MAINTENANCE MODELS FOR STOCHASTICALLY FAILING EQUIPMENT

BY

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>CONTENTS</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>A ONE MACHINE MAINTENANCE MODEL</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>2.1 The Model</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>2.2 Replacement Rules</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>2.3 IFR Markov Chains</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>2.4 The Optimality of Control Limit Laws</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>2.5 A Parametric Analysis of the Minimum Long-Run Cost</td>
<td>49</td>
</tr>
<tr>
<td></td>
<td>2.6 Computation</td>
<td>67</td>
</tr>
<tr>
<td>III</td>
<td>A K MACHINE MAINTENANCE MODEL</td>
<td>73</td>
</tr>
<tr>
<td></td>
<td>3.1 The Model</td>
<td>73</td>
</tr>
<tr>
<td></td>
<td>3.2 A Factorization of the Model</td>
<td>76</td>
</tr>
<tr>
<td></td>
<td>3.3 The One Machine Model $\phi^f(i)(S_i)$</td>
<td>107</td>
</tr>
<tr>
<td></td>
<td>3.4 Conclusion</td>
<td>118</td>
</tr>
<tr>
<td>REFERENCES</td>
<td></td>
<td>120</td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

In reliability theory the concept of increasing failure rate (IFR) distributions plays a large role. If \( F(t) \) is the failure distribution of a particular machine, the failure rate is defined as

\[
\frac{F(t+x) - F(t)}{1 - F(t)}
\]

and, intuitively, is the probability that a machine of age \( t \) will fail in the next \( x \) time units given that it has not failed up to time \( t \). The failure rate of an IFR distribution is increasing in \( t \) \((t \geq 0, F(t) < 1)\) for all \( x > 0 \). Consequently a machine with an IFR failure distribution deteriorates with age and is more likely to fail the older it gets.

In 1965 Derman [6] developed an analogue of increasing failure rates for Markov chains. If \( X_t \) is a Markov chain with transition matrix \((p_{ij})_{i,j=0,...,M} \), \( X_t \) is said to be IFR if

\[
\mathbb{P}(X_{t+1} \in B | X_t = k) = \sum_{j=2}^{M} P_{kj}
\]

is non-decreasing in \( k \) for all sets \( B \) of the form
 Obviously the concept of deterioration through aging has been retained. If the states \( \{0, 1, \ldots, M\} \) represent progressively greater states of deterioration, the higher the state \( k \) at time \( t \) (i.e., the greater the deterioration), the greater the probability the entering a still higher state (i.e., a still greater state of deterioration) at time \( t+1 \).

In the case where the IFR Markov chain describes the deterioration of a piece of machinery at each inspection time \( t \), Derman was able to show that there exists a very simple optimal maintenance policy. If at each inspection there is the option of replacing the machine (returning it to state 0) at a cost \( c \) with a penalty cost \( A \) of replacing a failed machine (a machine in state \( M \)), Derman was able to prove that a control limit replacement rule is the optimal form of the maintenance policy. That is, the machine should only be replaced when it has deteriorated beyond a certain point, or more specifically, when its state is above the specified control limit.

In 1966 Kolesar [14] expanded Derman's model by generalizing the cost structure. He was able to apply operating costs, which depended upon the state of the machine, to Derman's model without altering the basic conclusion of the model. The form of the optimal policy remained intact. Khanna [12] and Kalyman [11] developed other aspects of the same model, again retaining the optimality of the control limit replacement policies.
This thesis builds upon the foundation provided by Derman and Kolesar, but develops a different aspect of the problem. Derman's original model was limited by the fact that he provided for only one mode of failure. In Chapter II a maintenance model is developed in which two modes of failure are feasible. In this new model a machine is inspected at discrete time intervals and is classified as being in any of \( L+M+1 \) states, \( (-L, -L+1, \ldots, -1, 0, 1, \ldots, M) \), where \(-L\) and \(M\) both represent a failed machine. For example, \(-L\) might represent a machine failure that could be repaired, where as \(M\) would then represent a failure that could not be repaired but called for actual replacement by a new machine. Zero \((0)\) represents a machine in perfect working order, \(\{-1, -2, \ldots, -L+1\}\) intermediate states of deterioration leading to failure at state \(-L\), and \(\{1, 2, \ldots, M-1\}\) intermediate states of deterioration leading to failure at state \(M\).

\[
\begin{array}{c}
\text{FAILURE} \leftarrow \text{NEW} \rightarrow \text{FAILURE} \\
\hline
-L \ldots -2 -1 0 1 2 \ldots M
\end{array}
\]

As in Derman's model transition between states is governed by a Markov chain and at each inspection the option of returning the machine to state \(0\) is available. The costs of repair \(C(s)\) (of returning the machine to state \(0\)) are dependent upon the state of the machine \(s\), as are the operating costs \(A_s\), the costs incurred when no repair work is done. Chapter II develops conditions on the cost functions and the transition probability matrix that allow for a characterization of the
optimal maintenance policy. If the Markov chain governing the state transitions is analogous to Derman's IFR Markov chains, the optimal maintenance policy is also a control limit policy, but with two control limits, one for the negative states and one for the positive ones. The characterization of the form of the optimal policy makes the calculation of the actual policy very easy. Chapter II develops a linear program which specifies the two control limits that define the optimal maintenance policy.

The model developed in Chapter II can also be interpreted within the framework of quality control theory. Instead of having the states of the Markov chain represent different levels of deterioration of the machine, let them represent the quality of the output of the machine. For example, if the machine is designed to drill holes of a certain diameter, the states of the Markov chain would represent the actual diameter of the holes drilled. State 0 would be a perfectly drilled hole, state 1 a slightly larger hole, state 2 an even larger one, up to state M, a hole too large to be used. Similarly states -1 through -L would represent holes progressively smaller than the required diameter. The costs A_s, formerly called the operating costs, could represent the costs of remachining the holes so that they could be used, while the repair costs C(s) could represent the costs of shutting down the machine to readjust controls governing the size of the holes drilled. As proved in Chapter II, under certain conditions on the costs and transition matrix, it is optimal to reset the machine controls only when a hole is drilled larger than a certain specified diameter or smaller than another specified diameter.
The conditions outlined in Chapter II which lead to a characterization of the optimal maintenance policies as control limit replacement rules are easily generalized to the cases where there are more than two modes of failure. These additional modes of failure enable one to model more complex systems composed of many components. For instance, suppose a system is composed of two machines. If each machine can be described by a Markov chain with transition probabilities \( \{p_{ij}^1\}_{i,j=0_1, \ldots, M_1} \) and \( \{p_{ij}^2\}_{i,j=0_2, \ldots, M_2} \) and states \( \{0_1 \ldots M_1\} \) and \( \{0_2 \ldots M_2\} \) where \( 0_1 \) and \( 0_2 \) represent new machines and \( M_1 \) and \( M_2 \) failed machines, then the system can be modeled using the techniques developed in Chapter II. The states of the system can be represented by the vector \((X_1, X_2)\) with transitions between states governed by the Markov chain

\[
(p_{x_1,x_2}^1, p_{x_2,x_2}^2)_{x_1, x_1'=0_1 \ldots M_1, x_2, x_2'=0_2 \ldots M_2}
\]
Pictorially the system is being modeled as a wheel where each spoke represents states of deterioration leading away from a new system \((0,0)\) up to failure of machine 2 at state \((X_1, M_2)\). Thus the two component system has been modeled as a system with \(M_1\) modes of failure (\(M_1\) spokes of the wheel). If the two components were connected in series so that the failure of either component would result in the failure of the whole system, the operating and replacement costs could be adjusted so that replacement of the system (return to state \((0,0)\)) would be required for all states \((X_1 M_2), X_1 < M_1\) and for the whole spoke \((M_1 X_2)\). If the two components were connected in parallel so that system failure would only occur when both components failed, then the costs could again be adjusted to require replacement at state \((M_1, M_2)\). In either case, if the required conditions were satisfied, the optimal maintenance policy would be a control limit replacement policy with a control limits corresponding to each spoke of the wheel.

In Chapter III multi component systems are developed from a different perspective. In the above example of a two component system the model led to a system with \(M_1\) modes of failure and \(M_1\) control limits. It is easy to see that a system with additional components would require many more control limits to describe its optimal maintenance policy. Therefore, from a computational point of view, it would be advantageous if it were possible to treat each component as a separate entity so that each component would have its own control limit. Unfortunately in multi component systems the interactions between the individual components frequently play a large role in determining the operating and replacement costs. For example, if economies of scale are present,
the per unit cost of replacing a component decreases with the number of components replaced.

A model is developed in Chapter III which accounts for certain of these interactions. In this model there are \( r \) cost functions reflecting the cost of replacing individual components of the system. Furthermore, there is a rule \( f \), depending upon time and the states of all of the components, which specifies which replacement cost function is assigned to which component. Sufficient conditions on the assignment rule \( f \) and the cost functions, which allow the model to be factored into many one-machine models, are developed in Chapter III. These one-machine models are then analyzed through the techniques developed in Chapter II to yield control limits for each individual component.
CHAPTER II
A ONE MACHINE MAINTENANCE MODEL

2.1. The Model.

As was previously stated in the Introduction, this chapter will investigate those conditions which guarantee the optimality of certain simple replacement rules known as control limit laws.

The model considered will be concerned with equipment that is operating continuously but is inspected at discrete time intervals \( t = 0, 1, 2, \ldots \). At each inspection the machine is classified as being in one of \( L + M + 1 \) possible states labeled by the integers \( \{-L, -L+1, \ldots, 0, \ldots, M\} \). The label 0 denotes new equipment and \( M \) and \( -L \) denote failed equipment. Let \( X_t \) be the state of the machine at the inspection at time \( t \). If \( X_t = -L \) or \( X_t = M \), that is, if the machine has failed, it may be advantageous to replace or repair the machine so that at the next inspection at time \( t+1 \), the machine will be in state 0. Thus, the assumption that machine replacement or repair takes one time period is being made.

Furthermore, it is also assumed that the infinite sequence \( \{X_t\}_{t = 0, 1, 2, \ldots} \) is a finite state Markov chain with stationary transition probabilities. Let \( \{p_{ij}\}_{i, j = -L \ldots M} \) be defined as

\[
p_{ij} = \mathbb{P}(X_t = j | X_{t-1} = i).
\]
The matrix \( [p_{ij}]_{i,j=-L\ldots M} \) governs the state transitions of the machine between inspections, and by the assumption of stationarity is independent of \( t \). The further constraint that \( [p_{ij}]_{i,j=-L\ldots M} \) be irreducible will be placed on the matrix to guarantee that the equipment will eventually fail regardless of its initial state.

At the time of each inspection there is the option of replacing the machine even if it has not yet failed. In all cases it takes one time period to replace the equipment so that at the time of the next inspection the equipment will be in state 0. Let a replacement rule \( R \) be defined by specifying those states in which the machine will be replaced. A replacement rule \( R \) modifies the Markov chain, \( (X_t | t=0, 1, 2, \ldots) \). This new chain will be denoted by \( (X_t(R) | t = 0, 1, 2, \ldots) \), and if it is a Markov chain it will be characterized by the matrix \( [p_{ij}(R)]_{i,j=-L\ldots M} \) where

\[
p_{ij}(R) = P(X_t(R) = j | X_{t-1}(R) = i)
\]

The \( p_{ij}(R) \) evolve from the natural transition probabilities \( p_{ij} \) according to the following rules:

\[
p_{ij}(R) = p_{ij} \quad \text{if the machine is not replaced in state } i \text{ under replacement rule } R
\]

\[
p_{ij}(R) = \begin{cases} 
1, & j = 0 \\
0, & j \neq 0
\end{cases} \quad \text{if the machine is replaced in state } i \text{ under replacement rule } R
\]
A maintenance policy of replacing the equipment before failure can be instituted for many reasons. This model will investigate what maintenance policies, if any, will be used when certain costs are placed upon the operation and repair of the equipment. If the equipment is in state \( X_t \) at inspection \( t \) and is not replaced, then an operating cost of \( A_{X_t} \) will be incurred for operating the equipment until the next inspection at time \( t+1 \). If the machine is replaced, a cost \( C(X_t) \) is charged to cover the cost of repair or replacement and failure to operate the machine between the inspections at time \( t \) and \( t+1 \).

In the subsequent sections of this chapter the expected long run average cost and the expected discounted total cost of different replacement rules will be compared to determine which maintenance policy minimizes these costs. More specifically, if

\[
\phi_R(X_0, \alpha) = \begin{cases} 
  C(X_t) & \text{if } R \text{ replaces the equipment in state } X_t \\
  A_{X_t} & \text{if } R \text{ does not replace the equipment in state } X_t 
\end{cases}
\]

then the expected total \( \alpha \)-discounted cost with \( X_0 \) given as the initial state of the equipment is

\[
\phi_R(X_0, \alpha) = \mathbb{E} \left\{ \sum_{t=0}^{\infty} \alpha^t \phi_R(X_t) | X_0 \right\}
\]

and the expected long run average cost is

\[
v_R(X_0) = \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \phi_R(X_t) | X_0 \right].
\]
The problem will then be to determine which maintenance policy minimizes these two quantities.

2.2. Replacement Rules.

The class of maintenance policies considered in minimizing the expected total $\alpha$-discounted cost and the expected long run average cost will be limited to non-randomized stationary replacement rules. As Ross [18], Derman [5], Wagner [19], and Blackwell [2] have all shown, such a restriction does not result in any loss of generality. In fact if $\pi^*$ represents the class of all replacement rules and $\pi'$ the class of all stationary non-randomized replacement rules, the following theorems are true.

**Theorem 2.1:** There exists $R^* \in \pi'$ such that

$$\rho^{(X_0, \alpha)}_{R^*} = \min_{R \in \pi} \rho^{(X_0, \alpha)}_R = \min_{R \in \pi'} \rho^{(X_0, \alpha)}_R.$$  

**Proof:** See Ross [18], Chapter 6.

**Assumption 1:** For all $R \in \pi'$, $(p_{ij}(R))_{i,j=-L\ldots M}$ is irreducible.

**Theorem 2.2:** If Assumption 1 holds, then there exists $R^* \in \pi'$ such that

$$v^{(X_0)}_{R^*} = \min_{R \in \pi} v^{(X_0)}_R = \min_{R \in \pi'} v^{(X_0)}_R.$$  

**Proof:** See Ross [18], Chapter 6.
A subclass of the stationary non-randomized replacement rules $\pi'$ will prove to be of great interest in the subsequent sections of this paper. Christened control limit laws by Derman [6], this class of maintenance policies is intuitively appealing and of practical importance.

**Definition:** A control limit replacement rule denoted by $R_{ij}$ is a replacement rule of the form: Replace the equipment if and only if $X_t \geq j$ or $X_t \leq i$ where $i$ and $j$, the control limits, are integers such that $-L-1 \leq i \leq 0$ and $0 \leq j \leq M+1$.

If the labeling of the states of the equipment corresponds to increasing levels of deterioration in the machine, it is intuitively appealing to replace the machine in state $i_1 \leq 0$ if replacement is also called for in state $i_2$ where $i_2 < i_1$. Furthermore, if it is known that a control limit maintenance policy $R_{ij}$ is optimal, the problem of determining the control limits is relatively simple since there are only $L+2$ possible values for $i$ and $M+2$ possible values for $j$. And as will be indicated subsequently, in the case where the transition probabilities are known, there is an efficient algorithm for computing the control limits $i$ and $j$.

2.3. **IFR Markov Chains.**

In 1963 Derman [6] developed a generalized conception of aging in a stochastic environment. If for a Markov chain $\{Y_t\}$ with states $\{0, 1, \ldots, N\}$ it is assumed that the greater the value of the state,
the greater the amount of deterioration, then this concept of aging, known as increasing failure rate (IFR), can be given an intuitively appealing interpretation. A Markov chain is said to be IFR if the greater the value of the state, the greater the probability of further deterioration.

**Definition:** A Markov chain \( \{Y_t | t = 0, 1, \ldots \} \) with states \( \{0, 1, \ldots, N\} \) is said to be IFR if

\[
P(Y_{t+1} \in B | Y_t = i)
\]

is non-decreasing in \( i, i \in \{0, 1, \ldots, N\} \), for all sets \( B \) of the form \( B = \{k, k+1, \ldots, N\} \) for any \( k = 0, 1, \ldots, N \). Or equivalently, \( \{Y_t\} \) is said to be IFR if the function

\[
r_k(i) = \sum_{j=k}^{N} q_{ij}
\]

is non-decreasing in \( i \) for all \( k, k = 0, 1, \ldots, N \) where \( \{q_{ij}\}_{i,j=0,\ldots,N} \) is the transition probability matrix of \( \{Y_t\} \).

**Definition:** A Markov chain is said to be strictly IFR if

\[
r_k(i) = \sum_{j=k}^{N} q_{ij}
\]

is strictly increasing in \( i \) for all \( k, k = 0, 1, \ldots, N \).
Lemma 2.3: 1. A Markov chain $Y_t$ with transition matrix $(q_{ij})_{i,j=0\ldots N}$ is IFR if and only if for every non-decreasing function $h(j), j = 0, \ldots, N$, the function

$$K(i) = \sum_{j=0}^{N} q_{ij} h(j), \quad i = 0, \ldots, N$$

is also non-decreasing.

2. A Markov chain $Y_t$ with transition matrix $(q_{ij})_{i,j=0\ldots N}$ is strictly IFR if and only if for every non-decreasing non-constant function $h(j), j = 0, 1, \ldots, N$, the function

$$K(i) = \sum_{j=0}^{N} q_{ij} h(j)$$

is strictly increasing.

Proof: (Kolesar [14]). The proof of 1. will be omitted, being similar to the proof of 2.

Assume that $K(i) = \sum_{j=0}^{N} q_{ij} h(j)$ is strictly increasing for all non-decreasing non-constant functions $h(j)$. For each $k = 0, 1, \ldots, N$ let $h_k(j)$ be defined as

$$h_k(j) = \begin{cases} 0 & \text{if } j < k, \\ 1 & \text{if } j \geq k. \end{cases}$$

Then by assumption
\[ K(i) = \sum_{j=0}^{N} q_{ij} h_k(j) = \sum_{j=k}^{N} q_{ij} = r_k(i) \]

is strictly increasing in \( i \). Since \( r_k(i) \) is strictly increasing for each \( k \), \( \{ Y_t \mid t = 0, 1, \ldots \} \) is IFR.

Now assume that \( Y_t \) with transition matrix \( \{ q_{ij} \}_{i,j=0 \cdots N} \) is strictly IFR. Let \( h(j), j = 0, \ldots, N \), be a non-decreasing non-constant function. Choose \( \beta \geq 0 \) large enough so that \( \beta + h(j) \geq 0 \) for all \( j, j = 0 \cdots N \). Then there exist constants \( c_i \geq 0, i = 0, \ldots, N \), not all \( c_i = 0 \), such that

\[
\beta + h(j) = \sum_{i=0}^{N} c_i h_i(j)
\]

where

\[
 h_i(j) = \begin{cases} 
 0 & \text{if } i > j \\
 1 & \text{if } i \leq j.
\end{cases}
\]

Then

\[
\beta + K(i) = \beta + \sum_{j=0}^{N} q_{ij} h(j) = \sum_{j=0}^{N} q_{ij} [\beta + h(j)] \\
= \sum_{j=0}^{N} \sum_{k=0}^{N} q_{ij} c_k h_k(j) = \sum_{k=0}^{N} c_k [\sum_{j=0}^{N} q_{ij} h_k(j)] \\
= \sum_{k=0}^{N} c_k \sum_{j=k}^{N} q_{ij} = \sum_{k=0}^{N} c_k r_k(i).
\]

15
Since $Y_t$ is IFR, $r_k(i)$ is strictly increasing for all $k, k = 0 \cdots N$, and therefore, $\beta + K(i)$, the positive sum of strictly increasing functions, is also strictly increasing. Therefore, $K(i) = \sum_{j=0}^{N} q_{ij} h(j)$ is strictly increasing for all non-decreasing non-constant functions $h(j)$. QED

The model to be considered in this paper permits deterioration in two directions. That is, from the new state 0, the machine may deteriorate towards failure in the state M or towards failure in the state -L. For this reason simple IFR Markov chains will not be sufficient for the model.

The matrix $\{p_{ij}\}_{i,j=-L \cdots M}$ will be divided into four submatrices,

I. $\{p_{ij}\}_{i=0 \cdots M, j=0 \cdots M}$

II. $\{p_{ij}\}_{i=0 \cdots M, j=0 \cdots -L}$

III. $\{p_{ij}\}_{i=0 \cdots -L, j=0 \cdots M}$

IV. $\{p_{ij}\}_{i=0 \cdots -L, j=0 \cdots -L}$

Since deterioration increases from 0 up to -L and M, it will be assumed that the states are ordered from 0 up to M and from 0 up to -L. In other words, the ordering of the states is determined by their distance from the state 0, that is, $0 < 1 < 2 < \cdots < M-1 < M$ and $0 < -1 < -2 < \cdots < -L+1 < -L$. Then it will be assumed that each submatrix I, II, III and IV is IFR. More specifically conditions A and B will be of use in the next section.
**Condition A:** Let \( \{X_t \mid t = 0, 1, \ldots \} \) be a Markov chain with transition matrix \( [p_{ij}]_{i,j=-L \ldots M} \) satisfying the following conditions:

1. \( r^1_k(i) = \sum_{j=k}^{M} p_{ij} \) is non-decreasing in \( i, i = 0 \cdots M \), for all \( k, k = 0 \cdots M \)

2. \( r^2_k(i) = \sum_{j=k}^{-L} p_{ij} \) is non-decreasing in \( i, i = 0 \cdots M \), for all \( k, k = 0 \cdots -L \)

3. \( r^1_k(i) = \sum_{j=k}^{M} p_{ij} \) is non-decreasing in \( i, i = 0 \cdots -L \), for all \( k, k = 0 \cdots M \)

4. \( r^2_k(i) = \sum_{j=k}^{-L} p_{ij} \) is non-decreasing in \( i, i = 0 \cdots -L \), for all \( k, k = 0 \cdots -L \).

**Condition B:** Let \( \{X_t \mid t = 0, 1, \ldots \} \) be a Markov chain with transition matrix \( [p_{ij}]_{i,j=-L \ldots M} \) satisfying the following conditions:

1. \( r^1_k(i) = \sum_{j=k}^{M} p_{ij} \) is strictly increasing in \( i, i = 0 \cdots M \), for all \( k, k = 0 \cdots M \)

2. \( r^2_k(i) = \sum_{j=k}^{-L} p_{ij} \) is strictly increasing in \( i, i = 0 \cdots M \), for all \( k, k = 0 \cdots -L \)

3. \( r^1_k(i) = \sum_{j=k}^{M} p_{ij} \) is strictly increasing in \( i, i = 0 \cdots -L \), for all \( k, k = 0 \cdots M \)

4. \( r^2_k(i) = \sum_{j=k}^{-L} p_{ij} \) is strictly increasing in \( i, i = 0 \cdots -L \), for all \( k, k = 0 \cdots -L \).
Pictorially Conditions A and B can be illustrated by representing the transition matrix \( \{p_{ij}\}_{i,j=-L \ldots M} \) by the diagram below where the arrows represent directions of increasing (non-decreasing) values of \( r_k^1(i) \) and \( r_k^2(i) \).

\[ \begin{array}{cccc}
- L & k & 0 & k & M \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
- L & IV & III & \downarrow & \downarrow \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & r_k^2(i) & r_k^1(i) & 0 & r_k^2(i) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
M & II & I & M & II \\
\end{array} \]
Definition: A function $f(j), j = -L \cdots M$, is said to be unimodal with minimum at $j = 0$ if

\[ \Delta f(j) \geq 0 \quad \text{for} \quad j = 1 \cdots M \]

\[ \Delta f(j) \leq 0 \quad \text{for} \quad j = 0 \cdots -L+1 \]

where $\Delta f(j) = f(j) - f(j-1)$.

The graph below is an example of a unimodal function with minimum at $j = 0$. Notice that unimodal functions with minima at $j = 0$ are non-decreasing as $j$ travels from $0$ to $M$ and from $0$ to $-L$.

\[ \text{Graph} \]

Definition: A function $f(j), j = -L \cdots M$, is said to be strictly unimodal with minimum at $j = 0$ if

\[ \Delta f(j) > 0 \quad \text{for} \quad j = 1 \cdots M \]

\[ \Delta f(j) < 0 \quad \text{for} \quad j = 0 \cdots -L+1 \]
Lemma 2.4: 1. If Condition A holds, then $r^1_k(i)$ is unimodal with minimum at $i = 0$ for all $k = 0 \cdots M$ and $r^2_k(i)$ is unimodal with minimum at $i = 0$ for all $k = 0 \cdots -L$.

2. If Condition B holds, then $r^1_k(i)$ is strictly unimodal with minimum at $i = 0$ for all $k = 0 \cdots M$ and $r^2_k(i)$ is strictly unimodal with minimum at $i = 0$ for all $k = 0 \cdots -L$.

Proof: By definition of Condition A, $r^1_k(i)$ is non-decreasing in $i$ for $i = 0 \cdots M$ for all $k = 0 \cdots M$. Thus,

$$\Delta r^1_k(i) = r^1_k(i) - r^1_k(i-1) \geq 0.$$ 

Also $r^1_k(i)$ is non-decreasing in $i$, $i = 0 \cdots -L$, for $k = 0 \cdots M$.

Note, though, that for $i = 0 \cdots -L$ the states are ordered so that $0 < -1 < -2 < \cdots < -L$ and, consequently, non-decreasing means that $r^1_k(0) \leq r^1_k(-1) \leq r^1_k(-2) \leq \cdots \leq r^1_k(-L)$. Therefore, $\Delta r^1_k(i) = r^1_k(i) - r^1_k(i-1) \leq 0$.

Obviously since $\Delta r^1_k(i) \geq 0$ for $i = 0 \cdots M$ and $\Delta r^1_k(i) \leq 0$ for $i = 0 \cdots -L$, $r^1_k(i)$ has minimum at $i = 0$. Therefore $r^1_k(i)$ is unimodal with minimum at $i = 0$. The unimodality of $r^2_k(i)$ follows in exactly the same way. Also Condition B implies strict unimodality in an analogous way.

QED

Lemma 2.5: Let $f(j), j = -L \cdots M$, be a unimodal function with minimum at $j = 0$. 

20
1. If Condition A holds, then

\[ K(i) = \sum_{j=-L}^{M} p_{ij} f(j), \quad i = -L \cdots M \]

is unimodal with minimum at \( i = 0 \).

2. If Condition B holds and \( f(j) \) is non-constant, then

\[ K(i) = \sum_{j=-L}^{M} p_{ij} f(j), \quad i = -L \cdots M \]

is strictly unimodal with minimum at \( i = 0 \).

**Proof:** If Condition A holds, then the four submatrices of \( \{p_{ij}\}_{i,j=-L}^{M} \) are all IFR. That is

I. \( \{p_{ij}\}_{i=0}^{M}, j=0 \cdots M \)

II. \( \{p_{ij}\}_{i=0}^{M}, j=0 \cdots -L \)

III. \( \{p_{ij}\}_{i=-L}^{0}, j=0 \cdots M \)

IV. \( \{p_{ij}\}_{i=-L}^{0}, j=0 \cdots -L \)

are all IFR where the states \( \{0, \ldots, -L\} \) are ordered so that \( 0 < -1 < -2 < \cdots < -L \).

Since \( f(j) \) is a unimodal function with minimum at \( j = 0 \), \( f(j) \) is non-decreasing for \( j = 0 \cdots M \) and non-decreasing for
j = 0 \ldots -L \) (with the assumed ordering). Therefore by Lemma 2.3, since matrix \( I \) is IFR, \( \sum_{j=0}^{M} p_{ij} f(j) \) is non-decreasing in \( i \) for \( i = 0 \ldots M \).

Since matrix III is IFR, \( \sum_{j=0}^{M} p_{ij} f(j) \) is also non-decreasing in \( i \) for \( i = 0 \ldots -L \) (with the assumed ordering). Therefore \( \sum_{j=0}^{M} p_{ij} f(j) \) is unimodal with minimum at \( i = 0 \).

Similarly by Lemma 2.3, since matrices II and IV are IFR,

\[
\sum_{j=-L}^{-1} p_{ij} f(j) \text{ is unimodal with minimum at } i = 0.
\]

The sum of two unimodal functions with minimum at \( i = 0 \) is also unimodal with minimum at \( i = 0 \). Therefore

\[
K(i) = \sum_{j=0}^{M} p_{ij} f(j) + \sum_{j=-L}^{-1} p_{ij} f(j)
\]

is unimodal with minimum at \( i = 0 \).

The proof of 2. is similar and will be omitted. QED

2.4. The Optimality of Control Limit Laws:

Control limit laws as defined in a previous section are intuitively appealing, but are not optimal in all cases. Certain restrictions must be placed on the transition probabilities governing deterioration and on the operating and replacement costs. This section will investigate those conditions which, when applied to the long run average cost and the \( \alpha \)-discounted cost models, result in the optimality of control limit maintenance policies.
The expected long run average cost function

\[ v_R(X_0) = \lim_{T \to \infty} E \left( \frac{1}{T} \sum_{t=0}^{T-1} g_R(X_t) | X_0 \right) \]

where

\[ g_R(X_t) = \begin{cases} C(X_t) & \text{if } R \text{ replaces the machine in state } X_t \\ A_X & \text{if } R \text{ does not replace the machine in state } X_t \end{cases} \]

will not be treated directly, but as the limit of the expected \( \alpha \)-discounted total cost

\[ \varrho_R(X_0, \alpha) = E \left( \sum_{t=0}^{\infty} \alpha^t g_R(X_t) | X_0 \right) \quad 0 < \alpha < 1. \]

If we denote by \( R^*_\alpha \) that replacement policy which minimizes the \( \alpha \)-discounted cost, then by Theorem 2.1

\[ \varrho(X_0, \alpha) = \varrho_{R^*_\alpha}(X_0, \alpha) = \min_{R \in \pi'} \varrho_R(X_0, \alpha) \]

where \( \varrho(X_0, \alpha) \) is the minimum \( \alpha \)-discounted cost and \( \pi' \) the class of all stationary non-randomized replacement rules.

A dynamic programming approach will be employed in searching for \( R^*_\alpha \). Therefore, \( \varrho(X_0, \alpha) \) must satisfy the following functional equation:

\[ \varrho(X_0, \alpha) = \min \{ A_X + \alpha \sum_{j=-L}^{M} p_X j \varrho(j, \alpha), C(X_0) + \alpha \varrho(0, \alpha) \} \]

23
Let
\[
\delta(X_0, \alpha, N) = \min_{R \in \pi} \mathbb{E} \left[ \sum_{t=0}^{N} \alpha_t g_R(X_t) | X_0 \right].
\]

This N-stage problem will be used to investigate the \(\infty\)-stage \(\alpha\)-discounted problem.

**Lemma 2.6**: Let \(A(j)\) and \(B(j)\), \(j = -L \cdots M\), be unimodal functions with minima at \(j = 0\). Then
\[
C(j) = \min\{A(j), B(j)\}
\]
is also unimodal with minimum at \(j = 0\).

**Proof.** Since both \(A(j)\) and \(B(j)\) are minimized at \(j = 0\), \(C(j)\) is also minimized at \(j = 0\).

\[
\Delta C(j) = C(j) - C(j-1) = \min\{A(j), B(j)\} - \min\{A(j-1), B(j-1)\}
\]
\[
= \min\{A(j) - \min\{A(j-1), B(j-1)\}, B(j) - \min\{A(j-1), B(j-1)\}\}
\]
\[
\geq \min\{A(j) - A(j-1), B(j) - B(j-1)\} \geq 0 \quad \text{for } j = 1 \cdots M.
\]

The above inequality follows from the assumption that \(\Delta A(j) = A(j) - A(j-1) \geq 0\) and \(\Delta B(j) = B(j) - B(j-1) \geq 0\) for \(j = 1 \cdots M\),
\[ \Delta C(j) = \min\{A(j), B(j)\} - \min\{A(j-1), B(j-1)\} \]
\[= \max\{\min\{A(j), B(j)\} - A(j-1), \min\{A(j), B(j)\} - B(j-1)\} \]
\[\leq \max\{A(j) - A(j-1), B(j) - B(j-1)\} \leq 0 \quad \text{for } j = 0 \cdots -L+1. \]

Again the final inequality follows from the assumption of unimodality.

Therefore \( \Delta C(j) \geq 0 \) for \( j = 1 \cdots M \) and \( \Delta C(j) \leq 0 \) for \( j = -1 \cdots -L+1 \).

Consequently, \( C(j) \) is unimodal with minimum at \( j = 0 \). QED

**Lemma 2.7:** If \( f(j), g(j) \) and \( d(j), j = -L \cdots M \) are all unimodal functions with minima at \( j = 0 \) such that \( |\Delta f(j)| \geq |\Delta g(j)| \) where \( \Delta f(j) = f(j) - f(j-1) \), then \( g(j) + d \), where \( d \) is a constant, crosses \( f(j) + d(j) \) at most twice, from below if \( j \leq 0 \) and from above if \( j \geq 0 \).

**Proof:** For \( j \leq 0 \), \( \Delta f(j) \leq 0 \), \( \Delta g(j) \leq 0 \) and \( \Delta d(j) \leq 0 \) since all three functions are unimodal. Therefore,

\[ \Delta f(j) \leq \Delta g(j) \quad \text{for } j \leq 0. \]

Consequently

\[ \Delta[(f(j) + d(j)) - (g(j) + d)] \]
\[= [\Delta f(j) - \Delta g(j)] + \Delta d(j) \leq 0 \quad \text{for } j \leq 0 \]
since \( \Delta d(j) \leq 0 \) and \( \Delta f(j) \leq \Delta g(j) \). Thus the function \((f(j) + d(j)) - (g(j) + d)\) is a non-increasing function and can cross zero at most once, and if so from positive to negative.

\[
f(j) + d(j) - g(j) - d
\]

Therefore \( g(j) + d \) crosses \( f(j) + d(j) \) at most once for \( j \leq 0 \) and if so from below.

For \( j \geq 0 \) the proof is exactly the same once it is noted that \( \Delta f(j) \geq \Delta g(j) \) and \( \Delta d(j) \geq 0 \) for \( j \geq 0 \). \( \text{QED} \)

**Theorem 2.8:** Assume that \( A_X \) and \( C(X), X = -L \cdots M \), are both unimodal with minimum at \( X = 0 \) and that \( |\Delta C(X)| \leq |\Delta A_X| \). Then if Condition A holds, for each \( N \geq 0 \) there exists control limits \( i_N \) and \( j_N \) such that the control limit law \( \rho_{i_N, j_N} \) minimizes the \( N \)-stage \( \alpha \)-discounted problem. That is,

\[
\rho(X_0, \alpha, N) = \rho_{i_N, j_N} (X_0, \alpha, N) = \min_{R \in \pi'} E \left\{ \sum_{t=0}^{N} \alpha^t G_R(X_t | X_0) \right\} .
\]

Furthermore, \( \rho(X_0, \alpha, N) \) is unimodal with minimum at \( X_0 = 0 \).
Proof: The proof will be by induction. First consider the 0-stage problem

\[ \phi(x_0, \alpha, 0) = \min \{ A_{x_0}, C(x_0) \} \]

By Lemma 2.7 with \( d(j) = 0 \), \( C(X) \) crosses \( A_X \) at most twice, from below if \( X \leq 0 \) and from above if \( X \geq 0 \). Therefore, let \( i_0 \) be the first (closest to 0) \( X, X \leq 0 \), such that \( A_X \geq C(X) \) and let \( i_0 = -L-1 \) if \( A_X < C(X) \) for all \( X = 0 \cdots -L \). Let \( j_0 \) be the first \( X \geq 0 \) such that \( A_X \geq C(X) \) and let \( j_0 = M+1 \) if \( A_X < C(X) \) for all \( X = 0 \cdots M \).

Then

\[ \phi(x_0, \alpha, 0) = \begin{cases} A_{x_0} & \text{if } i_0 \leq x_0 < j_0 \\
C(x_0) & \text{if } x_0 \leq i_0 \text{ or } x_0 \geq j_0 \end{cases} \]
Therefore, it is optimal to replace the equipment if $X_0 \geq j_0$ or $X_0 \leq i_0$ and, consequently, the control limit replacement rule $R_{i_0, j_0}$ is optimal in the 0-stage problem.

Furthermore, since $A_X$ and $C(X)$ are unimodal with minima at $X = 0$, Lemma 2.6 implies that

$$\hat{\varphi}(X_0, \alpha, 0) = \min \{ A_X, C(X_0) \} \quad \text{for} \quad X_0 = -L \ldots M$$

is also unimodal with minimum at $X_0 = 0$.

Thus the theorem is true for the 0-stage model.

Assume, as an inductive hypothesis, that control limits $i_j$ and $j_j$ exist such that $R_{i_j, j_j}$ minimizes the N-stage $\alpha$-discounted cost and that the minimum cost $\hat{\varphi}(X_j, \alpha, N)$ is unimodal in $X$ with minimum at $X = 0$. Furthermore,

$$\hat{\varphi}(X_0, \alpha, N+1) = \min \{ A_X + \alpha \sum_{j=-L}^{M} p_{X,j} \hat{\varphi}(j, \alpha, N), C(X_0) + \alpha \hat{\varphi}(0, \alpha, N) \}.$$ 

By assumption $\hat{\varphi}(j, \alpha, N)$ is unimodal with minimum at $j = 0$. Therefore, by Lemma 2.5, $\alpha \sum_{j=-L}^{M} p_{X,j} \hat{\varphi}(j, \alpha, N)$ is unimodal in $X$ with minimum at $X = 0$. Consequently since $|\Delta X| \geq |\Delta C(X)|$, Lemma 2.7 guarantees that $C(X) + \alpha \hat{\varphi}(0, \alpha, N)$ crosses $A_X + \alpha \sum_{j=-L}^{M} p_{X,j} \hat{\varphi}(j, \alpha, N)$ at most twice, from below if $X \leq 0$ and from above if $X \geq 0$. Let $i_{N+1}$ be the first $X \leq 0$ such that

$$A_X + \alpha \sum_{j=-L}^{M} p_{X,j} \hat{\varphi}(j, \alpha, N) \geq C(X) + \alpha \hat{\varphi}(0, \alpha, N)$$
and let $i_{N+1} = -L-1$ if

$$A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j,\alpha,N) < C(X) + \alpha \phi(0,\alpha,N) \quad \text{for all } X = 0 \cdots L.$$ 

Let $j_{N+1}$ be the first $X \geq 0$ such that

$$A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j,\alpha,N) \geq C(X) + \alpha \phi(0,\alpha,N)$$

and let $j_{N+1} = M+1$ if

$$A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j,\alpha,N) < C(X) + \alpha \phi(0,\alpha,N)$$

for all $X = 0 \cdots M$. Therefore, it is optimal to replace the equipment if $X_0 \geq j_{N+1}$ or $X_0 \leq i_{N+1}$ and, consequently, the control limit law $R_{i_{N+1},j_{N+1}}$ is optimal for the $N$-stage problem.
Since $A_X$, $C(X)$, $\alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N)$ and $\phi(0, \alpha, N)$ are all unimodal with minima at $X = 0$, $A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N)$ and $C(X) + \alpha \phi(0, \alpha, N)$ are both unimodal with minima at $X = 0$. Then Lemma 2.6 implies that

$$\phi(X_0, \alpha, N+1) = \min \{ A_{X_0} + \alpha \sum_{j=-L}^{M} P_{X_0j} \phi(j, \alpha, N), C(X_0) + \alpha \phi(0, \alpha, N) \}$$

is unimodal with minimum at $X_0 = 0$.

Thus it has been shown by induction that for all $N \geq 0$ there exist control limits $i_N$ and $j_N$ such that the control limit law $R_{i_Nj_N}$ minimizes the $\alpha$-discounted $N$-stage problem and that the minimum cost $\phi(X_0, \alpha, N)$ is unimodal with minimum at $X_0 = 0$. QED

**Corollary 2.9:** Assume that $A_X$ is unimodal with minimum at $X = 0$ and that $C(X) \equiv C$ is constant. Then if Condition A holds, for each $N \geq 0$ there exist control limits $i_N$ and $j_N$ such that the control limit maintenance policy $R_{i_Nj_N}$ minimizes the $N$-stage $\alpha$-discounted cost.

**Proof.** Note that $|\triangle C(X)| = |\Delta C| = 0$. Consequently $|\Delta A_X| \geq |\Delta C(X)|$ and Theorem 2.8 is now applicable. QED

**Corollary 2.10** (Kolesar [14]). If the occupancy costs $A_X$, $X = 0 \cdots M$, are a non-decreasing function, if the replacement costs are constant, i.e., $C(X) \equiv C$ for $X = 0 \cdots M$, and if $\{p_{ij}\}_{i,j=0 \cdots M}$ is IFR, then
for all $N \geq 0$ there exist control limits $j^*_N$ such that the control limit maintenance policy $R^*_N$ (replace the machine if and only if $X_0 \geq j^*_N$) is optimal.

**Proof:** When $-L = 0$ the possible states are limited to $\{0 \cdots M\}$. Then the unimodality of $A_X^*$ is equivalent to $A_X$ being non-decreasing. Furthermore, Condition A reduces to $\{p_{ij}\}_{i,j=0\cdots M}$ being IFR. Now apply Corollary 2.9. QED

**Theorem 2.11:** Assume that $A_X$ and $C(X)$ are both unimodal with minima at $X = 0$, that $|\Delta C(X)| \leq |\Delta A_X|$, and that $A_X = C(X)$ for no more than one $X \leq 0$ and one $X \geq 0$. Then if Condition B holds, for each $N \geq 0$ the optimal replacement rule for the $N$-stage $\alpha$-discounted problem must be a control limit law.

**Proof:**

$$\phi(X_0, \alpha, 0) = \min\{A_{X_0}, C(X_0)\}.$$  

By Theorem 2.8, there exists an optimal control limit law for the 0-stage problem. Suppose there is a non-control limit law that is also optimal. Then either there exist states $X$ and $Y$ such that $-L \geq X > Y \geq 0$ and

$$C(X) \geq A_X \quad \text{where a replacement is not made}$$

$$C(Y) \leq A_Y \quad \text{where a replacement is made}$$

or there exist states $\bar{X}$ and $\bar{Y}$ such that $0 \leq \bar{Y} < \bar{X} \leq M$ and
\[ C(\bar{X}) \geq A_X \quad \text{where a replacement is not made} \]

\[ C(\bar{Y}) \leq A_Y \quad \text{where a replacement is made} . \]

In the first case, since \( A_X \geq A_Y \), either \( C(X) - C(Y) > A_X - A_Y \), or \( C(X) = A_X \) and \( C(Y) = A_Y \). The assumption that \( |\triangle C(S)| \leq |\triangle A_S| \) contradicts the first of these possibilities since

\[ C(X) - C(Y) = C(X) - C(X+1) + C(X+1) - C(X+2) + C(X+2) \]

\[ + \cdots + C(Y-1) - C(Y) \]

\[ = \sum_{j=X}^{X} |\triangle C(j)| \leq \sum_{j=Y}^{X} |\triangle A_j| \]

\[ = A_X - A_Y . \]
The second possibility contradicts the assumption that $A_S = C(S)$ for no more than one $S \leq 0$. Thus both possibilities are actually impossible and, therefore, there exists no $X$ and $Y$, $-L \geq X > Y > 0$ such that the machine is replaced at $Y$ but not at $X$. The second case of $\bar{X}$ and $\bar{Y}$ follows analogously. Consequently no optimal non-control limit laws can exist for the 0-stage problem.

The assumption that $A_S = C(S)$ for no more than one $S \leq 0$ and one $S \geq 0$ guarantees that $\phi(X, \alpha, 0)$ is not a constant function of $X$. By Theorem 2.8, $\phi(X, \alpha, 0)$ is unimodal with minimum at $X = 0$.

Therefore if Condition B holds, Lemma 2.5 implies that $\alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, 0)$ is strictly unimodal with minimum at $X = 0$. It can then be easily shown by an inductive argument that $\phi(X, \alpha, N)$ is not a constant function and, consequently, that $\alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N)$ is strictly unimodal with minimum at $X = 0$.

Now assume, for the $N$-stage problem, that there exists a non-control limit replacement rule that is optimal. Then either there exist states $X$ and $Y$ such that $-L \geq X > Y > 0$ and

$$C(X) + \alpha \phi(0, \alpha, N) \geq A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N)$$

where a replacement is not made

$$C(Y) + \alpha \phi(0, \alpha, N) \leq A_Y + \alpha \sum_{j=-L}^{M} P_{Yj} \phi(j, \alpha, N)$$

where a replacement is made

or there exist states $\bar{X}$ and $\bar{Y}$ such that $M \geq \bar{X} > \bar{Y} > 0$ and
\[
C(\tilde{X}) + \alpha \phi(0, \alpha, N) \geq A_{\tilde{X}} + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N)
\]

where a replacement is not made.

\[
C(\tilde{Y}) + \alpha \phi(0, \alpha, N) \leq A_{\tilde{Y}} + \alpha \sum_{j=-L}^{M} P_{Yj} \phi(j, \alpha, N)
\]

where a replacement is made.

Since \( \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N) \) is unimodal in \( X \) with minimum at \( X = 0 \), it follows that

\[
A_{\tilde{X}} + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N) \geq A_{\tilde{Y}} + \alpha \sum_{j=-L}^{M} P_{Yj} \phi(j, \alpha, N).
\]

Therefore, in the first case, either

\[
[C(\tilde{X}) + \alpha \phi(0, \alpha, N)] - [C(\tilde{Y}) + \alpha \phi(0, \alpha, N)]
\]

\[
> [A_{\tilde{X}} + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N)] - [A_{\tilde{Y}} + \alpha \sum_{j=-L}^{M} P_{Yj} \phi(j, \alpha, N)]
\]
or

\[
C(\tilde{X}) + \alpha \phi(0, \alpha, N) = A_{\tilde{X}} + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N)
\]

and

\[
C(\tilde{Y}) + \alpha \phi(0, \alpha, N) = A_{\tilde{Y}} + \alpha \sum_{j=-L}^{M} P_{Yj} \phi(j, \alpha, N).
\]

Since \( \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N) - \sum_{j=-L}^{M} P_{Yj} \phi(j, \alpha, N) \geq 0 \), the first possibility again implies that \( C(\tilde{X}) - C(\tilde{Y}) \geq A_{\tilde{X}} - A_{\tilde{Y}} \), contradicting the assumption that \( |\Delta C(S)| \leq |\Delta A_S| \). Assume that the second possibility is true. Since \( \sum_{j=-L}^{M} P_{Sj} \phi(j, \alpha, N) \) is strictly unimodal,
\[
\sum_{j=-L}^{M} p_{Xj} \phi(j, \alpha, N) > \sum_{j=-L}^{M} p_{Yj} \phi(j, \alpha, N)
\]

Therefore,

\[
C(X) + \alpha \phi(0, \alpha, N) = A_X + \alpha \sum_{j=-L}^{M} p_{Xj} \phi(j, \alpha, N)
\]

\[
C(Y) + \alpha \phi(0, \alpha, N) = A_Y + \alpha \sum_{j=-L}^{M} p_{Yj} \phi(j, \alpha, N)
\]

implies that \( C(X) - C(Y) > A_X - A_Y \), again contradicting the assumption that \( |\Delta C(S)| \leq |\Delta A_S| \). Thus the second possibility is also impossible and, consequently, there exist no points \( X \) and \( Y \). It can be similarly shown that no points \( \bar{X} \) and \( \bar{Y} \) exist. Therefore, no non-control limit law can be optimal for the N-stage problem. QED

**Corollary 2.12 (Kolesar [14]).** If the occupancy costs \( A_X, X = 0 \cdots M, \) are non-decreasing and the replacement costs \( C(X) \equiv C \) are constant and if \( \{p_{ij}\}_{i,j=0 \cdots M} \) is strictly IFR, then for \( N \geq 0 \) only control limit laws are optimal for the N-stage \( \alpha \)-discounted model.

**Theorem 2.13:** Assume that \( A_X \) and \( C(X), X = -L \cdots M, \) are both unimodal with minima at \( X = 0 \) and that \( |\Delta C(X)| \leq |\Delta A_X| \). Then if Condition A holds, there exist control limits \( i_{\alpha} \) and \( j_{\alpha} \) such that the control limit law \( R_{i_{\alpha} j_{\alpha}} \) minimizes the \( \omega \)-stage \( \alpha \)-discounted problem. That is,

\[
\phi(X_0, \alpha) = \phi_{R_{i_{\alpha} j_{\alpha}}}(X_0, \alpha) = \min_{R \in \pi} E \left\{ \sum_{t=0}^{\infty} \alpha^t e_R(X_t) | X_0 \right\}.
\]
Furthermore \( \varphi(X_0, \alpha) \) is unimodal with minimum at \( X_0 = 0 \).

**Proof:** Theorem 2.9 states that for all \( N \geq 0 \), \( \varphi(X_0, \alpha, N) \) is unimodal with minimum at \( X_0 = 0 \). It is known that \( \varphi(X_0, \alpha) = \lim_{N \to \infty} \varphi(X_0, \alpha, N) \).

Then, since \( \varphi(X_0, \alpha) \) is a limit of a sequence of unimodal functions each with minimum at \( X_0 = 0 \), it itself is a unimodal function with minimum at \( X_0 = 0 \).

Furthermore, by the principles of dynamic programming, \( \varphi(X, \alpha) \) must satisfy the functional equation

\[
\varphi(X_0, \alpha) = \min \{ A_{X_0} + \alpha \sum_{j=-L}^{M} P_{X_0j} \varphi(j, \alpha); C(X_0) + \alpha \varphi(0, \alpha) \}.
\]

Since \( \varphi(j, \alpha) \) is unimodal with minimum at \( j = 0 \), Lemma 2.5 implies that \( \alpha \sum_{j=-L}^{M} P_{Xj} \varphi(j, \alpha) \) is also unimodal with minimum at \( X = 0 \).

Therefore, as a consequence of Lemma 2.7, \( C(X) + \alpha \varphi(0, \alpha) \) crosses \( A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \varphi(j, \alpha) \) at most twice, from below if \( X \leq 0 \) and from above if \( X \geq 0 \). Let \( i_\alpha \) be the first (closest to 0), \( X \leq 0 \) such that

\[
A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \varphi(j, \alpha) \geq C(X) + \alpha \varphi(0, \alpha)
\]

and let \( i_\alpha = -L-1 \) if

\[
A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \varphi(j, \alpha) < C(X) + \alpha \varphi(0, \alpha)
\]
for all $X = 0 \cdots -L$. Let $j_{\alpha}$ be the first $X \geq 0$ such that

$$A_X + \alpha \sum_{j=-L}^{M} P_X j \phi(j, \alpha) \geq C(X) + \alpha \phi(0, \alpha)$$

and let $j_{\alpha} = M+1$ if

$$A_X + \alpha \sum_{j=-L}^{M} P_X j \phi(j, \alpha) < C(X) + \alpha \phi(0, \alpha)$$

for all $X = 0 \cdots M$. Then

$$\phi(X_0, \alpha) = \min \{ A_{X_0} + \alpha \sum_{j=-L}^{M} P_{X_0} j \phi(j, \alpha), C(X_0) + \alpha \phi(0, \alpha) \}$$

$$= \begin{cases} 
A_{X_0} + \alpha \sum_{j=-L}^{M} P_{X_0} j \phi(j, \alpha) & \text{if } i_{\alpha} < X_0 < j_{\alpha} \\
C(X_0) + \alpha \phi(0, \alpha) & \text{if } X_0 \leq i_{\alpha} \text{ or } X_0 \geq j_{\alpha}.
\end{cases}$$

Therefore, it is optimal to replace the equipment if $X_0 \geq j_{\alpha}$ or $X_0 \leq i_{\alpha}$ and, consequently, the control limit maintenance policy $R_{i_{\alpha} \cdots j_{\alpha}}$ is optimal.

**Corollary 2.14** (Kolesar [14]). If the occupancy costs $A_X, X = 0 \cdots M$, are non-decreasing and the replacement costs $C(X) \equiv C$ are constant and if $(P_{ij})_{i,j=0 \cdots M}$ is IFR, then there exists a control limit $j_{\alpha}$ such that the control limit law $R_{j_{\alpha}}$, replace if and only if $X \geq j_{\alpha}$, is optimal for the $\infty$-stage $\alpha$-discounted model.
Theorem 2.15: Assume that $A_X$ and $C(X)$ are both unimodal with minima at $X = 0$, that $|\triangle C(X)| \leq |\triangle A_X|$, and that $A_X = C(X)$ for not more than one $X \leq 0$ and one $X \geq 0$. Then if Condition B holds, only control limit laws are optimal for the $\infty$-stage $\alpha$-discounted model.

Proof: The proof is the same as that of Theorem 2.11.

Corollary 2.16: (Kolesar [14]). If the occupancy costs $A_X$, $X = 0 \cdots M$ are non-decreasing and the replacement costs $C(X) = C$ are constant and if $\{p_{ij}\}_{i,j=0\cdots M}$ is strictly IFR, then only control limit rules $R_j^\alpha$ are optimal for the $\infty$-stage $\alpha$-discounted problem.

Theorem 2.17: Assume that $A_X$ and $C(X)$, $X = -L \cdots M$, are unimodal with minimum at $X = 0$ and that $|\triangle C(X)| \leq |\triangle A_X|$. Then if Condition A and Assumption 1 hold, there exist control limits $i^*$ and $j^*$ such that the control limit law $R_{i^*, j^*}$ minimizes the expected long run average cost. That is

$$v(X_0) = v_{R_{i^*, j^*}}(X_0)$$

$$= \min_{R \in \Pi} \left\{ \lim_{T \to \infty} E\left[ \frac{1}{T} \sum_{t=0}^{T-1} g_R(X_t^t | X_0) \right] \right\}.$$ 

Proof: Theorem 2.15 implies that for each $\alpha$, $0 < \alpha < 1$, there exists a control limit law $R_{i^*, j^*}^{\alpha}$ which minimizes the $\infty$-stage $\alpha$-discounted cost problem. That is, $\phi(X_0, \alpha) = \phi_{R_{i^*, j^*}^{\alpha}}(X_0, \alpha)$. Choose a sequence of $\phi_{R_{i^*, j^*}^{\alpha}}$. 38
\(\alpha\)'s, \([\alpha_k]_{k=0}^{\infty}\), such that \(\lim_{k \to \infty} \alpha_k = 1\) and such that for each \(\alpha_k\) in the sequence the same control limit law \(R_{i^*, j^*}\) is optimal. It is possible to construct such a sequence since the number of control limit rules is finite. Consequently, at least one rule must be repeated infinitely often in any sequence of \(\alpha\)'s.

Let \(R^*\) denote the control limit law \(R_{i^*, j^*}\) and let \(R\) denote any replacement rule. Then since \(R^*\) is optimal for each \(\alpha_k\)

\[
\rho_{R^*}(x_0, \alpha_k) \leq \rho_R(x_0, \alpha_k)
\]

Then we have

\[
(1-\alpha_k) \rho_{R^*}(x_0, \alpha_k) \leq (1-\alpha_k) \rho_R(x_0, \alpha)
\]

It is a known fact that

\[
v_R(x_0) = \lim_{\alpha \to 1} (1-\alpha) \rho_R(x_0, \alpha)
\]

Therefore, taking limits

\[
V_{R^*}(x_0) = \lim_{k \to \infty} (1-\alpha_k) \rho_{R^*}(x_0, \alpha) \leq \lim_{k \to \infty} (1-\alpha_k) \rho_R(x_0, \alpha) = v_R(x_0)
\]

Consequently, the control limit law \(R^* = R_{i^*, j^*}\) minimizes the expected long run average cost.

QED
Corollary 2.18: (Kolesar [14]). Assume that the occupancy costs \( A_x \), \( x = 0 \cdots M \), are non-decreasing and that the replacement costs \( C(x) = C \) are constant. Then if \( \{p_{ij}\}_{i,j=0 \cdots M} \) is IFR and if Assumption 1 holds, there exists a control limit \( j^* \) such that the control limit replacement policy \( R_{j^*} \) minimizes the expected long-run average cost.

In order to prove that only control limit rules are optimal for the long run average cost problem, it will be necessary to review some of the basic results of Markov Decision Theory.

Lemma 2.19: (Blackwell [2]). There exists a replacement rule \( R^* \in \pi' \) and a constant \( \alpha^* \), \( 0 \leq \alpha^* < 1 \) such that for all \( x_0 \)

\[
\phi_{R^*}(x_0, \alpha^*) = \min_{R \in \pi'} \phi_R(x_0, \alpha)
\]

for all \( \alpha \in (\alpha^*, 1) \) where

\[
\phi_R(x_0, \alpha) = E\left\{ \sum_{t=0}^{\infty} \alpha^t g_R(x_t) | x_0 \right\}.
\]

Lemma 2.20: (Blackwell [2]). For any rule \( R \in \pi' \) let \( \mu(i, \alpha) \) be the solution of

\[
(2.4.1) \quad \mu(i, \alpha) = g_R(i) + \alpha \sum_{j=-L}^{M} p_{ij}(R) \mu(j, \alpha), \quad i = -L \cdots M
\]

where \( g_R(i) \) is the cost of taking action \( R \) when in state \( i \). Then replacement rule \( R \) is optimal for given \( \alpha \) if
\[ T_i(\alpha, R) = T_i = g_R(i) + \alpha \sum_{j=-L}^{M} p_{ij}(R) \mu(j, \alpha) - g_{R'}(i) \]

\[ - \alpha \sum_{j=-L}^{M} p_{ij}(R') \mu(j, \alpha) \leq 0 \quad \text{for} \ i = -L \cdots M \]

where \( R' \) is the complementary rule of \( R \). That is, if \( R \) replaces the machine in state \( i \), \( R' \) does not and visa versa. Furthermore, \( R \) is the only optimal rule for a given \( \alpha \) if \( T_i(\alpha, R) < 0 \) for \( i = -L \cdots M \).

**Lemma 2.21**: (Blackwell [2]). For \( R \in \pi' \) let \([g, v_i]\) be the solution of

\[ (2.4.2) \quad g + v_i = g_R(i) + \sum_{j=-L}^{M} p_{ij}(R) v_j, \quad i = -L \cdots M \]

\[ v_0 = 0 \]

Then \( R \) is long run average cost optimal \( (v_R(X_0) = \min_{R \in \pi'} \lim_{T \to \infty} \mathbb{E}[\frac{1}{T} \sum_{t=0}^{T-1} s_R(X_t) | X_0]) \) if

\[ T_i(R) = T_i = g_R(i) + \sum_{j=-L}^{M} p_{ij}(R) v_j - g_{R'}(i) \]

\[ - \sum_{j=-L}^{M} p_{ij}(R') v_j \leq 0 \]

for \( i = -L \cdots M \) where \( R' \) is the complementary rule of \( R \). Furthermore, \( R \) is the only long run average cost optimal rule if \( T_i < 0 \) for \( i = -L \cdots M \).
Lemma 2.22: (Blackwell [2]). Let \( R \in \pi' \) and let \( R' \) be its complementary rule. Let \( \{g, v_j\} \) satisfy equation (2.4.2) and \( \{\mu(i, \alpha)\} \) satisfy equation (2.4.1). Then

1. \( \mu(i, \alpha) = \frac{g}{1-\alpha} + v_i + \varepsilon_R(i, \alpha), \; i = -L \cdots M \) where \( \varepsilon_R(i, \alpha) \to 0 \) as \( \alpha \to 1 \).

2. Let \( G(R) \) be the set of all rules \( r \in \pi' \) such that the following inequality

\[
g_r(i) + \sum_{j=-L}^{M} p_{ij}(r) v_j \leq g_R(i) + \sum_{j=-L}^{M} p_{ij}(R) v_j
\]

holds for all \( i = -L \cdots M \) with a strict inequality for at least one \( i \). If \( G(R) \) is empty then \( R \) is average cost optimal.

Lemma 2.22: (Howard [10], pp. 43). Let \( R, S \in \pi' \) and let Assumption 1 hold. Then the difference in the expected long run average costs under policies \( R \) and \( S \) is

\[
\Delta g = g^R - g^S = \sum_{j=-L}^{M} \pi_j(R) \theta_j(R, S)
\]

where

\[
\theta_j(R, S) = g_R(i) + \sum_{j=-L}^{M} p_{ij}(R) v_j(S) - g_S(i) - \sum_{j=-L}^{M} p_{ij}(S) v_j(S)
\]

where \( \{g^R, v_j(R)\} \) and \( \{g^S, v_j(S)\} \) satisfy equation (2.4.2) and \( \{\pi_j(R)\} \) are the steady state probabilities of the stochastic matrix \( \{p_{ij}(R)\}, i, j = -L \cdots M \).
Lemma 2.24: Assume that $A_X$ and $C(X)$ are unimodal with minimum at $x = 0$ such that $|\Delta C(X)| \leq |\Delta A_X|$ and such that $A_X$ and $C(X)$ are not identical and constant. Assume that Condition A holds. Let $R^* \in \pi'$ be any replacement rule such that there exists $\alpha^*, 0 \leq \alpha^* < 1$, where $\rho_{R^*}(X_0, \alpha^*) = \min_{R \in \pi} \rho_R(X_0, \alpha)$ for all $\alpha \in (\alpha^*, 1)$. Then $\{g_R^{R*}, v_i^{R*}\}$, the solution of equations (2.4.2) under rule $R^*$, is non-constant and unimodal in $i, i = -L \cdots M$, with minimum at $i = 0$.

Proof: The principles of dynamic programming imply that $\rho_{R^*}(X_0, \alpha)$ satisfies equations (2.4.1). Therefore, as a consequence of Lemma 2.22,

$$\rho_{R^*}(i, \alpha) = \frac{g_{R^*}}{1-\alpha} + v_i^{R^*} + \epsilon_{R^*}(i, \alpha), \quad i = -L \cdots M.$$ 

Thus

$$\Delta v_i^{R^*} = v_i^{R^*} - v_{i-1}^{R^*} = \Delta \rho_{R^*}(i, \alpha) - \Delta \epsilon_{R^*}(i, \alpha).$$

By Theorem 2.13, $\rho_{R^*}(i, \alpha)$ is unimodal with minimum at $i = 0$ for $\alpha \in (\alpha^*, 1)$. Therefore, $\Delta \rho_{R^*}(i, \alpha) \geq 0$ for $i = 1 \cdots M$ and $\Delta \rho_{R^*}(i, \alpha) \leq 0$ for $i = 0 \cdots -L+1$. Assume that there exists $i \geq 0$ such that $\Delta v_i^{R^*} < 0$. Since $\Delta \epsilon_{R^*}(i, \alpha) \to 0$ as $\alpha \to 1$, it is possible to choose $\alpha \in (\alpha^*, 1)$ large enough so that $|\Delta v_i^{R^*}| > \Delta \epsilon_{R^*}(i, \alpha)$. But then

$$\Delta \rho_{R^*}(i, \alpha) = \Delta v_i^{R^*} + \Delta \epsilon_{R^*}(i, \alpha) < 0.$$

43
which contradicts the fact that $\Delta \phi_{\star}(i, \alpha) \geq 0$ for $i = 1 \ldots M$.

Hence $\Delta v_i(R^\star) \geq 0$ for $i = 1 \ldots M$. Similarly, it is easily shown
that $\Delta v_i(R^\star) \leq 0$ for $i = 0 \ldots -L+1$. Therefore $v_i(R^\star)$ is unimodal
in $i$, $i = -L \ldots M$, with a minimum at $i = 0$.

Now assume that $v_i(R^\star)$ is constant for $i = -L \ldots M$. Equation
(2.4.2) then reduce to $g = g_{R^\star}(i)$ or, equivalently

$$g = A_i \quad \text{when } R^\star \text{ does not replace in state } i,$$

$$g = C(i) \quad \text{when } R^\star \text{ does replace in state } i.$$

But this is impossible unless $A_i = C(i) = \text{constant}$, which contradicts
one of the assumptions of the lemma. Hence $v_i(R^\star)$ is non-constant.

QED

**Theorem 2.25:** Assume that $A_X$ and $C(X)$ are unimodal with minima at $X = 0$ such that $A_X = C(X)$ for not more than one $X \leq 0$ and one $X \geq 0$.

Furthermore assume that $|\Delta C(X)| \leq |\Delta A_X|$ and that Condition B and Assumption 1 hold. Then only control limit replacement rules are optimal for the expected long run average cost problem.

**Proof:** Lemma 2.19 proves that there exists an $\alpha^\star$, $0 \leq \alpha^\star < 1$, and an $R^\star \in \pi^\star$ such that

$$\phi_{R^\star}(X_0, \alpha) = \min_{R \in \pi^\star} \phi(X_0, \alpha) \quad \text{for all } \alpha \in (\alpha^\star, 1).$$

But then $R^\star$ must be a control limit law (Theorem 2.15). Assume that the control limits are $i^\star$ and $j^\star$, $-L \leq i^\star \leq 0$, $0 \leq j^\star \leq M+1$. Then
the solution to equation (2.4.2) under rule $R^*$, $(v_i(R^*))_{i=-L}^M$ is non-constant and unimodal with minimum at $i = 0$ (Lemma 2.24). Therefore, as a consequence of Lemma 2.5, $\sum_{j=-L}^M p_{ij} v_i(R^*)$ is strictly unimodal in $i$ with minimum at $i = 0$. Obviously $R^*$ is also optimal for the long run average cost problem (see Proof of Theorem 2.17). Therefore, by Lemma 2.21

$$T_i \equiv T_i(R^*) \leq 0$$

for $i = -L \cdots M$

where

$$T_i = g_{R^*}(i) + \sum_{j=-L}^M p_{ij}(R^*) v_j(R^*) - g_{R^*}(i) - \sum_{j=-L}^M p_{ij}(R^{*'}) v_j(R^*).$$

$R^*$ replaces the machinery if $i \leq i^*$ or $i \geq j^*$ and does not replace it otherwise. Therefore

$$p_{ij}(R^*) = p_{ij} \quad \text{if } i^* < i < j^*$$

$$p_{ij}(R^{*'}) = \begin{cases} 
1, & j = 0 \\
0, & j \neq 0
\end{cases} \quad \text{if } i \leq i^* \text{ or } i \geq j^*$$

$R^{*'}$ is the complement of $R^*$ and, therefore, replaces the machinery for $i^* < i < j^*$ and does not replace otherwise. Consequently

$$T_i = \begin{cases} 
C(i) - A_i - \sum_{j=-L}^M p_{ij} v_j(R^*) \quad \text{if } i \leq i^* \text{ or } i \geq j^* \\
A_i - C(i) + \sum_{j=-L}^M p_{ij} v_j(R^*) \quad \text{if } i^* < i < j^*.
\end{cases}$$
Since $A_X$ and $C(X)$ are both unimodal with minima at $X = 0$ and since $|\Delta C(X)| \leq |\Delta A_X|$, $A_i - C(i)$ is also unimodal with minimum at $i = 0$. Therefore since $\sum_{j=-L}^{M} p_{ij} v_j(R^*)$ is strictly unimodal, $A_i - C(i) + \sum_{j=-L}^{M} p_{ij} v_j(R^*)$ is also strictly unimodal with minimum at $i = 0$. Consequently $T_i$ is a strictly increasing function of $i$ for $0 < i < j^*$ and $-L \leq i \leq i^*$ and a strictly decreasing function of $i$ for $i^* < i \leq 0$ and $j^* \leq i \leq M$. Since $T_i \leq 0$, if $0 < j^* \leq M$

$$0 \geq T_{j^*} > T_{j^*+1} > T_{j^*+2} > \cdots > T_{M-1} > T_M$$

(2.4.3) $$0 \geq T_{j^*-1} > T_{j^*-2} > T_{j^*-3} > \cdots > T_1 > T_0$$

$$-T_{j^*} > T_{j^*-1}$$

and if $-L \leq i^* < 0$

$$0 \geq T_{i^*-1} > T_{i^*-2} > T_{i^*-3} > \cdots > T_1 > T_0$$

(2.4.4) $$0 \geq T_{i^*} > T_{i^*-1} > T_{i^*-2} > \cdots > T_{-L+1} > T_{-L}$$

$$-T_{i^*} > T_{i^*-1}$$

Obviously $T_{j^*} \neq T_{j^*-1}$ and $T_{i^*} \neq T_{i^*-1}$.

For each set of equations, (2.4.3) and (2.4.4), there are three possible ways in which the inequalities can hold. For inequalities (2.4.3) one of the following must be true.
I. $T_{j^*} = 0, \quad T_i < 0$ for all $i, 0 \leq i \leq M, i \neq j^*$

II. $T_{j^*-1} = 0, \quad T_i < 0$ for all $i, 0 \leq i \leq M, i \neq j^*-1$

III. $T_i < 0$, for all $i, 0 \leq i \leq M$.

For inequalities (2.4.4) one of the following must also be true.

I. $T_{i^*} = 0, \quad T_i < 0$ for all $i, -L \leq i \leq 0, i \neq i^*$

II. $T_{i^*+1} = 0, \quad T_i < 0$ for all $i, -L \leq i \leq 0, i \neq i^*+1$

III. $T_i < 0$, for all $i, -L \leq i \leq 0$.

Now assume that there exists a non-control limit replacement rule $R$ which minimizes the long run average cost. Lemma 2.23 gives the difference in the expected average costs of the two policies, $R$ and $R^*$, as

$$\Delta g = g^R - g^{R^*} = \sum_{j=-L}^{M} \pi_j(R) \Theta_j(R,R^*)$$

where

$$\Theta_j(R,R^*) = g^R(i) + \sum_{j=-L}^{M} p_{ij}(R) v_j(R^*) - g^{R^*}(i) - \sum_{j=-L}^{M} p_{ij}(R^*) v_j(R^*)$$

and where $\pi_i(R)$ are the steady state probabilities for the stochastic matrix $(p_{ij}(R))_{i,j=-L\cdots M}$. Now Assumption 1 implies that $\pi_i(R) > 0$ for all $i = -L \cdots M$.

Consider Case I for $j^*$.

$$\Theta_i(R,R^*) = 0 \quad \text{if } R(i) = R(i^*), \text{ that is, if } R \text{ and } R^* \text{ both replace the machine in state } i \text{ or both do not replace the machine in state } i.$$
\[ \theta_i(R, R^*) = 0 \quad \text{if } i = j^*, \text{ since, by Assumption, } T_{j^*} = 0, \]

\[ \theta_i(R, R^*) > 0 \quad \text{otherwise, since } T_i < 0. \]

Therefore, unless \( i = j^* \), \( R(i) \neq R^*(i) \) implies that \( \Delta g > 0 \) and, therefore, \( R \) is not average cost optimal. Hence, consider the case where \( R(i) = R^*(i) \) for \( i \neq j^* \). If \( R(j^*) \neq R^*(j^*) \), then \( R \) does not replace the machine at \( j^* \), but is identical to \( R^* \) for all \( i \neq j^* \). But in this case \( R \) is also a control limit rule, but with control limit \( j^*-1 \). Thus under Case I, if \( R \) is not a control limit replacement rule, \( R \) is not optimal.

Now consider Case II for \( j^* \).

\[ \theta_i(R, R^*) = 0 \quad \text{if } R(i) = R^*(i). \]

\[ \theta_i(R, R^*) = 0 \quad \text{if } i = j^*-1, \text{ since } T_{j^*-1} = 0. \]

\[ \theta_i(R, R^*) > 0 \quad \text{if } R(i) \neq R^*(i), i \neq j^*-1, \text{ since } T_i < 0. \]

Since \( \pi_i(R) > 0 \) for all \( i \), if \( R(i) \neq R^*(i) \) for some \( i, i \neq j^*-1 \), \( R \) is not average cost optimal. On the other hand, if \( R \) is identical to \( R^* \), except in state \( j^*-1 \) where \( R \) replaces the machine, then \( \Delta g = 0 \) and \( R \) is optimal for the average cost problem. But in this case, \( R \) is again a control limit law, with control limit \( j^*-1 \).

In Case III,

\[ \theta_i(R, R^*) = 0 \quad \text{if } R(i) = R^*(i), \]

\[ \theta_i(R, R^*) > 0 \quad \text{otherwise.} \]
Thus \( \Delta g > 0 \) unless \( R(i) = R^*(i) \) for all \( i = -L \cdots M \). Therefore, it \( R \neq R^* \), \( R \) can not be average cost optimal.

The proof is exactly the same for the three cases for the control limit \( i^* \) where \( 0 > i^* \geq -L \).

Now consider the case where \( j^* = 0 \) or \( j^* = M+1 \). If \( j^* = 0 \), then

\[
0 > T_1 > T_2 > T_3 > \cdots > T_M .
\]

If \( j^* = M+1 \), then

\[
0 > T_M > T_{M-1} > T_{M-2} > \cdots > T_0 .
\]

For \( j^* = 0 \) there are two possibilities:

1. \( T_1 = 0, T_i < 0 \) for \( i = 2, \ldots, M \),

2. \( T_i < 0 \) for \( i = 1, \ldots, M \).

The proof that a non-control limit replacement rule can not be optimal under these two cases is the same as above.

Similarly, it can be shown that a non-control limit replacement rule can not be optimal for \( i^* = 0 \) or \( i^* = -L-1 \).

QED

2.5. A Parametric Analysis of the Minimum Long Run Average Cost.

In the previous section the expected long run average cost

\[
v_R(X_0) = \lim_{T \to \infty} E\left( \frac{1}{T} \sum_{t=0}^{T-1} g_R(X_t) | X_0 \right)
\]
where

\[ g_R(X_t) = \begin{cases} 
  C(X_t) & \text{if rule } R \text{ replaces the equipment in state } X_t \\
  A_{X_t} & \text{if rule } R \text{ does not replace the equipment in state } X_t
\end{cases} \]

was shown to be minimized by a control limit replacement policy under certain conditions. This section will explore how these control limit maintenance policies vary as the operating and replacement costs change.

In minimizing \( v_R(X_0) \) in Section 2.4 consideration was limited to the class of stationary non-randomized replacement rules. Theorem 2.2 was used to justify limiting \( R \) to such a small class. But in order to carry out the analysis of this section stationary randomized replacement rules must also be allowed.

Let \( \{D_t(R) \mid t = 0, 1, \ldots\} \) denote the sequence of decisions under rule \( R \) where

\[ D_t(R) = \begin{cases} 
  0 & \text{if the machine is not replaced at time } t \\
  1 & \text{if the machine is replaced at time } t
\end{cases} \]

Then in order to characterize a stationary randomized replacement rule, it is sufficient to specify

\[ d_{ij} = P(D_t = j \mid X_t = i) \quad i = -L, \ldots, M; j = 0, 1; t = 0, 1, \ldots \]
Any stationary randomized replacement rule \( R \) generates a Markov chain \( \{X_t(R) | t = 0, 1, 2, \ldots\} \) with transition probabilities \( \{p_{ij}(R)\}_{i,j=-L \ldots M} \). From these transition probabilities, the \( t \)-step transition probabilities \( \{p_{ij}^t(R)\}_{i,j=-L \ldots M} \) can be calculated using the Chapman-Kolmogorov equations. Since Assumption 1 is assumed to hold, \( \{X_t(R)\} \) has a single ergodic class and, therefore, the Caesaro limit

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} p_{ij}^t(R) = \pi_j(R)
\]

is independent of \( i \) and can be interpreted as the steady state frequency of state \( j \) under replacement rule \( R \).

It is a well known fact (Chung [3], pp. 87) that for a Markov chain with a single ergodic class the expected long run average cost for any replacement rule \( R \) can be written as

\[
(2.5.1) \quad v_R = \lim_{T \to \infty} E\left( \frac{1}{T} \sum_{t=0}^{T-1} g_R(X_t) \right) = \sum_{j=-L}^{M} \pi_j(R) \tilde{g}_R(j)
\]

where the \( \pi_j(R) \)'s are the steady state frequencies which satisfy

\[
(2.5.2) \quad \pi_j(R) = \sum_{i=-L}^{M} p_{ij} \pi_i(R), \quad j = -L \ldots M
\]

\[
(2.5.3) \quad 1 = \sum_{j=-L}^{M} \pi_j(R)
\]

\[
(2.5.4) \quad 0 \leq \pi_j(R) \leq M
\]

51
and where \( \bar{g}_R(j) \) denotes the expected cost of being in state \( j \) if rule \( R \) is used. This fact allowed Dantzig and Wolfe \([4]\), Derman \([5]\), and Manne \([16]\) to formulate the problem of determining the replacement rule that minimizes the expected long run average cost as a linear program.

Let \( x_{ij}^R = d_{ij}(R) \pi_i^R(R) \). In using rule \( R \) characterized by \( d_{ij}(R) \), the expected cost of being in state \( j \) is

\[
\bar{g}_R(j) = \sum_{k=0}^{1} d_{jk} g_k(j)
\]

where

\[
g_k(j) = \begin{cases} 
C(j) & \text{if } k = 1 \quad \text{(replacement)} \\
A_j & \text{if } k = 0 \quad \text{(no replacement)}.
\end{cases}
\]

Therefore

\[
v_R = \sum_{j=-L}^{M} \pi_j^R(R) \bar{g}_R(j) = \sum_{j=-L}^{M} \pi_j^R(R) \sum_{k=0}^{1} d_{jk} g_k(j)
\]

\[
= \sum_{j=-L}^{M} \sum_{k=0}^{1} x_{jk}^R g_k(j) = \sum_{j=-L}^{M} x_{j0}^R A_j + \sum_{j=-L}^{M} x_{j1}^R C(j).
\]

Furthermore, under the randomized replacement rule characterized by \( d_{ik}(R) \),

\[
(2.5.5) \quad p_{ij}(R) = \sum_{k=0}^{1} d_{ik}(R) p_{ij}(k)
\]

52
where \( p_{ij}(k) = \mathcal{P}(X_{t+1} = j | X_t = i; k) \) is the probability of going from state \( i \) to state \( j \) given that action \( k \) is taken. In the model treated in this section, if \( k = 1 \) and the machine is replaced, then \( X_{t+1} = 0 \) with probability 1. Therefore,

\[
p_{ij}(1) = \begin{cases} 
1 & \text{if } j = 0 \\
0 & \text{if } j \neq 0
\end{cases}
\]

And if \( k = 0 \) and the machine is not replaced, the natural transition probabilities still apply. Therefore \( p_{ij}(0) = p_{ij} \). Consequently equation

\[(2.5.2) \quad \pi_j(R) = \sum_{i = -L}^{M} \pi_i(R) p_{ij}(R)\]

reduces to

\[
\pi_j(R) = \sum_{i = -L}^{M} \sum_{k = 0}^{1} \pi_i(R) d_{ik}(R) p_{ij}(k) = \sum_{i = -L}^{M} \sum_{k = 0}^{1} x_{ik}^{R} p_{ij}(k).
\]

But since \( \sum_{k = 0}^{1} d_{ik}(R) = 1 \), \( \sum_{k = 0}^{1} x_{jk}^{R} = \pi_j(R) \) and equation (2.5.2) becomes

\[
\sum_{k = 0}^{1} x_{jk}^{R} - \sum_{k = 0}^{1} \sum_{i = -L}^{M} x_{ik}^{R} p_{ij}(k) = 0, \quad j = -L \ldots M.
\]

Substituting for \( p_{ij}(k) \), yields

\[(2.5.6) \quad x_{j0} + x_{j1} - \sum_{i = -L}^{M} x_{i0}^{R} p_{ij} = 0, \quad j \neq 0\]

\[(2.5.7) \quad x_{00} + x_{01} - \sum_{i = -L}^{M} x_{i0}^{R} p_{i0} - \sum_{i = -L}^{M} x_{i1}^{R} p_{i1} = 0.\]
Similarly equations (2.5.3) and (2.5.4) reduce to

\[
\sum_{j=-L}^{M} \sum_{k=0}^{1} x_{ik}^R = 1
\]  

(2.5.8)

\[
x_{ik}^R \geq 0 \quad i = -L \cdots M, \; k = 0, 1.
\]  

(2.5.9)

If we drop the superscript \( R \) on \( x_{ij}^R \), the problem of determining the optimal long run average cost is equivalent to solving the linear program:

\[
\text{MIN} \quad \sum_{j=-L}^{M} x_{j0} A_j + \sum_{j=-L}^{M} x_{jl} C(j)
\]

\text{SUBJECT TO}

\[
x_{00} + x_{01} - \sum_{j=-L}^{M} x_{j0} p_{j0} - \sum_{j=-L}^{M} x_{j1} = 0
\]

\[
x_{i0} + x_{i1} - \sum_{j=-L}^{M} x_{j0} p_{ji} = 0 \quad i = -L \cdots -1, 1 \cdots M
\]

\[
\sum_{k=0}^{1} \sum_{j=-L}^{M} x_{jk} = 1
\]  

(2.5.10)

\[
x_{jk} \geq 0 \quad j = -L \cdots M, \; k = 0, 1
\]

The solution of this linear program will consist of values \( x_{jk} \), from which it will be possible to determine \( d_{jk} \) and the optimal decision rule

\[
\pi_j = x_{j0} + x_{j1}
\]

\[
d_{jk} = \frac{x_{jk}}{\pi_j} = \frac{x_{jk}}{x_{j0} + x_{j1}} \quad \text{if} \; \pi_j \neq 0
\]
\[ d_{jk} \text{ is arbitrary if } \pi_j = 0. \]

The linear program (2.5.10) can be simplified further. Adding equations (2.5.6) gives

\[
\sum_{i=-L}^{M} (x_{i0} + x_{i1}) - \sum_{i=-L}^{M} \sum_{j=-L}^{M} x_{j0} p_{j1} = 0
\]

which reduces to

\[
1 - x_{00} - x_{01} - \sum_{j=-L}^{M} x_{j0}(1-p_{j0}) = 0
\]
or

\[
1 - \sum_{j=-L}^{M} x_{j0} + \sum_{j=-L}^{M} x_{j0} p_{j0} = x_{00} + x_{01}
\]
or

\[
\sum_{j=-L}^{M} x_{j1} + \sum_{j=-L}^{M} x_{j0} p_{j0} = x_{00} + x_{01}
\]

Thus equation (2.5.7) is redundant and can be dropped from the linear program.

Theorem 2.17 specifies sufficient conditions for control limit laws to minimize the expected long run average cost. In carrying out a parametric analysis of the solution, it is necessary to vary the parameters in such a way that the sufficient conditions are always satisfied.
Lemma 2.26: Let

\[ A_j = a_j A^+ \quad \text{for } j = 1 \cdots M \]
\[ A_j = a_j A^- \quad \text{for } j = -1 \cdots -L \]
\[ A_0 = a_0 A \]

where \( a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_M \), \( a_0 \leq a_{-1} \leq a_{-2} \leq \cdots \leq a_{-L} \), \( A^+ \geq 0 \), 
\( A^- \geq 0 \) and

\[ A = \max\{A^+, A^-\} \quad \text{if } a_0 < 0 \]
\[ A = \min\{A^+, A^-\} \quad \text{if } a_0 \geq 0 . \]

Then \( A_j, j = -L \cdots M \), is a unimodal function with minimum at \( j = 0 \) for all values of \( A^+ \geq 0 \) and \( A^- \geq 0 \).

Proof: Obviously

\[ a_1 \leq a_2 \leq \cdots \leq a_M \]

and

\[ a_{-1} \leq a_{-2} \leq \cdots \leq a_{-L} \]

since \( a_1 \leq a_2 \leq \cdots \leq a_M \) and \( a_{-1} \leq a_{-2} \leq \cdots \leq a_{-L} \). If \( a_0 < 0 \), then

\[ a_0 A = a_0 \max\{A^+, A^-\} = \min\{a_0 A^+, a_0 A^-\} \leq \min\{a_1 A^+, a_{-1} A^-\} \]
\[ = \min\{A_1, A_{-1}\} . \]
Therefore $A_0 \leq A_1$ and $A_0 \leq A_{-1}$. If $a_0 \geq 0$, then

$$a_0A = a_0 \min(A^+, A^-) = \min(a_0A^+, a_0A^-) \leq \min(a_1A^+, a_{-1}A^-) = \min(A_1, A_{-1}).$$

Therefore $A_0 \leq A_1$ and $A_0 \leq A_{-1}$. Consequently, $A_j$, $j = -L \cdots M$ is a unimodal function with minimum at $j = 0$. QED

Let $C(X) = c$ and let $A_x$ be parametrized as in Lemma 2.26:

- $A_j = a_jA^+$ if $j = 1 \cdots M$
- $A_j = a_jA^-$ if $j = -1 \cdots -L$
- $A_0 = a_0A$

Then $A_x$ and $C(X)$ are both unimodal with minimum at $X = 0$ and $|\Delta A_x| \geq |\Delta C(X)|$. Therefore, Theorem 2.17 guarantees that there exist control limit maintenance policies which minimize the expected long run average cost.

Substituting the above parametric representation in the linear program, yields the following:

$$\text{MIN } A_0x_{00} + A^+ \sum_{j=1}^{M} x_{jo}a_j + A^- \sum_{j=-L}^{-1} x_{jo}a_j + c \sum_{j=-L}^{M} x_{j1}$$

$$\text{SUCH THAT } x_{10} + x_{11} - \sum_{j=-L}^{M} x_{jo}p_{ji} = 0 \quad i = -L \cdots -1, 1 \cdots M$$
\[
\sum_{k=0}^{1} \sum_{j=-L}^{M} x_{jk} = 1
\]

\[
x_{jk} \geq 0 \quad k = 0, 1; \; j = -L \cdots M.
\]

This linear program can be put in a tableau and this tableau appears in Figure 1.

The control limit replacement rule \( R_{00} \) specifies that the equipment is to be replaced at each inspection, no matter what its state is. Consequently, in the linear programming formulation, \( R_{00} \) is represented by the tableau in Figure 2 in which the basic variables are \( x_{-L1}, x_{-L+1}, \ldots, x_{01}, x_{11}, \ldots, x_{M1} \). In Figure 2, \( m_{j0} \) represent the simplex multipliers of the variables \( x_{j0} \).

The simplex multipliers are the negative of the priced out cost coefficients. Therefore, from Figure 2, it is obvious that

\[
m_{j0} = -[a_j^A^- + c \sum_{i=-L}^{M} p_{ji} - 2c] = c - a_j^A^- \quad \text{for } j = -1 \cdots -L
\]

\[
m_{00} = -[a_0^A^+ + c \sum_{i=-L}^{M} p_{0i} - 2c] = c - a_0^A
\]

\[
m_{j0} = -[a_j^A^+ + c \sum_{i=-L}^{M} p_{ji} - 2c] = c - a_j^A^+ \quad \text{for } j = 1 \cdots M,
\]
<table>
<thead>
<tr>
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<th>X_{L1}</th>
<th>X_{L+1,0}</th>
<th>X_{L+1,1}</th>
<th>\cdots</th>
<th>X_{j,0}</th>
<th>X_{j+1}</th>
<th>\cdots</th>
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<th>X_{0,1}</th>
<th>X_{1,0}</th>
<th>X_{1,1}</th>
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<th>X_{j,0}</th>
<th>X_{j+1}</th>
<th>\cdots</th>
<th>X_{m,0}</th>
<th>X_{m,1}</th>
</tr>
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<td>\cdots</td>
<td>a_{j}</td>
<td>c</td>
<td>\cdots</td>
<td>a_{M}</td>
<td>c</td>
</tr>
<tr>
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<td>-p_{-j,-L}</td>
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<td>-p_{1,-L}</td>
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<td>-p_{j,-L}</td>
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</tr>
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<td>-p_{-L+1,-L+1}</td>
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<td>-p_{j,-L+1}</td>
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<td>\cdots</td>
<td>-p_{M,-L+1}</td>
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<td></td>
</tr>
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</tr>
<tr>
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<td>-p_{-L+1,1}</td>
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<td>\cdots</td>
<td>-p_{-j,-j}</td>
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<td>-p_{1,-j}</td>
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<td>\cdots</td>
<td>-p_{j,-j}</td>
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<td>\cdots</td>
<td>-p_{M,-j}</td>
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</tr>
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<tr>
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<td>0</td>
<td>\cdots</td>
<td>-p_{-j,1}</td>
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<td>\cdots</td>
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<td>\cdots</td>
<td>-p_{j,1}</td>
<td>0</td>
<td>\cdots</td>
<td>-p_{M,1}</td>
<td>0 = 0</td>
</tr>
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<td>-p_{1,j}</td>
<td>0</td>
<td>\cdots</td>
<td>1-p_{j,j}</td>
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<td>\cdots</td>
<td>-p_{M,j}</td>
<td>0 = 0</td>
</tr>
<tr>
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</tr>
<tr>
<td>-p_{-LM}</td>
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<td>\cdots</td>
<td>-p_{-j,M}</td>
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<td>\cdots</td>
<td>-p_{0,M}</td>
<td>0</td>
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<td>0</td>
<td>\cdots</td>
<td>1-p_{j,M}</td>
<td>1</td>
<td>\cdots</td>
<td>-p_{M,M}</td>
<td>1 = 0</td>
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<tr>
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<td>1</td>
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</table>

**FIGURE 1**
<table>
<thead>
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<th>X_{-L0}</th>
<th>X_{-L1}</th>
<th>X_{-L1,0}</th>
<th>X_{L1,1}</th>
<th>...</th>
<th>X_{-j0}</th>
<th>X_{-j1}</th>
<th>X_{00}</th>
<th>X_{01}</th>
<th>X_{10}</th>
<th>X_{11}</th>
<th>...</th>
<th>X_{j0}</th>
<th>X_{j1}</th>
<th>...</th>
<th>X_{M0}</th>
<th>X_{M1}</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_{-L}^A</td>
<td>c</td>
<td>a_{-L+1}^A</td>
<td>c</td>
<td>...</td>
<td>a_{-j}^A</td>
<td>c</td>
<td>a_0^A</td>
<td>a_1^A</td>
<td>c</td>
<td>a_{-j}^A</td>
<td>c</td>
<td>a_M^A</td>
<td>c</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccccccccccccccccc}
1-p_{-L,-L} & 1 & -p_{-L+1,-L} & 0 & ... & -p_{-j,-L} & 0 & ... & -p_{0,-L} & 0 & -p_{1,-L} & 0 & ... & -p_{j,-L} & 0 & ... & -p_{M,-L} & 0 = 0 \\
-p_{-L,-L+1} & 0 & 1-p_{-L+1,-L+1} & 1 & ... & -p_{-j,-L+1} & 0 & ... & -p_{0,-L+1} & 0 & -p_{1,-L+1} & 0 & ... & -p_{j,-L+1} & 0 & ... & -p_{M,-L+1} & 0 = 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ...
\end{array}
\]

\[
\begin{array}{cccccccccccccccccc}
-p_{-L,-j} & 0 & -p_{-L+1,-j} & 0 & ... & 1-p_{-j,-j} & 1 & ... & -p_{0,-j} & 0 & -p_{1,-j} & 0 & ... & -p_{j,-j} & 0 & ... & -p_{M,-j} & 0 = 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ...
\end{array}
\]

\[
\begin{array}{cccccccccccccccccc}
-p_{-L} & 0 & -p_{-L+1,0} & 0 & ... & -p_{-j1} & 0 & ... & -p_{01} & 0 & 1-p_{11} & 1 & ... & -p_{j1} & 0 & ... & -p_{M1} & 0 = 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ...
\end{array}
\]

\[
\begin{array}{cccccccccccccccccc}
-p_{-L} & 0 & -p_{-L+1,j} & 0 & ... & -p_{-jj} & 0 & ... & -p_{0j} & 0 & -p_{1j} & 0 & ... & 1-p_{jj} & 1 & ... & -p_{Mj} & 0 = 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ...
\end{array}
\]

\[
\begin{array}{cccccccccccccccccc}
-p_{-L} & 0 & -p_{-L+1,M} & 0 & ... & -p_{-jM} & 0 & ... & -p_{0M} & 0 & -p_{1M} & 0 & ... & -p_{jM} & 0 & ... & 1-p_{jM} & 1 = 0 \\
1-p_{-L,0} & 0 & 1-p_{-L+1,0} & 0 & ... & 1-p_{jO} & 0 & ... & 2-p_{0O} & 1 & 1-p_{1O} & 0 & ... & 1-p_{jO} & 0 & ... & 1-p_{jO} & 0 = 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
\]

FIGURE 2
Lemma 2.27: Assume that $C(X) = c$ and that $A_X$ is parametrized as follows:

$$A_X = a_X A^+ \quad X = 1, 2, \ldots, M$$

$$A_X = a_X A^- \quad X = -1, -2, \ldots, -L$$

$$A_0 = a_0 A^-$$

$$a_0 \leq a_1 \leq \cdots \leq a_M \quad A^+ \geq 0 \quad A = \max\{A^+, A^-\} \text{ if } a_0 < 0$$

$$a_0 \leq a_{-1} \leq \cdots \leq a_{-L} \quad A^- \geq 0 \quad A = \min\{A^+, A^-\} \text{ if } a_0 \geq 0.$$

Furthermore, assume that Condition A and Assumption 1 hold. Then the long run average cost is minimized by policy $R_{00}$ for the following values of $A^+$ and $A^-:

for $a_0 > 0$, $c \geq 0$ \qquad $A^+ \geq \frac{c}{a_0} \geq 0 \quad \text{and} \quad A^- \geq \frac{c}{a_0} \geq 0$,

for $a_0 < 0$, $c \leq 0$ \qquad $\frac{c}{a_0} \geq A^+ \geq 0 \quad \text{and} \quad \frac{c}{a_0} \geq A^- \geq 0$,

for $a_0 \geq 0$, $c \leq 0$ \qquad $A^+ \geq 0 \quad \text{and} \quad A^- \geq 0$.

Proof: $R_{00}$ is optimal if its simplex multipliers $m_j$ are non-positive for all $j = -L \cdots M$. In other words,

$$c - a_j A^- \leq 0 \quad j = -L \cdots -1$$

$$c - a_0 A^- \leq 0$$

$$c - a_j A^+ \leq 0 \quad j = 1 \cdots M.$$
If $a_0 > 0$ then $a_j > 0$ for all $j = -L \cdots M$ and inequalities (2.5.11) reduce to

$$A^- \geq \frac{c}{a_j} \quad j = -1 \cdots -L$$

$$\min\{A^+, A^-\} = A \geq \frac{c}{a_0}$$

$$A^+ \geq \frac{c}{a_j} \quad j = 1 \cdots M.$$  

Since $a_0 \leq a_1 \leq \cdots \leq a_M$ and $a_0 \leq a_{-1} \leq \cdots \leq a_{-L}$, for $c \geq 0$,

$$\max_{j=-1 \cdots -L} \left\{ \frac{c}{a_j} \right\} = \frac{c}{a_0} \quad \text{and} \quad \max_{j=1 \cdots M} \left\{ \frac{c}{a_j} \right\} = \frac{c}{a_1} \leq \frac{c}{a_0}.$$  

Therefore inequalities (2.5.11) are equivalent to $A^+ \geq c/a_0$ and $A^- \geq c/a_0$ where $c \geq 0$, $a_0 > 0$. If $c \leq 0$ and $a_0 \geq 0$ then all $A^+ \geq 0$ and $A^- \geq 0$ satisfy inequalities (2.5.11). If $a_0 < 0$ then the inequalities become

$$\frac{c}{a_j} \geq A^- \quad \text{for} \quad a_j < 0, \quad j = -1 \cdots -L$$

$$\frac{c}{a_j} < A^- \quad \text{for} \quad a_j > 0, \quad j = -1 \cdots -L$$

$$\frac{c}{a_0} \geq A = \max\{A^+, A^-\}$$

$$\frac{c}{a_j} \geq A^+ \quad \text{for} \quad a_j < 0, \quad j = 1 \cdots M$$

$$\frac{c}{a_j} \leq A^+ \quad \text{for} \quad a_j > 0, \quad j = 1 \cdots M.$$
Since $A^+, A^- \geq 0$, it is necessary that $c \leq 0$ if the above inequalities are to have any solution. Since $a_0 \leq a_1 \leq \cdots \leq a_M$ and
\[ a_0 \leq a_{-1} \leq \cdots \leq a_{-L}, \quad \text{for} \quad c \leq 0 \quad \min\left\{ \frac{c}{a_j} \mid a_j < 0, \ j = 0 \cdots -L \right\} = \frac{c}{a_0} \]
and \[ \min\left\{ \frac{c}{a_j} \mid a_j < 0, \ j = 0 \cdots M \right\} = \frac{c}{a_0}. \] Therefore inequalities (2.5,11) are equivalent to
\[ \frac{c}{a_0} \geq A^+ \geq 0 \quad \text{and} \quad \frac{c}{a_0} \geq A^- \geq 0 \quad \text{for} \quad a_0 < 0, \ c \leq 0. \]
QED

**Theorem 2.28**: Assume that $C(X) = c$ and that $A_X$ is parametrized as follows:

\[ A_X = a_X A^- \quad x = -1 \cdots -L \]
\[ A_0 = a_0 A \]
\[ A_X = a_X A^+ \quad x = 1 \cdots M \]
\[ a_0 \leq a_1 \leq \cdots \leq a_M \quad A^+ \geq 0 \quad A = \max\{A^+, A^-\} \quad \text{if} \quad a_0 < 0 \]
\[ a_0 \leq a_{-1} \leq \cdots \leq a_{-L} \quad A^- \geq 0 \quad A = \min\{A^+, A^-\} \quad \text{if} \quad a_0 \geq 0 \]

And assume that Condition A and Assumption 1 hold. Then for $a_0 \leq 0$, $c \leq 0$ there exists a partition of the real line \{\(\tilde{A}_j^+\)\}_{j=0}^{M+2} and \{\(\tilde{A}_j^-\)\}_{j=0}^{-L-2}, where
\[ 0 = \tilde{A}_0^+ \leq \tilde{A}_1^+ \leq \cdots \leq \tilde{A}_M^+ \leq \tilde{A}_{M+2}^+ = \infty \]
and
\[ 0 = \tilde{A}_0^- \leq \tilde{A}_{-1}^- \leq \cdots \leq \tilde{A}_{-L-1}^- \leq \tilde{A}_{-L-2}^- = \infty, \] such that the control limit law $R_{ij}$ minimizes the long run average cost if $\tilde{A}_i^- \leq A^- \leq \tilde{A}_{i-1}^-$, $i = 0, \ldots, -L-1, \tilde{A}_j^+ \leq A^+ \leq \tilde{A}_{j+1}^+$, $j = 0, \ldots, M+1$. For $a_0 > 0,$
c \geq 0$, there exists partitions of the real line \( \{\tilde{A}_j^+\}_{j=0}^{M+2} \) and \( \{\tilde{A}_j^-\}_{j=0}^{L-2} \) where \( 0 = \tilde{A}_{M+2}^+ \leq \tilde{A}_{M+1}^+ \leq \cdots \leq \tilde{A}_0^+ = \infty \) and \( 0 = \tilde{A}_{L-2}^- \leq \tilde{A}_{L-1}^- \leq \cdots \leq \tilde{A}_1^- \leq \tilde{A}_0^- = \infty \), such that the control limit law \( R_{ij} \) minimizes the long run average cost if \( \tilde{A}_{i-1}^- \leq A^- \leq \tilde{A}_i^- \) and \( \tilde{A}_{j+1}^+ \leq A^+ \leq \tilde{A}_j^+ \). Furthermore, if Condition B holds, the sequences are strictly increasing or decreasing.

\[ a_0 \leq 0, \quad c \leq 0 \]

\[ a_0 > 0, \quad c \geq 0 \]
Proof: First assume that $a_0 \leq 0$, $c \geq 0$. Then Lemma 2.27 specifies that $R_0^0$ is optimal for $A^+ \in [0, \frac{c}{a_0}], A^- \in [0, \frac{c}{a_0}]$. Therefore, let $\bar{A}_0^+ = \bar{A}_0^- = 0$ and $\bar{A}_1^+ = \bar{A}_1^- = c/a_0$. Fix $A^- = 0$. For $A^+ > c/a_0$, $R_0^0$ is no longer optimal. Therefore, one or more of the simplex multipliers $m_{j0}$ are zero at $A^+ = c/a_0$ and are positive for $A^+ > c/a_0$. Assume that only one multiplier is zero at $c/a_0$. Then a parametric linear programming algorithm states that one iteration of the simplex method will produce a new optimal basis. Since there exist optimal control limit laws for all values of $A^+ \geq 0$, the unique variable to enter the basis at this iteration must be $x_{00}$ with $x_{01}$ becoming non-basic.

(Note that since $A^-$ is fixed, the costs $A_x, x < 0$, have not varied so that $x_{-1,0}$ is not a candidate to enter the basis.) Therefore, for $A^+ = \bar{A}_1^+$, $R_{01}$ is also optimal. Now proceed in the same manner to determine that value, $\bar{A}_2^+$, of $A^+$ at which $R_{01}$ becomes non-optimal.

If more than one simplex multiplier becomes zero at $\bar{A}_1^+$, let $k$ designate the largest index of the candidate variables $x_{00}, x_{10}, \ldots, x_{k0}$ to enter the basis. In each case

\begin{equation}
\begin{aligned}
m_{00} & = c - a_0 \bar{A} = c - a_0 \bar{A}^+ = 0 \\
m_{10} & = c - a_1 \bar{A}_{1}^+ = 0 \\
& \vdots \\
m_{k0} & = c - a_k \bar{A}_{k-1}^+ = 0
\end{aligned}
\end{equation}

and $m_{10}(A^+) \geq 0$, $i = 0, \ldots, k$, for $A^+ > \bar{A}_1^+$. Note that equations (2.5.12) imply that $a_0 = a_1 = \cdots = a_k$ and, therefore,
\[ m_{00}(A^+) = m_{10}(A^+) = \cdots = m_{k0}(A^+) \]. From Figure 2 it is obvious that adding \( x_{j0} \) to the basis while dropping \( x_{j1} \) from the basis changes the simplex multipliers to

\[
m_{10}'(A^+) = m_{10}(A^+), \quad j \neq 0,
\]

\[
m_{10}'(A^+) = m_{10}(A^+) - \frac{1 - p_{10}}{2 - p_{00}} m_{00}(A^+), \quad j = 0.
\]

Therefore, if \( x_{00} \) becomes basic, \( m_{10}'(A^+)_1 = 0 \) for all \( i = 1, \ldots, k \) but

\[
m_{10}'(A^+) = m_{10}(A^+)(1 - \frac{1 - p_{10}}{2 - p_{00}}) = m_{10}(A^+) (\frac{1 - p_{00} + p_{10}}{2 - p_{00}}) > 0
\]

if \( A^+ > \tilde{A}_1^+ \). Consequently, \( R_{01} \) is optimal for \( A^+ = \tilde{A}_1^+ \) but not for \( A^+ > \tilde{A}_1^+ \). Similarly \( R_{02} \) is optimal for \( A^+ = \tilde{A}_2^+ \) but possibly not for \( A^+ > \tilde{A}_1^+ \). This can continue until \( R_{0k} \) is reached. Then \( R_{0k} \) will be optimal for \( A^+ \geq \tilde{A}_1^+ \). So in this case let \( \tilde{A}_2^+ = \tilde{A}_1^+ = \cdots = \tilde{A}_k^+ \). Then \( R_{00} \) is optimal for \( 0 \leq A^+ \leq \tilde{A}_1^+ \) and \( R_{0i}, i = 1, \ldots, k-1 \) is optimal for \( \tilde{A}_1^+ \leq A^+ \leq \tilde{A}_1^+ \) where, in this case, \( \tilde{A}_1^+ = \tilde{A}_1^+ = \tilde{A}_1^+, i = 1, \ldots, k \).

Then the algorithm will proceed in the same manner to determine the value \( \tilde{A}_{k+1}^+ \) at which \( R_{0k} \) becomes non-optimal.

This same process can be repeated for all values of \( A^- \). Since the values of \( A^- \) do not affect the simplex multipliers \( m_{10}(A^+), i = 1, \ldots, M \), the values \( \tilde{A}_i^+ \), \( i = 2, \ldots, M+2 \) are unaffected by \( A^- \).

And by Lemma 2.27, \( \tilde{A}_1^+ = \frac{c}{d_0} \) for all values of \( A^- \in [0, a_0] \). Therefore when this same procedure is carried out by varying \( A^- \), a set of values
\( \tilde{A}_i^-, i = 0 \ldots -L-2, \) is obtained such that \( R_{ij} \) is optimal for 
\( \tilde{A}_i^- \leq A^- \leq \tilde{A}_{i-1}^- \) for a specified \( j. \)

If \( a_0 > 0 \) and \( c \geq 0, \) then for \( A^+ = A^- = A = 0 \) the linear program seeks to minimize \( c \sum_{j=-L}^{M} x_{j1}. \) Obviously this is minimized by making \( x_{j1} = 0 \) for all \( j = -L \ldots M. \) Thus with \( x_{j1} \) non-basic, the optimal solution is the control limit law \( R_{-L-1,M+1}, \) never replace the machinery. Now the same parametric linear program can be applied. \( A^+ \)
will be increased until \( R_{-L-1,M+1} \) is no longer optimal at \( A^+ = \tilde{A}_{M+1}^+, \) etc.

If \( \tilde{A}_j^+ = \tilde{A}_{j+1}^+ \) rules of the type, replace machines in state \( j \)
but not in state \( j+1, \) are optimal. But if Condition B holds only control limit laws can be optimal. Therefore \( \tilde{A}_j^+ \neq \tilde{A}_{j+1}^+ \) if Condition B holds.

\textit{QED}

2.6. \textbf{Computation:}

The original problem consisted of minimizing the expected long run average cost over all possible replacement rules,

\[
v(X_0) = \min_{R \in \pi} v_R(X_0) = \min_{R \in \pi} \lim_{T \to \infty} \mathbb{E}\left(\frac{1}{T} \sum_{t=0}^{T-1} g_R(X_t) | X_0\right) \]

where

\[
g_R(X_t) = \begin{cases} 
C(X_t) & \text{if } R \text{ replaces the equipment in state } X_t \\
\tilde{A}^+_X \text{ } X_t & \text{if } R \text{ does not replace the equipment in state } X_t 
\end{cases}
\]

67
If Assumption 1 is true, then Theorem 2.2 guarantees that there exists a stationary non-randomized policy that minimizes \( v_R(X_0) \). The class of policies is finite, but can be quite large \( 2^{L+M+1} \) members. But if Condition A or B is true and \( C(X) \) and \( A_X \) are both unimodal with minimum at \( X = 0 \) such that \( |\triangle C(X)| \leq |\triangle A_X| \), then Theorem 2.17 guarantees that a control limit law minimizes \( v_R(X_0) \). The class of control limit laws is small \( (L+2)(M+2) \) members) compared to the class of stationary non-randomized replacement policies.

Therefore, if the values of \( A_X \) and \( C(X) \) and \( \{p_{ij}\}_{ij=-L\cdots M} \) are known and if \( (L+2)(M+2) \) is small enough, it would be efficient to iterate all possible control limit maintenance policies and then compute their long run average costs. For each policy \( R_{k\ell} \) the modified transition probabilities can be derived:

\[
p_{ij}(R_{k\ell}) = \begin{cases} p_{ij} & \text{for } k < i \leq 0 \text{ and } 0 \leq i < \ell \\ 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \\ \end{cases}
\]

The steady state frequencies \( \pi_j(R_{k\ell}) \) can then be determined from the Chapman-Kolmogorov equations and

\[
v_{R_{k\ell}}(X_0) = \sum_{j=-L}^{M} \pi_j(R_{k\ell}) g_{R_{k\ell}}(j)
\]

can be minimized over all possible values of \( k = 0 \cdots -L-1 \) and \( \ell = 0 \cdots M+1 \). The next lemma shows that not all values of \( k \) and \( \ell \) need be considered.
Lemma 2.29: Let $R_{k,l}$ represent a control limit law and let $k^*$ and $l^*$ represent the optimal control limits for the long run average cost (N-stage $\alpha$-discounted, $\infty$-stage $\alpha$-discounted) problem. Then $v_{R_{k,l}}(X_0)$ \( \phi_{R_{k,l}}(X_0, \alpha, N), \phi_{R_{k,l}}(X_0, \alpha) \) is unimodal in $l$ with minimum at $l = l^*$ for fixed $k$ and is unimodal in $k$ with minimum at $k = k^*$ for fixed $l$.

Proof:

\[ \phi(X_0, \alpha, N) = \min \{ A_{X_0} + \alpha \sum_{j=-L}^{M} P_{X_0j} \phi(j, \alpha, N-1), C(X_0) + \alpha \phi(0, \alpha, N-1) \} \]

Therefore

\[ \phi_{R_{k,l}}(X_0, \alpha, N) = \begin{cases} 
A_{X_0} + \alpha \sum_{j=-L}^{M} P_{X_0j} \phi(j, \alpha, N-1) & \text{if } k < X_0 < l \\
C(X_0) + \alpha \phi(0, \alpha, N-1) & \text{if } X_0 \leq k \\
& \text{or } X_0 \geq l
\end{cases} \]

Fix $k$ and let $l^*$ represent the optimal control limit. Then

\[ A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N-1) \geq C(X) + \alpha \phi(0, \alpha, N-1) \quad \text{if } X \geq l^* \]

\[ A_X + \alpha \sum_{j=-L}^{M} P_{Xj} \phi(j, \alpha, N-1) < C(X) + \alpha \phi(0, \alpha, N-1) \quad \text{if } 0 \leq X < l^* \]

Now fix $X$. Then $\phi_{R_{k,l}}(X, \alpha, N) > \phi_{R_{k,l+1}}(X, \alpha, N)$ if $l < l^*$ since the value of each function is the same except at $X = l$. At $X = l$

\[ \phi_{R_{k,l}}(l, \alpha, N) = C(l) + \alpha \phi(0, \alpha, N-1) \]

\[ > A_l + \alpha \sum_{j=-L}^{M} P_{l,j} \phi(j, \alpha, N-1) = \phi_{R_{k,l+1}}(l, \alpha, N) \]
since $k < k^*$. Similarly $\phi_{R_{k^*}}(X_{\alpha}, N) \leq \phi_{R_{k^*+1}}(X_{\alpha}, N)$ if $k \geq k^*$ since the functions agree on all values except for $X = k+1$. Therefore, $\phi_{R_{k^*}}(X_{\alpha}, N)$ is unimodal in $k$ with minimum at $k = k^*$ for fixed $k$. The same can be proved for $k$.

$$\phi_{R_{k^*}}(X_{\alpha}, N) = \lim_{N \to \infty} \phi_{R_{k^*}}(X_{\alpha}, N).$$

Since each $\phi_{R_{k^*}}(X_{\alpha}, N)$ is unimodal in $k$ and $\ell$ with minima at $k = k^*$ and $\ell = \ell^*$, $\phi_{R_{k^*}}(X_{\alpha}, N)$ is also unimodal in $k$ and $\ell$ with minima at $k = k^*$, $\ell = \ell^*$ where $R_{k^*}^*_{\ell^*}$ is the optimal control limit law for the $\infty$-stage $\alpha$-discounted problem.

Similarly, since $v_{R_{k^*}}(X_{\alpha}) = \lim_{\alpha \to 1} (1-\alpha) \phi_{R_{k^*}}(X_{\alpha}, N)$, $v_{R_{k^*}}(X_{\alpha})$ is unimodal in $k$ and $\ell$ with minima at $k = k^*$, $\ell = \ell^*$ where $R_{k^*}^*_{\ell^*}$ is the optimal control limit law for the long run average cost problem.

QED

In iterating the costs of different control limit maintenance policies, it is obvious when a minimum has been reached. Since $v_{R_{k^*}}(X_{\alpha})$ is unimodal in each variable, $k^*$ and $\ell^*$ are minima if for any $j$, $v_{R_{k^*+1,j}}(X_{\alpha}) \geq v_{R_{k^*,j}}(X_{\alpha})$ and $v_{R_{k^*,j}}(X_{\alpha}) \leq v_{R_{k^*-1,j}}(X_{\alpha})$ and for any $i$, $v_{R_{i^*,\ell^*+1}}(X_{\alpha}) \geq v_{R_{i^*,\ell^*}}(X_{\alpha})$ and $v_{R_{i^*,\ell^*}}(X_{\alpha}) \leq v_{R_{i^*,\ell^*-1}}(X_{\alpha})$.

Therefore, to efficiently locate an optimal set of control limits, $k^*$ and $\ell^*$, it is sufficient to compute the cost of policy $R_{ij}$, $v_{R_{ij}}(X_{\alpha})$, starting with $j = M+1$ and $i = -L-1$ and then reduce $j$ one state at a time until a state $j^*$ is reached at which $v_{R_{L-1,j^*-1}}(X_{\alpha}) \geq v_{R_{L-1,j^*}}(X_{\alpha})$. Then similarly increase $i$ one state at a time until
a state \( i^* \) is reached at which \( v_{R_{i^*, j^*}, j^*} (X_0) \geq v_{R_{i^*, j^*}, j^* + 1} (X_0) \). In this case \( j^* = \ell^* \) and \( i^* = k^* \).

If \( L \) and \( M \) are too large to efficiently iterate the costs, it is also possible to utilize the linear program of Section 5:

\[
\begin{align*}
\text{MIN} & \quad \sum_{j=-L}^{M} x_{j0} A_j + \sum_{j=-L}^{M} x_{j1} C(j) \\
\text{SUBJECT TO} & \quad x_{10} + x_{11} - \sum_{j=-L}^{M} x_{j0} p_{ji} = 0, \quad i = -L \cdots -1, 1 \cdots M \\
& \quad \frac{1}{L} \sum_{k=0}^{M} \sum_{j=-L}^{M} x_{jk} = 1 \\
& \quad x_{jk} \geq 0, \quad k = 0, 1, j = -L \cdots M.
\end{align*}
\]

The optimal policy can then be determined from the solution of the linear program by setting the probability of replacing a machine when in state \( j \) equal to \( d_{j1} \) where

\[
d_{j1} = \frac{x_{j1}}{x_{j0} + x_{j1}} \quad \text{if} \quad x_{j0} + x_{j1} \neq 0
\]

and \( d_{j1} \) is arbitrary if \( x_{j0} + x_{j1} = 0 \).

In some instances it has been shown that a linear program proceeds along a long path on its route to an optimum (Quandt and Kuhn [17]). When it is known that a control limit law is optimal, this inefficiency can be eliminated by changing the entry and exit rules for the variables \( x_{jk} \). Let the original basis correspond to the control limit rule \( R_{-L-1, M+1} \), that is, \( x_{-L, 0}, x_{-L+1, 0}, \ldots, x_{00}, x_{10}, \ldots, x_{M1} \).
are basic. If this basis is non-optimal select either $x_{M1}$ to enter the basis with $x_{MO}$ leaving the basis or $x_{L_1,1}$ to enter the basis with $x_{L_0,0}$ leaving the basis. Continue in this way, proceeding from bases corresponding to $R_{ij}$ to bases corresponding to either $R_{i+1,j}$ or $R_{i,j-1}$. Following this modified procedure for introducing new variables, the maximum number of pivots until an optimum is obtained is $L+M+2$. 
CHAPTER III

A K MACHINE MAINTENANCE MODEL

3.1. The Model:

A simple one machine maintenance model was presented in Chapter II. It was shown that the intuitively appealing control limit maintenance policies were optimal under certain specified conditions on the cost functions and the transition probabilities. Unfortunately, most systems are not composed of one machine and, therefore, cannot be subsumed under the model of Chapter II. Yet the simple control limit maintenance rules still retain their intuitive appeal even for a complex system composed of many subcomponents. This chapter will investigate such a system and will specify sufficient conditions for the control limit maintenance policies to be optimal.

The model consists of a system of K identical components operating continuously but inspected at discrete time intervals \( t = 0, 1, 2, \ldots \). As in the model of Chapter II, at each inspection each component will be classified as being in any of \( L+M+1 \) possible states, \([-L \cdots 0 \cdots M]\) where 0 represents a new component and \(-L\) and \( M\) failed components. Also at each inspection there will be the option of replacing any or all of the system's components. Note that the assumption that repair or replacement takes one time period is being made.
Let \( X_t = (S_1^t, S_2^t, \ldots, S_K^t) \) denote the states of the \( K \) components at time \( t \). Assume that each \( S_j^t, j = 1, \ldots, K \), is a finite state Markov chain with stationary transition probabilities. Let 
\[
\{p_{ij}\}_{i,j=-L}^{M} 
\]
represent these transition probabilities,
\[
p_{ij} = \mathbb{P}(S_{k}^{t+1} = j | S_{k}^{t} = i).
\]
Assume that the Markov chains \( \{S_j^t | j = 1 \cdots K\} \) are all independent. Then \( \{X_t | t = 0, 1, 2, \ldots\} \) is also a Markov chain with stationary transition probabilities where the transition probabilities are given by
\[
\mathbb{P}(X_{t+1} = (j_1 \cdots j_K) | X_t = (i_1 \cdots i_K)) = \prod_{\ell=1}^{K} p_{i_{\ell} j_{\ell}}.
\]
To guarantee that each component will eventually fail, regardless of its initial state, each Markov chain \( \{S_j^t | t = 0, 1, \ldots ; j = 1, \ldots, K\} \) will be assumed to be irreducible.

In this \( K \) machine model a replacement policy \( R \) must specify, for each separate component, those states in which the component is to be replaced. Each replacement rule \( R \) modifies the Markov chain \( \{X_t | t = 0, 1, \ldots\} \). The new chain will be denoted by \( \{X_t(R) | t = 0, 1, \ldots\} \). If this modified chain is Markovian, the transition probabilities will be
\[
\mathbb{P}(X_{t+1}(R) = (j_1 \cdots j_K) | X_t(R) = (i_1 \cdots i_K)) = \prod_{\ell=1}^{K} p_{i_{\ell} j_{\ell}}(R)
\]
where the transition probabilities of the individual components under policy \( R \) are \( \{p_{ij}(R)\}_{i,j=-L}^{M} \). These modified transition probabilities
can be derived from the original probabilities by the following rules:

\[
\begin{align*}
    p_{ij}^{(R)} &= p_{ij} & \text{if component } \ell \text{ in state } i \ell \text{ is not replaced by rule } R \\
    p_{ij}^{(R)} &= \begin{cases} 
        1 & \text{if } j \ell = 0 \\
        0 & \text{if } j \ell \neq 0
    \end{cases} & \text{if component } \ell \text{ in state } i \ell \text{ is replaced by rule } R
\end{align*}
\]

So far there is little difference between this model and that of Chapter II. In fact, if the same cost structure as in the model of Chapter II were placed on each component, there would be no significant difference between the models. Since each component would be stochastically and cost-wise independent of the other components, it could be analyzed separately and the results of Chapter II applied.

In multicomponent systems interactions between components frequently play a large role in the cost of repair or replacement. For example, a shortage of replacement parts or of repair labor will have a great bearing on the repair and replacement policy of the system. For if the number of replacement parts is limited and the number of machines to be replaced exceeds the number of spares, the cost of replacement will be extremely high. Similarly, if there are different methods of repair, each with a certain range of states in which it functions most efficiently, the assignment of repair methods to individual machines will effect the overall efficiency of the repair operation and, hence, the cost of repair. Economies of scale, in which the cost per component of repair work is a decreasing function of the number of components repaired at one time,
can also have a great bearing on the cost of repair and, consequently, on the maintenance policy.

To take into account these contingencies, the cost of repair will be measured by a set of functions \( \{ C(S, i), i = 1 \cdots r \} \). At the time of inspection each machine will be assigned one of these functions as a measure of the cost of repairing it. If \( f(i,t) \) designates the cost function applied to machine \( i \) at inspection \( t \), then the cost of repairing component \( i \) in state \( S_i \) at time \( t \) is \( C(S_i, f(i,t)) \).

As in the model of Chapter II, there will be an operating cost \( a_{S_i} \) charged to each machine \( i \) in state \( S_i \) which does not undergo repair.

In the following sections the expected long run average cost and the expected total discounted cost will be examined under different policies \( R \) to determine which maintenance policies minimize these costs.

3.2. A Factorization of the Model:

As in the analysis of Chapter II, the \( N \)-stage \( \alpha \)-discounted problem will be investigated first as a preliminary step in the analysis of the long run average cost problem.

A replacement policy \( R \) can be represented as a partition of the set of states \( \{ S_1 \cdots S_K \} \). Let \( \pi, \pi \in \{ S_1 \cdots S_K \} \), represent those machines to be replaced and \( \pi' \) those machines that are not to be replaced. Then the cost of repair can be denoted by \( g(\pi) \) where
\[ g(\pi) = \sum_{S_1 \in \Pi} c(S_1, f(i, t)) . \]

Similarly the operating costs can be represented as

\[ A_{\pi'} = \sum_{S_1 \in \Pi} A_{S_1} . \]

Let \( \phi(S_1 \ldots S_K), \alpha, N \) denote the minimum \( \alpha \)-discounted cost for an \( N \)-stage problem where the initial states of the machines are \( (S_1 \ldots S_K) \). Then by dynamic programming

\[
\phi((S_1 \ldots S_K), \alpha, N) = \min_{\pi} \left( g(\pi) + A_{\pi'} + h_{N-1}^\alpha(\pi', \tilde{\sigma}) | \pi \cap \pi' = \emptyset, \pi \cup \pi' = \{S_1 \ldots S_K\} \right)
\]

where

\[
h_{N-1}^\alpha(\pi', \tilde{\sigma}) = \alpha \sum_{S_1 \in \Pi} \left( \prod_{S_1} p_{S_1 S_1'} \phi((0, S_1'), \alpha, N-1) \right) .
\]

The vector \( (\pi', \tilde{\sigma}) \) represents the vector of states with \( 0 \) in place of all machines replaced by policy \( \pi \). Similarly \( \phi((\tilde{\sigma}, S_i), \alpha, N-1) \) represents the minimum cost for an \( \alpha \)-discounted \( N-1 \)-stage problem where the initial state of machine \( i \) is \( S_i' \) if \( \pi \) did not replace it and \( 0 \) if \( \pi \) did replace it. Consequently, if \( \pi = (S_1 \ldots S_{i_1}) \),

\[
h_{N-1}^\alpha(\pi', \tilde{\sigma}) = h_{N-1}^\alpha(0 \ldots 0, S_{i_1}', 0 \ldots 0, S_{i_2}', 0 \ldots 0, S_{i_{i_1}}', 0 \ldots 0)
\]

\[
= \alpha \sum_{S_{i_1}' = -L}^M \sum_{S_{i_p}' = -L}^M p_{S_{i_1}' S_{i_1}} p_{S_{i_2}' S_{i_{i_1}}} \ldots p_{S_{i_{i_1}}' S_{i_{i_1}}'} \phi((0 \ldots 0, S_{i_1}', 0 \ldots 0, S_{i_2}', 0 \ldots 0, S_{i_{i_1}}', 0 \ldots 0), \alpha, N-1) .
\]
By definition \( h_{0}^{\alpha}(\pi', \delta) = 0 \).

For a given \((S_{1} \cdots S_{K})\) assume that there are \( \beta \) machines in non-negative states and \( K-\beta \) in negative states. Let \( S_{(1)} \) be the state of the machine in the highest non-negative state and \( (1) \) the number of that machine, \( S_{(2)} \) the state of the machine in the second highest state and \( (2) \) its number, etc. Similarly, if \( \beta \neq K \), let \( S_{(K)} \) represent the state of the machine in the lowest negative state and \( (K) \) its number, \( S_{(K-1)} \) the second lowest and \( (K-1) \) its number, etc. Then,

\[
S_{(1)} \geq S_{(2)} \geq S_{(3)} \geq \cdots \geq S_{(\beta)} \geq 0 \geq S_{(\beta+1)} \geq \cdots \geq S_{(K)}.
\]

**Lemma 3.1**: Let \( d(S) \), \( S = -L \cdots M \), be a unimodal function with minimum at \( S = 0 \). Let \((S_{1} \cdots S_{K})\) be given and for a fixed \( i \), \( i = 1 \cdots K \), define \( D(S_{1} \cdots S_{K}) = d(S_{(i)}) \). Then \( D(S_{1} \cdots S_{K}) \) is unimodal in each of its elements \( S_{j} \), \( j = 1 \cdots K \), with minimum at \( S_{j} = 0 \).

**Proof**: \( \Delta_{j} D(S_{1} \cdots S_{K}) = D(S_{1} \cdots S_{j-1} S_{j} S_{K}) - D(S_{1} \cdots S_{j-1} S_{K}) \).

Assume that \( S_{(i)} = S_{k_{1}} \) for \((S_{1} \cdots S_{j} \cdots S_{K})\) and that

\( S_{(i)} = S_{k_{2}} \) for \((S_{1} \cdots S_{j-1} \cdots S_{K})\). Then since \( S_{j} \geq S_{j-1} \), \( S_{k_{1}} \geq S_{k_{2}} \).

Furthermore, \( S_{k_{1}} - S_{k_{2}} = 0 \) if \( k_{1} \neq j \) and \( S_{k_{1}} = S_{k_{2}} = 0 \) or \( 1 \) if \( k_{1} = j \). Therefore,

\[
\Delta_{j} D(S_{1} \cdots S_{K}) = d(S_{k_{1}}) - d(S_{k_{2}}).
\]

Since \( d(S) \) is unimodal with minimum at \( S = 0 \) and since \( S_{k_{1}} \geq S_{k_{2}} \),
\[ d(S_{k_1}) - d(S_{k_2}) \geq 0 \quad \text{if } S_{k_1} > 0 , \]

and

\[ d(S_{k_1}) - d(S_{k_2}) \leq 0 \quad \text{if } S_{k_1} \leq 0 . \]

Therefore,

\[ \Delta_j D(S_1 \cdots S_K) \geq 0 \quad \text{if } S_{k_1} > 0 , \]

and

\[ \Delta_j D(S_1 \cdots S_K) \leq 0 \quad \text{if } S_{k_1} \leq 0 . \]

and consequently,

\[ \Delta_j D(S_1 \cdots S_K) = 0 \quad \text{if } S_j \neq S_{k_2} , \]

\[ \Delta_j D(S_1 \cdots S_K) \geq 0 \quad \text{if } S_j = S_{k_1} > 0 , \]

\[ \Delta_j D(S_1 \cdots S_K) \leq 0 \quad \text{if } S_j = S_{k_1} \leq 0 . \]

Therefore \( D(S_1 \cdots S_K) \) is unimodal in \( S_j, j = 1 \cdots K \), with minimum at \( S_j = 0 \).

QED

**Lemma 3.2:** Assume that the following conditions hold:

1. \( \{p_{ij}\}_{i,j=1\cdots M} \) satisfies Condition A.

2. \( A_S \) is unimodal in \( S \) with minimum at \( S = 0 \).

3. \( C(S,i) \) is unimodal in \( S \) with minimum at \( S = 0 \) for \( i = 1 \cdots r \).

4. For all \( (S_1 \cdots S_K) \), \( C(S_{(i)}, f((i),t)) - A_S(i) \) is non-decreasing in \( i \) for \( i = 1 \cdots \beta \) and non-increasing in \( i \) for \( i = \beta + 1 \cdots K \),

where \( \beta \) is the number of non-negative states in \( \{S_1 \cdots S_K\} \).
\[ C(S_{(i)}, f((i), t)) - \Lambda_{S_{(i)}} \]

Then for fixed \( S_j, j = 1 \cdots K, j \neq i, \emptyset(S_1 \cdots S_i \cdots S_K), \alpha, \eta \) is unimodal in \( S_i, i = 1 \cdots K, \) with minimum at \( S_i = 0 \) for all \( N = 0, 1, 2, \ldots \). If \( S_1 \in \pi' \),

\[ h_N^{\alpha, \beta} = \alpha \sum_{\{j \mid S_j \in \pi'\}} \sum_{S_j = -L}^{M} \prod_{\{S_j \in \pi'\} \setminus j} p_{S_j S_j} \emptyset(\hat{\beta}, S_i), \alpha, \eta \]

is also unimodal in \( S_i \) with minimum at \( S_i = 0 \).

Furthermore, if \( \ell^+ \) machines in non-negative states are to be replaced, it is optimal to replace those \( \ell^+ \) machines in the \( \ell^+ \) highest states. And if \( \ell^- \) machines in negative states are to be replaced, it is optimal to replace those \( \ell^- \) machines in the \( \ell^- \) lowest states. Therefore, if among \( (S_1 \cdots S_K) \) there are \( \beta \) non-negative states and \( \ell^+ \leq \beta, \ell^- \leq K - \beta \),
\[ \theta((S_1 \cdots S_K), \alpha, N) = \min \left\{ \sum_{i=1}^{K} A_{S_i}^{(i)} + h_{N-1}^{\alpha}(S_1 \cdots S_K) \right\}; \]

\[ C(S_1, f((1), N)) + \sum_{i=2}^{K} A_{S_i}^{(i)} + h_{N-1}^{\alpha}(O, S_2, \ldots, S_K); \]

\[ C(S_K, f((K), N)) + \sum_{i=1}^{K-1} A_{S_i}^{(i)} + h_{N-1}^{\alpha}(S_1, \ldots, S_{K-1}, C); \ldots; \]

\[ \frac{\ell^+}{K} \sum_{i=1}^{\ell^+} C(S_i, f((1), N)) + \sum_{i=K-\ell^+}^{K-1} C(S_i, f((1), N)) + \sum_{i=\ell^+}^{K} A_{S_i}^{(i)} \]

\[ + h_{N-1}^{\alpha}(O \cdots O, S_{\ell^+}, \ldots, S_{K-\ell^+}, \ldots, O \cdots O); \ldots; \]

\[ \sum_{i=1}^{K} C(S_i, f((1), N)) + h_{N-1}^{\alpha}(O \cdots O); \]

Proof: Let \( N = 1, \)

\[ \theta((S_1 \cdots S_K), \alpha, 1) = \min \left\{ g(\pi) + A_{\pi}, |\pi \cup \pi'| = (S_1 \cdots S_K), \pi \cap \pi' = \emptyset \right\}; \]

where \[ g(\pi) = \sum_{S_i \in \pi} C(S_i, f(i, t)); \]

\[ A_{\pi'} = \sum_{S_i \in \pi'} A_{S_i}. \]

If no machines are replaced the cost is

\[ A_{(S_1 \cdots S_K)} = \sum_{i=1}^{K} A_{S_i}. \]

81
If one machine, machine \( j \) in state \( S_j \), is replaced, the cost is

\[
g(S_j) + \sum_{i=1}^{K} A_{S_i} = C(S_j, f(j,1)) + \sum_{i=1}^{K} A_{S_i} \\
= C(S_j, f(j,1)) - A_j + \sum_{i=1}^{K} A_{S_i}.
\]

By assumption \( C(S_{(i)}, f((i),1)) - A_{S_{(i)}} \) is non-decreasing in \( i \) for \( i = 1 \cdots \beta \) and non-increasing in \( i \) for \( i = \beta+1 \cdots K \). Therefore, \( C(S_j, f(j,1)) - A_j \) is minimized by choosing \( j = 1 \) (\( \beta \neq 0 \)) or \( j = K \) (\( \beta \neq K \)). Thus, if one machine is to be replaced, it will be the one in the highest non-negative state \( S_{(1)} \) at cost \( C(S_{(1)}, f((1),1)) + \sum_{i=2}^{K} A_{S_{(i)}} \) or the one in the lowest negative state \( S_{(K)} \) at cost \( C(S_{(K)}, f((K),1)) + \sum_{i=1}^{K-1} A_{S_{(i)}} \).

If two machines, \( j_1 \) and \( j_2 \) in states \( S_{j_1} \) and \( S_{j_2} \), are to be replaced the cost is

\[
g(S_{j_1}, S_{j_2}) + \sum_{i=1}^{K} A_{S_i} \\
= C(S_{j_1}, f(j_1,1)) - A_{j_1} + C(S_{j_2}, f(j_2,1)) - A_{j_2} + \sum_{i=1}^{K} A_{S_i}.
\]

Again by assumption, \( C(S_{(i)}, f((i),1)) - A_{S_{(i)}} \) is non-decreasing for \( i = 1 \cdots \beta \), and non-increasing for \( i = \beta+1 \cdots K \). Consequently, \( C(S_{j_1}, f(j_1,1)) - A_{j_1} + C(S_{j_2}, f(j_2,1)) - A_{j_2} \) is minimized by choosing...
$S_{j_1}$ and $S_{j_2}$ to be the two machines with the two highest non-negative states, with the two lowest negative states, or with the highest non-negative and the lowest negative state. Thus, the cost of replacing two machines is either

$$C(S_{(1)}, f((1), 1)) + C(S_{(2)}, f((2), 1)) + \sum_{i=3}^{K} A_{S_{(i)}},$$

or

$$C(S_{(K)}, f((K), 1)) + C(S_{(K-1)}, f((K-1), 1)) + \sum_{i=1}^{K-2} A_{S_{(i)}},$$

or

$$C(S_{(1)}, f((1), 1)) + C(S_{(K)}, f((K), 1)) + \sum_{i=2}^{K-1} A_{S_{(i)}}.$$  

Continuing in this way it is obvious that if $\ell$ machines are to be replaced, $\ell^+$ in non-negative states and $\ell^-$ in negative states, $\ell = \ell^+ + \ell^-$, then it is optimal to replace those $\ell^+$ machines in the $\ell^+$ highest non-negative states and those $\ell^-$ machines in the lowest negative states. Therefore,

$$\rho((S_1 \cdots S_K), \alpha, 1)$$

$$= \min \left\{ \sum_{i=1}^{K} A_{S_{(i)}}, C(S_{(1)}, f((1), 1)) + \sum_{i=2}^{K} A_{S_{(i)}}, C(S_{(K)}, f((K), 1)) + \sum_{i=1}^{K-1} A_{S_{(i)}}, C(S_{(1)}, f((1), 1)) + C(S_{(2)}, f((2), 1)) + \sum_{i=3}^{K} A_{S_{(i)}}, C(S_{(1)}, f((1), 1)) + C(S_{(K)}, f((K), 1)) + \sum_{i=2}^{K-2} A_{S_{(i)}}, \cdots; \sum_{i=1}^{K} C(S_{(i)}, f((i), 1)) \right\}. $$

83
Since $A_S$ and $C(S, i)$ are unimodal with minimum at $S = 0$, Lemma 3.1 guarantees that all of the terms of the form

$$C(S_{(i)}, f((i),1)) + C(S_{(j)}, f((j),1)) + \cdots + \sum A_S(i)$$

are also unimodal. Therefore, by Lemma 2.6, $\phi((S_1 \cdots S_K), \alpha, 1)$ is unimodal in each $S_i, i = 1 \cdots K$, with minimum at $S_i = 0$,

$$h_1^\alpha(\pi', \tilde{\delta}) = \sum_{i \in \pi'} \sum_{S_i = -L}^M \prod_{S_i' \in \pi'} P_{S_i i, S_i'} \phi((\tilde{\delta}, S_i), \alpha, 1)$$

$$= \sum_{S_{i_1} = -L}^M \cdots \sum_{S_{i_p} = -L}^M \prod_{j=1}^p P_{S_{i_j} i_j, S_{i_j}}$$

- $\phi((0 \cdots 0 S_{i_1}', 0 \cdots 0 S_{i_2}', 0 \cdots 0 S_{i_p}', 0 \cdots 0), \alpha, 1))$

if $\pi' = \{S_{i_1} \cdots S_{i_p}\}$. Since $\phi((\tilde{\delta}, S_{i}'), \alpha, 1)$ is unimodal in each $S_{i}' \in \pi'$ with minimum at $S_{i}' = 0$, Lemma 2.5 guarantees that $h_1^\alpha(\pi', \tilde{\delta})$ is also unimodal in each $S_{i} \in \pi'$ with minimum at $S_{i} = 0$. In other words if $\pi' = \{S_{i_1} \cdots S_{i_p}\}$

$$h_1^\alpha(\pi', \tilde{\delta}) = h_1^\alpha(0 \cdots 0, S_{i_1}', 0 \cdots 0, S_{i_2}', 0 \cdots 0 \cdots 0, S_{i_p}', 0 \cdots 0)$$

is unimodal in each $S_{i_j}, j = 1 \cdots p$, with minimum at $S_{i_j} = 0$. Note that $h_1^\alpha(\pi', \tilde{\delta})$ depends only on the values of the states $\{S_{i_1} \cdots S_{i_p}\}$ and not upon the assignment of the states $\{S_{i_1} \cdots S_{i_p}\}$ to the individual machines $i_1 \cdots i_p$. Therefore, $h_1^\alpha(\pi', \tilde{\delta})$ is minimized by
choosing $\pi'$ to be the machines in the $\xi^+$ lowest non-negative states
and the $\xi^-$ highest negative states where $\xi = \xi^+ + \xi^-$ is the number
of machines that are not replaced. That is, if $\ell = K - \xi^+ - \xi^-$
machines are to be replaced, $h_1^{\alpha}(\pi', \bar{\delta})$ is minimized by replacing the
machines in the $\ell^+ = \beta - \xi^+$ highest non-negative states and the
$\ell^- = K - \beta - \xi^-$ lowest negative states where $\ell = \ell^+ + \ell^-$ or, more
succinctly, $h_2^{\alpha}(\pi', \bar{\delta})$ is minimized by choosing $\pi$ to be

$\{S(1), \ldots S(\ell^+), S(K-\ell^-+1), \ldots S(K)\}.$

Thus the lemma is true for $N = 1$. Assume that it is also true
for $N = \eta$ and that $h_2^{\alpha}(\pi', \bar{\delta})$ is minimized by choosing $\pi$ to be in
the form $\{S(1), \ldots S(\ell^+), S(K-\ell^-+1), \ldots S(K)\}$ if $\ell = \ell^+ + \ell^-$ machines
are to be replaced,

$\phi((S_1, \ldots S_K), \alpha, \eta+1)$

$= \min_{\pi} \{g(\pi) + A_{\pi}, + h_2^{\alpha}(\pi', \bar{\delta}) \mid \pi \cup \pi' = \{S_1, \ldots S_K\}, \pi \cap \pi' = \emptyset\}.$

If no machines are replaced the cost is

$$\sum_{i=1}^{K} A_{S(i)} + h_2^{\alpha}(S(1), \ldots S(K))$$.

If one machine, machine $j$ in state $S_j$, is replaced the cost is

$$g(S_j) + \sum_{i=1}^{K} A_{S_i} + h_2^{\alpha}(S_1, \ldots S_{j-1}, 0, S_{j+1}, \ldots S_K)$$

$$= C(S_j, f(j, \eta+1)) - A_{S_j} + \sum_{i=1}^{K} A_{S_i} + h_2^{\alpha}(S_1, \ldots S_{j-1}, 0, S_{j+1}, \ldots S_K).$$
By assumption, $C(S_{(i)}, f((i), \eta+1)) - A_{S_{(i)}}$ is non-decreasing in $i$ for $i = 1 \cdots \beta$ and non-increasing in $i$ for $i = \beta+1 \cdots K$. Therefore $C(S_{j}, f(j, \eta+1) - A_{S_{j}}$ is minimized by choosing $j = (1)$ ($\beta \neq 0$) or $j = (K)$ ($\beta \neq K$). Also by the inductive hypothesis $h_{\eta}(S_{1} \cdots S_{j-1}^{'}, 0, S_{j+1} \cdots S_{K})$ is minimized by choosing $\pi$ to be $S_{(1)}$ or $S_{(K)}$. Thus, if one machine is to be replaced, it will be the one in the highest non-negative state, $S_{(1)}$, at cost $C(S_{(1)}, f((1), \eta+1)) + \sum_{i=2}^{K} A_{S_{(i)}} + h_{\eta}^{\alpha}(0, S_{(2)} \cdots S_{(K)})$, or the one in the lowest negative state, $S_{(K)}$, at cost $C(S_{(K)}, f((K), \eta+1)) + \sum_{i=1}^{K-1} A_{S_{(i)}} + h_{\eta}^{\alpha}(S_{(1)} \cdots S_{(K-1)}, 0)$. Consequently, the cost of replacing one machine is

$$\min\{C(S_{(1)}, f((1), \eta+1)) + \sum_{i=2}^{K} A_{S_{(i)}} + h_{\eta}^{\alpha}(0, S_{(2)} \cdots S_{(K)})\};$$

$$C(S_{(K)}, f((K), \eta+1)) + \sum_{i=1}^{K-1} A_{S_{(i)}} + h_{\eta}^{\alpha}(S_{(1)} \cdots S_{(K-1)}, 0)\};$$

Continuing this type of argument yields

$$\phi [(S_{1} \cdots S_{K}), \alpha, \eta+1]$$

$$= \min\{ \sum_{i=1}^{K} A_{S_{(i)}} + h_{\eta}^{\alpha}(S_{(1)} \cdots S_{(K)});$$

$$C(S_{(1)}, f((1), \eta+1)) + \sum_{i=2}^{K} A_{S_{(i)}} + h_{\eta}^{\alpha}(0, S_{(2)} \cdots S_{(K)});$$

$$C(S_{(K)}, f((K), \eta+1)) + \sum_{i=1}^{K-1} A_{S_{(i)}} + h_{\eta}^{\alpha}(S_{(1)} \cdots S_{(K-1)}, 0); \cdots;$$

$$\sum_{i=1}^{K} C(S_{(i)}, f((i), \eta+1)) + h_{\eta}^{\alpha}(0 \cdots 0)\}.$$
Lemma 3.1 guarantees that $C(S(i), f((i), \eta+1))$ and
\[ \sum A_S(i) \] are unimodal. Since $h^{\eta}(S_1 \cdots S_K)$ is unimodal by the inductive hypothesis, Lemma 3.1 also proves that $h^{\eta}(0 \cdots 0, S(\ell^+1) \cdots S(K-\ell^-), 0 \cdots 0)$ is also unimodal in $(S_1 \cdots S_K)$. Therefore by Lemma 2.6, $\phi((S_1 \cdots S_K), \alpha, \eta+1)$ is unimodal in each $S_j$ with minimum at $S_j = 0$.

It can now be shown, exactly as in the case $N = 1$, that $h^{\eta+1}(\pi', \bar{\delta})$ is unimodal in each $S_j \in \pi'$ with minimum at $S_j = 0$ and that $h^{\eta+1}(\pi', \bar{\delta})$ is minimized by choosing $\pi$ to be in the form
\[ \{S(1) \cdots S(\ell^+1), S(K-\ell^-1) \cdots S(K)\} \] if $\ell^+$ machines in non-negative states and $\ell^-$ machines in negative states are to be replaced.

Thus the lemma has been proved by induction. QED

**Theorem 3.3.** Assume that the following conditions are true:

1. $\{p_{ij}\}_{i,j=-L \cdots M}$ satisfies condition A.
2. $A_S$ is unimodal in $S$ with minimum at $S = 0$.
3. $C(S,i)$ is unimodal in $S$ with minimum at $S = 0$ for $i = 1 \cdots r$.
4. For all $(S_1 \cdots S_K)$, $C(S(i), f((i), t)) - A_S(i)$ is non-decreasing in $i$ for $i = 1 \cdots \beta$ and non-increasing in $i$ for $i = \beta+1 \cdots K$ for all $t = 1, 2, \ldots$ where $\beta$ is the number of non-negative states in $(S_1 \cdots S_K)$.
5. $f(i,t) = 1$ if the machine was not replaced at the previous inspection.

Then $\phi((S_1 \cdots S_K), \alpha, N) = \sum_{i=1}^{K} \phi^f(i,N)(S_i, \alpha, N)$ where $\phi^f(i,N)(S_i, \alpha, N)$ represents the minimum cost for a one machine $N$-stage $\alpha$-discounted problem with the cost of replacement $C(S) \equiv C(S_i, f(i,t))$, $t = 1 \cdots N$. 87
Proof: The proof will be by induction. By Lemma 3.2 for \( N = 1 \),
\( \ell^+ \leq \beta, \ell^- \leq \nu - \beta \),

\[
\hat{\rho}(\{S_1 \cdots S_K\}, \alpha, 1) = \min\left\{ \sum_{i=1}^{K} A_{S_i(1)} \cdot C(S_{(1)}, f((1), 1)) + \sum_{i=2}^{K} A_{S_i(1)} , \right.
\]
\[
\sum_{i=1}^{K} C(S_{(K)}, f((K), 1)) + \sum_{i=1}^{K-1} A_{S_i(1)} ; \cdots ;
\]
\[
\ell^+ \sum_{i=1}^{K} C(S_{(i)}, f((i), 1)) + \sum_{i=K-\ell^-+1}^{K} C(S_{(i)}, f((i), 1)) + \sum_{i=\ell^++1}^{K-\ell^-} A_{S_i(1)} ; \cdots
\]
\[
\sum_{i=1}^{K} C(S_{(i)}, f((i), 1)))
\]

\[
= \sum_{i=1}^{K} A_{S_i(i)} + \min\{0; C(S_{(1)}, f((1), 1)) - A_{S_{1}(1)}, C(S_{(K)}, f((K),1)) - A_{S_{(K)}} \}
\]
\[
\cdots \ell^+ \sum_{i=1}^{K} [C(S_{(i)}, f((i),1)) - A_{S_{i}(i)}] + \sum_{i=K-\ell^-+1}^{K} [C(S_{(i)}, f((i),1)) - A_{S_{i}(i)}] ; \cdots
\]
\[
\sum_{i=1}^{K} [C(S_{(i)}, f((i), 1)) - A_{S_{i}(i)}] .
\]

By Assumption 4, \( C(S_{(i)}, f((i), 1)) - A_{S_{i}(i)} \) is non-decreasing in \( i \)
for \( i = 1 \cdots \beta \) where \( \beta \) is the number of machines among \( \{S_1 \cdots S_K\} \)
in non-negative states. Therefore, \( C(S_{(i)}, f((i), 1)) \) crosses \( A_{S_{i}(i)} \)
at most once and then from below.
Therefore, there exists an $i_1^* \leq \beta$ such that

$$C(S_{(i)}, f((i), 1)) \leq A_{S_{(i)}}$$

for $i \leq i_1^*$,

$$C(S_{(i)}, f((i), 1)) > A_{S_{(i)}}$$

for $\beta \geq i > i_1^*$.  

89
If \( C(S(i), f((i), 1)) > A_S(i) \) for \( i = 1 \ldots \beta \) let \( i_1 = 0 \). Similarly \( C(S(i), f((i), 1)) \) can cross \( A_S(i) \) for \( i = \beta + 1 \ldots K \) at most once and then from above, consequently there exists \( J_1^* > \beta \) such that

\[
C(S(i), f((i), 1)) > A_S(i)
\]

for \( \beta < i < J_1^* \)

\[
C(S(i), f((i), 1)) \leq A_S(i)
\]

for \( i \geq J_1^* \)

If \( C(S(i), f((i), 1)) > A_S(i) \) for \( i = \beta + 1 \ldots K \) let \( J_1^* = K + 1 \). Therefore,

\[
C(S(i), f((i), 1)) - A_S(i) \leq 0
\]

for \( i \leq i_1^* \) and \( i \geq J_1^* \)

\[
C(S(i), f((i), 1)) - A_S(i) > 0
\]

for \( i_1^* < i < J_1^* \).

Obviously, then

\[
\min \{ 0; C(S_1, f((1), 1)) - A_S(1), C(S_K, f((K), 1)) - A_S(K) \}
\]

\[
\cdots \sum_{i=1}^{\ell_+} [C(S(i), f((i), 1)) - A_S(i)]
\]

\[
+ \sum_{i=K+1}^{K} [C(S(i), f((i), 1)) - A_S(i)] ;
\]

\[
\cdots \sum_{i=1}^{i_1^*} [C(S(i), f((i), 1)) - A_S(i)]
\]

\[
= \sum_{i=1}^{i_1^*} [C(S(i), f((i), 1)) - A_S(i)] + \sum_{i=j_1^*}^{K} [C(S(i), f((i), 1)) - A_S(i)]
\]

since the minimum must be attained by the sum of all the non-positive terms of the form \( C(S(i), f((i), 1)) - A_S(i) \), which occurs when \( \ell_+ = i_1^* \), \( \ell_- = j_1^* \). But since \( C(S(i), f((i), 1)) - A_S(i) > 0 \) for \( i_1^* < i < j_1^* \).
\[
\sum_{i=1}^{i^*} \left[ C(S(i), f((i), 1)) - A_{S(i)} \right] + \sum_{i=j_1^*}^{K} \left[ C(S(i), f((i), 1)) - A_{S(i)} \right] \\
= \sum_{i=1}^{K} \min\{C(S(i), f((i), 1)) - A_{S(i)} ; 0\}.
\]

Therefore,

\[
\phi((S_1 \ldots S_K), \alpha, 1) \\
= \sum_{i=1}^{K} A_{S(i)} + \sum_{i=1}^{K} \min\{C(S(i), f((i), 1)) - A_{S(i)} ; 0\} \\
= \sum_{i=1}^{K} \min\{C(S(i), f((i), 1)), A_{S(i)} \} \\
= \sum_{i=1}^{K} \min\{C(S_i, f(i, 1)), A_{S_i} \} = \sum_{i=1}^{K} \phi^{f(i, 1)}(S_i, \alpha, 1).
\]

Therefore the theorem is true for \( N = 1 \).

Assume it is true for \( N = \eta \). Then by the inductive hypothesis for \( \pi = \{S(1) \ldots S(\ell^+), S(\ell^-+1) \ldots S(K)\}, \)

\[
h^{\alpha}_{\eta}(\pi', \delta) = h^{\alpha}_{\eta}(0 \ldots 0, S(\ell^-+1) \ldots S(K-\ell^-), 0 \ldots 0) \\
= \sum_{S'(\ell^-+1)=-L}^{M} \ldots \sum_{S'(K-\ell^-)=-L}^{M} \prod_{i=\ell^-+1}^{K-\ell^-} p_{S(i)} S'(i) \\
\cdot \phi((0\ldots0), S'(\ell^-+1) \ldots S'(K-\ell^-), 0\ldots0), \alpha, \eta)
\]

continued
\[
\sum_{i=1}^{\ell+1} \phi^{f(i), \eta}(0, \alpha, \eta) + \sum_{i=K-\ell}^{K} \phi^{f((i), \eta)}(0, \alpha, \eta)
\]

where the last equality follows as a result of assumption 5, that is, as

\[
\phi^{f(i), \eta}(S_{(i)}, \alpha, \eta) = \phi^{l(S_{(i)}, \alpha, \eta)}
\]

for all those machines that were not replaced. Therefore,

\[
\sum_{i=1}^{\ell+1} \phi^{f((i), \eta)}(0, \alpha, \eta) - \sum_{i=K-\ell}^{K} \phi^{f((i), \eta)}(0, \alpha, \eta)
\]
\[
H^\alpha_\eta(S(1), f((i), \eta)) = \sum_{S(i) = 1}^M p_{S(i)} S'(i) \phi^f((i), \eta)(S'(i), \alpha, \eta)
\]
corresponds to \( h^\alpha_\eta(S(i)) \) for a one machine problem with cost of replacement \( C(S) = C(S(i), f((i), t)) \). Then

\[
\phi(S_1 \cdots S_K, \alpha, \eta+1)
\]

\[
= \min \left\{ \sum_{i=1}^K A_{S(i)} + h^\alpha_\eta(S_1 \cdots S_K); C(S_1, f((1), \eta+1)) + \sum_{i=2}^K A_{S(i)} + h^\alpha_\eta(S_2, \cdots, S_K), C(S_K, f((K), \eta+1)) + \sum_{i=1}^{K-1} A_{S(i)} + h^\alpha_\eta(S_1 \cdots S_{K-1}, 0); \ldots; \sum_{i=1}^K C(S(i), f((i), \eta+1)) + h^\alpha_\eta(0 \cdots 0) \right\}
\]

\[
= \sum_{i=1}^K A_{S(i)} + h^\alpha_\eta(S_1 \cdots S_K)
\]

\[
+ \min \left\{ 0; C(S_1, f((1), \eta+1)) - A_{S(1)} + \phi^f((1), \eta)(0, \alpha, \eta) \right\}
\]

\[
- H^\alpha_\eta(S_1, f((1), \eta)), C(S_K, f((K), \eta+1)) - A_S(K)
\]

\[
+ \phi^f((K), \eta)(0, \alpha, \eta) + h^\alpha_\eta(S_K, f((K), \eta))
\]

\[
\ldots; \sum_{i=1}^K \left[ C(S(i), f((i), \eta+1)) - A_{S(i)} + \phi^f((i), \eta)(0, \alpha, \eta)
\right.
\]

\[
- H^\alpha_\eta(S(i), f((i), \eta)) \right\}
\]

95
Again by Assumption 4, \( C(S(I), f((i), \eta + 1)) - A_{S(I)} \) is non-decreasing in \( i \) for \( i = 1 \cdots \beta \) and non-increasing in \( i \) for \( i = \beta + 1 \cdots K \). Furthermore, note that by Assumption 5, if a machine has not been repaired at the previous inspection \( f((i), \eta) = 1 \). Therefore,

\[
H^\alpha_{\eta}(S(I), f((i), \eta)) = \sum_{S'_I = -L}^{M} p_{S(I)} S'_I \sum_{S'_I = -L}^{M} p_{S(I)} S'_I \phi^f((i), \eta)(S'_I, \alpha, \eta) \\
= \sum_{S'_I = -L}^{M} p_{S(I)} S'_I \phi^1(S'_I, \alpha, \eta) \\
= H^\alpha_{\eta}(S'_I, 1) .
\]

Thus, since \( S(1) \geq S(2) \geq \cdots \geq S(\beta) \geq 0 > S(\beta + 1) \geq \cdots \geq S(K) \), Condition A guarantees that \( -H^\alpha_{\eta}(S(I), 1) \) is non-decreasing in \( i \) for \( i = 1 \cdots \beta \) and non-increasing in \( i \) for \( i = \beta + 1 \cdots K \).

Finally \( \phi^f((i), \eta)(0, \alpha, \eta) \) can be shown to be non-decreasing in \( i, i = 1 \cdots \beta \), and non-increasing in \( i, i = \beta + 1 \cdots K \), by a simple inductive argument:

\[
\phi^f((i), 1)(0, \alpha, 1) = \min[C(0, f((i), 1)) - A_0, 0] + A_0 .
\]

By Assumption 4, for all \( (S(1) \cdots S(K)) \), \( C(S(I), f((i), 1)) - A_S(I) \) is non-decreasing in \( i \) for \( i = 1 \cdots \beta \) and non-increasing in \( i \) for \( i = \beta + 1 \cdots K \). Let \( S(I) = \cdots = S(K) = 0 \). Then \( C(0, f((i), 1)) - A_0 \) is non-decreasing in \( i, i = 1 \cdots \beta \) and non-increasing in \( i \) for \( i = \beta + 1 \cdots K \). Now assume that the result is true for \( N = \eta \). Then
as a consequence of the inductive hypothesis and assumption 4,

\[ \phi^{f(i), \eta+1}(0, \alpha, \eta+1) \]

\[ = \min\{C(0, f((i), \eta+1)) + \phi^{f(i), \eta}(0, \alpha, \eta), A_0 + H_{\eta}^{\alpha}(0, 1)\} \]

\[ = A_0 + H_{\eta}^{\alpha}(0, 1) + \min\{0, C(0, f((i), \eta+1)) - A_0 + \phi^{f((i), \eta)}(0, \alpha, \eta) - H_{\eta}^{\alpha}(0, 1)\} \]

is obviously non-decreasing in \( i \) for \( i = 1 \cdots \beta \) and non-increasing in \( i \), for \( i = \beta+1 \cdots K \).

Therefore

\[ C(S(i), f((i), \eta+1)) - A_S(i) + \phi^{f((i), \eta)}(0, \alpha, \eta) - H_{\eta}^{\alpha}(S(i), f((i), \eta)) \]

is non-decreasing in \( i \) for \( i = 1 \cdots \beta \) and non-increasing in \( i \) for \( i = \beta+1 \cdots K \). Consequently, there exist states \( i_{\eta+1}^* \leq \beta \) and \( j_{\eta+1}^* > \beta \) such that

\[ C(S(i), f((i), \eta+1)) - A_S(i) + \phi^{f((i), \eta)}(0, \alpha, \eta) - H_{\eta}^{\alpha}(S(i), f((i), \eta)) \leq 0 \quad \text{for} \quad i \leq i_{\eta+1}^* \quad \text{and} \quad i \geq j_{\eta+1}^* \]

\[ C(S(i), f((i), \eta+1)) - A_S(i) + \phi^{f((i), \eta)}(0, \alpha, \eta) - H_{\eta}^{\alpha}(S(i), f((i), \eta)) > 0 \quad \text{for} \quad i_{\eta+1}^* < i < j_{\eta+1}^*. \]

Therefore,
\[ \phi[(s_1 \ldots s_K), \alpha, \eta+1] \]
\[ = \sum_{i=1}^{K} A_{s(i)} + \sum_{i=1}^{K} H_{\eta}(s(i), f((i), \eta)) \]
\[ + \sum_{i \leq i' \leq \eta+1} \mathbb{C}(s(i), f((i), \eta+1)) - A_{s(i)} + \phi_{\eta}(i, \eta)(0, \alpha, \eta) \]
\[ - H_{\eta}(s(i), f((i), \eta)) \]
\[ = \sum_{i=1}^{K} A_{s(i)} + \sum_{i=1}^{K} H_{\eta}(s(i), f((i), \eta)) \]
\[ + \sum_{i=1}^{K} \min \left\{ 0, \mathbb{C}(s(i), f((i), \eta+1)) - A_{s(i)} + \phi_{\eta}(i, \eta)(0, \alpha, \eta) \right\} \]
\[ - H_{\eta}(s(i), f((i), \eta)) \]
\[ = \sum_{i=1}^{K} \min \left\{ A_{s(i)} + H_{\eta}(s(i), f(i, \eta)), \mathbb{C}(s(i), f(i, \eta+1)) + \phi_{\eta}(i, \eta)(0, \alpha, \eta) \right\} \]
\[ = \sum_{i=1}^{K} \phi_{\eta}(i, \eta+1)(s(i), \alpha, \eta+1) . \]

Hence the theorem has been proved by induction. QED

Assumption 5, that \( f(i, t) = 1 \) if the machine was not repaired at the previous inspection, is open to a variety of interpretations. For instance, if a machine had deteriorated to a state \( s \) and was then repaired, there is some probability that the repair work would not have been effective. In that case, at the next inspection the machine may again have deteriorated enough to require actual replacement rather
than repair. It could be assumed that the probability that a machine, repaired at the previous inspection, will need actual replacement at the next inspection is well correlated with the machine's state of deterioration prior to its repair at the last inspection. The costs, $C(S_{(i)}, f((i, t)))$ would then represent the actual cost of repair plus the expected cost of replacement at the next inspection. Furthermore, once a machine has lasted one inspection period without repair, it might be assumed that the probability of actual replacement is constant, hence, $f(i, t) = 1$ if the machine was not repaired at the previous inspection.

**Theorem 3.4:** Assume that the following conditions hold:

1. $(p_{ij})_{i,j=-L}^{+M}$ satisfies Condition A.

2. $A_s$ is unimodal in $S$ with minimum at $S = 0$.

3. $C(S, i)$ is unimodal in $S$ with minimum at $S = 0$ for $i = 1 \ldots r$.

4. For all $(S_1 \cdots S_K)$, $C(S_{(i)}, f((i)) - A_{S_{(i)}}$ is non-decreasing in $i$ for $i = 1 \cdots \beta$ and non-increasing in $i$ for $i = \beta+1 \cdots K$ where $\beta$ is the number of non-negative states in $(S_1 \cdots S_K)$.

5. $f((i)) = 1$ if the machine was not repaired at the previous inspection.

Then the minimum cost for the $\omega$-stage $\alpha$-discounted problem with initial states $(S_1 \cdots S_K)$, $\hat{\rho}((S_1 \cdots S_K), \alpha)$, can be factored into $K$ one machine problems. That is,

$$\hat{\rho}((S_1 \cdots S_K); \alpha) = \sum_{i=1}^{K} \hat{\rho}^{f(i)}(S_i, \alpha)$$
where $\phi^f(i)(S_i, \alpha)$ represents the minimum cost for an $\infty$-stage $\alpha$-discounted problem with cost of replacement $C(S_i) \equiv C(S_i, f(i))$.

**Proof:** It is known that

$$\phi((S_1 \cdots S_K), \alpha) = \lim_{N \to \infty} \phi((S_1 \cdots S_K), \alpha, N).$$

But, by Theorem 3.3,

$$\phi((S_1 \cdots S_K), \alpha, N) = \sum_{i=1}^{K} \phi^f(i)(S_i, \alpha, N).$$

Therefore,

$$\phi((S_1 \cdots S_K), \alpha) = \lim_{N \to \infty} \sum_{i=1}^{K} \phi^f(i)(S_i, \alpha, N) = \sum_{i=1}^{K} \lim_{N \to \infty} \phi^f(i)(S_i, \alpha, N) = \sum_{i=1}^{K} \phi^f(i)(S_i, \alpha).$$

**Theorem 3.5:** Assume that Conditions 1-5 of Theorem 3.4 are true. Then let $\phi((S_1 \cdots S_K))$ represent the minimum expected average cost of the $K$ machine problem where the initial states are $(S_1 \cdots S_K)$, and let $\phi^f(i)(S_i)$ represent the minimum expected average cost for the one machine problem with initial state $S_i$ and cost of replacement $C(S_i, f(i))$. Then

$$\phi((S_1 \cdots S_K)) = \sum_{i=1}^{K} \phi^f(i)(S_i).$$
Proof:
\[
\phi((S_1, \ldots, S_K)) = \lim_{\alpha \to 1} (1-\alpha) \phi((S_1, \ldots, S_K), \alpha)
\]
\[
= \lim_{\alpha \to 1} (1-\alpha) \sum_{i=1}^{K} \phi^f(i)(S_i, \alpha)
\]
\[
= \sum_{i=1}^{K} \lim_{\alpha \to 1} (1-\alpha) \phi^f(i)(S_i, \alpha) = \sum_{i=1}^{K} \phi^f(i)(S_i)
\] QED

In their present form Conditions 2-5 are difficult to verify.

The following lemmas will provide necessary and sufficient conditions on the costs $A_S$ and $C(S,i)$, $i = 1 \cdots r$, and on the assignment rule $f(i,t)$ for Conditions 2-5 to hold.

Lemma 3.6: If Conditions 4 holds, for all $j = 1 \cdots r$, $C(S,j) - A_S$ is unimodal in $S$ with maximum at $S = 0$. 

![Diagram showing unimodal trend of $C(S,j) - A_S$ with maximum at $S = 0$.]
Proof: By Condition 4, for all \((S_1 \cdots S_K), C(S(i), f((i), t)) - A_{S(i)}\) is non-decreasing for \(i = 1 \cdots \beta\) and non-increasing for \(i = \beta+1 \cdots K\) where \(\beta\) is the number of non-negative states in \((S_1 \cdots S_K)\). Since the cost functions \(C(S,i), i = 1 \cdots r\), are given prior to the designation of the rule \(f(i,t)\) by the machine operator, the costs \(C(S,i), i = 1 \cdots r\), must be compatible with all rules. Hence let \(f(i) = k\). Then let \(S(1) = M, S(2) = M-1\). By Condition 4,

\[
C(M,k) - A_M \leq C(M-1,k) - A_{M-1}
\]

Repeat this procedure with \(S(1) = M-1, S(2) = M-2\), etc., to show that \(C(S,k) - A_S\) is non-increasing for \(S \geq 0\). Similarly, for \(S < 0\) let \(S(K) = -L, S(K-1) = -L+1\). Condition 4 then implies that

\[
C(-L,k) - A_{-L} \leq C(-L+1, k) - A_{-L+1}
\]

Repeating this procedure shows that \(C(S,k) - A_S\) is non-decreasing in \(S\) for \(S < 0\).

QED

Lemma 3.7: If Condition 4 holds, then \(C(S, f((i), t))\) is non-decreasing in \(i\) for \(S \geq 0\) and non-increasing in \(i\) for \(S < 0\).
Proof: Let $S_1 = S_2 = \cdots = S_K = S \geq 0$. Then by Condition 4,

\[ C(S(i), f((i), t)) - A_S(i) = C(S, f((i), t)) - A_S \]

is non-decreasing in $i$ for $i = 1 \cdots \beta$ where $\beta = K$. Thus $C(S, f((i), t))$ is non-decreasing in $i$ for $i = 1 \cdots K$. Similarly if $S < 0$,

\[ C(S(i), f((i), t)) - A_S(i) = C(S, f((i), t)) - A_S \]

is non-increasing in $i$ for $i = \beta + 1 \cdots K$ where $\beta = 0$ and, therefore, $C(S, f((i), t))$ is non-increasing in $i$ for $i = 1 \cdots K$ if $S < 0$. \[ \text{QED} \]

Lemma 3.8: Necessary and sufficient conditions for Condition 4 to be valid are

a. $C(S, j) - A_S$ is unimodal in $S$ with maximum at $S = 0$ for $j = 1 \cdots r$.

b. For $S \geq 0$, $C(S, f((i), t))$ is non-decreasing in $i$, $i = 1 \cdots K$.

For $S < 0$, $C(S, f((i), t))$ is non-increasing in $i$, $i = 1 \cdots K$.  

101
Proof: Necessity has been shown in Lemmas 5.6 and 5.7. Assume that Conditions a and b are true. Let \( (S_1 \cdots S_K) \) be given such that \( S_1 > S_2 > 0 \). Then

\[
C(S_1, f((1), t)) - A_S(1) \leq C(S_2, f((1), t)) - A_S(2)
\]
by Condition a. Furthermore Condition b implies that

$$C(S(2), f((1, t)) - A_S(2) \leq C(S(2), f((2, t)) - A_S(2).$$

Therefore

$$C(S(1), f((1, t)) - A_S(1) \leq C(S(2), f((2, t)) - A_S(2).$$

Continuing in this way it is easily shown that $C(S(i), f((i, t)) - A_S(i)$
is non-decreasing in $i$ for $i = 1 \cdots \beta$ where $\beta$ is the number of
non-negative terms in $(S_1 \cdots S_k)$. Similarly it can be shown that

$C(S(i), f((i, t)) - A_S(i)$ is non-increasing in $i$ for $i = \beta+1 \cdots k$.

QED

If we assume that the cost functions are indexed so that

$C(M,1) \leq C(M,2) \leq \cdots \leq C(M,r)$ and $C(-L,r) \leq C(-L, r-1) \leq \cdots \leq C(-L,1),

then the following lemma can be proved.

**Lemma 3.2:** Sufficient conditions that Condition 4 be valid are

a. $C(S,j) - A_S$ is unimodal with maximum at $S = 0$ for $j = 1 \cdots r$.

b. $C(S,i) \leq C(S, i+1)$ for $S \geq 0$, $i = 1 \cdots r-1$.

$c. f((1)) \leq f((2)) \leq \cdots \leq f((K)).$

**Proof:** Assume that $j_1 = f((i))$ and $j_2 = f((i+1))$. Then by
Condition c, $j_1 \leq j_2$. Furthermore Conditions b implies that

$C(S,j_1) \leq C(S,j_2)$ if $S \geq 0$ and $C(S,j_1) \geq C(S,j_2)$ if $S < 0$. Thus
\[
C(S, f((i))) \leq C(S, f((i+1))) \quad \text{for } S \geq 0
\]
and
\[
C(S, f((i))) \geq C(S, f((i+1))) \quad \text{for } S < 0.
\]

Hence \(C(S, f((i)))\) is non-decreasing in \(i\) for \(S \geq 0\) and non-increasing in \(i\) for \(S < 0\). Therefore by Lemma 3.8, Conditions a, b and c are sufficient conditions for Condition 4 of Theorem 3.3 to be valid.

QED

**Theorem 3.10:** Assume that the following conditions hold:

1. \(\{p_{ij}\}_{i,j=-L}^M\) satisfies Condition A.
2. \(A_S\) is unimodal in \(S\) with minimum at \(S = 0\).
3. \(C(S, i)\) is unimodal in \(S\) with minimum at \(S = 0\) for \(i = 1 \cdots r\).
4. \(C(S, i) - A_S\) is unimodal in \(S\) with maximum at \(S = 0\) for \(i = 1 \cdots r\).
5. \(C(S, i) \leq C(S, i+1)\) for \(S \geq 0, i = 1 \cdots r-1,\)
   \(C(S, i+1) \leq C(S, i)\) for \(S < 0, i = 1 \cdots r-1,\)
6. \(f(i) \leq f(2) \leq \cdots \leq f(K)\).
7. \(f(i) = 1 (f(i,t) = 1)\) if the machine was not repaired at the previous inspection.

Then
\[
\phi([S_1 \cdots S_K], \alpha, N) = \sum_{i=1}^K \phi^{f(i,N)}(S_i, \alpha, N),
\]
\[
\phi([S_1 \cdots S_K], \alpha) = \sum_{i=1}^K \phi^{f(i)}(S_i, \alpha),
\]
\[
\phi([S_1 \cdots S_K]) = \sum_{i=1}^K \phi^{f(i)}(S_i).
\]
When Condition 5 of Theorem 3.3 is included along with Conditions 2-4 of Theorem 3.3 a further simplification takes place. For those machines that were not replaced at the previous inspection the cost of replacement will be \( C(,1) \). But if the machine was repaired at the previous inspection, it is in state 0, when it is inspected at the next inspection. Thus the cost of replacement of a machine that was repaired at the previous inspection is \( C(0, f(i)) \). Therefore the properties of \( C(S, f(i)) \) where \( S \neq 0 \), \( f(i) \neq 1 \) are unimportant since these values are never encountered. Consequently Condition 4 of Theorem 3.3 can be simplified.

**Theorem 3.11:** Assume that the following conditions hold:

1. \( \{p_{ij}\}_{i,j=1}^{M} \) satisfies Condition A.
2. \( A_{S} \) is unimodal in \( S \) with minimum at \( S = 0 \).
3. \( C(S,1) \) is unimodal in \( S \) with minimum at \( S = 0 \).
4. \( C(S,1) - A_{S} \) is unimodal in \( S \) with maximum at \( S = 0 \).
5. \( C(0, f((i))) \) is non-decreasing in \( i \) such that for \( i = 1 \ldots K \), \( C(0,1) \leq C(0, f((i))) \).
6. \( f(i) = 1 \) (\( f(i,t) = 1 \)) if the machine was not repaired at the previous inspection.

Then

\[
\phi([S_1 \ldots S_K], \alpha, N) = \sum_{i=1}^{K} \phi^{f(i,N)}(S_i, \alpha, N),
\]

\[
\phi([S_1 \ldots S_K], \alpha) = \sum_{i=1}^{K} \phi^{f(i)}(S_i, \alpha),
\]

\[
\phi([S_1 \ldots S_K]) = \sum_{i=1}^{K} \phi^{f(i)}(S_i).
\]
Proof: Condition 4 of Theorems 3.3, 3.4 and 3.5 reads: for all \((S_1 \cdots S_K), C(S_{(i)}, f((i), t)) - A_{S_{(i)}}\) is non-decreasing in \(i\) for \(i = 1 \cdots \beta\) and non-increasing in \(i\) for \(i = \beta + 1 \cdots K\) where \(\beta\) is the number of non-negative states. But Condition 6 states that all machines that were not replaced at the previous inspection are replaced at cost \(C(\ , 1)\). Furthermore, all machines that were replaced are in state 0. Thus the initial states \((S_1 \cdots S_K)\) must be in the form:

\[(S_{i_1} \cdots S_{i_\gamma}, 0 \cdots 0, S_{i_\delta} \cdots S_{i_\xi})\]

where \(S_{i_1} \cdots S_{i_\gamma} \geq 0, S_{i_\delta} \cdots S_{i_\xi} < 0\) and the machines in states \(0 \cdots 0\) were replaced at the previous inspection. Therefore, the differences in the replacement and operating costs is

\[
C(S_{(i)}, f((i), t)) - A_{S_{(i)}} = \begin{cases} 
C(S_{(i)}, 1) - A_{S_{(i)}} & \text{for } S_{(i)} = S_{i_1} \cdots S_{i_\gamma} \\
C(0, f((i), t)) - A_{S_{(i)}} & \text{for } S_{(i)} = 0
\end{cases}
\]

Thus a necessary and sufficient condition that Condition 4 hold is that \(C(S, 1) - A_S\) be unimodal in \(S\) with a maximum at \(S = 0\) and that \(C(0, f((i), t))\) be non-decreasing in \(i\) with

\[
C(0, 1) - A_0 = \max \{C(S, 1) - A_S\} \leq C(0, f((i), t)) - A_0 .
\]

QED
Lemma 3.12: If \( C(0,1) < C(0,2) < \cdots < C(0,r) \), then Condition 5 of Theorem 3.11, that \( C(0, f((i))) \) be non-decreasing in \( i \), implies that

\[
f((1)) \leq f((2)) \leq \cdots \leq f((K))
\]

Proof: If \( f((i)) > f((i+1)) = j \) for some \( i \) and \( j \), then, since \( C(0,k) < C(0, k+1) \) for all \( k \), \( C(0,j) = C(0, f((i+1))) < C(0, j+1) \).

But \( f((i)) > f((i+1)) = j \) implies that \( f((i)) \geq j+1 \). Therefore \( C(0, f((i))) \geq C(0, j+1) \). Thus \( f((i)) > f((i+1)) \) implies that \( C(0, f((i+1))) < C(0, f((i))) \) contradicting Condition 5. QED

3.3. The One Machine Model \( \phi^{f(i)}(S_1) \):

The one machine models of Chapter II differ from the one machine model \( \phi^{f(i)}(S_1) \) in the assignment of the costs of replacement. In the models of Chapter II the costs of replacement are \( C(S), S = -L \cdots M \).

But in the model \( \phi^{f(i)}(S_1) \) the cost of replacing a machine in state \( S = 0 \) is dependent upon the rule \( f(i) \). It can be any one of \( r \) possible values, \( C(0,1), C(0,2), \ldots, C(0,r) \), depending upon the rule \( f(i) \). The differences between these two models is easily resolved, though. Augmenting the state space of the model \( \phi^{f(i)}(S_1) \) practically reduces it to the form of Chapter II.

Instead of limiting \( S \) to the states \( \{-L \cdots -1, 0 \cdots M\} \), assume that \( S \) can be in any of \( L+M+r \) states, \( \{-L \cdots -1, 0, 0_2, \ldots, 0_r, 1 \cdots M\} \).

The states \( \{-L \cdots -1, 1 \cdots M\} \) are exactly as before and the state \( 0 \) has been expanded into \( r \) separate states, each corresponding to a
replacement cost $C(0,i)$, $i = 1 \cdots r$. Thus, the cost of replacing a
machine in state $S \in \{-L \cdots -1, 0_1 \cdots 0_r, 1 \cdots M\}$ is $C(S,1)$ if
$S \in \{-L \cdots -1, 1 \cdots M\}$ and $C(S,i) = C(0,i)$ if $S = 0_i$, $i = 1 \cdots r$.
The operating costs are $A_S$ if $S \in \{-L \cdots -1, 1 \cdots M\}$ and
$A_S = A_0$ if $S = 0_i$, $i = 1 \cdots r$. If $\{p_{ij}\}_{i,j=-L}^M$ represents the
transition probability matrix of the original model, the augmented
model will have transition probability matrix $\{p'_{ij}\}_{i,j=-L}^M_{0_1} \cdots 0_r \cdots M$
where

$$
\begin{align*}
(3.3.1) \quad p'_{ij} = \begin{cases} 
    p_{ij} & \text{if } i \neq 0_k, j \neq 0_k, k = 1 \cdots r \\
    p_{0j} & \text{if } i = 0_k, j \neq 0_k, k = 1 \cdots r \\
    p_{00} & \text{if } i = 0_k, j = 0_1, k = 1 \cdots r \\
    0 & \text{if } i = 0_k, k = 1 \cdots r, j = 0_\ell, \ell = 2 \cdots r \\
    p_{i0} & \text{if } i \neq 0_k, k = 1 \cdots r, j = 0_1 \\
    0 & \text{if } i \neq 0_k, k = 1 \cdots r, j = 0_\ell, \ell = 2 \cdots r
\end{cases}
\end{align*}
$$

Note that the states $0_2 \cdots 0_r$ can only be entered by machine replace-
ment.

**Lemma 3.13:** If $\{p_{ij}\}_{i,j=-L}^M$ satisfies Condition A, then
$\{p'_{ij}\}_{i,j=-L}^M_{0_1} \cdots 0_r \cdots M$ also satisfies Condition A.
Proof: We must show that

1. \( r_k^1(i) = \sum_{j=k}^{M} p_{ij} \) is non-decreasing in \( i, i = 0_1 \cdots M \) for all \( k = 0_1 \cdots M \)

2. \( r_k^2(i) = \sum_{j=k}^{L} p_{ij} \) is non-decreasing in \( i, i = 0_1 \cdots M \) for all \( k = 0_1, -1 \cdots -L \)

3. \( r_k^1(i) = \sum_{j=k}^{M} p_{ij} \) is non-decreasing in \( i, i = 0_1, -1 \cdots -L \) for all \( k = 0_1 \cdots M \)

4. \( r_k^2(i) = \sum_{j=k}^{L} p_{ij} \) is non-decreasing in \( i, i = 0_1 \cdots -L \) for all \( k = 0_1 \cdots -L \)

where in cases 3 and 4, \( r_k^1(i) \) and \( r_k^2(i) \) are non-decreasing as \( i \) travels from \( 0_1 \) to \( -1 \) to \( -2 \) up to \( -L \). Consider case 1,

\[
\begin{align*}
    r_k^1(i) &= \sum_{j=k}^{M} p_{ij} \\
    &= \begin{cases} 
    \sum_{j=k}^{M} p_{ij} & \text{if } k \geq 1, i \geq 1 \\
    \sum_{j=1}^{M} p_{ij} & \text{if } k = 0_1, i \geq 1 \\
    \sum_{j=k}^{M} p_{ij} & \text{if } k = 0_1, i = 0_\ell, \ell = 1 \cdots r \\
    \sum_{j=1}^{M} p_{ij} & \text{if } k = 0_1, i = 0_\ell, \ell = 1 \cdots r \\
    \sum_{j=0}^{M} p_{ij} & \text{if } k \geq 1, i = 0_\ell, \ell = 1 \cdots r \\
    \sum_{j=0}^{M} p_{ij} & \text{if } k = 0_1, i = 0_\ell, \ell = 1 \cdots r 
    \end{cases}
\end{align*}
\]
Therefore,

\[
\begin{align*}
r_k^{\frac{1}{i}} &= \begin{cases} 
\sum_{j=k}^{M} p_{ij} & i \geq 1 \\
\sum_{j=0}^{M} p_{0j} & i = 0_1 \cdots 0_r
\end{cases} \\
\sum_{j=1}^{M} p_{ij} & i \geq 1 \\
\sum_{j=0}^{M} p_{0j} & i = 0_1 \cdots 0_r
\end{align*}
\]

if \( k \geq 1 \)

if \( k = 0_2 \cdots 0_r \)

if \( k = 0_1 \)

Obviously for all values of \( k = 0_1 \cdots M \), \( r_k^{\frac{1}{i}} \) is non-decreasing in \( i, i = 0_1, 0_2 \cdots M \), since \( \{p_{ij}\}_{i,j=0}^{M} \) satisfies Condition A.

Cases 2, 3 and 4 are shown to be true in an analogous way. QED

In order that a control limit replacement rule be optimal for the problem \( \beta^{(i)}(S) \), it is necessary for the costs of replacement to be unimodal with minimum at \( S = 0 \). Consequently, with the augmented state space the condition that \( C(0,i) \leq C(1,1), i = 1 \cdots r \) will be imposed.
Theorem 3.14: Assume that the following conditions hold:

1. \( \{p_{ij}\}_{i,j=-L}^{M} \) satisfies Condition A.

2. \( A_{S} \) is unimodal with minimum at \( S = 0 \).

3. \( C(S,1) \) is unimodal with minimum at \( S = 0 \),

\[
C(0,1) \leq C(0,2) \leq \cdots \leq C(0,r) \leq C(1,1) .
\]

4. \( |\Delta C(S,1)| \leq |\Delta A_{S}| \) where \( \Delta C(S,1) = C(S,1) - C(S-1,1) \).

Then for the N-stage \( \alpha \)-discounted problem with initial, state \( S_{i} \),

\( \left( \beta_{f}(i,N)(S_{i}, \alpha, N) \right) \) arising from the factorization of the K machine,

N-stage \( \alpha \)-discounted problem, there exist control limits \( i_{N}, k_{N}, j_{N} \)

such that the replacement rule \( R_{i_{N}, k_{N}, j_{N}} \) is optimal where \( R_{i_{N}, k_{N}, j_{N}} \)

is interpreted as: replace machine \( i \) in state \( S_{i} \) if and only if

\( S_{i} \leq i_{N} \) or \( S_{i} \geq j_{N} \), \( S_{i} \in \{-L \cdots -1, 1 \cdots M\} \), and replace machine \( i \)

in state \( S_{i} = 0 \) if and only if \( f(i,N) \leq k_{N} \). Furthermore, \( \beta_{f}(i,N)(S_{i}, \alpha, N) \)

is unimodal with minimum at \( S_{i} = 0 \).

Proof: As outlined above the problem \( \beta_{f}(i,N)(S_{i}, \alpha, N) \) is equivalent
to another problem with an augmented state space \( \{-L \cdots -1, O_{1} \cdots O_{r}, \)

\( 1 \cdots M\} \). In this equivalent problem the transition matrix

\( \{p_{ij}\}_{i=-L}^{M}, j=O_{1}, O_{2}, \cdots, M \) is given in equations (3.3.1). The operating
costs are

\[
A_{S}' = \begin{cases} 
A_{S} & \text{if } S \in \{-L \cdots -1, 1 \cdots M\} \\
A_{0} & \text{if } S \in \{O_{1} \cdots O_{r}\}
\end{cases}
\]
and the replacement costs are

\[ C'(S) = \begin{cases} 
  C(S, l) & \text{if } S \in \{-L, \cdots, -1, 1, \cdots, M\} \\
  C(0, l) = C(0, i) & \text{if } S \in \{0, \cdots, 0_r\}
\end{cases} \]

Condition 1 and Lemma 3.13 guarantee that \( \{p_{ij}\}_{i=-L, \cdots, L} \) satisfies Condition A. Conditions 2 and 3 imply that \( A'_S \) and \( C'(S) \) for \( S = -L, \cdots, -1, 0, \cdots, 0_r \cdots M \) are both unimodal with minimum at \( S = 0 \).

Finally, \( \Delta C'(S) \leq \Delta A'_S \) for \( S \in \{-L, \cdots, -1, 2, \cdots, M\} \). Thus the augmented problem is in the same form as the one machine model of Chapter II except for the fact that replacement is equivalent to transfer to state \( O_{f(i, t)} \) determined by rule \( f(i, t) \), \( t = 1, \cdots, N \).

Also the condition \( \Delta C'(S) \leq \Delta A'_S \) does not hold for \( S \in \{0, \cdots, 0_r\} \). Consequently this proof will be very similar to that of Theorem 2.8.

It will be by induction.

Let the minimum cost for the N-stage \( \alpha \)-discounted augmented problem be denoted by \( \phi^{f(i, N)}(S_1, \alpha, N) \). Then

\[ \phi^{f(i, 1)}(S_1, \alpha_1, 1) = \min \{A'_S, C'(S_1)\} \]

First note that for \( S_1 \in \{0, \cdots, 0_r\} \), \( A'_S = A_0 \) and if \( S_1 = 0_j \), \( C'(S_1) = C(0, j) \), \( j = 1, \cdots, r \). \( C(0, j) \) is non-decreasing in \( j \) and, therefore, either there exists \( k_1 \) such that
\[C(0, j) \leq A_0 \quad \text{for } j \leq k_1\]
\[C(0, j) > A_0 \quad \text{for } j > k_1\]

or \(C(0, j) > A_0\) for all \(j\). In the second case let \(k_1 = 0\). Therefore replacement is optimal if \(S_i \leq 0_{k_1}\) and non-replacement is optimal if \(S_i > 0_{k_1}\) where \(S_i \in \{0_1, \ldots, 0_r\}\). Furthermore by Lemma 2.7, \(C'(S)\) crosses \(A_S^i\), \(S = -L \cdots -1, 1 \cdots M\) at most twice, from below for \(S < 0\) and from above for \(S > 0\). Therefore there exist \(i_1 \leq -1\) and \(j_1 \geq 1\) such that

\[C'(S) \leq A_S^i \quad \text{if } S \leq i_1\]
\[C'(S) > A_S^i \quad \text{if } i_1 < S \leq -1\]
\[C'(S) \leq A_S^i \quad \text{if } j_1 \leq S\]
\[C'(S) > A_S^i \quad \text{if } 1 \leq S < j_1\]

Therefore, replacement is optimal if and only if \(S \leq i_1\) or \(S \geq j_1\) where \(S \in \{-L \cdots -1, 1 \cdots M\}\). Thus, the control limit rule \(R_{i_1, k_1, j_1}\) is optimal for the one stage problem \(\check{\rho}^{f(i,1)}(S_i, \alpha, 1)\). Furthermore, since \(A_S^i\) and \(C'(S)\) are both unimodal with minimum at \(S = 0_1\), \(\check{\rho}^{f(i,1)}(S_i, \alpha, 1) = \min\{A_S^i, C'(S)\}\) is also unimodal with minimum at \(S = 0_1\) (Lemma 2.6).
Now assume that the Theorem is true for $N = \eta$, that is,

\[ \phi^F(i, \eta)(S_1, \alpha, \eta) \]

has an optimal control limit rule $R_{i, \eta}^{k_\eta, j_\eta}$ and that $\phi^F(i, \eta)(S_1, \alpha, \eta)$ is unimodal with minimum at $S = O_1$.

\[
\phi^F(i, \eta + 1)(S_1, \alpha, \eta + 1)
\]

\[
= \min\{A^r_{S_1} + \alpha \sum_{j = -L}^{M} p_{S_1}^r \phi^F(i, \eta)(j, \alpha, \eta), c'(S_1) + \alpha' \phi^F(i, \eta)(O_\eta(i, \eta + 1), \alpha, \eta)\}.
\]

By the inductive hypothesis $\phi^F(i, \eta)(j, \alpha, \eta)$ is unimodal with minimum at $j = O_1$. Therefore Lemma 2.5 and 3.13 guarantee that

\[
\alpha \sum_{j = -L}^{M} p_{S_1}^r \phi^F(i, \eta)(j, \alpha, \eta)
\]
is also unimodal with minimum at $S_i = O_i$. Consequently, if 
$S_i \in \{-L \ldots -1, 1 \ldots M\}$ Lemma 2.7 implies that

$$C'(S_i) + \alpha' \phi^f(i, \eta)(O_f(i, \nu(i+1), \alpha, \eta)$$

crosses

$$A_{S_i}' + \alpha \sum_{j=-L}^{M} p_{S_i j}' \phi^f(j, \alpha, \eta)$$

at most twice, from below for $S_i \leq -1$ and from above for $S_i \geq 1$.
Therefore there exists $i_{\eta+1} \leq -1$ and $j_{\eta+1} \geq 1$ such that

$$C'(S_i) + \alpha' \phi^f(i, \eta)(O_f(i, \nu(i+1), \alpha, \eta) \leq A_{S_i}' + \alpha \sum_{j=-L}^{M} p_{S_i j}' \phi^f(j, \alpha, \eta)$$

if and only if $S_i \leq i_{\eta+1}$ or $S_i \geq j_{\eta+1}$ where $S_i \in \{-L \ldots -1, 1 \ldots M\}$.
Consequently replacement is optimal for $S_i \in \{-L \ldots -1, 1 \ldots M\}$ if
and only if $S_i \leq i_{\eta+1}$ or $S_i \geq j_{\eta+1}$.

If $S_i \in \{O_1 \ldots O_x\}$,

$$C'(S_i) + \alpha' \phi^f(i, \eta)(O_f(i, \nu(i+1), \alpha, \eta)$$

$$= C(0, \xi) + \alpha' \phi^f(i, \eta)(O_f(i, \nu(i+1), \alpha, \eta)$$

and

$$A_{S_i}' + \alpha \sum_{j=-L}^{M} p_{S_i j}' \phi^f(j, \alpha, \eta)$$

$$= A_0 + \alpha \sum_{j=-L}^{M} p_0 (j, \alpha, \eta)$$
Since \( C(0, \ell) \) is a non-decreasing function of \( \ell \),

\[
C'(S_i) + \alpha \sum_{j=-L}^{M} p_{S_i, j}^{i} \phi^f(i, \eta)(j, \alpha, \eta)
\]
crosses the constant function

\[
A_{S_i}^i + \alpha \sum_{j=-L}^{M} p_{S_i, j}^{i} \phi^f(i, \eta)(j, \alpha, \eta)
\]
at most once and then from below as \( S_i \) travels from \( 0_1 \) to \( 0_r \).

Therefore there exists \( k_{\eta+1} \) such that if \( S_i \in \{0_1 \cdots 0_r\} \) replacement is optimal if and only if \( S_i \leq k_{\eta+1} \). Note that if replacement is not optimal for any \( S_i \in \{0_1 \cdots 0_r\}, k_{\eta+1} = 0 \). Therefore the control limit replacement rule \( R_{i, \eta+1, k_{\eta+1}, j} \) is optimal for the problem \( \phi^f(i, \eta+1)(S_i, \alpha, \eta+1) \).

Finally

\[
\phi^f(i, \eta+1)(S_i, \alpha, \eta+1) = \min\{A_{S_i}^i + \alpha \sum_{j=-L}^{M} p_{S_i, j}^{i} \phi^f(i, \eta)(j, \alpha, \eta), C'(S_i)
\]

\[
+ \alpha \phi^f(i, \eta)(0(i, \eta+1), \alpha, \eta)\}
\]
is unimodal with minimum at \( S_i = 0_1 \) since \( A_{S_i}^i \), \( C'(S_i) \) and

\[
\alpha \sum_{j=-L}^{M} p_{S_i, j}^{i} \phi^f(i, \eta)(j, \alpha, \eta)
\]
are all unimodal with minima at \( S_i = 0_1 \).

Thus the theorem has been proved by induction for the augmented problem. Consequently, the theorem is true for the equivalent problem \( \phi^f(i, \eta)(S_i, \alpha, \eta) \).

QED
Theorem 3.15: Assume that the following conditions hold:

1. \((P_{ij})_{i,j=-L \cdots M}\) satisfies Condition A.
2. \(A_S\) is unimodal with minimum at \(S = 0\).
3. \(C(S,1)\) is unimodal with minimum at \(S = 0\),
\[
C(0,1) \leq C(0,2) \leq \cdots \leq C(0,r) \leq C(1,1).
\]
4. \(|\Delta C(S,1)| \leq |\Delta A_S|\).

Then there exist optimal control limit replacement rules, \(R_{i,\alpha}^k\), \(j,\alpha\) and \(R_{i,\alpha}^{k*,j*}\) which minimize the \(\infty\)-stage \(\alpha\)-discounted problem \((\rho^f(i)(S_i, \alpha))\) and the long run average cost problem \((\rho^f(i)(S_i, \alpha))\)
respectively where both of these problems arise from the factorization of the \(K\) machine problems. The replacement rule \(R_{i,\alpha}^k\), \(j,\alpha\) means:
replace machine \(i\) in state \(S_i\) if and only if \(S_i \leq i,\alpha\) or \(S_i \geq j,\alpha\)
where \(S_i \in \{-L \cdots -1, 1 \cdots M\}\) and replace machine \(i\) in state \(S_i = 0\) if and only if \(f(i) \leq k,\alpha\).

Proof: The proof is exactly the same as the proofs of Theorems 2.13 and 2.17. QED

Analogous theorems to those in Chapter II using Condition B can also be proved for the problems,
\[
\rho^f(i,N)(S_i, \alpha, N), \quad \rho^f(i)(S_i, \alpha) \quad \text{and} \quad \rho^f(i)(S_i).
\]
Condition 7 of Theorem 3.10 and Condition 6 of Theorem 3.11, that is, the requirement that \( f(i,t) = 1 \) if the machine was not replaced at the previous inspection, reduces the importance of the rule \( f(i,t) \) to the times when the machine has just been replaced, i.e., when the \( i \)th machine is already in state 0. Since all machines that actually make use of the rule \( f(i,t) \) are all in state 0, the order statistics \((i), (i+1), \ldots, \) can be assigned arbitrarily among those machines in state 0. Thus, machine \( i \) in state \( S_i = 0 \) can be assigned any replacement cost among \( C(0,1), C(0,2), \ldots, C(0,r) \) by rule \( f(i,t) \).

Consequently, Condition 5 of Theorem 3.11, the condition that \( C(0,f((i),t)) \) be non-decreasing, and Lemma 3.12 requiring \( f((i),t) \leq f((i+1),t) \), are essentially meaningless. Therefore, the states of the other machines, as far as they would affect the order statistics, do not hinder the arbitrary assignment of the costs of replacement by the rule \( f(i,t) \).

To actually calculate the control limits for each machine it is necessary to know \( f(i,t) \) for those times when machine \( i \) is in state 0. Once this rule is known it would be possible to actually calculate the control limits using dynamic programming methods or linear programming depending upon the character of the rule.

3.4. Conclusion:

As was noted in Chapter I, a system composed of \( K \) components can be modeled in a generalized form of the models presented in Chapter II. Unfortunately the maintenance policies derived from such a model are quite complex even though they are control limit maintenance policies.
For example, the simple two component system presented in the Introduction required $M_1$ control limits to characterize its maintenance policy where $M_1$ was the number of states of one of the components. Furthermore, in this generalized form of the model of Chapter II, replacement is defined as replacement of all components.

In order to circumvent this restrictive form of the replacement policies and their complexity, Chapter III analyzed a $K$ machine system from a different point of view. Its aim was to factor the problem into $K$ subproblems, each of whose maintenance policies would be characterized by one control limit. Such a factorization results in simple repair policies which allow system components to be replaced individually. As was to be expected, such simple optimal maintenance policies only occur under the most restrictive conditions. These conditions essentially reduce the $K$ machine model to a simple one in which the discretionary assignment of replacement costs by rule $f(i,t)$ is limited to those times at which the machine has just been replaced. Other conditions sufficiently strong to result in the factorization of the $K$ machine model are likely to exist. But it is also likely that they will be equally restrictive, leading to a similar simplification of the inherent complexities of the $K$ machine model.
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**Maintenance Models for Stochastically Failing Equipment**

Two maintenance models are developed for machines subject to two distinct modes of failure. In the single machine model, the equipment is inspected at discrete time intervals and classified according to its deterioration, positive states representing deterioration towards one mode of failure and negative states towards the other. Transition among states of deterioration is governed by a Markov chain. At each inspection there is the option of replacing the machine at a cost dependent upon the state of the equipment. If no replacement is made, an operating cost, also state dependent, is charged. Sufficient conditions on the Markov chain and the replacement and operating costs are developed which guarantee that the optimal maintenance policy minimizing total discounted and long run average costs is a control limit replacement rule: replace the machine if and only if its state is higher than a designated positive state or lower than a designated negative state. Characterization of the optimal policy allows for efficient computation through linear programming. A parametric linear programming analysis indicates how the policy varies with different costs.

A multicomponent system in which different maintenance procedures can be assigned to individual components is also formulated. Factorization of the system into individual single component systems enables one to formulate optimal control limit maintenance rules for each component.
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</tr>
</thead>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MARKOV CHAINS</td>
<td></td>
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<tr>
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<td>LINEAR PROGRAMMING</td>
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