ASYMPTOTIC OPTIMAL POLICIES FOR THE STOCHASTIC
SEQUENTIAL ASSIGNMENT PROBLEM

by

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Summary

In [1] the following sequential assignment problem is treated. We have \( n \) men with values, or efficiencies, \( p_1 \leq \ldots \leq p_n \), and these men must be paired with \( n \) jobs which come in sequentially. The values of the jobs are \( X_1, \ldots, X_n \), assumed to be independent, identically distributed random variables with a known distribution function \( F \). It is assumed that if a "p" man is assigned to an "x" job, a reward of \( px \) is obtained. The main result is that, with this special reward function, the optimal assignment is independent of the \( p_i \)'s. Namely, if there are \( n \) men left, there exist \( n-1 \) numbers, \( a_1,n \leq a_2,n \leq \ldots \leq a_{n-1},n' \) dividing the real line into \( n \) intervals, such that if the next job has a value \( x \) falling into the \( i \)th interval, it is best to assign the man with value \( p_i \) to this job. An interpretation of these \( a_{i,n} \)'s is also given. If there are \( n \) men left, \( a_{i,n+1} \) is the expected value of the job to which the man with value \( p_i \) is assigned. This allows a recursive formula for the \( a_{i,n} \)'s:

\[
(1) \quad a_{i,n+1} = \int_{a_{i-1,n}}^{a_{i,n}} xF(dx) + a_{i-1,n} F(a_{i-1,n}) + a_{i,n} (1 - F(a_{i,n}))
\]

for \( 1 \leq i \leq n \), where \( a_{0,n} \) and \( a_{n,n} \) are defined appropriately in the next section.

The present paper is concerned with the limiting behavior of these \( a_{i,n} \)'s as \( n \) gets large. The first results are obtained trivially. If we fix \( i \) and look at \( a_{i,n} \) or \( a_{n-i,n'} \), these go to the upper and lower endpoints of support of the distribution function \( F \). A more interesting problem is to let \( i = [n\pi] \) for some \( 0 < \pi < 1 \).
and find the limiting behavior of \( a_{[n\pi],n+1} \) as \( n \) gets large. Two results in this spirit are obtained under suitable restrictions on \( F \).

First it is shown that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lceil n\pi \rceil + 1}^{n} a_{i,n+1} = \int_{F^{-1}(\pi)}^{\infty} xF(dx).
\]

Then this result is used to show that

\[
\lim_{n \to \infty} a_{[n\pi],n+1} = F^{-1}(\pi)
\]

**Limits of** \( a_{i,n} \) **and** \( a_{n-i,n} \) **for fixed** \( i \).

Define the following numbers:

- \( \bar{R}_F = \inf \{ x : F(x) = 1 \} \), which may be \( +\infty \)
- \( R_F = \sup \{ x : F(x) = 0 \} \), which may be \( -\infty \)

**Lemma 1.** Suppose \( X_1, X_2, \ldots \) are independent, identically distributed random variables with distribution function \( F \). Suppose also that \( E|X_1| < \infty \) and that \( \bar{R}_F > \bar{R}_F \). Then

\[
a_{0,n} \leq a_{1,n} \leq a_{2,n} \leq \cdots \leq a_{n-1,n} < \bar{R}_F \leq a_{n,n}.
\]

**Proof:** \( a_{1,2} = \int_{\bar{R}_F}^{\bar{R}_F} xF(dx) = EX_1 \) by (1), and \( E_F < EX_1 < \bar{R}_F \),

so (2) holds for \( n=2 \). Now assume (2) holds for \( n \). Then for \( n+1 \), we have

\[
a_{1,n+1} = \int_{\bar{R}_F}^{a_{1,n}} xF(dx) + a_{1,n}(1-F(a_{1,n})).
\]
Integrating by parts gives

\[ a_{i,n+1} = a_{i,n} - \int_{a_{i-1,n}}^{a_{i,n}} F(x) \, dx > R_F. \]

since \( F(a_{i,n}) < 1 \). Similarly,

\[ a_{n,n+1} = \int_{a_{n-1,n}}^{a_{n,n}} x F(x) \, dx + a_{n-1,n} F(a_{n-1,n}), \]

and integrating by parts gives

\[ a_{n,n+1} = a_{n-1,n} + \int_{a_{n-1,n}}^{a_{n,n}} (1-F(x)) \, dx < R_F, \]

since \( 1-F(a_{n-1,n}) < 1 \). Note that integrating by parts is allowed in both cases because \( E|X_1| < \infty \).

This completes the induction step, and thus the lemma is proved.

**Theorem 1.**

Under the same assumptions as in lemma 1, we have, for any fixed integer \( i \geq 1 \),

\[ \lim_{n \to \infty} a_{i,n} = R_F, \quad \text{and} \quad \lim_{n \to \infty} a_{n-i,n} = R_F. \]

**Proof:** By convention we again set \( a_{0,n} = R_F, \quad a_{n,n} = R_F \) for all \( n \).

Then lemma 1 permits integration by parts of equation (1) to get:

\[ a_{i,n+1} = a_{i,n} - \int_{a_{i-1,n}}^{a_{i,n}} F(x) \, dx \quad \text{for} \quad i \neq n, \]

since in this case we know \( a_{i,n} < R_F < \infty \), and
(4) \[ a_{i,n+1} = a_{i-1,n} + \int_{a_{i-1,n}}^{a_{i,n}} (1-F(x))dx \quad \text{for } i \neq 1, \]

since in this case we know \( a_{i-1,n} > \frac{R_F}{2} > -\infty. \)

These imply the relations:

(i) \( a_{i-1,n} \leq a_{i,n+1} \leq a_{i,n} \) if \( 1 < i < n, \)

(ii) \( \frac{R_F}{2} = a_{0,n} < a_{1,n+1} < a_{1,n} \) if \( i = 1 \)

(iii) \( a_{n-1,n} \leq a_{n,n+1} < a_{n,n} = \frac{R_F}{2} \) if \( i = n. \)

Now observe that \( a_{0,n} = \frac{R_F}{2} \) for all \( n, \) and assume that \( a_{i-1,n} \leq \frac{R_F}{2} \) as \( n \to \infty. \) Since \( a_{i,n} \) is monotone decreasing in \( n, \) it has a limit, say \( B > \frac{R_F}{2}. \) Then taking limits in (3) gives

\[ B = B - \int_{\frac{R_F}{2}}^{B} F(x)dx < B, \]

a contradiction. So \( \lim_{n \to \infty} a_{i,n} = \frac{R_F}{2} \) also, and the induction step is completed.

Similarly we can use (4) to show that \( \lim_{n \to \infty} a_{n-1,n} = \frac{R_F}{2}. \) This completes the proof of theorem 1.

**Limiting behavior of** \( a_{[n\pi],n+1} \), \( 0 < \pi < 1 \)

Suppose \( 0 < \pi < 1 \) and that in the original problem, \( n-[n\pi] \) of the \( p_i \)'s are 1 and the rest of the \( p_i \)'s are 0. Then we know the optimal total expected reward is

\[ \sum_{i=[n\pi]+1}^{n} a_{i,n+1}. \]
We use this fact to find the limiting behavior of the average expected reward per job,

\[
\frac{1}{n} \sum_{i=[n\theta]+1}^{n} a_{i,n+1}.
\]

In this section we use the following random variables. Let \( X_i \) be the value of the \( i \)th job, which comes from a distribution \( F \), has a finite mean, and which we now assume to be non-negative. Next, let

\[
Y_i(a) = X_i I_{\{X_i > a\}}, \quad \text{and} \quad Z_i(a) = I_{\{X_i > a\}},
\]

where \( I_{\{X_i > a\}} \) is the indicator random variable of the event \( \{X_i > a\} \).

First we establish the following lemma.

**Lemma 2.** Let \( U_r \) be a pascal random variable with parameters \( r \) and \( \rho \). That is,

\[
\mathbb{P}(U_r = k) = \binom{k-1}{r-1} \rho^{r} (1-\rho)^{k-r}, \quad k \geq r,
\]

and \( \mathbb{E}U_r = r/\rho, \mathbb{V}ar\ U_r = r(1-\rho)/\rho^2 \). Let \( n > r \) be any integer, and truncate \( U_r \) to get \( N_r \) as follows:

\[
N_r = \begin{cases} 
U_r & \text{if } r \leq U_r \leq n \\
n & \text{if } U_r > n.
\end{cases}
\]

Finally, let \( r = [n\theta] \), where \( 0 < \theta < \rho < 1 \) and \([\cdot]\) denotes greatest integer. Then we have
\[
\lim_{n \to \infty} \frac{E_N}{n} \geq \frac{\theta}{\rho}
\]

**Proof:** By definition we have

\[
E_N_r = \sum_{k=r}^{n} kP(U_r = k) + nP(U_r > n) \\
\geq \sum_{k=r}^{\infty} kP(U_r = k) - \sum_{k=n+1}^{\infty} kP(U_r = k) \\
= \frac{r}{\rho} - \sum_{k=n+1}^{\infty} kP(U_r = k)
\]

Consider the second term:

\[
\sum_{k=n+1}^{\infty} kP(U_r = k) = \sum_{k=n+1}^{\infty} \frac{k(k-1)!}{(k-1)!(k-r)!} \rho^{r(1-\rho)^{k-r}} \\
= \frac{r}{\rho} \sum_{k=n+1}^{\infty} \binom{k}{r} \rho^{r+1(1-\rho)^{k-r}} \\
= \frac{r}{\rho} P(U_{r+1} > n+1) \\
= \frac{r}{\rho} P(U_{r+1} - \bar{E}U_{r+1} > n+1 - (r+1)/\rho).
\]

Now, if \( n+1 - (r+1)/\rho > 0 \), we have by Chebyshev's inequality that the above expression is

\[
\leq \frac{r}{\rho} \frac{\Var U_{r+1}/(n+1 - (r+1)/\rho)^2}{(n+1-(r+1)/\rho)^2} \\
= \frac{r}{\rho} \cdot \frac{(r+1)(1-\rho)/\rho^2}{(n+1-(r+1)/\rho)^2} \\
= \frac{r}{\rho} \cdot \frac{(r+1)(1-\rho)/(\rho(n+1) - (r+1))^2}{(n+1-(r+1)/\rho)^2}
\]
Since \( r = [n\theta] = n\theta + \tau \), for some \( 0 \leq \tau < 1 \),

\[
\frac{n+1 - (r+1)}{\rho} = \frac{n+1 - (n\theta + \tau + 1)}{\rho} = \frac{n(1-\theta/\rho) + 1-(\tau+1)}{\rho} > 0
\]

for large enough \( n \), since \( \theta < \rho \). Then we have

\[
\frac{1}{n} \mathbb{E}_r \leq \frac{r}{n\rho} \left[ 1-(r+1)(1-\rho)/(\rho(n+1)-(r+1)) \right]^2
\]

\[
= \theta/\rho \cdot (1-A) + \tau/n\rho \cdot (1-A),
\]

where

\[
A = (n\theta + \tau + 1)(1-\rho)/(\rho(n+1)-(n\theta + \tau + 1))^2 + 0
\]

as \( n \to \infty \). Hence, letting \( n \to \infty \) in the above inequality gives

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}_r \geq \frac{\theta}{\rho}.
\]

Now we compare the optimal assignment policy with a non-optimal policy to get a lower bound on

\[
\frac{1}{n} \sum_{i=\lceil n\pi \rceil + 1}^{n} a_{i,n+1}.
\]

**Lemma 3.** Assuming that \( F \) is continuous, we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lceil n\pi \rceil + 1}^{n} a_{i,n+1} \geq \int_{F^{-1}(\pi)}^{\infty} x F(dx)
\]

**Proof:** Let \( \varepsilon > 0 \) be arbitrary. Compare the optimal assignment of the \( n-[n\pi] \) \( p_i \)'s which are 1 with a policy which assigns to the \( n \) jobs at most \( n-[n\pi]-1 \) \( p_i \)'s equal to 1, and assigns a 1 to every job with a value \( x > F^{-1}(\pi-\varepsilon) \). This policy is not even feasible
since the \( n-[n\pi]-1 \) \( p_i \)'s equal to 1 may not all be assigned. However, since \( X_i \geq 0 \) for each \( i \), this policy is worse than the feasible policy which assigns a 1 to a job if \( x > F^{-1}(\pi-\varepsilon) \) until there are as many \( p_i \)'s equal to 1 left as there are jobs remaining, at which point it assigns a 1 to every remaining job.

Now we notice that the reward from \( n \) jobs under the infeasible policy is simply

\[
\sum_{i=1}^{r} Y_i(F^{-1}(\pi-\varepsilon)),
\]

where \( r = \lfloor n\theta \rfloor \), with \( \theta = 1-\pi \), and

\[N_r = \begin{cases} 
\min \{k \leq n : \sum_{i=1}^{k} Z_i(F^{-1}(\pi-\varepsilon)) = r \} & \text{if } \sum_{i=1}^{n} Z_i(F^{-1}(\pi-\varepsilon)) \geq r \\
n & \text{if } \sum_{i=1}^{n} Z_i(F^{-1}(\pi-\varepsilon)) < r
\end{cases}
\]

Notice also that this \( N_r \) is the same as in lemma 2, with \( r = \lfloor n-\lfloor n\pi \rfloor \rfloor - 1 \) and \( \rho = 1-\pi+\varepsilon \). Since \( N_r \) is a finite stopping rule for the \( Y_i \)'s, we have by Wald's equation,

\[
E\left( \sum_{i=1}^{N_r} Y_i \right) = (EN_f)(EY_1) = (EN_f) \int_{F^{-1}(\pi-\varepsilon)}^{\infty} xF(dx).
\]

Now we use the fact that the reward from the optimal policy is better than the reward from the infeasible policy, and lemma 2, to get

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lfloor n\pi \rfloor+1}^{n} a_i, n+1 \geq \lim_{n \to \infty} \left( \frac{EN_r}{n} \right) \int_{F^{-1}(\pi-\varepsilon)}^{\infty} xF(dx)
\]

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\[ \geq \frac{1-\pi}{1-\pi+\varepsilon} \int_{F^{-1}(\pi-\varepsilon)}^{\infty} x F(dx). \]

Since this is true for each \( \varepsilon > 0 \), we may let \( \varepsilon \to 0 \) and use continuity of \( F \) to get the result of the lemma.

We now compare the optimal policy with an infeasible, but certainly better-than-optimal, policy in order to get an upper bound.

**Lemma 4.** Under the same assumptions as in lemma 3, we have

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=\lceil n\pi \rceil + 1}^{n} a_{i,n+1} \leq \int_{F^{-1}(\pi)}^{\infty} x F(dx). \]

**Proof:** Let \( X_{(1)} \leq \ldots \leq X_{(n)} \) be the order statistics associated with \( X_1, \ldots, X_n \). Then certainly we have

\[ \frac{1}{n} \sum_{i=\lceil n\pi \rceil + 1}^{n} a_{i,n+1} \leq \frac{1}{n} \mathbb{E} \left( \sum_{i=\lceil n\pi \rceil + 1}^{n} X_{(i)} \right). \]

since no sequential assignment can do better than assigning the \( p_i \)'s equal to 1 to the highest order statistics. But now we show that

\[ \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \sum_{i=\lceil n\pi \rceil + 1}^{n} X_{(i)} \right) \leq \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \sum_{i=1}^{n} Y_i (F^{-1}(\pi)) \right). \]

To do this, first define

\[ T_n = \sum_{i=1}^{n} Z_i (F^{-1}(\pi)). \]

Then we have
\begin{align}
1/n & \ E \left( \sum_{i=[n\pi]+1}^{n} X(i) - \sum_{i=1}^{n} Y_i \right) \\
& = 1/n \ \left[ \int_{\{T_n < n - [n\pi]\}} \left( \sum_{i=[n\pi]+1}^{n} X(i) - \sum_{i=1}^{n} Y_i \right) \ dP \right] \\
& \quad + \int_{\{T_n > n - [n\pi]\}} \left( \sum_{i=[n\pi]+1}^{n} X(i) - \sum_{i=1}^{n} Y_i \right) \ dP \\
& = 1/n \ \left[ \int_{\{T_n < n - [n\pi]\}} \left( X([n\pi]+1) + \cdots + X_{(n-T_n)} \right) \ dP \right] \\
& \quad - \int_{\{T_n > n - [n\pi]\}} \left( X_{(n-T_n+1)} + \cdots + X_{([n\pi])} \right) \ dP \\
\end{align}

In the first integral there are \( n-T_n-[n\pi] \) terms, all \( \leq F^{-1}(\pi) \). In the second integral there are \(-n-T_n-[n\pi]\) terms, all \( \geq F^{-1}(\pi) \). Thus continuing (6), the left side is

\begin{align}
\leq F^{-1}(\pi)/n & \left[ \int_{\{T_n < n - [n\pi]\}} (n-T_n-[n\pi]) \ dP \right] \\
& \quad - \int_{\{T_n > n - [n\pi]\}} -(n-T_n-[n\pi]) \ dP \\
& = F^{-1}(\pi)/n \ E(n-T_n-[n\pi]) \to 0 \ as \ n \to \infty ,
\end{align}

since \( ET_n = n(1-\pi) \).

However, for each \( n \),

\[ 1/n \ E \left( \sum_{i=1}^{n} Y_i \left( F^{-1}(\pi) \right) \right) = \int_{F^{-1}(\pi)}^{\infty} xF(dx). \]
Hence, from (5) and (6), we have

\[
\lim_{n \to \infty} \sum_{i=[n\pi]+1}^{n} a_{i,n+1} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=[n\pi]+1}^{n} X(i) \\
\leq \int_{F^{-1}(\pi)}^{\infty} xF(dx).
\]

**Theorem 2.** If the \( X_i \)'s are non-negative, have a finite mean, and have a continuous distribution function \( F \), then for each \( 0 < \pi < 1 \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=[n\pi]+1}^{n} a_{i,n+1} = \int_{F^{-1}(\pi)}^{\infty} xF(dx).
\]

**Proof:** The proof follows directly from lemmas 3 and 4.

**Corollary 1.** Under the same assumptions as in theorem 2, we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{[n\pi]} a_{i,n+1} = \int_{0}^{F^{-1}(\pi)} xF(dx).
\]

**Proof:** In the original assignment problem, if we let \( p_1 = \ldots = p_n = 1 \), we see trivially that the total expected reward is \( nEX_1 \), that is,

\[
\frac{1}{n} \sum_{i=1}^{n} a_{i,n+1} = EX_1.
\]

Thus from theorem 2,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{[n\pi]} a_{i,n+1} = EX_1 - \int_{F^{-1}(\pi)}^{\infty} xF(dx) \\
= \int_{0}^{F^{-1}(\pi)} xF(dx).
\]
Corollary 2. Under the same assumptions as in theorem 2, we have

\[ \lim_{n \to \infty} \frac{1}{n} E \left( \sum_{i=1}^{\lfloor n \pi \rfloor+1} X(i) \right) = \int_{F^{-1}(\pi)}^{\infty} xF(dx) \]

and

\[ \lim_{n \to \infty} \frac{1}{n} E \left( \sum_{i=1}^{\lfloor n \pi \rfloor} X(i) \right) = \int_{0}^{F^{-1}(\pi)} xF(dx) \]

**Proof:** Lemma 4 shows that

\[ \lim_{n \to \infty} \frac{1}{n} E \left( \sum_{i=1}^{\lfloor n \pi \rfloor+1} X(i) \right) \leq \int_{F^{-1}(\pi)}^{\infty} xF(dx). \]

The opposite inequality follows by using theorem 2 and the fact that

\[ \sum_{i=1}^{\lfloor n \pi \rfloor+1} a_{i,n+1} \leq E \left( \sum_{i=1}^{\lfloor n \pi \rfloor+1} X(i) \right). \]

Finally, (8) follows by using (7) and the fact that

\[ E \left( \sum_{i=1}^{n} X(i) \right) = n E X_{1}. \]

We are now in a position to determine the limiting behavior of \( a_{\lfloor n \pi \rfloor, n+1} \).

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1. The authors do not mean to imply that this is a new result.
Theorem 3. In addition to the assumptions of theorem 2, assume that $F$ is absolutely continuous with density $f$. Then for each $0 < \pi < 1$,

$$\lim_{n \to \infty} a_{[n\pi]}, n+1 = F^{-1}(\pi).$$

Proof: Suppose $0 < \pi' < \pi < 1$. Then from theorem 2,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lceil n\pi' \rceil + 1}^{\lfloor n\pi \rfloor} a_{i,n+1} \geq \int_{F^{-1}(\pi')}^{F^{-1}(\pi)} xF(dx).$$

Since we have $a_{1,n+1} \leq \ldots \leq a_{n,n+1}$ for each $n$, it follows that

$$\frac{1}{n} ([n\pi]-[n\pi']) a_{[n\pi],n+1} \geq \frac{1}{n} \sum_{i=\lceil n\pi' \rceil + 1}^{\lfloor n\pi \rfloor} a_{i,n+1}$$

for large enough $n$ such that $[n\pi] \geq [n\pi'] + 1$.

Now let $n \to \infty$ to obtain

$$(\pi - \pi') \lim_{n \to \infty} a_{[n\pi],n+1} \geq \int_{F^{-1}(\pi')}^{F^{-1}(\pi)} xF(dx),$$

or, dividing by $\pi - \pi'$,

$$\lim_{n \to \infty} a_{[n\pi],n+1} \geq \frac{1}{\pi - \pi'} \int_{F^{-1}(\pi')}^{F^{-1}(\pi)} xF(dx).$$

Finally, let $\pi' \succ \pi$:

$$\lim_{\pi' \to \pi} \frac{1}{\pi - \pi'} \int_{F^{-1}(\pi')}^{F^{-1}(\pi)} xF(dx) = d/d\pi \int_{0}^{\pi} xF(dx).$$
\[
= (d/d\pi \ F^{-1}(\pi))F^{-1}(\pi) \ f(F^{-1}(\pi))
\]
\[
= F^{-1}(\pi),
\]
so that
\[
\lim_{n \to \infty} a_{\lfloor n\pi \rfloor, n+1} \geq F^{-1}(\pi).
\]

Similarly, letting \( 0 < \pi < \pi' < 1 \) and going through the same argument, we obtain
\[
\lim_{n \to \infty} a_{\lfloor n\pi \rfloor, n+1} \leq F^{-1}(\pi).
\]

Combining these inequalities yields the desired result.
REFERENCES

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13. ABSTRACT
Suppose we have \( n \) men available to do \( n \) jobs. The men have values \( p_1 \leq \ldots \leq p_n \) and the jobs come in sequentially and take on values \( x_1, \ldots, x_n \) are independent, identically distributed random variables with CDF \( F \). The reward from assigning a "p" man to an "x=x" job is assumed to be \( p x \). The optimal assignment is as follows. There exist numbers \( a_{1:n} \leq \ldots \leq a_{n-1,n} \), independent of the \( p_i \)'s, dividing the real line into \( n \) intervals, such that if the first job falls into the \( i \)th interval, it is best to assign man \( p_i \) to this job.

Limiting results for these \( a_{1:n} \)'s are found. First we fix \( i \) and find
\[
\lim_{n \to \infty} a_{i,n} \quad \text{and} \quad \lim_{n \to \infty} a_{n-i,n}.
\]
Then we let \( 0 < \pi < 1 \) and find
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lceil \pi n \rceil+1}^{n} a_{i,n+1} \quad \text{and} \quad \lim_{n \to \infty} a_{\lceil \pi n \rceil,n} \quad \text{in terms of the CDF} \ F.\]
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**KEY WORDS**

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