}
2 \leq n_2 \leq N_2
\end{cases}
\end{align*}

And we may immediately write for \(2 \leq n_1 \leq N_1, 2 \leq n_2 \leq N_2, n_1 + n_2 \leq N_1 + N_2 - 2:\)

\begin{align*}
\tag{4.42} & \quad P_{2, n_1, n_2} = c_{n_1, n_2} \frac{1}{\mu} - \frac{1}{\lambda} (1-F^*(\lambda)) \\
& \quad P_{1, n_1, n_2} = c_{n_1, n_2} \frac{1}{\lambda} (1-F^*(\lambda)) \\
& \quad P_{0, n_1, n_2} = c_{n_1, n_2} \frac{1}{\lambda} F^*(\lambda)
\end{align*}

by \((4.36.14), (4.36.17), \text{ and } (4.36.5), \text{ and}

\begin{align*}
\tag{4.43} & \quad P_{1, N_1 - 1, N_2} = c_{N_1 - 1, N_2} \frac{1}{\lambda} (1-F^*(\lambda)) \\
& \quad P_{1, N_1, N_2 - 1} = c_{N_1, N_2 - 1} \frac{1}{\lambda} (1-F^*(\lambda)) \\
& \quad P_{0, N_1 - 1, N_2} = c_{N_1 - 1, N_2} \frac{1}{\lambda} F^*(\lambda) \\
& \quad P_{0, N_1, N_2 - 1} = c_{N_1, N_2 - 1} \frac{1}{\lambda} F^*(\lambda) \\
& \quad P_{0, N_1, N_2} = c_{N_1, N_2} \frac{1}{\lambda}
\end{align*}

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where the $C_{n_1, n_2}$ are as given in (4.41) and all probabilities are defined as multiples of $C$.

We now derive recursive equations for $P_{0, n_1, l}$ and $P_{1, n_1, l}$ (similar results hold for $P_{0, l, n_2}$ and $P_{1, l, n_2}$):

\[(4.44)\]

\[P_{0, n_1, 1} = \frac{1}{\lambda} \left( C_{n_1, 1} \frac{F^*(\lambda)}{1} + C_{n_1, 1}'' \right) \]

\[= \frac{1}{\lambda} \left( C_{n_1, 1} \frac{F^*(\lambda)}{1} + \lambda_1 P_{0, n_1+1, 1} + \lambda_2 P_{0, n_1+2, 1} - C_{n_1, 1}' \right) \]

\[= \frac{1}{\lambda} \left( -C_{n_1, 1} \frac{1-F^*(\lambda)}{1} + \lambda_2 P_{0, n_1+2, 1} + \lambda_1 P_{0, n_1+1, 1} \right) \]

\[= \frac{1}{\lambda} \left( \bar{\lambda}_2 C_{n_1, 2} \frac{1-F^*(\lambda)}{1} + \bar{\lambda}_1 C_{n_1, 2} \frac{F^*(\lambda)}{1} + \lambda_1 P_{0, n_1+1, 1} \right) \]

\[= \frac{1}{\lambda} \bar{\lambda}_2 C_{n_1, 2} + \bar{\lambda}_1 P_{0, n_1+1, 1} \]

\[(4.45)\]

\[P_{1, n_1, 1} = C_{n_1, 1} \frac{1}{\lambda} \left( 1-F^*(\lambda) \right) + C_{n_1, 1}'' \left( \frac{1}{\mu} \right) \]

\[= C_{n_1, 1} \frac{1}{\lambda} \left( 1-F^*(\lambda) \right) + \frac{1}{\mu} \lambda_1 P_{0, n_1+1, 1} \]

\[+ \frac{1}{\mu} \lambda_2 P_{0, n_1+2, 1} - \frac{1}{\mu} C_{n_1, 1}' \]

\[= \frac{1}{\lambda} \frac{1}{\mu} \left( 1 - \frac{1}{\lambda} (1-F^*(\lambda)) + \frac{1}{\mu} \lambda_2 P_{0, n_1+2, 1} + \frac{1}{\mu} \lambda_1 P_{0, n_1+1, 1} \right) \]

\[= \bar{\lambda}_2 C_{n_1, 2} \frac{1}{\mu} \left( 1-F^*(\lambda) \right) + \frac{1}{\lambda} \lambda_2 C_{n_1, 2} \frac{F^*(\lambda)}{1} \]

\[+ \frac{1}{\mu} \lambda_1 P_{0, n_1+1, 1} \]

\[= \bar{\lambda}_2 C_{n_1, 2} \frac{1}{\mu} (1+F^*(\lambda)) - \frac{1}{\lambda} (1-F^*(\lambda)) + \bar{\lambda}_1 \frac{1}{\mu} P_{0, n_1+1, 1} \]

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Solving the recursive equation (4.44), for $P_{0,n_1,1}$ gives

$$P_{0,n,1} = \frac{1}{\lambda_2} \sum_{i=0}^{N_1-n_1} C_{n_1+i,2} \frac{\tilde{\lambda}_2^i}{\tilde{\lambda}_1^i}$$

and since

$$C_{n_1+i,2} = \binom{(N_1-n_1)-1 + (N_2-2)}{N_1-n_1-1} \frac{N_1-n_1}{N_1-n_1-1} \frac{N_2-2}{\lambda_1 \lambda_2} c,$$

we have

$$P_{0,n_1,1} = \frac{1}{\lambda} \frac{N_2-1}{\lambda_2} \frac{N_1-n_1}{\lambda_1} \sum_{i=0}^{N_1-n_1} \binom{(N_1-n_1) + (N_2-2)-i}{N_2-2} c$$

$$= \frac{1}{\lambda} \frac{N_2-1}{\lambda_2} \frac{N_1-n_1}{\lambda_1} \sum_{i=0}^{N_2-2} \binom{N_2-2+i}{N_2-2} c$$

$$= \frac{1}{\lambda} \frac{N_2-1}{\lambda_2} \frac{N_1-n_1}{\lambda_1} \binom{(N_1-n_1) + (N_2-1)}{N_2-1} c$$

since $\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{n+1}$. Defining

$$C_{n_1,1} = \frac{N_1-n_1}{\lambda_1} \frac{N_2-1}{\lambda_2} \binom{(N_1-n_1) + (N_2-1)}{N_2-1} c,$$

to be consistent with the $C_{n_1,n_2}$ given by (4.41), we have

$$(4.46) \quad P_{0,n_1,1} = \frac{1}{\lambda} C_{n_1,1} \quad 2 \leq n_1 \leq N_1$$

and similarly
\[ P_{0,1,n_2} = \frac{1}{\lambda} \ C_{1,n_2} \quad 2 \leq n_2 \leq N_2. \]

From (4.36.7), \( C_{11} = \lambda_1 P_{02} + \lambda_2 P_{012} = \tilde{\lambda}_1 C_{21} + \tilde{\lambda}_2 C_{12} = C_{11} \)
where

\[ C_{11} = \begin{pmatrix} N_1+N_2-2 \\ N_1-1 \end{pmatrix} \begin{pmatrix} N_1-1 \\ \tilde{\lambda}_1 \end{pmatrix} \begin{pmatrix} N_2-2 \\ \tilde{\lambda}_2 \end{pmatrix} C \]

is again defined consistently with (4.41). Thus

\[ P_{011} = \frac{1}{\lambda} \ C_{11} \]

\[ P_{111} = \frac{1}{\mu} \ C_{11}. \]

It remains only to calculate the values of \( P_{1,n_1,1} \) and \( P_{1,1,n_2} \).

From the recursion relation (4.45):

\[ P_{1,n_1,1} = \tilde{\lambda}_2 C_{1,n_1,2} \frac{1}{\mu} (1-F^*(\lambda)) - \frac{1}{\lambda} (1-F^*(\lambda)) + \tilde{\lambda}_1 \frac{\lambda}{\mu} P_{0,n_1+1,1} \]

\[ = \tilde{\lambda}_2 C_{1,n_1,2} \frac{1}{\mu} (1-F^*(\lambda)) - \frac{1}{\lambda} (1-F^*(\lambda)) + \tilde{\lambda}_1 C_{n_1+1,1} \left( \frac{1}{\mu} \right). \]

Upon expansion of the values of \( C_{n_1,2}, C_{n_1+1,1} \), and \( C_{n_1,1} \), we get

\[ P_{1,n_1,1} = C_{n_1,1} \left[ \left( \frac{N_2-1}{N_1-n_1+N_2-1} \right) \left( \frac{1}{\mu} (1-F^*(\lambda)) - \frac{1}{\lambda} (1-F^*(\lambda)) \right) \right. \]

\[ + \left. \left( \frac{N_1-n_1}{N_1-n_1+N_2-1} \right) \frac{1}{\mu} \right] \]

\[ = C_{n_1,1} \left[ \frac{1}{\mu} + \frac{N_2-1}{N_1-n_1+N_2-1} \left( F^*(\lambda) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda} \right) \right]. \]

Similarly,
\[ P_{1,1,n_2} = C_{1,n_2} \left[ \frac{1}{\mu} + \frac{N_1 - 1}{N_1 - 1 + N_2 - n_2} \right] \left( F^*(\lambda) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda} \right). \]

Equations (4.42) (4.43) and (4.46) - (4.50) together give us the limiting probabilities for the case of a general repair distribution, which proves

**Theorem 4.3.** If component \( i \) fails at a constant failure rate \( \lambda_i \), repair is completed in \( t \) units of time with probability \( F(t) \), \( F(0) = 0 \);

\[ F^*(s) = \int_0^\infty e^{-st} dF(t), \]

then the limiting probabilities in the steady state are given by:

\[ P_{2,n_1,n_2} = C_{n_1,n_2} \left( \frac{1}{\mu} - \frac{1}{\lambda} \right) \left( 1 - F^*(\lambda) \right) \]
\[ \text{if } 2 \leq n_1 \leq N_1, \quad 2 \leq n_2 \leq N_2, \quad n_1 + n_2 \leq N_1 + N_2 - 2 \]

\[ P_{1,n_1,n_2} = C_{n_1,n_2} \frac{1}{\lambda} \left( 1 - F^*(\lambda) \right) \]
\[ \text{if } 2 \leq n_1 \leq N_1, \quad 2 \leq n_2 \leq N_2, \quad n_1 + n_2 \leq N_1 + N_2 - 2 \]

\[ P_{1,1,n_2} = C_{1,n_2} \left[ \frac{1}{\mu} + \frac{N_1 - 1}{N_1 - 1 + N_2 - n_2} \right] \left( F^*(\lambda) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda} \right) \]
\[ \text{if } 2 \leq n_2 \leq N_2 \]

\[ P_{1,n_1,1} = C_{n_1,1} \left[ \frac{1}{\mu} + \frac{N_2 - 1}{N_1 - n_1 + N_2 - 1} \right] \left( F^*(\lambda) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda} \right) \]
\[ \text{if } 2 \leq n_1 \leq N_1 \]

\[ P_{1,1,1} = C_{1,1} \frac{1}{\mu} \]

\[ P_{0,n_1,n_2} = C_{n_1,n_2} \frac{1}{\lambda} F^*(\lambda) \]
\[ \text{if } 2 \leq n_1 \leq N_1, \quad 2 \leq n_2 \leq N_2, \quad n_1 + n_2 \leq N_1 + N_2 - 2 \]

\[ P_{0,1,n_2} = C_{1,n_2} \frac{1}{\lambda} \]
\[ \text{if } 1 \leq n_2 \leq N_2 \]

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\[(4.51.8) \quad P_{0,n_1,1} = C_{n_1,1} \frac{1}{\lambda} \quad 1 \leq n_1 \leq N_1 \]
\[(4.51.9) \quad P_{0,1,1} = C_{1,1} \frac{1}{\lambda} \]
\[(4.51.10) \quad P_{0,N_1,N_2} = C_{N_1,N_2} \frac{1}{\lambda} \]
\[(4.51.11) \quad P_{r,n_1,n_2} = 0 \quad \text{otherwise} \]

where

\[
C_{n_1,n_2} = \left( \frac{(N_1-n_1) + (N_2-n_2)}{(N_1-n_1)} \right) \frac{N_1-n_1}{\lambda_1} \frac{N_2-n_2}{\lambda_2} C ,
\]

and \( C \) is defined so that

\[
\sum_{r=0}^{2} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \sum_{n_1 \geq r}^{n_2 \geq r} \sum_{n_1+n_2 \leq N_1+N_2-r} P_{r,n_1,n_2} = 1 .
\]

We show these probabilities graphically in Figure 4.1. The shaded areas again indicate the states in which the system is out of operation. It may be instructive at this point to relate the general case to the case of exponential repair time.

If \( F(t) = 1 - e^{-\mu t} \), then \( F^*(s) = \frac{1}{\mu + s} \) so \( F^*(\lambda) = \frac{1}{\mu + \lambda} \). The expression \( \frac{1}{\mu} - \frac{1}{\lambda} (1-F^*(\lambda)) \) in (4.51.1) becomes \( \frac{\lambda}{\mu} \frac{1}{\mu + \lambda} \); in (4.51.2), we have \( \frac{1}{\lambda} (1-F^*(\lambda)) = \frac{1}{\mu + \lambda} \), and in (4.51.3), the last term in parentheses,
Figure 4.1

\[ ((F^*(\lambda)) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda} = 0 \]
and thus \( P_{1,1,n_2} = \frac{1}{\mu} C_{1,n_2} \). We exhibit these probabilities in Figure 4.2 and can see that, with a permitted change in scale of \( \lambda \), they are equal to those given for the exponential case in Theorem 3.5 and Figure 3.2.

The procedure here is similar to Section 3.3. As in the exponential case, we have

\[ F(N_1, N_2) = \frac{\sum \text{operating states } \in S p_{r, n_1, n_2}}{\sum_S p_{r, n_1, n_2}} = \frac{E(X_1)}{E(Z_1)}. \]
Again define

\[ p'_{r, n_1, n_2} = \frac{1}{C} p_{r, n_1, n_2} \]

and the five subsets \( A, B, C, D, E \) of \( S \):

\[ A = \{ (r, n_1, n_2) \in S \mid r = 0, \min n_i = 1 \} \]

\[ B = \{ (r, n_1, n_2) \in S \mid r = 0, \min n_i \geq 2 \} \]

\[ C = \{ (r, n_1, n_2) \in S \mid r = 1, \min n_i = 1 \} \]

\[ D = \{ (r, n_1, n_2) \in S \mid r = 1, \min n_i \geq 2 \} \]

\[ E = \{ (r, n_1, n_2) \in S \mid r = 2, \min n_i \geq 2 \} \]

The sets were illustrated in Figure 3.3. By definition of the \( p'_{r, n_1, n_2} \)

\[ C^{-1} = p'(A) + p'(B) + p'(C) + p'(D) + p'(E) \]

and since \( A, B, \) and \( D \) contain only operating states and \( C, E \)
contain only non-operating states,

\[ F(n_1, n_2) = \frac{p'(A) + p'(B) + p'(D)}{p'(A) + p'(B) + p'(C) + p'(D) + p'(E)} \]
Let \( \tilde{C}_{n_1, n_2} = \frac{1}{C} C_{n_1, n_2} \). Then the relations (4.51.1) - (4.51.11) in Theorem 4.5 hold also with \( P_{r, n_1, n_2} \) replaced by \( P'_{r, n_1, n_2} \), and \( C_{n_1, n_2} \) replaced by \( \tilde{C}_{n_1, n_2} \), and

\[
(4.53) \quad \tilde{C}_{n_1, n_2} = \left( \frac{N_1-n_1 + N_2-n_2}{\lambda_1 \lambda_2} \right)^{\tilde{C}} \frac{N_1-n_1}{\lambda_1} \frac{N_2-n_2}{\lambda_2} \quad 1 \leq n_1 \leq N_1, 1 \leq n_2 \leq N_2
\]

Consider first the sets \( A, B, \) and \( D \). State probabilities given in Theorem 4.3 for the general case are of the same form as those given in Theorem 3.9 for the exponential case. We have by Theorem 4.3,

\[
(4.54) \quad P'(A) = \sum_{j=1}^{N_2} P'_{01j} + \sum_{i=1}^{N_1} P'_{0i1} - P'_{011}
\]

\[
= \frac{1}{\lambda} \left( \sum_{i=1}^{N_1} \tilde{C}_{i1} + \sum_{j=1}^{N_2} \tilde{C}_{1j} - \tilde{C}_{11} \right)
\]

\[
= \frac{1}{\lambda} g(N_1-1, N_2-1)
\]

where \( g(\cdot) \) was defined in (3.55), Section 3.3. Similarly

\[
(4.55) \quad P'(B) = \frac{1}{\lambda} F*(\lambda) \left[ h(N_1-2, N_2-2) - 1 \right] + \frac{1}{\lambda}
\]

\[
= \frac{1}{\lambda} F*(\lambda) h(N_1-2, N_2-2) + \frac{1}{\lambda} (1-F*(\lambda))
\]

where \( h(\cdot) \) was defined in (3.52), Section 3.3. Also

\[
(4.56) \quad P'(D) = \frac{1}{\lambda} (1-F*(\lambda)) \left[ h(N_1-2, N_2-2) - 1 \right]
\]

\[
= \frac{1}{\lambda} (1-F*(\lambda)) h(N_1-2, N_2-2) - \frac{1}{\lambda} (1-F*(\lambda))
\]

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and thus

**Lemma 4.4.** For the case of general repair time distribution with $I = 2$,

$$P'(A) + P'(B) + P'(D) = h(N_{i-1}, N_{j-1}) [\frac{1}{\lambda}] .$$

**Proof:** By (4.54) - (4.56), we have for $P'(A) + P'(B) + P'(D)$:

$$\frac{1}{\lambda} g(N_{1-1}, N_{2-1}) + \frac{1}{\lambda} [F*(\lambda) + 1-F*(\lambda)] h(N_{1-2}, N_{2-2})$$

$$+ \frac{1}{\lambda} (1-F*(\lambda)) - \frac{1}{\lambda} \frac{1}{\lambda} (1-F*(\lambda))$$

$$= \frac{1}{\lambda} [g(N_{1-1}, N_{2-1}) + h(N_{2-2}, N_{2-2})]$$

$$= \frac{1}{\lambda} h(N_{1-1}, N_{2-1}) .$$

For the set $E$ we have

$$P'(E) = (\frac{1}{\mu} - \frac{1}{\lambda} (1-F*(\lambda))) (h(N_{1-2}, N_{2-2}) - 1 - \tilde{\lambda}_1 - \tilde{\lambda}_2)$$

$$= (\frac{1}{\mu} - \frac{1}{\lambda} (1-F*(\lambda))) h(N_{1-2}, N_{2-2}) - 2(\frac{1}{\mu} - \frac{1}{\lambda} (1-F*(\lambda))) .$$

The calculations for $A, B, C,$ and $E$ were similar to those for the exponential case. However, the calculation for $C$ is of a different form and we shall go into it in more detail. We have by definition of $C$: 
\[ P'(C) = \sum_{n_1=2}^{N_1} P'_{1,n_1,1} + \sum_{n_2=2}^{N_2} P'_{1,1,n_2}. \]

By (4.51.5),

\[ P'_{111} = \tilde{c}_{11} \left( \frac{1}{\mu} \right) = \left( \frac{1}{\mu} \right) \left( \begin{array}{c} N_1-1 + N_2-1 \\ N_1 \end{array} \right) \frac{N_1-1}{\lambda_1} \frac{N_2-1}{\lambda_2}. \]

For \( 2 \leq n_1 \leq N_1 \),

\[ P'_{1,n_1,1} = \left[ \frac{1}{\mu} - \frac{N_2-1}{N_1-n_1+N_2-1} \right] (F*(\lambda) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda}) \left( \begin{array}{c} N_1-n_1+N_2-1 \\ N_1-n_1 \end{array} \right) \frac{N_1-n_1}{\lambda_1} \frac{N_2-1}{\lambda_2}. \]

so

\[
\sum_{n_1=2}^{N_1} P'_{1,n_1,1} = \frac{1}{\mu} \sum_{i=0}^{N_2-2} \left( \begin{array}{c} N_2-1+i \\ i \end{array} \right) \frac{i}{\lambda_1} \frac{N_2-1}{\lambda_2} \\
+ (F*(\lambda) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda}) \sum_{i=0}^{N_2-2} \left( \begin{array}{c} N_2-2+i \\ i \end{array} \right) \frac{i}{\lambda_1} \frac{N_2-1}{\lambda_2} \\
= t_1 + t_2
\]

where the first term in (4.58) is \( t_1 \) and the second \( t_2 \). Similarly,

\[
\sum_{n_2=2}^{N_2} P'_{1,1,n_2} = \frac{1}{\mu} \sum_{j=0}^{N_1-1} \left( \begin{array}{c} N_1-1+j \\ j \end{array} \right) \frac{N_1-1}{\lambda_1} \frac{i}{\lambda_2} \\
+ (F*(\lambda) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda}) \sum_{j=0}^{N_1-2} \left( \begin{array}{c} N_1-2+j \\ j \end{array} \right) \frac{N_1-1}{\lambda_1} \frac{j}{\lambda_2} \\
= t_3 + t_4
\]

Then we have
\[ P'(C) = P'_{111} + t_1 + t_2 + t_3 + t_4. \]

Lemma 4.5. For \( t_2, t_4 \) defined above, \( t_2 + t_4 = (F^*(\lambda)) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda}. \)

Proof: By definition, we have

\[
t_2 + t_4 = (F^*(\lambda)) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda} \left[ \sum_{j=0}^{N_1-1} \frac{N_2-2}{\lambda_1} \left( \frac{N_1-2+j}{j} \right) \frac{N_2-1}{\lambda_2} \right]
\]

and the bracketed term is equal to \( 1 \) by Lemma 3.12.

Lemma 4.6.

\[ P'(C) = \left( \frac{1}{\mu} \right) g(N_1-1, N_2-1) + (F^*(\lambda)) \left( \frac{1}{\mu} + \frac{1}{\lambda} \right) - \frac{1}{\lambda}. \]

Proof: By (4.59), \( P'(C) = (P'_{111} + t_1 + t_2) + (t_2 + t_4). \) The sum \( (P'_{111} + t_1 + t_2) \) is equal to \( \left( \frac{1}{\mu} \right) g(N_1-1, N_2-1) \) by definition of \( g(\cdot) \), and the result of Lemma 4.5 completes the proof.

The combination of (4.57), Lemma 4.6, and Lemma 4.8 gives us the values of \( F(N_1, N_2) \) and \( C \):

Theorem 4.7. For the process with repair distribution \( F(t) \) with expectation \( \frac{1}{\mu} \), \( I = 2 \), and \( N_i \geq 2 \) parts initially provided of type \( i \), \( i = 1, 2 \); the limiting probability that a system will be operating is
(4.60) \[ F(N_1, N_2) \]
\[ = \frac{h(N_1-1, N_2-1)}{[h(N_1-1, N_2-1)(\rho-(1-F^*(\lambda))+g(N_1-1, N_2-1)(1-F^*(\lambda))-(\rho(2-F^*(\lambda))-(1-F^*(\lambda)))]} \]

(4.61) \[ C = \lambda h(N_1-1, N_2-1) (\rho-(1-F^*(\lambda))+g(N_1-1, N_2-1)(1-F^*(\lambda))) \]
\[ - (\rho(2-F^*(\lambda))-(1-F^*(\lambda)))^{-1} \cdot \]

**Proof:** By (4.57), Lemma 4.4, and Lemma 4.6, we have

(4.62) \[ F(N_1, N_2) \]
\[ = \frac{\frac{1}{\lambda} h(N_1-1, N_2-1)}{\frac{1}{\lambda} h(N_1-1, N_2-1)+h(N_1-2, N_2-2)(\frac{1}{\mu} - \frac{1}{\lambda} (1-F^*(\lambda)))+ \frac{1}{\mu} g(N_1-1, N_2-1)-\frac{1}{\mu}(2-F^*(\lambda)-\frac{1}{\lambda}(1-F^*(\lambda)))} \]

and \( C^{-1} \) equals the denominator of \( F(N_1, N_2) \). Using the fact that \( g(N_1, N_2) = h(N_1, N_2) - h(N_1-1, N_2-1) \), recalling \( \rho = \frac{\lambda}{\mu} \) and rearranging in (4.62) gives (4.60) and (4.61).

We note that with the value of \( C \) given by (4.61), if \( F(t) = 1 - e^{-\mu t} \), then this case reduces to the exponential case given in Chapter 3.
CHAPTER 5

VARIANTS OF THE MODEL

In this chapter some variations of the model are considered. Section 5.1 analyzes by means of Markov renewal theory the case where repair is permitted, but no spare system is available for replacement. In Section 5.2 the original model of Section 1.2 is extended to include the resupply of spare parts, and a set of equations in the limiting state probabilities is derived. It is shown that a special case of the model is a system of two finite queues in series.

5.1. No Spare System is Provided.

Consider the model of Section 1.2 with the modification that no spare system is provided for use during repair. The procedure of the process is thus as follows. Part failure causes the system to discontinue operation while repair is completed. This interval is followed by an interval of operation, and so forth until the supply of spares is exhausted. We allow the repair distribution to depend upon the failed part type; repair of a failure of part type \( i \) is completed according to the d.f. \( F_i(t) \). Part failure of type \( i \) is again governed by the exponential d.f. \( 1 - e^{-\lambda_i t} \).

This model may be formulated as a semi-Markov process. Let \( T_n \) denote the time when repair of the \( n \)th failure is completed. Let \( X_{in} \) denote total number of parts of type \( i \) remaining at epoch \( T_n \).
Assume $X_{i0} = N_i$ for $i = 1, \ldots, I$; and let $X_n = (X_{1n}, \ldots, X_{In})$.

The process terminates whenever some $X_{in} = 0$.

The state space $S = \{(n_1, \ldots, n_I) | 0 \leq n_i \leq N_i, i = 1, \ldots, I\}$.

We may relabel the states in lexicographical order if desired and refer to a state generically as $j$. It is to be understood that this refers to an $I$ vector $(n_1, \ldots, n_I)$. Let $X(t)$ denote the state of the system at time $t$.


We again follow Cinlar [8] in our presentation.

Let $j, k \in S$ and $t \geq 0$. Then by definition of a Markov renewal process the quantity

\[
P(X_{n+1} = k, T_{n+1} \leq t+u | X_n = j, T_n = u)
\]

does not depend on $n$ or $u$. Call the expression (5.1) $A_{jk}(t)$ and define $B_j(t) = \sum_{k \in S} A_{jk}(t)$, $\widetilde{A}_{jk} = A_{jk}(\infty)$, $\widetilde{B}_j = B_j(\infty)$. Let $L$ denote the total number of failures including the first assumed at $T_0 = 0$ and let $Z = T_{L-1}$. Then $Z$ is the lifetime of the system. Let

\[
A_{jk}^{(0)}(t) = \delta_{jk}, \quad A_{jk}^{(1)}(t) = A_{jk}(t), \quad \text{and} \quad A_{jk}^{(n)}(t) = \sum_{v} \int_{0}^{t} A_{jk}^{(n-1)}(t-u) \text{d}A_{vj}(u).
\]

Then $A_{jk}^{(n)}(t) = P(X_n = k, T_n \leq t | X_0 = j)$. The matrix of functions $A = [A_{jk}]$ is called the semi-Markov matrix.

Define

\[
R_{jk}(t) = \sum_{n=0}^{\infty} A_{jk}^{(n)}(t) \quad \tilde{R}_{jk} = R_{jk}(\infty).
\]
The matrix \( R(t) = [R_{jk}(t)] \) is called the Markov renewal matrix corresponding to \( A(t) \). For \( j, k \in S \), \( R_{jk}(t) \) represents the expected number of visits to state \( k \) in \([0,t]\), given that \( X(0) = j \).

For notational convenience, for any event \( E \), let \( P_j[E] = P[E|X(0) = j] \). Also let the state \( j = 0 \) represent \((N_1, \ldots, N_I)\).

We now calculate certain of the quantities defined above to obtain information about the reliability problem under consideration.

5.1.2. Distribution of the System Lifetime \( Z \).

The transition probabilities \( P_{jk}(t) = P[X(t) = k|X(0) = j] \) satisfy the Markov renewal equation

\[
P_{jk}(t) = g_{jk}[B_j(\infty) - B_j(t)] + \sum_{v=0}^{t} \int_0^{t} P_{vk}(t-u) \, dA_{jv}(u).
\]

Since the state space \( S \) is finite, a solution to (5.3) exists and is unique.

By (7.10) in Cinlar ([8], p. 159)

\[
P_0[Z \leq t, L = \infty] = \sum_k R_{0k}(t) (1 - \bar{E}_k)
\]

A Markov renewal process is said to be normal if \( R_{jk}(0) < \infty \) for all \( j, k \). It is trivial that the process under consideration is normal, since there are no instantaneous states (states with \( B_j(0) > 0 \)). This implies

\[
P_0[Z \leq t, L = \infty] = 0
\]
and it suffices to obtain (5.4) as the distribution of \( Z \).

Consider a state \( j = (n_1, \ldots, n_I) \) such that \( \min(n_i) \geq 2 \).

We have

\[
A_{jk}(t) = \int_0^t F_i(t-u) \lambda_i e^{-\lambda u} \, du
\]

for \( k = (n_1, \ldots, n_{i-1}, n_i-1, n_i+1, \ldots, n_I) \); \( i = 1, \ldots, I \), and

\( A_{jk}(t) = 0 \) otherwise. Since the function may take on only \( I \) possible values, calculations are greatly simplified. Let

\[
H_i(t) = \int_0^t F_i(t-u) \lambda_i e^{-\lambda u} \, du, \quad i = 1, \ldots, I.
\]

Then for \( j \) such that \( \min(n_i) = 2 \), and for any \( k \), either \( A_{jk}(t) = H_i(t) \) for some \( i \) or \( A_{jk}(t) = 0 \).

**Lemma 5.1.** For any \( j,k \) in \( S \), there exists at most one \( n \geq 0 \) such that

\[
A_{jk}^{(n)}(\infty) > 0.
\]

**Proof:** The value \( \tilde{A}_{jk}^{(n)} - A_{jk}^{(n)}(\infty) \) is the probability of going from \( j \) to \( k \) in the associated Markov chain in \( n \) steps. For any state \( j = (n_1, \ldots, n_I) \), let \( d(j) = n_1 + \ldots + n_I \) denote the total number of parts. For an \( n \)-step transition from \( j \) to \( k \), therefore, we must have \( d(j) - d(k) = n \). Thus if \( d(k) > d(j) \), \( \tilde{A}_{jk}^{(n)} = 0 \) for all \( n \). If \( d(k) = d(j) \) and \( k \neq j \), \( \tilde{A}_{jk}^{(n)} = 0 \) for all \( n \). If \( d(k) < d(j) \), then

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\( \tilde{A}_{jk}(d(j)-d(k)) > 0 \)

and \( \tilde{A}^{(n)}_{jk} = 0 \) for \( n \neq d(j) - d(k) \). Finally, if \( j = k \), then \( \tilde{A}^{(0)}_{jk}(t) = 1 \) for \( t \geq 0 \) and \( \tilde{A}^{(n)}_{jk} = 0 \) for every \( n > 0 \).

Thus for \( k \in S \), and \( 0 = (N_1, \ldots, N_I) \), we have

\[
R_{0k}(t) = A^{(N-d(k))}_{0k}(t)
\]

where \( N = N_1 + \cdots + N_I \).

Let \( a^{(n)}_{jk}(s) = \int_0^\infty e^{-st} dA^{(n)}_{jk}(t) \) denote the Laplace transform of \( A^{(n)}_{jk}(\cdot) \). Let \( a^{(n)}(s) = [a^{(n)}_{jk}(s)] \), \( a(s) = a^{(1)}(s) \) and \( a(0) = [a^{(0)}_{jk}(0)] \).

Then the matrix \( a^{(n)}(s) = [a(s)]^n \) for every \( s \), or in shorthand notation, \( a^{(n)} = a^n \). Let \( h_{ik}(s) \) be the Laplace transform of \( H_{ik}(\cdot) \),

\[
h_{ik}(s) = \int_0^\infty e^{-st} dH_{ik}(t).
\]

Also define

\[
r_{jk}(s) = \int_0^\infty e^{-st} dR_{jk}(t).
\]

**Theorem 5.2.** Let \( j, k \in S \) be such that

(i) \( d(j) - d(k) = \tilde{n} > 0 \)

(ii) \( j = (n_1, \ldots, n_I) \) and \( \min n_i \geq 2 \)

(iii) \( k = (n_1', \ldots, n_I') \) and \( \min n_i' \geq 1 \)

(iv) \( m_i = n_i - n_i' \geq 0 \) for each \( i \).
Then \( \tilde{n} = \sum m_i \) and

\[
(5.7) \quad a_{nk}^{(n)}(s) = \binom{n}{m_1 \ldots m_1} h_1(s)^{m_1} \ldots h_1(s)^{m_1} \quad n = \tilde{n}
\]

\[
= 0 \quad \text{otherwise}
\]

\textbf{Proof:} For \( n \neq d(j) - d(k) \), \( A_{jk}^{(n)} = a_{jk}^n = 0 \). For \( n = d(j) - d(k) = \tilde{n} \) transition from \( j \) to \( k \) in \( n \) steps requires \( m_i \) failures of type \( i \) for each \( i \) and there are

\[
\binom{n}{m_1 \ldots m_1} = \frac{(\Sigma m_i)!}{(m_1)! \cdots (m_i)!}
\]

such paths from \( j \) to \( k \). For each such path, the distribution of total transition time is the convolution \( H_{1}^{(m_1)} * H_{2}^{(m_2)} * \cdots * H_{I}^{(m_I)}(t) \), which implies (5.7).

\textbf{Corollary 5.3.} If \( j = (n_1, \ldots, n_I) \) and \( \min n_i \geq 2 \), then \( \tilde{B}_j = 1 \).

\textbf{Proof:} For \( j = (n_1, \ldots, n_I) \),

\[
\tilde{B}_j = B_j(\omega) = \sum_{k \in S} A_{jk}(\omega) = \sum_{k \in S} a_{jk}(0)
\]

There are exactly \( I \) states \( k \in S \) such that \( a_{jk}(0) > 0 \); for \( k = (n_1, \ldots, n_{i-1}, \ldots, n_I) \), \( a_{jk}(0) = \lambda_i(0) = (1 \ldots 0 1 0 \cdots 0)^{t} h_i(0) = H_i(\omega) = \tilde{\lambda}_i \), where \( \tilde{\lambda}_i = \lambda_i/\lambda \). Since \( \sum \tilde{\lambda}_i = 1 \), Corollary 5.3 holds.
Let us now examine states $j = (n_1, \ldots, n_L)$ such that $n_i = 1$ for some $i$. If $n_i = 1$ and a part of type $i$ fails, the process stops. Thus for $j \in S$ define $D(j) = \{i | n_i = 1\}$ to be the set of parts whose failure would cause termination of the system. We have for the defect,

\[
1 - \bar{B}_j = \sum_{i \in D(j)} \bar{\lambda}_i .
\]

Note: Since $L$ is the number of transitions before termination and $Z = T_{L-1}$, $Z$ actually represents the completion of repair on the last component to be repaired and is less than the system lifetime by exactly one failure free interval. This poses no problems for the analysis, however.

From (5.6) - (5.8), using (5.4), we calculate the distribution of $Z$. Let

\[
Z(t) = P_0\{Z \leq t\} ,
\]

\[
z(s) = \int_0^\infty e^{-st} dz(t) .
\]

By (5.4) and (5.5),

\[
Z(t) = \sum_k R_{0k}(t)(1 - \bar{B}_k) ,
\]

thus

\[
z(s) = \sum_k r_{0k}(s)(1 - \bar{B}_k) .
\]
From (5.6) and (5.8)

\[ z(s) = \sum_{\{k \mid D(k) \neq \emptyset \}} a^{(N-d(k))}(s) \left( \sum_{i \in D(k)} \frac{\lambda_i}{\lambda_1} \right) \]

from which follows, by (5.7)

\[ z(s) = \sum_{\{k \mid D(k) \neq \emptyset \}} \binom{N-d(k)}{(N_1-n_1) \ldots (N_I-n_I)} h_1(s)^{N_1-n_1} \cdots h_I(s)^{N_I-n_I} \left( \sum_{i \in D(k)} \frac{\lambda_i}{\lambda_1} \right). \]  

(5.9)

Let us write (5.9) explicitly in the case \( I = 2 \). The set

\[ \{k \mid D(k) \neq \emptyset \} = \{k \mid n_1 = 1 \text{ or } n_2 = 1 \} \]

\[ = \{(1,1), \ldots, (N_1,1), (1,2), \ldots, (1,N_2)\} \]

so

\[ z(s) = \lambda_1 \sum_{j=1}^{N_2} \binom{N_1-1+N_2-j}{N_2-j} h_2(s)^{N_2-j} h_1(s)^{N_1-1} \]

\[ + \lambda_2 \sum_{i=1}^{N_1} \binom{N_1-i+N_2-1}{N_1-i} h_1(s)^{N_1-i} h_2(s)^{N_2-1}. \]

Since \( h_1(0) = H_1(\omega) = \tilde{\lambda}_1 \), we have by Lemma 3.8

\[ Z(\omega) = z(0) = \tilde{\lambda}_1 \sum_{j=1}^{N_2} \binom{N_1-1+N_2-j}{N_1-1} \tilde{\lambda}_1^{N_1-1} \tilde{\lambda}_2^{N_2-j} \]

\[ + \tilde{\lambda}_2 \sum_{i=1}^{N_1} \binom{N_1-i+N_2-1}{N_1-i} \tilde{\lambda}_1^{N_1-i} \tilde{\lambda}_2^{N_2-1} \]

\[ = 1 \]

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which is to be expected. The first moment of $Z$ is $\frac{\partial}{\partial s} z(s)|_{s=0}$, thus

$$E(Z) = \tilde{\lambda}_1 \sum_{j=1}^{N_2} \left( \begin{array}{c} N_1-1 + N_2-j \\ N_1-1 \end{array} \right) \frac{\partial}{\partial s} [h_2^{N_2-j} (s) \ h_1^{N_1-1} (s)]_{s=0}$$

$$+ \tilde{\lambda}_2 \sum_{i=1}^{N_1} \left( \begin{array}{c} N_1-i + N_2-1 \\ N_1-i \end{array} \right) \frac{\partial}{\partial s} [h_1^{N_1-i} (s) \ h_2^{N_2-1} (s)]_{s=0}$$

or

$$E(Z) = \tilde{\lambda}_1 \sum_{j=1}^{N_2} \left( \begin{array}{c} N_1-1 + N_2-j \\ N_1-1 \end{array} \right) \tilde{\lambda}_1^{N_1-1} \tilde{\lambda}_2^{N_2-j} \left[ \frac{1}{\lambda} + \frac{1}{\mu_2} \right] (N_2-j) + \left( \frac{1}{\lambda} + \frac{1}{\mu_1} \right) (N_1-1)$$

$$+ \tilde{\lambda}_2 \sum_{i=1}^{N_1} \left( \begin{array}{c} N_1-i + N_2-1 \\ N_1-i \end{array} \right) \tilde{\lambda}_1^{N_1-i} \tilde{\lambda}_2^{N_2-1} \left[ \frac{1}{\lambda} + \frac{1}{\mu_1} \right] (N_1-i)$$

$$+ \left( \frac{1}{\lambda} + \frac{1}{\mu_2} \right) (N_2-1)$$

since $h_1'(0) = \tilde{\lambda}_1 (\frac{1}{\lambda} + \frac{1}{\mu_1})$. The expression (5.10) may be evaluated recursively to give

$$E(Z) = \frac{1}{\lambda} \ h(N_1-1, N_2-1) \left[ 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} \right]$$

where as before, $h(N_1, N_2) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \binom{i+j}{i} \tilde{\lambda}_1^i \tilde{\lambda}_2^j$.

**Probability of Reaching any Given State.** For a state $k = (n_1, \ldots, n_N)$, the probability $p(k)$ of reaching the state is, by Lemma 5.1,

$$p(k) = \sum_{n} \tilde{\lambda}_0^{(n)} = \bar{r}_{0k}.$$
Applying Theorem 5.2, let \( n = N - d(k) \) where \( N = N_1 + \cdots + N_I \)
and \( d(k) = n_1 + \cdots + n_I \). Let \( m_i = N_n_i, i = 1, \ldots, I \). Then

\[
(5.12) \quad p(k) = R_{0k} = a^a_{0k}(0) = \left( m_1 \cdots m_I \right) \prod_{i=1}^I \lambda_i^{m_i}.
\]

**Expected Duration of Time in a Given State.** Let \( k = (n_1, \ldots, n_I) \),
\( n = (N_1 + \cdots + N_I) - (n_1 + \cdots + n_I) \), and \( m_i = N_i - n_i \). Let \( Z_k \) denote
the time spent in state \( k \). The expectation is finite only for transient
states, thus assume \( n_i \geq 2 \) for all \( i \). We have

\[
E(Z_k) = E(Z_k \mid \text{state } k \text{ is entered}) \cdot p(k).
\]

If \( X_n = k \), the duration \( T_{n+1} - T_n \) depends on the type of part which
fails. With probability \( \lambda_i \), part type \( i \) fails. The expected failure
free interval is \( \frac{1}{\lambda_i} \), and the expected repair time is \( \frac{1}{\mu_i} \); and similarly
for the other part types. Thus

\[
E(Z_k \mid \text{state } k \text{ is entered}) = \lambda_1 \left( \frac{1}{\lambda} + \frac{1}{\mu_1} \right) + \cdots + \lambda_I \left( \frac{1}{\lambda} + \frac{1}{\mu_I} \right)
= \frac{1}{\lambda} \left( 1 + \frac{\lambda_1}{\mu_1} + \cdots + \frac{\lambda_I}{\mu_I} \right)
\]

from which follows, using the value \( p(k) \) from (5.12)

\[
E(Z_k) = \frac{1}{\lambda} \left( m_1 \cdots m_I \right) \prod_{i=1}^I \lambda_i^{m_i} \left( 1 + \sum_{i=1}^I \frac{\lambda_i}{\mu_i} \right).
\]

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Since \( E(Z) = \sum_{k \in S} E(Z_k) \), we see that this result is in agreement with (5.11) above.

5.2. **Resupply of Spare Parts.**

In Chapters 3 and 4, the reliability problem was modified by restarting the system from the beginning at the end of each process lifetime. While this method enabled us to evaluate certain aspects of the process, it yields a model which is difficult to interpret physically. In this section we allow for resupply of spare parts to the system according to a Poisson process when the number of parts of a type \( i \) is smaller than \( N_i \). In this system there is never a need to start over.

Consider a system composed of \( I \) parts in series, each subject to failure with d.f. \( G(t) = 1 - e^{-\lambda_i t} \). Repair is performed on the system, following the d.f. \( F(t) \) (\( F(0) = 0 \)), and using one of \( N_i - 2 \) spare parts initially provided. An interchangeable spare system is available for use during repair. Total failure occurs when both systems are in a failed condition. It is assumed that failures are independent of each other and of the repair time, that repair of only one system may be performed at one time, and that the repair d.f. \( F(t) \) is absolutely continuous. Let \( \lambda = \lambda_1 + \cdots + \lambda_I \) and \( \lambda_i^* = \lambda_i / \lambda \). This is the model of Section 1.2. We make the following addition. Resupply of spare parts occurs according to a Poisson process. The interval between successive arrivals has the d.f. \( 1 - e^{-vt} \) for \( v > 0 \), and is of type \( i \) with probability \( v_i \) where \( v_i > 0 \) and \( \sum_{i=1}^{I} v_i = v \).
Thus the interarrival distribution of part type \( i \) is \( 1 - e^{-\frac{\nu_i}{\Delta t}} \), and the probability of an arrival of part type \( i \) in the interval \([t, t + \Delta t]\) is equal to \( \nu_i \Delta t + o(\Delta t) \), where \( o(\Delta t)/\Delta t \to 0 \) as \( \Delta t \to 0 \). We allow arrivals of type \( i \) only when the total number of parts of that type currently on hand is less than \( N_i \); thus the state space \( S \) is finite and similar to that for the previous model. The state space

\[
S = \{(r_1, n_1, \ldots, n_I) | r = 0, 1, 2; r \leq n_i \leq N_i; (r, n_1, \ldots, n_I) \neq (0, 0, \ldots, 0)\}
\]

and is thus composed of \( \sum_{k=0}^{2} \prod_{i=1}^{I} (N_i+1-k)-1 \) elements.

We will consider the case \( I = 2 \) and \( F(t) = 1 - e^{-\mu t} \), and, using the method of differential difference equations, derive a set of linear equations in the limiting probabilities \( P_k \). Comments about the solution of the equations and about a special case will be given.

### 5.2.1 State Transitions.

Compared to the model of Section 1.2, there are more "exceptional" states; i.e., states where the basic rules do not apply. For example, a transition from state \((1, n_1, n_2)\) may be caused by an arrival of part type 1, except when \( n_1 = N_1 \). To avoid writing a large number of separate equations, define for a state \((r, n_1, n_2)\)

\[
\delta_i^k(r, n_1, n_2) = \begin{cases} 
1 & n_i = k \quad i = 1, 2 \\
0 & n_i \neq k
\end{cases}
\]

In what follows, the argument \((r, n_1, n_2)\) will normally be dropped. It is to be understood that the \( \delta_i^k \) refers to the state on the left hand side of the equation.
Since the logic is similar to previous sections, only brief explanation of the equations will be given. Terms in \( o(\Delta t) \) will be collected immediately.

State \((1, n_1, n_2) \ (2 \leq n_1 \leq N_1)\). In this state, the next transition may be an arrival of type 1 or 2, a failure of type 1 or 2, or a repair completion. The state may be entered similarly, thus

\[
(5.13) \quad P_{1, n_1, n_2}(t+\Delta t) = P_{1, n_1, n_2}(t)[1-(\mu + \lambda v_1(1-\delta_1^{N_1}) + v_2(1-\delta_2^{N_2}))\Delta t] \\
+ P_{0, n_1+1, n_2}(t)[\lambda_1 \Delta t] + P_{0, n_1, n_2+1}(t)[\lambda_2 \Delta t] \\
+ P_{1, n_1-1, n_2}(t)[v_1(1-\delta_1)\Delta t] \\
+ P_{1, n_1, n_2-1}(t)[v_2(1-\delta_2)\Delta t] \\
+ P_{2, n_1, n_2}(t)[\mu \Delta t] + o(\Delta t) .
\]

State \((2, n_1, n_2) \ (2 \leq n_1 \leq N_1)\).

\[
(5.14) \quad P_{2, n_1, n_2}(t+\Delta t) = P_{2, n_1, n_2}(t)[1 - (\mu + v_1(1-\delta_1^{N_1}) + v_2(1-\delta_2^{N_2}))\Delta t] \\
+ P_{1, n_1+1, n_2}(t)[\lambda_1 \Delta t] + P_{1, n_1, n_2+1}(t)[\lambda_2 \Delta t] \\
+ P_{2, n_1-1, n_2}(t)[v_1 \Delta t] + P_{2, n_1, n_2-1}(t)[v_2 \Delta t] \\
+ P_{1, n_1-1, n_2}(t)[v_1(\delta_1^{2}) \Delta t] \\
+ P_{1, n_1, n_2-1}(t)[v_2(\delta_1^{2}) \Delta t] + o(\Delta t) .
\]

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The last two terms occur since if $X(t) = (1,1,n_2)$ and an arrival of type 2 occurs, it becomes possible to repair both systems; thus the next state is $(2, 2, n_2)$.

**State $(0, n_1, n_2)$, $(1 \leq n_1 \leq N_1)$**.

(5.15) \[ P_{0,n_1,n_2}(t+\Delta t) = P_{0,n_1,n_2}(t) \left[ 1 - (\lambda + v_1(1-\delta_1) + v_2(1-\delta_2))\Delta t \right] + P_{1,n_1,n_2}(t)[\mu \Delta t] + P_{0,n_1-1,n_2}(t)[v_1(1-\delta_1)\Delta t] + P_{0,n_1,n_2-1}(t)[v_2(1-\delta_2)\Delta t] + o(\Delta t) \]

**State $(1, 1, n_2)$, $(2 \leq n_2 \leq N_2)$ or $(1, n_1, 1)$, $(2 \leq n_1 \leq N_1)$**. State $(1, 1, n_2)$ is non-operating. An arrival of type 1 leads to $(2, 2, n_2)$ and of type 2 to $(1, 1, n_2+1)$. Entry by arrivals is made from $(0, 0, n_2)$ or $(1, 1, n_2-1)$,

(5.16) \[ P_{1,1,n_2}(t+\Delta t) = P_{1,1,n_2}(t) \left[ 1 - (\mu + v_1 + v_2(1-\delta_2))\Delta t \right] + P_{0,2,n_2}(t)[\lambda_1 \Delta t] + P_{0,1,n_2+1}(t)[\lambda_2 \Delta t] + P_{1,2,n_2}(t)[\lambda_1 \Delta t] + P_{0,0,n_2}(t)[v_1 \Delta t] + P_{1,1,n_2-1}(t)[v_2 \Delta t] + o(\Delta t) \]
\[
(5.17) \quad P_{1, n_1, 1}(t+\Delta t) = P_{1, n_1, 1}(t)[1 - (\mu + \nu_1(1-\delta_1) + \nu_2)\Delta t]
+ P_{0, n_1+1, 1}(t)[\lambda_1\Delta t] + P_{0, n_1, 2}(t)[\lambda_2\Delta t]
+ P_{1, n_1, 2}(t)[\lambda_2\Delta t]
+ P_{0, n_1, 0}(t)[\nu_2\Delta t] + P_{1, n_1-1, n_2}(t)[\nu_1\Delta t]
+ o(\Delta t).
\]

**State (1,1,1).**

\[
(5.18) \quad P_{1, 1, 1}(t+\Delta t) = P_{1, 1, 1}(t)[1 - (\mu + \nu)\Delta t]
+ P_{0, 2, 1}(t)[\lambda_1\Delta t] + P_{0, 1, 2}(t)[\lambda_2\Delta t]
+ P_{0, 0, 1}(t)[\nu_1\Delta t] + P_{0, 1, 0}(t)[\nu_2\Delta t] + o(\Delta t).
\]

**State \((0,0,n_2), (1 \leq n_2 \leq N_2)\) or \((0, n_1, 0), (1 \leq n_1 \leq N_1)\).** An arrival of type 1 allows repair of one system and thus leads from \((0,0,n_2)\) to \((1,1,n_2)\). An arrival of type 2 leads to \((0,0,n_2+1)\). This state cannot be entered via a repair completion or a failure of type 2, nor obviously by an arrival of type 1. Thus

\[
(5.19) \quad P_{0, 0, n_2}(t+\Delta t) = P_{0, 0, n_2}(t)[1 - (\nu_1 + \nu_2(1-\delta_2))\Delta t]
+ P_{0, 1, n_2}(t)[\lambda_1\Delta t]
+ P_{0, 0, n_2-1}(t)[\nu_2\Delta t] + o(\Delta t).
\]
\[(5.20) \quad P_{0,n_1,0}(t+\Delta t) = P_{0,n_1,0}(t)[1 - (v_2 + v_1(1-\delta_1^1))\Delta t] + P_{0,n_1,1}(t)[\lambda_2\Delta t] + P_{0,n_1-1,0}(t)[v_1\Delta t] + o(\Delta t) .\]

**State (0,0,0).** There is no way to enter this state, thus

\[(5.21) \quad P_{0,0,0}(t) = 0 .\]

To complete the set of equations,

\[(5.22) \quad P_{r,n_1,n_2}(t) = 0 \quad (n_1 > N_1 \text{ or } n_2 > N_2)\]

\[(5.23) \quad \sum_S P_{r,n_1,n_2}(t) = 1 .\]

5.2.2. **Solution of the Equation.**

To obtain a set of linear equations in the limiting state probabilities, \(P_{r,n_1,n_2}\), the first term on the right is subtracted from both sides of (5.13) - (5.20), and each equation is divided by \(\Delta t\). Letting \(\Delta t \to 0\) and \(t \to \infty\) gives

**Theorem 5.4.** The limiting state probabilities satisfy

\[(5.24.1) \quad P_{1,n_1,n_2}(\mu + \lambda + v_1(1-\delta_1^1) + v_2(1-\delta_2^2))\]

\[= \lambda_1 P_{0,n_1+1,n_2} + \lambda_2 P_{0,n_1,n_2+1} + v_1(1-\delta_1^1)P_{1,n_1-1,n_2}

+ v_2(1-\delta_2^2)P_{0,n_1,n_2-1} + \mu P_{2,n_1,n_2} \quad (2 \leq n_1 \leq N_1)\]
\begin{align*}
\text{(5.24.2) } & \quad P_{2,n_1,n_2}^{N_1}(\mu + v_1(1-\delta_{1}) + v_2(1-\delta_{2})) \\
& = \lambda_1 P_{1,n_1+1,n_2} + \lambda_2 P_{1,n_1, n_2+1} + v_1 P_{2,n_1-1,n_2} \\
& \quad + v_2 P_{2,n_1, n_2-1} + v_1 \delta_1^2 P_{1,n_1-1,n_2} + v_2 \delta_2^2 P_{1,n_1,n_2-1} \\
& \quad (2 \leq n_1 \leq N_1)
\end{align*}

\begin{align*}
\text{(5.24.3) } & \quad P_{0,n_1,n_2}^{N_1}(\lambda + v_1(1-\delta_{1}) + v_2(1-\delta_{2})) \\
& = \mu P_{1,n_1,n_2} + v_1(1-\delta_1^1) P_{0,n_1-1,n_2} + v_2(1-\delta_2^1) P_{0,n_1,n_2-1} \\
& \quad (1 \leq n_1 \leq N_1)
\end{align*}

\begin{align*}
\text{(5.24.4) } & \quad P_{1,1,n_2}^{N_2}(\mu + v_1 + v_2(1-\delta_{2})) \\
& = \lambda_1 P_{0,2,n_2} + \lambda_2 P_{0,1,n_2+1} + \lambda_1 P_{1,2,n_2} \\
& \quad + v_1 P_{0,0,n_2} + v_2 P_{1,1,n_2-1} \quad (2 \leq n_2 \leq N_2)
\end{align*}

\begin{align*}
\text{(5.24.5) } & \quad P_{1,n_1,1}^{N_1}(\mu + v_1(1-\delta_{1}) + v_2) \\
& = \lambda_1 P_{0,n_1+1,1} + \lambda_2 P_{0,n_1,2} + \lambda_1 P_{1,n_1,2} \\
& \quad + v_2 P_{0,n_1,0} + v_1 P_{1,n_1-1,2} \quad (2 \leq n_1 \leq N_1)
\end{align*}

\begin{align*}
\text{(5.24.6) } & \quad P_{1,1,1}(\mu + v) = \lambda_1 P_{0,2,1} + \lambda_2 P_{0,1,2} + v_1 P_{0,0,1} \\
& \quad + v_2 P_{0,1,0} 
\end{align*}
(5.24.7) \( P_{0,0,n_2}(v_1 + v_2(1-\delta_2)) = \lambda_1 P_{0,1,n_2} + v_2 P_{0,0,n_2-1} \) 
\((1 \leq n_2 \leq N_2)\)

(5.24.8) \( P_{n_1,0}(v_2(1-\delta_1) + v_2) = \lambda_2 P_{0,n_1,1} + v_1 P_{0,n_1-1,0} \) 
\((1 \leq n_1 \leq N_1)\)

(5.24.9) \( P_{0,0,0} = 0 \)

(5.24.10) \( P_{r,n_1,n_2} = 0 \) 
\((n_1 > N_1 \text{ or } n_2 > N_2)\)

(5.24.11) \( \sum_{s} P_{r,n_1,n_2} = 1 \)

Theorem 5.4 gives a system of \( N+1 \) equations in \( N \) unknowns, where \( N = \sum_{k=0}^{2} \left( \prod_{i=1}^{2} (N_i+1-k) \right) - 1 \) is the number of possible states.

Hillier and Boling ([16], p. 291) refer to a number of efficient procedures for solving large systems of linear equations with a high proportion of zero coefficients, with the diagonal element of the coefficient matrix dominating, such as the Gauss-Seidel or Liebmann extrapolated method ([19], p. 39) or the Aitken convergence accelerating procedure ([19], p. 123).

Since we have assumed an exponential repair distribution, the successive states entered form a Markov chain. Thus an alternative procedure for determining an exact solution is to obtain the transition
probability matrix $Q$. Then, denoting the vector of limiting state probabilities in the Markov chain by $\pi$, the equations $\pi = \pi Q$, $\pi \cdot 1 = 1$ may be solved for $\pi$. These equations, of course, will be similar to (5.24.1)-(5.24.11) above. Once the $\pi_k$ are obtained, the limiting probability $P_k$ in the reliability problem (semi-Markov process) is given by

$$P_k = \frac{\pi_k m_k}{\sum_i \pi_i m_i}$$

where $m_k$ is the expected sojourn time in state $k$ (Çinlar [8], p. 160). This procedure is also applicable to the model of Chapter 3.

**Finite Queues in Series.** For $I \geq 2$, the model of this section is very close to a system of two finite queues in series. If $I = 1$, however, the model is a system of two queues in series, with finite waiting room of $N_1 - 1$ in the first and 0 in the second queue. Arrivals to the first queue are the arrivals of spare parts. Service time there corresponds to system operation. Departures from the first queue are failures. Service at the second queue is equivalent to repair of failed units. If service at the first queue is completed (failure) and the second queue is not empty (a repair is in process), the system is blocked. Thus the probability of system operation in the reliability model is equal to the queueing system probability that the first server is busy and unblocked.

If $I \geq 2$ the difference between the reliability model and the queueing model is that one of each of $I$ different types of customers
must be present at the first queue for service to be performed. Alternatively, the system may be viewed as separate service stations for each customer type at the first queue.

Hillier and Boling [16] have given procedures for obtaining exact and approximate results for systems of finite queues in series. In the case $I = 1$, the exact procedure is that described above; alternatively, the approximating procedure may be used.
BIBLIOGRAPHY


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23. Srinivasan, V. S., "First Emptiness in the Spare Parts Problem for 

Oliver and Boyd (1960).

25. Wilken, D. R., and E. S. Langford, "Failure Probability Formulas 
COMPREHENSIVE BIBLIOGRAPHY

Following is a list of books and articles dealing with the analysis of the stochastic behavior of complex systems. The criterion for inclusion in this bibliography is the analysis in a quantitative fashion, or the optimization under various criteria, of the reliability of specified complex systems. Normally, technical reports or memoranda have not been included unless they are especially important and have not been repeated in journal articles.

BOOKS


ARTICLES


______, "A Simple Rule for the Consolidation of Allowance Lists," 


Morrison, D. F., "Cost Functions for Systems with Spare Components,"

_______, "The Optimum Allocation of Spare Components in Testing,"


This paper concerns the description of stochastic behavior of a class of complex systems. The class of problems considered is that of semi-Markov processes with finite expected lifetimes. A simple reformulation permits the easy derivation of additional results for the model. The reformulation consists of treating the process mathematically as though it restarts from the beginning upon completion of each lifetime. In this way limiting distributions in the modified process may be used to determine relevant characteristics of the original process.

The application examined is a complex system composed of parts connected in series each with independent exponential failure distributions. An additional spare system and a finite number of spare parts are provided. The spare parts are used for repair; the systems are interchangeable so that one may operate while the other is being repaired. Failure occurs when both systems are out of operation simultaneously or when the supply of spare parts is exhausted. The model combines features of the spare parts problem and the repairable item problem, both of which have been previously studied in the theory of reliability. The expected duration of total lifetime, operating lifetime, and time spent in any state is derived for the cases of exponential and general repair distributions. Modifications and extensions considered include the resupply of spare parts of the system. Appended is an extensive bibliography of reliability theory work in the description of stochastic behavior of complex systems.
Reliability
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