EFFICIENT OPERATION OF OPTIONAL PRIORITY QUEUEING SYSTEMS

BY

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TECHNICAL REPORT NO. 138
AUGUST 1971

SUPPORTED BY THE ARMY, NAVY AND AIR FORCE
UNDER CONTRACT N00014-67-A-0112-0052  (NR-042-002)
WITH THE OFFICE OF NAVAL RESEARCH

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ABSTRACT

An $M/G/1$ queueing system with two classes of customers is studied. The per unit time cost of holding a customer differs for the two classes. The server has the option of serving the first customer in line at no extra charge or paying a fee to earn the right to serve any higher class customer further back in the queue. An optimal policy is found for the simple case where this fee is independent of the position in the queue of the promoted customer.

For the case where the server is restricted to serving either the first customer or the first higher class customer further back in the queue, an appealing conjectured from of an optimal policy is shown to be invalid.
Previous optimization work on priority queueing systems has been primarily concerned with the design of a system rather than the control of its operation. For example, Cobham [2] attacked the problem of assigning priorities to each of \( k \) classes of customers in an M/G/1 system when each class had different holding costs and service time distributions. Also for an M/G/1 system, Oliver and Pestalozzi [5] discussed the problem of assigning \( M \) priority classes to customers on the basis of their required service time which is observed when the customer arrives. Balachandran [1] determined optimal times for pre-empting service when either early or late pre-emption was allowed. A thorough bibliography on priority queues is given in Jaiswal [4].

In contrast to this previous work, we shall be concerned with a dynamic policy for operating a system in which there are frequent opportunities to change the operating procedure as a function of the system's current state. We shall seek an optimal policy for operating a single server queueing system when customers of two classes join a single queue. Customers will be denoted as 1customers and 2-customers (or simply 1's and 2's). 1-customers will be V.I.P.'s and this status will be reflected in a higher holding cost per unit time. The server, after completing a service, will have the option of serving the first customer in line (who may be a "1" or "2") or paying a fee (designed to be a crude measure of inconvenience, loss of goodwill, etc.) to earn the right to service a 1-customer who is not first in line. Thus an optimal policy for operating the service facility will strike a happy mix between excessive holding costs and
excessive charges for disrupting a first-come-first-served service
discipline. Changes in precise cost assumptions will yield a variety
of models but the analysis of each will have the common theme of
characterizing an optimal policy which determines, as a function of
the queue, whether the server should act according to a non-
preemptive priority discipline or should use a first-come-first-served
rule in deciding whom to serve next.

In the next section we shall analyze the basic model and discuss
some possible modifications in assumptions. Then in section II we
shall further restrict the basic model and find a counter example to
a conjectured optimal operating policy which appears to have intuitive
appeal.

I. The Basic Model

Customers of types 1 and 2 arrive according to independent Poisson
processes with rates $\lambda_1, \lambda_2$, respectively. A single server provides
independent services having a general service time distribution with
mean $\mu^{-1}$ and finite variance $\sigma^2$. We assume that $\lambda_1 + \lambda_2 \leq \mu$. We
should emphasize that service time distributions do not differ for
the two types of customers since this assumption will later simplify
the analysis.

Costs are incurred as follows: a holding cost of $h > 0$ per
unit time per 1-customer in the system including his time in service,
(this assumption will prove equivalent to assuming holding costs $h_1$
and $h_2$ for the two classes with $h_1 - h_2 = h$) and fixed charge of
R>0 incurred each time a customer is served who is not first in line. After a service is completed, the server has the right to serve either the first customer in line or any 1-customer present (including, for example, a "1" who is 7th in line with three "1"s and three "2"s ahead of him). The time horizon is assumed infinite and we seek a policy which minimizes expected average cost per unit time. We could have chosen to work with discounted cases with only minor changes in results.

States are described by the precise list of customers in the system—typically a list of 1's and 2's. Our convention is that a list of 1's and 2's from left to right represents a queue from last to first customer. For example, 11212222 represents the state in which three are eight customers in line with 1's in 5th, 7th and 8th positions. The available actions are to serve the first 2, to promote the 1 in 5th position, to promote the 1 in 7th or to promote the 1 in 8th. The empty state will be called 0. Since actions are taken at points in time when a service is completed rather than at equally spaced points in time, the problem in question is a Markov Renewal Programming Problem with countable state space and finitely many actions in each state.

The general issue of the existence of a stationary optimal policy in such problems is unsettled, particularly when one considers that expected costs between transitions are not bounded over (state, action) pairs. By considering a series of truncated
problems with finite but increasing waiting room capacities, we will show that at least an $\varepsilon$-optimal stationary policy exists for our problem. In the process of finding such a policy we will introduce some tools which will be used throughout the remainder of this paper, namely: the reduction of the average cost problem to an equivalent problem of minimizing total expected costs in the remainder of a busy cycle and the convention of tabulating the expected total excess cost above and beyond the expected total cost which would be incurred if all customers were served on a first-come-first-served basis.

Our queueing system alternates through busy and idle cycles with costs being incurred only during busy cycles. The choice of action in any particular state does not influence the distribution of the remaining length of a busy cycle. Since policy decisions affect the mixture of 1's and 2's in the queue but do not influence the total number of customers in the queue at any future time, a policy which, for every initial state, minimizes the total expected cost in the remainder of a busy cycle will thus minimize the expected average cost per unit time. In addition, it can be easily shown that because the empty state is positive recurrent under any policy, any policy which is optimal assuming that the system is empty initially is optimal from any initial state. Thus to find an optimal policy for our original problem we need only find one which minimizes the expected cost in the ensuing busy cycle assuming that the system starts empty.
Having reduced our problem to one of minimizing expected total costs in the remainder of a busy cycle we could equivalently minimize the excess of these costs over the expected costs which would be incurred if all customers were served in the normal order without any promotions. The expected cost of holding one 1-customer for one service is $h_2^{-1}$. Let us consider promoting a 1 with $n_1$ 1's and $n_2$ 2's ahead of him and the resulting contribution to the excess in expected costs over cost of service in the normal order. First, there is a charge of $R$; second, the expedited customer must wait through one service rather than $n_1+n_2+1$ and hence there is a savings in his expected total holding costs of $(n_1+n_2)h_2^{-1}$, third, each of the $n_1$ 1's ahead of the expedited customer must wait through an additional service, thus the promotion causes an additional cost of $n_1h_2^{-1}$. Summing these three contributions yields an increase in cost of $R-n_2h_2^{-1}$, a quantity which is independent of $n_1$.

Let us define a 1-busy period as the length of time between the start of service of a 1-customer and the next instant at which there is one less 1-customer in the system assuming that only 1-customers are served in the interim. We shall denote by $W_k$ the expected total excess costs over service in the normal order incurred in a 1-busy period which starts with $k$ 2's ahead of the only 1 in line. In the appendix we show that

$$W_k = -\frac{1}{2} h_2(2\mu+\lambda_1\sigma^2\mu \lambda^{-2} \lambda_1^{-1})(\mu-\lambda_1^{-3} - kh(\mu-\lambda_1)^{-1} + R\mu(\mu-\lambda_1)^{-1} \tag{1}$$
We point out that $W_k$ is linear in $k$ and that the first two terms on the right hand side of (1) represent the expected savings in holding costs over service in the normal order and the last term represents the expected sum of fixed charges for promotions.

To demonstrate the existence of an $\varepsilon$-optimal stationary policy, let us consider a sequence of problems with increasing finite waiting room capacities. We thus define the $N$-problem as one in which arrivals are not permitted into the queue if there are $N$ customers in the queue (not including the one in service). Each such problem has a finite state and action space and hence an optimal stationary policy $\pi^*_N$. We will let $\pi_N$ denote an arbitrary policy for the $N$ problem and $C_N(s, \pi_N)$ the expected total excess costs over service in the normal order incurred in the first busy cycle of the $N$-problem when policy $\pi_N$ is used and the system starts in state $s$. Associated with any policy $\pi_N$ is a whole class of policies for the original problem which we will denote by $A(\pi_N)$ and whose members are all policies which follow $\pi_N$ until the first time when there are $N$ or more customers in the system. Policies for the original problem will be denoted by $\pi$ and their expected total excess costs in the first busy cycle by $C(s, \pi)$.

We can now prove

**Lemma 1:** For any $s$ with $N$ customers present, any $\pi_N$ and any $\pi$ in $A(\pi_N)$ there are constants $c$ and $d$ such that

$$|C(s, \pi) - C_N(s, \pi_N)| \leq cN^2 + dN$$
Proof: Consider $C(s, \pi)$. $C(s, \pi)$ is less than or equal to the cost of promoting every customer for the remainder of the busy cycle. The expected number of customers served in $N$ busy periods (the remainder of the busy cycle) is $N \mu (\mu - \lambda_1 - \lambda_2)^{-1}$ and the expected fixed charges accumulated in promoting everyone are less than or equal to $NR\mu (\mu - \lambda_1 - \lambda_2)^{-1}$. Also $C(s, \pi) \leq NR\mu (\mu - \lambda_1 - \lambda_2)^{-1}$.

Now $C(s, \pi)$ is greater than or equal to the accumulated savings in holding costs associated with promoting every 1-customer, ignoring the fixed charges $R$. This is in turn greater than or equal to the savings which would be incurred if (1). $\mu^{-1}$ were saved for every 1 jumped over as well as every 2 and (2). The initial state consisted of $N$ 1's and (3). 1's arrived at a rate of $\lambda_1 + \lambda_2$ and 2's at rate 0 for the rest of the busy cycle. The expected excess costs under these assumptions can be found from (1). By substituting 0 for $R$, 0 for $\lambda_2$, $\lambda_1 + \lambda_2$ for $\lambda_1$ and $N$ for $k$ we get the expected savings while the queue size is being reduced from $N$ to $N-1$, then by substituting $N-1$ for $k$ we get the expected savings while the queue size is being reduced from $N-1$ to $N-2$ etc., and we have for the total expected savings:

$$\sum_{n=0}^{N} W_n = \sum_{n=0}^{N} -hN(\mu - \lambda_1 - \lambda_2)^{-1} = -\frac{1}{2} hN(N+1)(\mu - \lambda_1 - \lambda_2)^{-1}$$

Thus both $C(s, \pi)$ and $C_N(s, \pi_N)$ are greater than or equal to $-\frac{1}{2} hN(N+1)(\mu - \lambda_1 - \lambda_2)^{-1}$ and $|C(s, \pi) - C_N(s, \pi_N)| \leq NR(\mu - \lambda_1 - \lambda_2)^{-1}$. $\frac{1}{2} hN(N+1)(\mu - \lambda_1 - \lambda_2)^{-1} = cN^2 + dN$. Q.E.D.
Of course as \( N \to \infty \), the probability that the waiting room capacity constraint is ever binding in the first busy cycle (starting at state 0) becomes minute. It is this fact that allows us to prove

**Theorem 1**: There is an \( \varepsilon \)-optimal stationary policy for the original problem.

**Proof**: The probability that the system ever has \( N \) customers in it during the first busy cycle starting with the system empty is

\[
\frac{[\mu (\lambda_1 + \lambda_2)^{-1}] - 1}{[\mu (\lambda_1 + \lambda_2)^{-N} - 1]}.
\]

This is an application of the formula for quitting a winner in the gambler's ruin problem [4, p. 283]. Thus for any \( \pi_N \) and any \( \pi \) in \( A(\pi_N) \)

\[
|C_0(\pi) - C_0^*(\pi_N)| \leq (cN^2 + dN)[\mu (\lambda_1 + \lambda_2)^{-1}] - 1]/[\mu (\lambda_1 + \lambda_2)^{-N} - 1]
\] (2)

and the right hand side of (2) goes to 0 as \( N \) gets large.

For \( N \) sufficiently large, any stationary policy in \( A(\pi^*_N) \) comes within \( \varepsilon \) of minimizing the total expected excess costs in the first busy cycle and within \( \varepsilon / (\text{average busy cycle length}) \) of minimizing the average excess costs per unit time. Q.E.D.

We shall now seek an optimal policy which will turn out to be stationary. Recall that the excess expected costs of promoting any 1 with \( n_1 \) 1's and \( n_2 \) 2's ahead of him is \( R - n_2 \mu^{-1} \). A simple rule which has intuitive appeal is to promote a 1 when \( n_2 \) is sufficiently large that \( R - n_2 \mu^{-1} \)
is negative. This will turn out to be sound advice. An optimal policy will promote any 1-customer which has \( n_2^* \) or more 2's in front of them where \( n_2^* \) is the smallest integer satisfying \( R - n_2^* \bar{h}_2^{-1} < 0 \) and such a policy will promote the last 1 in line first. We shall prove this in three steps by first comparing the cost attainable from two initial states which differ only by having two adjacent customers in reverse order, then we use this result to show that no optimality is lost in considering promotions only from the rear of the line and finally easily find the optimal policy in this class.

Within our proofs we will use example states for clarity. We shall describe states both by symbols such as 1, s and by strings of 1's and 2's. Also an expression such as 1122212 will be used to describe future states where \( X \) stands for the random (possibly empty) stream of arrivals during a single service. Throughout the arguments below we are tabulating excess total expected costs over service in the normal order for the remainder of the busy cycle. Again we shall let \( C(s, \pi) \) be the expected excess costs using policy \( \pi \) from state \( s \). We define a 1-2 pair as a pair of adjacent customers in a queue with the customer farther from the front being a 1 and the other a 2. We can then prove

**Lemma 2:** Given any state \( s \) which contains at least one 1-2 pair, any policy \( \pi \) and state \( s' \) which is obtained from \( s \) by interchanging the members of the 1-2 pair, there is a policy \( \pi' \) such that
\[ C(s', \pi') \leq C(s, \pi) + h_{\mu}^{-1} \]

Proof: Call the 1 whose position is changed in going from \( s \) to \( s' \), Joe. From state \( s \), policy \( \pi \) (which need not be stationary) will tell you which if any 1's to promote throughout the remainder of the busy cycle. From state \( s' \) determine policy \( \pi' \) as follows: promote the same individuals (if any) at the same times as called for by policy \( \pi \) starting in \( s \) (except that one need not pay \( R \) if \( \pi \) calls for the promotion of Joe at a time when Joe is in the front of the line under \( \pi' \) from \( s' \)). Then:

1). It is possible that for part of the busy cycle Joe will have been served under \( \pi' \) from \( s' \) but not under \( \pi \) from \( s \). During this period of time the cost of promotion is lower by \( h_{\mu}^{-1} \) under \((s', \pi')\) than \((s, \pi)\). At all other times, the cost of promoting all customers other than Joe will be the same under the two policies.

2). The cost of promoting Joe will be 0 in both cases if Joe is not promoted, less by \( h_{\mu}^{-1} \) from \((s, \pi)\) than \((s', \pi')\) if Joe is promoted in both cases, less by \( h_{\mu}^{-1} - R \) from \((s, \pi)\) than \((s', \pi')\) if Joe is promoted under \((s, \pi)\) but not under \((s', \pi)\). These two findings imply that \( C(s', \pi') \leq C(s, \pi) + h_{\mu}^{-1} \), Q.E.D.

Now we can use Lemma 2 to restrict attention to policies which always choose between the alternatives "do not promote" and "promote the last 1 in line". We will make repeated use of the interchange of 1-2 pairs in the proof below.
Lemma 3: If policy \( \pi \) calls for the promotion of a 1 who is not last in line from \( s \) there is another policy \( \pi' \) which calls for the promotion of the last 1 in line and \( C(s,\pi') \leq C(s,\pi) \).

Proof: Assume that there are a 2's in front of the 1 promoted by \( \pi \) and b 2's in front of the 1 promoted by \( \pi' \) where \( b \geq a \). Let \( \pi' \) follow \( \pi \) after the first promotion. Let \( X, s(\pi) \) and \( X, s(\pi') \) denote the states which result under \( \pi \) and \( \pi' \) after exactly 1 service. \( X \) stands for the arrivals during the service; \( s(\pi) \) and \( s(\pi') \) represent state \( s \) with the 1 promoted under \( \pi \) and \( \pi' \) removed. Then

\[
C(s,\pi) = R - ah\mu^{-1} + E_X\{C((X,s(\pi)),\pi)\}
\]

\[
C(s,\pi') = R - bh\mu^{-1} + E_X\{C((X,s(\pi'),\pi)\}
\]

and subtracting yields

\[
C(s,\pi) - C(s,\pi') = (b-a)h\mu^{-1} + E_X\{C(X,s(\pi),\pi) - C(X,s(\pi'),\pi')\} \tag{3}
\]

Now for any particular arrival stream \( x \), the state \( X, s(\pi') \) can be obtained from \( x, s(\pi) \) in exactly \( b-a \) interchanges of 1-2 pairs. By repeated use of Lemma 2

\[
C((x,s(\pi)),\pi) - C((x,s(\pi')),\pi') \geq -(b-a)h\mu^{-1} \text{ for all } x \tag{4}
\]

Since (4) holds for every \( x \), the expectation on the right hand side of (3) is no less than \(-(b-a)h\mu^{-1}\) and the entire right hand side of (3) is non-negative which proves the theorem. As an illustration consider
\( s = 121221222 \) and let \( \pi \) promote the first 1. Then \( a = 3, b = 6 \), and (3) would read

\[
C(s, \pi) - C(s, \pi') = 3h_\mu^{-1} + E_X(C(X12122222, \pi) - C(X21222222, \pi'))
\]

and for any \( x \), \( x21222222 \) can be obtained from \( x12122222 \) in exactly 3 interchanges:

\[
x12122222 \rightarrow x21122222 \rightarrow x21212222 \rightarrow x21221222 \quad \text{Q.E.D.}
\]

Now since we are able to restrict attention to policies which in any state either promote the last 1 in line or nobody, we can prove:

**Theorem 2:** There is an optimal policy which promotes the last 1-customer if and only if there are \( k \) 2's in front of him where \( kh_\mu^{-1} > R \).

**Proof:** One characteristic of any policy is that the number of 2's ahead of any particular 1 will never increase over time. Thus a lower bound on the expected excess costs achievable would be the expected excess cost if all 1-customers who arrive to find \( k > R h_\mu^{-1} \) 2's ahead of them in line could be promoted before the number of 2's ahead of them is allowed to decrease. But our hypothesized policy attains this bound. Any 1-customer who has \( k > R h_\mu^{-1} \) 2's ahead of him in line when he arrives will eventually be a last 1 in line at the time of service completion with exactly \( k \) 2's ahead of him since in the interim only customers behind him in line will be served. Q.E.D.
Several modifications in assumptions can be made without radically changing the shape of the optimal policy. Other changes cause optimal actions to be a complex function of the precise state description. We will examine a few modifications in assumptions here.

First, let us allow the service time distributions to be different for the two customer classes with means $1/\mu_1$ and $1/\mu_2$ respectively where $\mu_1$ and $\mu_2$ are sufficiently high that the traffic intensity for the system as a whole is less than one (i.e. $\lambda_1 < \mu_1, \lambda_2 < \mu_2$ and $(\lambda_1 + \lambda_2)^2 < \lambda_1 \mu_1 + \lambda_2 \mu_2$). In addition let us allow holding costs $h_1$ and $h_2$ per unit time respectively for the two customer classes with $h_1 < h_2$. Again we can tabulate expected excess costs over service in the normal order and can look at the contribution to this excess of promoting a 1 with $n_1$ 1's and $n_2$ 2's ahead of him. First, there is a charge of $R$; second the expedited customer must wait through his own service rather than his own plus $n_1$ 1's and $n_2$ 2's. This yields a savings of $h_1(n_1^{1-1} + n_2^{1-1})$; third, every customer ahead of the expedited customer must wait through one more 1-service, yielding an additional cost of $\mu_1^{-1}(n_1 h_1 + n_2 h_2)$. The total contribution of these 3 effects is thus an excess expected cost of $R - n_2(h_1^{1-1} - h_2^{1-1})$. This is exactly the same expression which would result from the original model with holding cost $c(h_1^{-1} - h_2^{-1})$ and common service time distribution with mean $(c \mu_1^{-1} \mu_2)^{-1}$ where $c$ is any constant such that $\lambda_1 + \lambda_2 < c \mu_1 \mu_2$. Thus the modified model is equivalent to the original model with different $h$ and $\mu$ and an optimal policy says "promote the last 1 in line if and only if there are $k^*$ 2's ahead of him and $R - k^*(h_1^{1-1} - h_2^{1-1}) < 0".$
There is no difficulty in extending the original model into one with \( M > 2 \) customer classes each with the same service distribution but with differing holding costs \( h_1 > h_2 > \cdots > h_M \geq 0 \) in which only type 1 (most important) customers present may be promoted. Again if we tally costs in the same way and consider the expected excess costs incurred when a 1 is promoted with \( n_j \) 's, \( j = 1, 2, \cdots, M \) ahead of him we have first a cost of \( R \), second a savings of 
\[ h_1 \mu^{-1} \sum_{j=1}^{M} n_j \text{ and an additional cost of } h_2 \mu^{-1} \sum_{j=1}^{M} n_j n_j. \]
Summing these 3 contributing yields 
\[ R - h_1 \mu^{-1} \sum_{j=1}^{M} n_j \mu^{-1} \sum_{j=1}^{M} h_j n_j = R - h_1 \mu^{-1} \sum_{j=2}^{M} n_j \mu^{-1} \sum_{j=2}^{M} h_j n_j, \]
an expression which is again independent of \( n_1 \). As in the basic model, the total expected excess cost would be minimized if an arriving customer who finds \( n_j \) 's, \( j = 1, 2, \cdots, M \), ahead of him such that 
\[ R - h_1 \mu^{-1} \sum_{j=1}^{M} n_j \mu^{-1} \sum_{j=2}^{M} h_j n_j \]
is negative could be promoted at a time when the composition of the queue ahead of him is unchanged.
This is ensured by the rule "promote the last 1 in line if and only if 
\[ h_1 \mu^{-1} \sum_{j=2}^{M} n_j \mu^{-1} \sum_{j=2}^{M} h_j n_j \]
is negative". If we were to generalize this model so that any customer in line could be promoted, an optimal policy would typically be very complex.

Let us return to the basic model and consider replacing the fixed charge \( R \) by a promotion charge which is a function \( R(n_1, n_2) \) of the number of 1's and 2's jumped over. If we first assume 
\[ R(n_1, n_2) = R^1(n_2) \]
we can calculate the excess costs as before to be 
\[ R^1(n_2) - n_2 \mu^{-1} \]
when a 1 is promoted past \( n_1 \) 1's and \( n_2 \) 2's. As long as \( R^1(n_2) \) is chosen so that \( R^1(n_2) \mu^{-1} \) is non-increasing for all \( n_2 > n_2^* \) where \( n_2^* \) is the smallest positive integer which
makes the expression negative, an optimal policy promotes the last 1 if and only if there are \( n_2^* \) or more 2's ahead of him. Again, this follows from the realization that a lower bound on the expected excess cost is given by the expected excess cost if all 1's who arrive to find \( n_2 \geq n_2^* \) 2's ahead of them could be promoted upon arrival. As in the basic model, the above policy guarantees that for such 1's, the composition of the queue in front of them will be unchanged until the time of their promotion. Unfortunately, if \( R(n_1, n_2) \) is allowed to depend on \( n_1 \), a more complex optimal policy will generally result.

Although no model as simple as those that we have considered can adequately mirror a real-world situation, there is one assumption of our basic model which may be more often unrealistic then others, namely the flexibility to promote any 1-customer in line for a fee of \( R \). It might seem more appropriate to restrict one's options to not promoting anyone or promoting the first 1 in line. This restriction will be added in the next section.

II. A Modification of the Basic Model

In the basic model we showed that an optimal policy promoted any 1 for whom the savings in holding costs outweighed the fee \( R \). Now, if we add the restriction that only the first 1 may be promoted the form of an optimal policy may be changed. The effects of promoting a 1 with \( n_2 \) 2's ahead of him are again an expected excess cost of \( R - n_2 h u^{-1} \) but in addition such an act often earns one
the right to promote one or more other 1's in line. Thus it may be optimal to promote a 1-customer for whom $R-n_2h\mu^{-1}$ is positive in order to win the right to promote others further back in line. An optimal policy can clearly be a complicated function of the precise list of 1's and 2's in line. However, one might conjecture that if it is ever worth promoting a 1, it is worth continuing to promote 1's until there are no more in line. In other words, if it is worth taking a loss on the promotion of a 1 in order to win the right to promote other 1's on whom the loss is recouped, it would be senseless to stop promotions short of these gains. This conjecture is false as we shall illustrate by the counter-example below.

We shall use $\lambda_1 = 90, \lambda_2 = 1, \mu = 100, R = 1, h = 33$ and shall assume for simplicity that service times have an exponential distribution. Thus $3h\mu^{-1} = .99, 4h\mu^{-1} = 1.32$ and if we were free to promote anyone we would promote the last 1 with 4 or more 2's ahead of him. The expected excess cost of promoting a 1 with 3 2's ahead of him is only .01, however. Let us consider two states $s = 11222$ and $s' = 1222$. If our conjecture is accurate then if it is worth promoting from $s$ it must be worth promoting from $s'$ since $s'$ is the state which results if one promotes from $s$ and no arrivals occur during the first service. We shall show that promotion is the best action in $s$ but not in $s'$. The argument hinges on the fact that the probability of one or more 2-customer arrivals in the process of emptying the system of 1's from $s$ is sufficiently
large to make it worthwhile risking a few promotions on whom a loss of .01 is taken to get a chance at promotions on which a savings of .32 or more is earned. Whereas from $s'$ the expected time until the system is emptied of 1's is half as long as from $s$ and there is a much smaller probability that losses of .01 will ever be recouped.

Let us define a 1-busy period as the length of time from the start of a 1-customer service until the number of 1-customers in the system is reduced by one assuming that only 1-customers are being served. Let $W_k$ be the expected excess costs over service in the normal order accumulated during such a busy period assuming that every 1-customer is promoted and that the initial state has a single 1 behind $k$ 2's. From (1) above and the fact that an exponential random variable with mean $\mu^{-1}$ has variance $\mu^{-2}$ we get

$$W_k = -h\mu^{-1} \left[ \lambda_2 \mu^2 (\mu - \lambda_1)^{-3} + k\mu (\mu - \lambda_1)^{-1} \right] + R\mu (\mu - \lambda_1)^{-1}$$

which we notice is linear in $k$. In particular, when $k = 3$ and specific values of the other parameters are substituted as given above we get $W_3 = +.0967$.

Now let us consider $s$ and $s'$ and the 2 available actions in each state. If we take the action of serving the first 2 in line in either state there are no excess costs. Also, with the arrival rate of 2's being 1 and that of 1's being 90 the probability that there will ever be an occasion in the remainder of the busy period in which a 1 is behind 4 or more 2's is minute and the best total expected excess costs that we can hope to incur are for
all practical purposes 0. So from state s, serving the first
2 yields essentially 0 while "cleaning out the l's" yields
about +.0967 and the former action is preferable.

From state s if we "clean out the l's", the expected
excess cost during the first l-busy period is \( W_3 = +.0967 \) but
the expected excess cost during the second l-busy period is
\( E_A(W_{3+A}) \) where \( A \) is the number of 2 arrivals during the first
l-busy period. By linearity of \( W \):

\[
E_A(W_{3+A}) = W_3 - hE(A)(\mu-\lambda)^{-1}
\]

But \( E(A) = \lambda_2(\mu-\lambda_1)^{-1} \) and

\[
E_A(W_{3+A}) = W_3 - h\lambda_2(\mu-\lambda_1)^{-2} = .0967 - \frac{33 \times 1}{(100-90)^2}
\]

\[= -.2333\]

Thus the expected excess costs in "cleaning out the system of l's" from
state s is +.0967 -.2333 = -.1360. Since this is less than 0 it
is better to promote from s than not. Of course, an even smaller
expected excess cost can be attained from s if one stops promoting
if s' is reached.

We have thus looked at a series of rather simple models designed
to get a feeling for an optimal response to the touchy trade-off
between keeping important customers waiting and raising the anger of
other customers through promotions. Except in cases where one was
permitted to promote the last high priority customer in line and where
this provided the key to an optimal policy, we found that optimal policies, if they exist at all, are likely to have a complex description. In particular, we conjectured that if one were restricted to promoting the first high priority customer in line, optimal behavior would oscillate between periods in which no promotions occurred and periods in which all high priority customers in the system were promoted (one at a time). Yet this conjecture has been shown to be untrue.
Appendix

Here we derive expression (1) for $W_k$, the expected total excess costs over service in the normal order in a 1-busy period which starts with $k$ 2's in front of the only 1 in line. We shall first find $W_0$. The mean and variance of the number of 1-customers served in a 1-busy period are $\mu(\mu-\lambda_1)^{-1}$ and $(\lambda_1\mu^2+\lambda_2\mu^3)(\mu-\lambda_1)^{-3}$ respectively. Hence the expected sum of promotion fixed charges is $\rho\mu(\mu-\lambda_1)^{-1}$.

We denote the probability that exactly $\lambda$ customers are served in the 1-busy period by $P_\lambda$ and the expected number of 2-customers ahead of the $j$th 1-customer served by $N_j$. Then

$$W_0 = -\mu^{-1}\sum_{\lambda=1}^{\infty} P_\lambda \sum_{j=1}^{\lambda} N_j + \rho\mu(\mu-\lambda_1)^{-1}$$

Since $N_j$ is the expected number of 2-arrivals before the $j$-1st 1-arrival, $N_j$ is the mean of a negative binomial random variable which counts the number of failures preceding the $j$-1st success in independent Bernoulli trials with success probability $\lambda_1(\lambda_1+\lambda_2)^{-1}$. Thus $N_j = (j-1)\lambda_2\lambda_1^{-1}$ and

$$W_0 = -\mu^{-1}\lambda_2\lambda_1^{-1} \sum_{\lambda=1}^{\infty} P_\lambda \sum_{j=1}^{aw} (j-1) + \rho\mu(\mu-\lambda_1)^{-1}$$

$$= -\mu^{-1}\lambda_2\lambda_1^{-1} \sum_{\lambda=1}^{\infty} P_\lambda (\lambda-1)\lambda/2 + \rho\mu(\mu-\lambda_1)^{-1}$$

This expression depends only on the first and second moments of the distribution of the number of customers served in a 1-busy period.
Substitution and routine algebraic manipulation then yield

\[ W_0 = -\frac{1}{2} h\lambda_2 (2\mu + \lambda_1^2 \sigma^2 \mu^2 - \lambda_1) (\mu - \lambda_1)^{-3} + R\mu (\mu - \lambda_1)^{-1} \]

When \( k \) is positive, each customer served has an additional savings of \( k\mu^{-1} \); the expected total additional savings is thus \( \mu (\mu - \lambda_1)^{-1} k\mu^{-1} \) or \( kh(\mu - \lambda_1)^{-1} \). We thus have

\[ W_k = -\frac{1}{2} h\lambda_2 (2\mu + \lambda_1^2 \sigma^2 \mu^2 - \lambda_1) (\mu - \lambda_1)^3 - kh(\mu - \lambda_1)^{-1} + R\mu (\mu - \lambda_1)^{-1} \quad (1) \]
References


EFFICIENT OPERATION OF OPTIONAL PRIORITY QUEUEING SYSTEMS

An M/G/1 queueing system with two classes of customers is studied. The per unit time cost of holding a customer differs for the two classes. The server has the option of serving the first customer in line at no extra charge or paying a fee to earn the right to serve any higher class customer further back in the queue. An optimal policy is found for the simple case where this fee is independent of the position in the queue of the promoted customer.

For the case where the server is restricted to serving either the first customer or the first higher class customer further back in the queue, an appealing conjecture from of an optimal policy is shown to be invalid.
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