DETERIORATING MARKOV PROCESSES UNDER UNCERTAINTY

BY

DONALD BARRY ROSENFIELD

TECHNICAL REPORT NO. 162
MAY 6, 1974

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UNDER CONTRACT NO0014-67-A-0112-0052 (NR-042-002)
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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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Gerald J. Lieberman, Project Director

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CHAPTER 1

INTRODUCTION

1.1. Introduction

This paper is concerned with the general problem of finding optimal repair and inspection policies for deteriorating systems. The types of systems that we consider include deteriorating machinery, inventory systems, or any other system that involves periodic repair, replacement, or upgrading. We develop a mathematical model for these systems and seek mathematically optimal policies. Due to widespread applications, these types of deteriorating systems have been studied fairly extensively in the literature. Unfortunately, the existing results are incomplete for the case of costly inspection, or where the underlying process is only partially observable to the operator. Since it is felt that imperfect information and costly inspection are more realistic assumptions for such deteriorating systems, we have developed a mathematical model that incorporates this assumption.

Before discussing the details of the relevant models, we introduce several examples of failing processes. The first and most obvious example is a failing mechanical or electrical device. A factory machine, automobile, or a communications network is typical of this example. As the device deteriorates, it becomes increasingly expensive to operate. It then becomes beneficial (in terms of total cost) to repair the device. A second example is an inventory system. As stock in the system becomes depleted, the chances for losses due to shortages increase, and, even though inventory changes are smaller, overall operating costs might be increasing. In this sense the system is
deteriorating. Assuming for the moment that ordering is done up to a
certain level, it must be decided when to upgrade (order up) the system.
Any other object or facility that undergoes deterioration in terms of
operating costs can furnish an additional example.

To solve these types of problems, we use the technique of dynamic
programming and optimize successive stages recursively. We call the
process a Markovian decision process if the underlying state space is
Markovian. In the machine example, the optimal course of action is
formulated recursively in terms of Markovian states in future stages.
The Markovian states represent relative states of deterioration.
Several authors have studied the basic theory of dynamic programming
and Markovian decision processes, including Howard [11], Derman [5],
and Blackwell [2]. It is useful to note the three types of optimization
that exist in this theory: the n-period problem, the infinite-horizon
(discounted) problem, and the average-cost problem. Consider a multi-
stage decision problem lasting several time periods. The state i
measures the deterioration. A policy s stipulates an action for
each possible state and each time period. A discount factor \( \alpha \leq 1 \)
discounts future costs to the present by a factor of \( \alpha \) per period.
If \( \beta \) is the interest rate, then \( \alpha = 1/(1+\beta) \). The n-period cost
starting in state i, using policy s, and with discount factor \( \alpha \) is
denoted as \( C^n(i,s,\alpha) \). The n-period problem is to minimize this cost
over the set of possible s. The infinite-horizon (discounted) cost
for \( \alpha < 1 \) is \( \lim_{n \to \infty} C^n(i,s,\alpha) \). The infinite-horizon problem is to
minimize this cost over the set of policies. Finally, the average
cost starting in state i and using policy s is \( \lim_{n \to \infty} \frac{C^n(i,s,1)}{n} \).
The average-cost problem is to minimize this cost over the set of policies. The existence of the relevant limits and minima is part of the theory of dynamic programming and is discussed in [2], [5], and [21]. In Chapter 2, we discuss these issues as they relate to our model.

1.2. Previous Models of Deteriorating Processes

Many Markovian models of deteriorating processes with perfect information have been developed in the literature, and it is useful to outline some of these. For these models, the state of the process is a measure of the deterioration. Two types of cost exist. First, there are repair costs, which are the costs necessary to bring the process to the best state. Second, there are operating costs, which are the expected costs due to penalties for bad items (or due to shortages in inventory problems) when repair is not chosen. In general, one would expect these costs to increase with deterioration. The question that one asks is what are optimal policies (repair or operation for each state) for these models for the n-period, infinite-horizon and average-cost criteria. One relevant model is that of Derman [4 and 6]. In the Derman model, there are \( N+1 \) states, states \( 0, 1, \ldots, N \), repair costs \( C_j, j = 0, 1, \ldots, N \) and operating costs \( L_j, j = 0, 1, \ldots, N \), and a Markov matrix \( P \) that describes the deterioration of the process under operation. Under the assumptions that \( C_j \) is non-decreasing in \( j \), \( L_j \) is non-decreasing in \( j \), \( C_j - L_j \) is non-increasing in \( j \) (i.e., one-period repair costs decrease relative to
operating costs) and that \( \sum_{j=k}^{N} p_{ij} \) is non-decreasing in \( i \), for all \( k \).
(increasing failure rate), Derman shows that the optimal policy, under
any of the mentioned criteria, is to operate if the state \( j \) is less
than some number \( j^* \) (where \( j^* \) depends on the criterion) and to
repair otherwise. The increasing failure rate assumption (IFR) is

\[
\begin{aligned}
| & \quad \text{operate} \quad \text{operate} \quad | & \quad \text{repair} \quad \text{repair} \quad | \\
\hline
0 & 1 & 2 & 3 & j^*-1 & j^* & \cdots & N-1 & N \\
\end{aligned}
\]

Figure 1: General Optimal Policy in Derman Model

a general aging assumption developed by Derman. If the state is a
measure of deterioration, then in an IFR chain, the higher the state
the greater the chance of additional deterioration. The general
model that we develop has similar assumptions to the Derman model.
Extensions of the Derman model are found in [17] and [27].

Another relevant model is that of Girshick and Rubin [9]. In
their model, the process is in either a good state or a bad state. If
in the good state the process can enter the bad state with some fixed
probability. Once the process enters the bad state, it remains there
until repaired. There are repair costs and operating costs similar
to those of the Derman model. In addition, the operator observes some
measurable quality characteristic of the process each time period.
An example of such characteristic is the quality (good or bad) of the
produced items of a machine. This quality characteristic is a random variable whose distribution depends on the state. The operator can assess, at each time period, a Bayesian a posteriori probability that the process is in the bad state. Girshick and Rubin show that, under the average-cost criterion, there is some $p^*$ such that the optimal policy is to operate if $p \leq p^*$ and repair if $p > p^*$. Taylor [26] has also studied this model. Note that the optimal policy here is similar in form to that of the Derman model, as is shown in Figure 2.

![Diagram of operate and repair]

Bayesian probability

Figure 2: General Optimal Policy in Girshick and Rubin Model

The form of this policy is also similar to optimal policies in certain types of dynamic inventory problems. The policy here is called $(s, S)$ and stipulates to increase the inventory level to $S$ if the inventory falls below $s$. These types of problems are found, for example, in Arrow, Karlin, and Scarf [1]. There is also a great deal of other work in deteriorating processes and stochastically failing equipment, such as Eppen [7] and Jorgenson et al. [14].

Despite the breadth of elegant results for deteriorating processes such as those above, it is felt that the problems of imperfect and costly
information have not been properly incorporated in existing models. In the references above, the operator either knows the state with certainty (perfect information), as in the Derman model, or the inspection information has no cost, as in the Girshick and Rubin model. It is felt that perfect or no-cost information is an unrealistic assumption, for the real-world operator often has to pay an extra inspection cost to know the real state of his process or receive quality-control information. This paper is primarily concerned with the problem of incorporation of imperfect information into a model of a deteriorating process. Some work has also been done for deteriorating processes with imperfect information, but results are by no means complete.

Before introducing our model, we describe some of this work.

Girshick and Rubin analyzed a model with costly inspection, for they also considered their same model with the observable quality characteristic being an option with non-zero cost. Consequently, there are three options: repair, operation with inspection of the observable characteristic, and operation without inspection. Girshick and Rubin conjectured that there are numbers $p_1^*$ and $p_2^*$ such that the optimal policy is to operate without inspection if the Bayesian probability $p < p_1^*$, operate with inspection if $p_1^* \leq p < p_2^*$ and otherwise repair (see Figure 3). That is, they conjectured that the space $p \in [0,1]$ could be broken up into three regions. Several years later, Taylor [26] found a counterexample with four regions (see Figure 4). This general problem has not yet been solved. Tafeen [25], who found results for a similar problem for special cases, and Ray [20], have also studied the problem. Derman [6] has also
Figure 3: Conjectured Optimal Policy in General Girshick and Rubin Model

Figure 4: General Optimal Policy in Ross Model and in Taylor Counterexample
looked at a multi-state problem with imperfect information, but assumed inspection must be performed before repair. Iglehart and Morey [13] have studied inventory systems with imperfect asset information and devised the policy of a buffer stock of extra inventory.

Another relevant work in the area of imperfect information and costly inspection is by Ross [23]. Ross' approach is general, as is ours, and our model is structured in a manner similar to his. For each state in the countable state space, there is an inspection cost, a repair cost, and an operation without inspection cost. Upon repair, the process goes to the best state. Upon operation with inspection, the operator finds out with certainty the state of the process, and upon operation without inspection the operator finds out no new information. The operator can classify his knowledge by a Bayesian a posteriori probability vector for each state, and can devise a policy for each value of that vector. Ross sets up the structure for the general model and finds the structure of the optimal policy (for all three criteria) for the good-bad case. Note that the latter is the Girshick and Rubin model under the assumption that observation of the quality characteristic gives one the precise state. If $p$ is the probability of being in the bad state, the optimal policy in this case is operate without inspection if $0 \leq p < p_1^*$ or $p_2^* \leq p < p_3^*$, operate with inspection if $p_1^* \leq p \leq p_2^*$ and repair otherwise, for some choice of $0 \leq p_1^* \leq p_2^* \leq p_3^* \leq 1$. This is the four-region case of Taylor's counter-example.

Smallwood and Sondik [24] also examine a general finite-state model with imperfect information. They provide an algorithm for computation and some general results about the $n$-period cost but do not provide definitive policy results.
1.3 Introduction of Model

Our model is similar to both the Derman model and the Ross model. We assume that the underlying process is a discrete-state Markov process with several states and that the manager does not necessarily know the real state of the system. He can either pay an inspection cost to determine (with certainty) the present state, he can repair the system (that is, bring the process to the lowest, or best state), or he can leave the process alone for at least another period. As the inspection process is instantaneous, the manager can find out the present state and make a decision afterwards (but in the same time period) as to whether to repair the machine. We have attempted to characterize the form of optimal policies for the n-period problem, the infinite-horizon problem, and the average-cost problem for this type of structure.

We feel that such a model is a reasonable model for the examples that we have mentioned. For a deteriorating machine, there are often several states that the machine can be in. For example, if a machine consists of \( N \) equivalent components, the state is the number of failed components. Furthermore, the operator may not know the condition of the machine, but an inspection may give him the precise state. For the inventory example assume that demands are discrete and that whenever inventory is increased, it must be increased to some level \( S \). (When this constraint does not exist, such stipulation is still often optimal, as in the \((s,S)\) policy.) So the possible inventory levels are \((S, S-1, \ldots, 0)\), and the "repair" option brings the process up to \( S \). If the manager is not receiving information on the demands in each
period (when, for example, his office is in a different city than the warehouse), then the system has the structure of our model. It is felt that such a condition of imperfect information is reasonable.

We now introduce notation for our model. The underlying Markov process has states 0 through N, where 0 is the best state, the state to which the process returns after repair, and N is the worst state. (For the inventory model, state i represents an inventory level of N-i.) In similar studies state N is denoted to be failure, and upon reaching this state, repair is required. That assumption is not made here, and the manager may leave the process in state N. Associated with each state j, j = 0, 1, ..., N are operating costs \( L_j \geq 0 \) and repair costs \( C_j \geq 0 \). That is, if the process is not repaired, an operating cost \( L_j \) is incurred in the ensuing period, and the process proceeds to a new state by the transition matrix \( P = [p_{ij}] \). If the machine is repaired, a cost \( C_j \) is incurred in the ensuing time period and the process proceeds to state 0. We make the following assumptions about \( C_j \), \( L_j \), and \( P \):

1) \( C_j \) is non-decreasing in \( j \).
2) \( L_j \) is non-decreasing in \( j \).
3) \( C_j - L_j \) is non-increasing in \( j \).
4) \( C_0 \) is greater than \( L_0 \).
5) For the set of recurrent states, there is a number \( \delta > 0 \) and an integer \( r \) such that \( \min_{0 \leq i \leq n} [p^r]_{ij} \geq \delta \) for any recurrent state \( j \).
6) \( \sum_{j=k}^{N} p_{ij} \) is non-decreasing in \( i, k = 0, 1, \ldots, N \) (increasing failure rate, or IFR).

Assumption 5 assures that the Markov chain has a single ergodic class.
The six assumptions above are analogous to those of Derman. Derman, of course, developed the intuitive increasing failure rate concept. In addition to the above we make the additional assumption:

7) $p_{ij} = 0$ for $j$ less than $i$.

This assumption seems quite reasonable. The states $0, 1, \ldots, N$ represent increasing levels of deterioration, and it does not seem logical that the process can improve if one leaves it alone. We shall prove our results for a much more general condition than 7) in Chapter 5, but 7) represents the most important subclass of this general condition.

If we impose a stronger condition than Assumption 6, we obtain stronger results. In Chapter 4, we assume this assumption:

6') $P$ is totally positive of order two.

Increasing failure rate is a somewhat intuitive assumption. We shall further discuss the stronger 6') in Chapter 4.

The structure of the process (i.e., the states $0, 1, \ldots, N$ and the repair and holding costs) describe the process under perfect information, which we are not assuming. As far as the manager is concerned, two integers are necessary to describe the state: the first denoting the state of the Markov process that the manager last knew with certainty, and the second being the number of time units since that knowledge was gained. Thus, in the expanded description, observed state $(i, k)$ means that $k$ units ago the simple process was in state $i$ and the manager has not known (with certainty) the state
of the simple process since then. This state space is graphed in Figure 5. In observed state \((i,k)\) the manager can assert that the

![Graph of state space]

last real state known with certainty

**Figure 5: Observed State Space of Process**

simple process is in state \(m\) with probability \(p_{im}^k\) (where this denotes the \(i\)-th element of \(p^k\)). He then has three options open to him. He can leave the process alone (no action), in which case he incurs an expected operating cost of \(\sum_{m=0}^{N} p_{im}^k L_m\) and proceeds to state \((i, k+1)\). He can repair the machine (repair), in which case he incurs an expected repair cost of \(\sum_{m=0}^{N} p_{im}^k C_m\) and proceeds to observed state \((0,0)\). Finally, he can pay an inspection cost \(M > 0\) and determine exactly what state the simple process is in presently (inspection), and decide on the basis of this new information whether to repair or not.

An equivalent formulation of the inspection option is that, if one wants an inspection in state \((i, k+1)\), one orders an inspection
in state \((i,k)\). In that case, the manager pays an inspection cost \(M\) plus an expected holding cost \(\sum_{m=0}^{N} p_{im} L_{m}\) and proceeds to state \((m,0)\) with probability \(p_{im}^{k+1}\). Thus when we think of an instantaneous inspection report given, that report is actually ordered one period earlier. Since there is never a need for an instantaneous inspection in any state \((i,0)\), the new formulation is sufficient.

Given these considerations, we can now write the functional equations to determine the optimal \(n\)-period cost given an initial state \((i,k)\). Denote \(C_{n}^{n}(i,k)\) as this cost and let \(0 \leq \alpha \leq 1\) be the discount factor. Then

\[
C_{n}^{n}(i,k) = \min\left( \sum_{j=0}^{N} p_{ij} C_{j} + \alpha C_{n-1}^{n-1}(0,0), \sum_{j=0}^{N} p_{ij} L_{j} + \alpha C_{n-1}^{n-1}(i, k+1), \sum_{j=0}^{N} p_{ij} L_{j} + M + \alpha \sum_{j=0}^{N} p_{ij} C_{n-1}^{n-1}(j,0) \right)
\]

and \(C_{0}^{n}(0,0) = 0\). Note that the factors in the minimization correspond to repair, no action, and an inspection respectively. The existence of a non-randomized optimal policy inherent in this expression is immediate.

Our intentions are to find the forms of the optimal policies for the \(n\)-period problem, the infinite-horizon (discounted) problem, and the average-cost problem. Let \(C_{n}^{n}(i,k,\alpha,s)\) denote the \(n\)-period cost given an initial state of \((i,k)\), a discount factor \(\alpha\), and a policy \(s\). The infinite-horizon cost is \(C(i,k,\alpha,s) = \lim_{n \to \infty} C_{n}^{n}(i,k,\alpha,s)\). We are interested in finding the minima (over \(s\)) of \(C_{n}^{n}(i,k,\alpha,s)\), \(C(i,k,\alpha,s)\), and \(\lim_{n \to \infty} [C_{n}^{n}(i,k,1,s)/n]\). The latter is the average cost
per period under policy \( s \). A policy \( s \) specifies what action to take given a state \((i,k)\) and a number of periods from a given starting time.

In the next chapter, we develop some basic mathematical results for these optimality criteria. In Chapter 3, we derive the structure of regions where repair is optimal, and show that such optimal policy is of a certain intuitive type. In Chapter 4, we derive the complete structure of an optimal policy using Assumption 6' instead of 6. In Chapter 5, we derive results for a more general condition than \( \gamma \). In Chapter 6, we look at special cases, and in Chapter 7, we offer some conclusions. The results in the following chapters are mainly theoretical and state that an optimal policy is of a certain type. It is felt that such theoretical analysis is useful in a number of important ways. First, it often facilitates computation for particular examples by dictating the form of the answer. Second, it enhances our understanding of the general problem. Finally, it makes us aware of the types of policies that are cost-beneficial. We give the background for these theoretical results in the next chapter.
CHAPTER 2

BASIC BACKGROUND RESULTS

In this chapter we present some theoretical background, much of which relates to dynamic programming theory. The material is in three sections. In the first section, we present results relating to the infinite-horizon (discounted) problem. These results include recursive equations, limiting behavior, and possible existence of optimality of inspection. In the second section, we present results relating to the average-cost problem. In the third section we analyze limiting behavior of cost functions with variance in the uncertainty integer of the observed state. Except for non-negativity of costs, and except when otherwise explicitly stated, the results in this chapter do not depend on any of our assumptions.

Some of the material in the first two sections is based on the theory of dynamic programming. Many of the ideas are found in Blackwell [2], Taylor [26], Ross [21], and for dynamic inventory problems, Iglehart [12].

2.1. The Infinite-Horizon Problem

The general approach in solving the infinite-horizon problem is to extend the results of the n-period problem. In order to do this, we need recursive equations for the optimal infinite-horizon cost. We present these in the first part of this section and show how the n-period cost converges.
We first define \( C(i, k, \alpha, s) = \lim_{n \to \infty} C^n(i, k, \alpha, s) \) (and we will show that this limit exists). The optimal infinite-horizon cost is \( C(i, k) = \inf_s C(i, k, \alpha, s) \), where \( s \), the policy, specifies an action in every state for every time period. We show that \( C(i, k) \) is the limit of \( C^n(i, k) \) and satisfies recursive equations. The proof is similar to a proof for infinite-horizon costs for an inventory problem in [12]. A different proof of this, using a fixed point theorem, is found in Taylor [26]. We first show that \( C^n(i, k, \alpha, s) \) has a limit for \( 0 \leq \alpha < 1 \). We will later show that the infimum of \( C(i, k) \) is a minimum.

**Lemma 2.1.1.** As \( n \to \infty \), \( C^n(i, k, \alpha, s) \) converges uniformly in \( i, k, s \); where \( \alpha \) is less than one.

**Proof.** Note that

\[
C^n(i, k, \alpha, s) = \sum_{\ell=0}^{n-1} \alpha^\ell \sum_{m=0}^N p_{ikm}(\ell, s) F(i, k, m, \ell, s) ,
\]

where

\[
p_{ikm}(\ell, s) = \text{Pr}\{\text{being in state } m \text{ after } \ell \text{ transitions, starting from state } (i, k) \text{ under policy } s\} ,
\]

and

\[
F(i, k, m, \ell, s) = \{\text{cost of being in state } m \text{ after } \ell \text{ transitions starting from state } (i, k) \text{ under policy } s\} .
\]

Note that \( F(i, k, m, \ell, s) = C_m, L_m, \) or \( L_m + M \). It is obvious from this expression, since all costs are non-negative, that \( C^n(i, k, \alpha, s) \)
is non-decreasing in \( n \). In addition, the right hand side is bounded by

\[
\sum_{j=0}^{n-1} \alpha^j \max \{ I_N + M, C_N \} \leq \frac{\max \{ I_N + M, C_N \}}{1 - \alpha}, \quad \forall i, k, \alpha, s.
\]

Thus each \( C^n(i, k, \alpha, s) \) has a limit. From (1), this limit is

\[
C(i, k, \alpha, s) = \sum_{\ell=0}^{\infty} \alpha^\ell \sum_{m=0}^{N} P_{ikm}(\ell, s) F(i, k, m, \ell, s).
\]

The only thing left is to show that the convergence is uniform:

\[
C(i, k, \alpha, s) - C^n(i, k, \alpha, s)
\]

\[
= \sum_{\ell=n}^{\infty} \alpha^\ell \sum_{m=0}^{N} P_{ikm}(\ell, s) F(i, k, m, \ell, s)
\]

\[
\leq \sum_{\ell=n}^{\infty} \alpha^\ell \max \{ I_N + M, C_N \} = \frac{a \text{ constant}}{1 - \alpha} \alpha^n
\]

\[
= c \alpha^n \rightarrow 0
\]

uniformly, where \( c \) is a constant.

We now obtain a similar result for \( C^n(i, k) \) where \( C^n(i, k) = \min_s C^n(i, k, \alpha, s) \).

**Lemma 2.1.2.** \( C^n(i, k) \) converges up to \( C(i, k) \).
Proof: First note that \( C^1(i,k) \geq 0 = C^0(i,k) \). Assuming that
\( C^n(i,k) \geq C^{n-1}(i,k) \) inductively,

\[
C^{n+1}(i,k) = \min \left\{ \sum_{j=0}^{N} p_{ij} c_j + \alpha C^n(0,0), \sum_{j=0}^{N} p_{ij} L_j \sum_{j=0}^{N} \frac{k+1}{p_{ij}} C^n(j,0) \right\}
\geq \min \left\{ \sum_{j=0}^{N} p_{ij} c_j + \alpha C^{n-1}(0,0), \sum_{j=0}^{N} p_{ij} L_j \sum_{j=0}^{N} \frac{k+1}{p_{ij}} C^{n-1}(j,0) \right\} \text{ by induction}

= C^n(i,k).
\]

Thus \( C^n(i,k) \) is non-decreasing. From the proof of Lemma 2.1.1,

\[
C^n(i,k,\alpha,s) \leq \frac{\text{constant}}{1-\alpha}.
\]

Thus

\[
\min_{s} C^n(i,k,\alpha,s) = C^n(i,k) \leq \frac{\text{constant}}{1-\alpha},
\]

so \( C^n(i,k) \) is bounded above. Consequently, it has a limit. Thus we want to show that

\[
\lim_{n \to \infty} C^n(i,k) = C(i,k),
\]

or
\[ \lim_{n \to \infty} \min_{s} C^n(i, k, \alpha, s) = \inf_{s} \lim_{n \to \infty} C^n(i, k, \alpha, s). \]

Now

\[ C^n(i, k, \alpha, s) \geq \min_{s} C^n(i, k, \alpha, s), \]

so

\[ \lim_{n \to \infty} C^n(i, k, \alpha, s) \geq \lim_{n \to \infty} \min_{s} C^n(i, k, \alpha, s). \]

Since it is true for all \( s \), it is true for the infimum, so

\[ \inf_{s} \lim_{n \to \infty} C^n(i, k, \alpha, s) \geq \lim_{n \to \infty} \min_{s} C^n(i, k, \alpha, s). \quad (1) \]

By uniform convergence in Lemma 2.1.1, for any \( \epsilon \) we can make \( n \) big enough so that

\[ C^n(i, k, \alpha, s) \geq \lim_{n \to \infty} C^n(i, k, \alpha, s) - \epsilon, \]

so

\[ \min_{s} C^n(i, k, \alpha, s) \geq \inf_{s} \lim_{n \to \infty} C^n(i, k, \alpha, s) - \epsilon. \]

Since it is true for all sufficiently large \( n \), it is true for the limit, so

\[ \lim_{n \to \infty} \min_{s} C^n(i, k, \alpha, s) \geq \inf_{s} \lim_{n \to \infty} C^n(i, k, \alpha, s) - \epsilon. \]

But \( \epsilon \) was arbitrary so we can let \( \epsilon \to 0 \). So
\[
\lim_{n \to \infty} \min_s C^n(i,k,\alpha,s) \geq \inf_{s \to \infty} \lim_{n \to \infty} C^n(i,k,\alpha,s) .
\]

Combining this with (1) gives the desired result. \( \square \)

We now can show that \( C(i,k) \) satisfies a set of recursive equations. Let

\[
T(i,k,\Psi) = \min \left\{ \sum_{j=0}^{N} p_{ij}^k C_j + \alpha \Psi(0,0), \sum_{j=0}^{N} p_{ij}^k L_j + \alpha \Psi(i, k+1), \sum_{j=0}^{N} p_{ij}^k L_j + M + \alpha \sum_{j=0}^{N} p_{ij}^{k+1} \Psi(j,0) \right\},
\]

where the three terms correspond to repair, no action, and inspection respectively. Note that \( C^n(i,k) = T(i,k,C^{n-1}) \).

**Theorem 2.1.1.** (Taylor): \( C(i,k) \) satisfies the recursive equations \( C(i,k) = T(i,k,C) \), and hence the infimum of \( C(i,k) \) is also a minimum.

**Proof:** First note that \( T \) is monotonic with respect to \( \Psi \). Thus \( C^n(i,k) = T(i,k,C^{n-1}) \leq T(i,k,C) \) by Lemma 2.1.2. Taking limits of both sides,

\[
C(i,k) \leq T(i,k,C) . \quad (1)
\]

As \( C(i,k) \) is bounded from the proof of Lemma 2.1.2, we can find a \( B \) such that
\[ C^1(i,k) \geq C(i,k) - B. \]

Assuming inductively that \( C^{n-1}(i,k) \geq C(i,k) - \alpha^{n-2} B \),

\[ C^n(i,k) = T(i,k,C^{n-1}) \geq T(i,k,C - \alpha^{n-2} B) \]

\[ = T(i,k,C) - \alpha^{n-1} B \quad \text{by definition of } T \quad (2) \]

\[ \geq C(i,k) - \alpha^{n-1} B \quad \text{by (1)}. \]

So

\[ C^n(i,k) \geq C(i,k) - \alpha^{n-1} B. \]

By (2),

\[ C(i,k) \geq C^n(i,k) \geq T(i,k,C) - \alpha^{n-1} B. \quad (3) \]

By (1) and (3),

\[ T(i,k,C) - \alpha^{n-1} B \leq C(i,k) \leq T(i,k,C). \]

Taking limits of the above gives the result. \( \square \)

**Corollary 2.1.1.** \( C^n(i,k) \) converges up to \( C(i,k) \) uniformly in \( i \) and \( k \) with \( C(i,k) - C^n(i,k) \leq \alpha^n B. \)

**Proof:** This follows directly from equations (1) and (3). \( \square \)
At this point, we have a description of the behavior of \( C^n(i,k) \) and \( C(i,k) \). Furthermore, the terms in the minimization of the recursive equations indicate optimal policy. The existence of an optimal policy for the infinite-horizon case that is stationary (identical in every time period) is due originally to Blackwell. The existence of non-randomized optimal rules inherent in the recursive equations for the infinite-horizon case is also due originally to Blackwell. In the following chapters we try to find the form of the optimal policies.

In addition to existence of optimal policies and recursive equations, the other issue for the infinite-horizon problem (and n-period problem) is whether inspection is a real possibility. The rest of this section shows that inspection can be an optimal action. We first need two lemmas.

**Lemma 2.1.3.** For \( P = [p_{ij}] \) any transition matrix, and \( a_j \) and \( b_j \) functions; \( j = 0, 1, \ldots, N; i = 0, 1, \ldots, N, \)

\[
a) \sum_{j=0}^{N} p_{ij} \min(a_j, b_j) \leq \min(\sum_{j=0}^{N} p_{ij} a_j, \sum_{j=0}^{N} p_{ij} b_j),
\]

b) let

\[ J = \{ j : b_j < a_j \} \]

\[ K = \{ j : a_j < b_j \} \]

Then \( J \) and \( K \) are both accessible from state \( i \) under transition matrix \( P \) if

\[
\sum_{j=0}^{N} p_{ij} \min(a_j, b_j) < \min(\sum_{j=0}^{N} p_{ij} a_j, \sum_{j=0}^{N} p_{ij} b_j).
\]

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Proof: a) \[
\min(a_j, b_j) \leq b_j ,
\]
\[
\min(a_j, b_j) \leq a_j ,
\]
so
\[
\sum_{j=0}^{N} p_{ij} \min(a_j, b_j) \leq \sum_{j=0}^{N} p_{ij} b_j ,
\]
and
\[
\sum_{j=0}^{N} p_{ij} \min(a_j, b_j) \leq \sum_{j=0}^{N} p_{ij} a_j .
\]
Hence the assertion holds.

b) \[
\min(a_j, b_j) < b_j \quad \text{if and only if} \quad j \in K ,
\]
\[
\min(a_j, b_j) < a_j \quad \text{if and only if} \quad j \in J ,
\]
so
\[
\sum_{j=0}^{N} (p_{ij} \min(a_j, b_j)) < \sum_{j=0}^{N} p_{ij} a_j \quad \text{if and only if} \quad J \text{ accessible from } i ,
\]
and
\[
\sum_{j=0}^{N} (p_{ij} \min(a_j, b_j)) < \sum_{j=0}^{N} p_{ij} b_j \quad \text{if and only if} \quad K \text{ accessible from } i .
\]
So
\[
\sum_{j=0}^{N} p_{ij} \min(a_j, b_j) < \min(\sum_{j=0}^{N} p_{ij} a_j, \sum_{j=0}^{N} p_{ij} b_j)
\]
if and only if both \( J \) and \( K \) accessible.
Using this, we prove the next lemma:

**Lemma 2.1.4.** For all $i \in [0, 1, \ldots, N]$, $k \in [0, 1, \ldots, \infty)$, and $m \in [0, 1, \ldots, \infty)$,

$$ C^n(i, k+m) \geq \sum_{j=0}^{N} p_{ij}^k C^n(j, m), $$

$$ C(i, k+m) \geq \sum_{j=0}^{N} p_{ij}^k C(j, m). $$

**Proof:** We prove the first part by induction:

$n = 1$: Let

$$ a_i = \sum_{j=0}^{N} p_{ij}^m C_j, \quad b_i = \sum_{j=0}^{N} p_{ij}^m L_j, $$

and consider the transition matrix $P^k$. So

$$ C^1(i, k+m) = \min(\sum_{j=0}^{N} p_{ij}^{k+m} C_j, \sum_{j=0}^{N} p_{ij}^{k+m} L_j) $$

$$ = \min(\sum_{j=0}^{N} p_{ij}^k a_j, \sum_{j=0}^{N} p_{ij}^k b_j). $$

Since $\sum_{j=0}^{N} p_{ij}^k C^1(j, m) = \sum_{j=0}^{N} p_{ij}^k \min(a_j, b_j)$, the assertion follows by Lemma 2.1.3.
Case n: Let

\[ a_i = \sum_{j=0}^{N} p_{ij}^m c_j + \alpha c^{n-1}(0,0), \]

\[ b_i = \sum_{j=0}^{N} p_{ij}^m l_j + \min\{M + \alpha \sum_{j=0}^{N} p_{ij}^{m+1} c^{n-1}(j,0), \alpha c^{n-1}(i,\ell+1)\}. \]

Note that \( \sum_{j=0}^{N} p_{ij}^k c^{n}(j,m) = \sum_{j=0}^{N} p_{ij}^k \min(a_j, b_j) \) and apply Lemma 2.1.3.

Now

\[ \sum_{j=0}^{N} p_{ij}^k a_j = \sum_{j=0}^{N} p_{ij}^{k+m} c_j + \alpha c^{n-1}(0,0), \]

and

\[ \sum_{j=0}^{N} p_{ij}^k b_j = \sum_{j=0}^{N} p_{ij}^{k+m} l_j + \sum_{j=0}^{N} p_{ij}^k \min(M + \alpha \sum_{j=0}^{N} p_{ij}^{m+1} c^{n-1}(i,\ell,0), \alpha c^{n-1}(j,\ell+1)) \]

\[ \leq \sum_{j=0}^{N} p_{ij}^{k+m} l_j + \min(M + \alpha \sum_{j=0}^{N} p_{ij}^{k+m+1} c^{n-1}(j,0), \alpha c^{n-1}(i,\ell+1)) \]

by induction and Lemma 2.1.3.

So we see that

\[ \sum_{j=0}^{N} p_{ij}^k c^{n}(j,m) = \sum_{j=0}^{N} p_{ij}^k \min(a_j, b_j) \]

\[ \leq \min(\sum_{j=0}^{N} p_{ij}^k a_j, \sum_{j=0}^{N} p_{ij}^k b_j) \leq c^{n}(i, k+m) \]

by (1) and by definition.
The second part follows directly by letting $n \to \infty$ on both sides of the first part.

We are now ready to prove a theorem that tells you when you may find inspection points. The idea of the theorem is that inspection points are possible. The first part is for the $n$-period problem, and the second part is for the infinite-horizon problem.

**Theorem 2.1.2.** a) Assume that for any real state $i$ there are points $(j, 0)$ for which in the $(n-1)$ problem it is strictly optimal to repair and points $(j', 0)$ for which it is strictly suboptimal to repair. Assume also that the sets $[j]$ and $[j']$ are both accessible from $i$ in $k+1$ steps for some $k$ such that $(i, k)$ is not a repair-optimal point in the $n$-period problem. Then for $M$ small enough, there is a point $(i, k)$ for which inspection is optimal.

b) Assume that for any real state $i$ there are points $(j, 0)$ for which repair is strictly optimal in the infinite-horizon problem and points $(j', 0)$ for which repair is strictly suboptimal. Assume also that the sets $[j]$ and $[j']$ are both accessible from $i$ in $k+1$ steps such that $(i, k)$ is not a repair-optimal point. Then for $M$ small enough, there is a point $(i, k)$ for which inspection is optimal.

**Proof:** a) From the recursive expression it suffices to prove that for some $k$ such that $(i, k)$ is not a repair-optimal point,

$$
\sum_{j=0}^{N} \sum_{i,j}^{k+1} c_{i,j}^{n-1} < c_{i,k+1}^{n-1}.
$$
Let
\[ a_j = c_j + \alpha c^{n-2}(0,0), \]
\[ b_j = l_j + \min(M + \alpha \sum_{j=0}^{N} p_{ij} c^{n-2}(\ell,0), \alpha c^{n-2}(j,1)) . \]

Note that \( c^{n-1}(j,0) = \min(a_j, b_j) \). By hypothesis, there is a \( k \) not a repair-optimal point such that the hypotheses of Lemma 2.1.3 hold under transition matrix \( p^{k+1} \). So
\[
\sum_{j=0}^{N} p_{ij} c^{n-1}(j,0) = \sum_{j=0}^{N} p_{ij} c^{n-1}(j,0) \leq \min(\sum_{j=0}^{N} p_{ij} a_j, \sum_{j=0}^{N} p_{ij} b_j)
\]
\[
= \min(\sum_{j=0}^{N} p_{ij} c_j + \alpha c^{n-2}(0,0), \sum_{j=0}^{N} p_{ij} l_j + \sum_{j=0}^{N} p_{ij} \min(M + \sum_{\ell=0}^{N} p_{\ell j} c^{n-2}(0,0), \alpha c^{n-2}(j,1)))
\]
\[
\leq \min(\sum_{j=0}^{N} p_{ij} c_j + \alpha c^{n-2}(0,0), \sum_{j=0}^{N} p_{ij} l_j + \min(M + \alpha \sum_{j=0}^{N} p_{ij} c^{n-2}(j,0), \alpha c^{n-2}(i,k+2)))
\]

by successively applying Lemmas 2.1.3 and 2.1.4,

\[ = c^{n-1}(i, k+1) . \]

So
\[
\sum_{j=0}^{N} p_{ij} c^{n-1}(j,0) < c^{n-1}(i, k+1) .
\]
b) The proof of b) follows exactly as a) except that infinite-horizon expressions are used instead of expressions from the \((n-1)\)-period problem.

This completes the theoretical background for the infinite-horizon problem.

2.2. The Average-Cost Problem

Our procedure in finding optimal policies for the infinite-horizon (discounted) problem is to use the recursive equations. For the average-cost case, we extend the results of the infinite-horizon case. The key result in this extension is the following: Let \( \phi(i,k,s) \) be the average cost starting in observed state \((i,k)\) and using policy \(s\). That is,

\[
\phi(i,k,s) = \lim_{n \to \infty} \frac{C(i,k,1,s)}{n}.
\]

Then

\[
\phi(i,k,s) = \lim_{\alpha \to 1} \left(1 - \alpha\right) C(i,k,\alpha,s).
\]  

(See, for example, Hardy [10], p. 155 for the underlying results.) Consequently, we do not have explicit need for recursive equations for the average-cost case. In fact, recursive equations are not necessarily satisfied for the average-cost case. Existence of such equations is nevertheless of theoretical interest and is useful in
practical calculations. We will thus draw on a result of dynamic programming theory to establish the equations for certain cases. Let 
\( H_{\alpha}(i,k) = C(i,k) - C(0,0) \), for any \( \alpha \). Then by a simple manipulation

\[ H_{\alpha}(i,k) + (1-\alpha) C(0,0) = T(i,k,h_{\alpha}) . \]

The relevant theorem is the following:

**Theorem 2.2.1. (Ross)** If \( H_{\alpha}(i,k) \) is uniformly bounded for \( \alpha < 1 \) from above and below then there exists a bounded solution to the functional equations

\[ g + \psi(i,k) = T(i,k,\psi) \quad \text{for} \quad \alpha = 1 \quad . \quad (1) \]

Furthermore, there is a sequence \( \alpha_i \) such that

\[ H_{\alpha_i}(i,k) \to \psi(i,k) \quad \text{and} \quad (1-\alpha_i) C(0,0) \to g \]

and if \( s^* \) is a rule that minimizes \( T \) in (1) then \( s^* \) is the optimal policy for the average cost problem and \( g \) is the optimal average cost.

For a proof see Ross [21], Theorems 1.1 - 1.2, and Derman [5].

One does need the condition that \( H_{\alpha}(i,k) \) is uniformly bounded, and this is not necessarily the case. It should be noted that
\[ H_\alpha(i,k) = T(i,k,C) - C(0,0) \quad \text{by the recursive equations} \]

\[ \leq \sum_{j=0}^{N} p_{ij} C_j + \alpha C(0,0) - C(0,0) \quad \text{also by the recursive equations} \]

\[ \leq (N+1) \max_i C_i - (1-\alpha) C(0,0) \leq (N+1) \max_i C_i \]

by Lemma 2.1.2.

Also,

\[ H_\alpha(i,k) = C(i,k) - C(0,0) \geq \sum_{j=0}^{N} p_{ij} (C(j,0) - C(0,0)) \]

by Lemma 2.1.4.

As it will be shown, \( C(j,0) \geq C(0,0) \) under our assumptions and so \( H_\alpha(i,k) \) will be bound uniformly. Some assumptions are needed to show \( C(j,0) - C(0,0) \) is bounded below and so the hypothesis of Theorem 2.2.1 does not necessarily hold.

If the hypothesis of Theorem 2.2.1 is satisfied, a result analogous to Theorem 2.1.2 follows from the recursive equations for \( \Psi(i,k) \) for the average-cost case. We omit the result as the statement and proof are nearly identical to that of Theorem 2.1.2. The main point is that if \( H_\alpha(i,k) \) is uniformly bounded then inspection-optimal states are possible for the average-cost problem. Aside from this point, we will not use average-cost recursive equations in the following chapters.
2.3. Limiting Behavior of $C^n(i,k)$ and $C(i,k)$ in $k$.

One interesting aspect of the model is the infinite state space, as $k \in [0, 1, \ldots, \infty)$ for the state $(i,k)$. It is thus of interest how $C^n(i,k)$ and $C(i,k)$ behave as $k \to \infty$. This behavior is also important in showing the results of the following chapters. This section is intended to analyze this behavior. For this section, we assume Assumption 5, the ergodicity assumption. It then follows that under the transition matrix $P$, the $N+1$ real states have limiting state probabilities $\pi_j$. That is,

$$p^n_{ij} \to \pi_j \quad \text{as } n \to \infty, \forall i.$$ 

We start by defining recursively $C^n_{\ast}$:

$$C^n_{\ast} = \min \left\{ \sum_{j=0}^{N} \pi_j c_j + \alpha c^{n-1}_{\ast}(0,0), \sum_{j=0}^{N} \pi_j l_j + \alpha C^n_{\ast} \right\},$$

$$= \left\{ \sum_{j=0}^{N} \pi_j l_j + M + \alpha \sum_{j=0}^{N} \pi_j c^{n-1}(j,0) \right\}$$

$$= T^\ast(C_{\ast}^{n-1}, c^{n-1}).$$

We use the notation for $\pi_j$ throughout this paper.

**Lemma 2.3.1.** $C^n(i,k) \to C^n_{\ast}$ as $k \to \infty$.

**Proof: (Induction)**

$$C^1(i,k) = \min \left\{ \sum_{j=0}^{N} p_{ij} c_j, \sum_{j=0}^{N} p_{ij} l_j \right\}$$

$$\rightarrow \min \left\{ \sum_{j=0}^{N} \pi_j c_j, \sum_{j=0}^{N} \pi_j l_j \right\} = C^1_{\ast}.$$
$\lim_{k \to \infty} C_n(i,k)$

$= \lim_{k \to \infty} \min\left( \sum_{j=0}^{N} p_{ij} c_j + \alpha c_{n-1}(0,0), \sum_{j=0}^{N} p_{ij} l_j + \alpha c_{n-1}(i, k+1), \sum_{j=0}^{N} p_{ij} l_j + M + \alpha \sum_{j=0}^{N} p_{ij} c_{n-1}(j,0) \right)$.

By taking the limits inside and by induction, the result follows. □

**Lemma 2.3.2.** $C_* = \lim_{n \to \infty} C_n^*$ exists for $0 < \alpha < 1$.

**Proof:** By Lemma 2.1.2, $C_{n+1}(i,k) \geq C_n(i,k)$. Taking the limit of both sides of this,

$C_{n+1}^* \geq C_n^*$.

By the recursive expression,

$C_n^* \leq \sum_{j=0}^{N} \pi_j c_j + \alpha c_{n-1}(0,0)$

$\leq \sum_{j=0}^{N} \pi_j c_j + \frac{\text{constant}}{1-\alpha}$ from the proof of Lemma 2.1.2.

So $C_n^*$ is non-decreasing and is bounded, so it has a limit. □

**Theorem 2.3.1.** $\lim_{k \to \infty} C(i,k) = C_*$ and $C_*$ satisfies $C_* = T^*(C_*, C)$.  

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Proof: From Lemma 2.1.2 and Corollary 2.1.1,

\[ C^n(i,k) + \alpha^{n-1} B \geq C(i,k) \geq C^n(i,k) \, . \]

Take the \( \overline{\lim} \) and \( \underline{\lim} \) of the above equation as \( k \to \infty \). So

\[ C^n_\alpha + \alpha^{n-1} B \geq \overline{\lim}_{k \to \infty} C(i,k) \geq C^n_\alpha \, , \tag{1} \]

and

\[ C^n_\alpha + \alpha^{n-1} B \geq \underline{\lim}_{k \to \infty} C(i,k) \geq C^n_\alpha \, . \]

Now take the limit as \( n \to \infty \),

\[ C_\alpha \geq \overline{\lim} C(i,k) \geq C_\alpha \, , \]

and

\[ C_\alpha \geq \underline{\lim} C(i,k) \geq C_\alpha \, . \]

Thus

\[ C_\alpha = \lim_{k \to \infty} C(i,k) \, . \]

Now note that

\[ C^n_\alpha \geq C_\alpha - \alpha^{n-1} B \, , \tag{2} \]

from above. Also,

\[ C^n_\alpha = T^*(C_\alpha^{n-1} , C^{n-1}) \leq T^*(C_\alpha , C) \, , \tag{3} \]
since \( C_{i,k} \leq C(i,k) \), \( C_{\infty} \leq C_* \) (by Lemma 2.3.1), and \( T^* \) is monotonic in both of its arguments. So

\[
C_* = \lim_{n \to \infty} C^n_* \leq T^*(C_*^n, 0) \quad \text{by (3)},
\]

and

\[
C_* = \lim_{n \to \infty} C^n_\infty = \lim_{n \to \infty} T^*(C_*^{n-1}, C^{n-1}) \geq \lim_{n \to \infty} (T^*(C_*^n, C) - \alpha^n B) \quad \text{by (2) and Corollary 2.1.1}
\]

\[
= T^*(C_*^0, C),
\]

so

\[
T^*(C_*^0, C) \leq C_* \leq T^*(C_*^0, C) \Rightarrow C_* = T^*(C_*^0, C) \quad \Box.
\]

These results are somewhat simpler under Assumption 7, where \( p_{ij} = 0 \) for \( j < i \).

**Corollary 2.3.1.** Under Assumption 7,

\[
C^n(N,k) = C^n_* \quad \text{and} \quad C(N,k) = C_* \quad \text{all} \quad k,
\]

and thus

\[
C^n(i,k) \to C^n(N,0) \quad \text{and} \quad C(i,k) \to C(N,0) \quad \text{as} \quad k \to \infty.
\]

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Proof: By Assumption 5, the \( \pi_j \) exist. Using the backwards equations, we want a solution to

\[
\pi = \pi P.
\]  

(1)

Note that \( \pi = \{0, 0, \ldots, 1\} \) solves (1) since \( p_{ij} = 0, j < i \) and \( p_{NN} = 1. \) Since steady-state probabilities are unique (by the fact that there is one ergodic class due to Assumption 5), \( \pi = \delta_{iN}. \) In addition, \( p_{NN} = 1, \) and assuming inductively that \( p_{NN}^{k-1} = 1, p_{NN}^k = p_{NN}^{k-1} = 1. \) So \( \pi_N = p_{NN}^k = 1. \) Thus

\[
C^1(N,k) = \min\{C_N, L_N\} = C_*^1,
\]

and by induction,

\[
C^n(N,k) = \min\{C_N + \alpha C^{n-1}(O,0), L_N + \alpha C^{n-1}(N,1), L_N + M + \alpha C^{n-1}(N,0)\}
\]

\[
= \min\{C_N + \alpha C^{n-1}(O,0), L_N + \alpha C_*^{n-1}, L_N + M + \alpha C^{n-1}(N,0)\}
\]

\[
= C_*^n.
\]

Taking limits, \( C(N,k) = C_*, \) which completes the result.

Without dealing very much with assumptions, we have outlined the theoretical background for the different optimality criteria and cost functions. In the next chapter, we use our assumptions to determine the structure of the optimal policy.
CHAPTER 3

POLICY RESULTS UNDER ASSUMPTIONS 1 THROUGH 7

In this chapter we begin to develop structures of optimal policies. The results here depend critically on the monotonicity of certain functions that we define. Derman uses such monotonicity in his model, as do Wagner [27] and Kolesar [17] in their extensions. The difference is that we have two degrees of variation, and thus the monotonicities and policy structures become more difficult and complicated. In this chapter, we concentrate on where repair is optimal. (In the next chapter we derive a complete policy using Assumption 6'.) We shall now define what we shall call a monotonic policy. The major purpose of this chapter is to show that an optimal policy under Assumptions 1 through 7 is a monotonic one under our three criteria.

Definition. Let the observed state be \((i,k)\) and let \(n\) denote the number of remaining periods \((n = \infty\) for the infinite-horizon and average-cost cases). If there exist numbers \(k^*(i,n)\) (\(k^*(i)\) for the average-cost case) such that a policy consists of repair if \(k \geq k^*(i,n)\) and either inspection or no action otherwise, and furthermore, \(k^*(i,n)\) is non-decreasing in \(i\), then we call the \(k^*(i,n)\) critical numbers and call such a policy a monotonic policy. □

We feel that a monotonic policy is a relatively intuitive one. Notice that under such a policy the procedure is reasonably simple. Starting from some state \((i,0)\) (repair is \((0,0)\)), the manager performs no action until, for some state \((i,k)\), inspection is optimal. He then
inspects, going to a new state \((j,0)\). If no such state \((i,k)\) exists, he repairs at \((i, k^*(i,n))\). If \(k^*(i,n) = 0\), he immediately repairs.

A typical monotonic policy is graphed in Figure 6.

The first section of this chapter gives preliminary results under Assumptions 1 through 6. These results hold also in Chapter 5, and include basic lemmas dealing with monotonicity and conditions under which the optimal policy is a trivial one. In the following section we show that monotonic policies are optimal under the three optimality criteria and under Assumptions 1 through 7.

![Figure 6: A Typical Monotonic Policy](image-url)
3.1. Preliminary Results Under Assumptions 1 Through 6

The results in this section hold in Chapter 5 also since Assumption 7 does not hold. In Part A, we look at some basic monotonicity results and in Part B, we see when the optimal policy is always trivially no action. Assumptions 1 through 6 hold throughout this section.

A. Monotonicity Results

The results of this entire chapter depend on the monotonicity of certain functions. We now establish some of these monotonicities. Assumption 6, increasing failure rate, as mentioned before, is used by Derman and others. Intuitively, it means that the mass of the transition matrix goes toward (deteriorates to) the higher states. Assumption 6 is a critical assumption for these results.

**Lemma 3.1.1.** (Derman) Let $p_j(i)$ be a discrete probability function of a random variable, $j = 0, 1, \ldots, N$; that depends on an index $i$. That is $p_j(i) \geq 0$ and $\sum_{j=0}^{N} p_j(i) = 1$. Let $\sum_{j=0}^{N} p_j(i)$ be non-decreasing in $i$, and $F_j$ be non-increasing (non-decreasing) in $j$. Then $\sum_{j=0}^{N} p_j(i) F_j$ is non-increasing (non-decreasing) in $i$.

**Proof:** Let $\Delta_j = F_j - F_{j-1}$, $\Delta_0 = F_0$. So for $j > 0$, $\Delta_j \leq 0$ ($\geq 0$). So
\[ \sum_{j=0}^{N} p_j(i) F_j = \sum_{j=0}^{N} p_j(i) (\sum_{\ell=0}^{j} \delta_{\ell}) = \sum_{j=0}^{N} \sum_{\ell=0}^{j} p_j(i) \delta_{\ell} \]

\[ = \sum_{\ell=0}^{N} \sum_{j=\ell}^{N} p_j(i) \delta_{\ell} = \sum_{\ell=0}^{N} \delta_{\ell} \sum_{j=\ell}^{N} p_j(i) \]

\[ = \delta_0 + \sum_{\ell=1}^{N} \delta_{\ell} (\sum_{j=\ell}^{N} p_j(i)) = F_{ik} \]

As \[ \sum_{j=\ell}^{N} p_j(i) \] is non-decreasing and \( \delta_{\ell} \leq 0 (\geq 0) \), \( F_{ij} \) is non-increasing (non-decreasing).

Note that \( p_j(i) \) may or may not be \( p_{ij} \), an element of the transition matrix.

**Lemma 3.1.2.** If \[ \sum_{j=\ell}^{N} p_{ij} \] is non-decreasing in \( i \) then so is \[ \sum_{j=\ell}^{N} k \]

Proof: (induction) It is true for \( k = 1 \) trivially,

\[ p_{ij} = \sum_{m=0}^{k} p_{im} p_{mj} \]

so

\[ \sum_{j=\ell}^{N} p_{ij} = \sum_{j=\ell}^{N} \sum_{m=0}^{k} p_{im} p_{mj} = \sum_{m=0}^{k} p_{im} (\sum_{j=\ell}^{N} p_{mj}) \]

Let \( F_m = \sum_{j=\ell}^{N} p_{mj} \), which is non-decreasing in \( m \). So

\[ \sum_{j=\ell}^{N} p_{ij} = \sum_{m=0}^{k} p_{im} F_m \] is non-decreasing in \( i \) by Lemma 3.1.1. \( \square \)
Lemma 3.1.3: Given Assumption 6 on $P$, and assuming $F_j$ is non-increasing (non-decreasing), then $\sum_{j=0}^{N} p_{ij} F_j$ is non-increasing (non-decreasing) in $i$.

Proof: Lemmas 3.1.1 and 3.1.2, by direct application. \hfill \square

Lemma 3.1.4: Let $F_{ij}$ be a collection of functions that are non-decreasing (non-increasing) in $j$. Then

a) $\inf_i F_{ij}$ is non-decreasing (non-increasing),

b) If $i \in [0, \infty)$ then $\lim_{i \to \infty} F_{ij}$ is non-decreasing (non-increasing),

c) $\sup_i F_{ij}$ is non-decreasing (non-increasing).

Proof: a) $F_{i, j+1} \geq F_{ij} \geq \inf_i F_{ij}$, taking the inf of both sides,

$$\inf_i F_{i, j+1} \geq \inf_i F_{ij}.$$ If $F_{ij}$ is non-increasing then $\inf_i F_{i, j+1} \leq F_{i, j+1} \leq F_{ij}$. Taking the inf of both sides gives the result.

b) $F_{i, j+1} - F_{ij} \geq (\leq) 0$; taking the limit of this equation in $i$ gives the desired result.

c) Let $G_{ij} = -F_{ij}$. Applying the results of a) to $G_{ij}$ gives c). \hfill \square

These lemmas enable proof of the following theorem:
Theorem 3.1.1. C^n(i,k) and C(i,k) are non-decreasing in i.

Proof: We use induction to prove the assertion for C^n(i,k),

\[ C^1(i,k) = \min\{ \sum_{j=0}^{N} p_{ij}^k c_j, \sum_{j=0}^{N} p_{ij}^k l_j \} . \]

By Lemmas 3.1.3 and 3.1.4, C^1(i,k) is non-decreasing in i. Now assume C^{n-1}(i,k) is non-decreasing in i,

\[ C^n(i,k) = \min\{ \sum_{j=0}^{N} p_{ij}^k c_j + \alpha C^{n-1}(0,0), \sum_{j=0}^{N} p_{ij}^k l_j + \alpha C^{n-1}(i, k+1), \sum_{j=0}^{N} p_{ij}^k l_j + M + \alpha \sum_{j=0}^{N} p_{ij}^{k+1} C^{n-1}(j,0) \} . \]

By Lemmas 3.1.3 and 3.1.4, induction and the fact that the sum of two monotonic functions is monotonic, C^n(i,k) is non-decreasing in i. Furthermore, b) of Lemma 3.1.4 shows the monotonicity of C(i,k). □

Both the theorem and the lemmas will be used extensively in succeeding sections. They establish monotonicity of other functions also.

B. Trivial Optimal Policies

Under certain conditions, the optimal policy is trivially no action in all states for all time periods. In this section we establish necessary and sufficient conditions relating to such trivial
optimal policies. These conditions tell us whether we have any interesting
conditions at all. It should be noted, of course, that a trivial
optimal policy is also a monotonic policy. We start by proving a lemma.

Lemma 3.1.5. a) Assume that repair is never strictly optimal in
stages \(1, \ldots, n\) of the \(n\)-period problem. Then no action is always
optimal and

\[
C^n(i,k) = \sum_{\ell=0}^{n-1} \alpha^\ell \left( \sum_{j=0}^{N} p_{ij}^{k+\ell} L_j \right).
\]

b) Assume repair not strictly optimal in any state for the infinite-
horizon problem. Then no action is optimal in all states and

\[
C(i,k) = \sum_{\ell=0}^{\infty} \alpha^\ell \left( \sum_{j=0}^{N} p_{ij}^{k+\ell} L_j \right).
\]

Proof: a) The proof is by induction and we leave the repair term out
of the recursive expression for \(C^n(i,k)\), \(n = 1:\)

\[
C^1(i,k) = \min \left\{ \sum_{j=0}^{N} p_{ij}^k L_j, \sum_{j=0}^{N} p_{ij}^k L_j + M \right\} = \sum_{j=0}^{N} p_{ij}^k L_j
\]

and no action is optimal. Assume it is true for \(n-1:\)

\[
C^n(i,k) = \sum_{j=0}^{N} p_{ij}^k L_j + \alpha \min \{C^{n-1}(i, k+1), M + \sum_{j=0}^{N} p_{ij}^{k+1} C^{n-1}(j,0) \},
\]
where the first term corresponds to no action and the second term corresponds to inspection. So

\[
c^n(i,k) = \sum_{j=0}^{N} p_{ij}^k L_j + \alpha \min\left\{ \sum_{\ell=0}^{n-2} \sum_{j=0}^{N} p_{ij}^{k+\ell+1} L_j, \right. \\
M + \sum_{j=0}^{N} p_{ij}^{k+1} \left( \sum_{\ell=0}^{n-2} \sum_{m=0}^{N} p_{jm}^{\ell} L_m \right) \left. \right\}
\]

\[
= \sum_{j=0}^{N} p_{ij}^k L_j + \alpha \min\left\{ \sum_{\ell=0}^{n-2} \sum_{j=0}^{N} p_{ij}^{k+\ell+1} L_j, \right. \\
M + \sum_{\ell=0}^{n-2} \sum_{m=0}^{N} p_{im}^{k+\ell+1} L_m \left. \right\}.
\]

Clearly no action is minimum. Also,

\[
c^n(i,k) = \sum_{j=0}^{N} p_{ij}^k L_j + \alpha \sum_{\ell=0}^{n-2} \sum_{j=0}^{N} p_{ij}^{k+\ell+1} L_j \\
= \sum_{\ell=0}^{n-1} \alpha^{\ell} \sum_{j=0}^{N} p_{ij}^{k+\ell} L_j.
\]

b) Consider any initial state \((i,k)\) and any policy consisting of no actions and inspections. In any period in the future, the costs consist of an operating cost and a possible inspection cost. However, the probability distribution of the real state \(n\)-periods into the future is the same no matter what the future policy is. That distribution is simply \(p_{i1}^{n+k}\) since repair can always be ruled out. Therefore the operating costs cannot be affected by policy. Thus the optimal policy is that which minimizes inspection costs and that policy is continual no action. Thus
\[ C(i,k) = \sum_{\ell=0}^{\infty} \alpha^\ell \{ \text{expected operating costs in state } (i, k+\ell) \} \]

\[ = \sum_{\ell=0}^{\infty} \alpha^\ell \left( \sum_{j=0}^{N} p_{ij}^{k+\ell} L_j \right). \]

We also need the following lemma:

**Lemma 3.1.6.** Let \( F_j \) be non-decreasing in \( j \) and \( \pi_j \) be the limiting state probabilities under \( P \). Then

\( a) \quad \sum_{j=0}^{N} p_{Nj}^k F_j \left( \sum_{j=0}^{N} p_{0j}^k F_j \right) \) converges down (up) to \( \sum_{j=0}^{N} \pi_j F_j \) as \( k \to \infty \),

\( b) \quad C^N(N,k) \left( C^N(0,k) \right) \) converges down (up) to \( C^N_\pi \) as \( k \to \infty \),

\( c) \quad C(N,k) \left( C(0,k) \right) \) converges down (up) to \( C_\pi \) as \( k \to \infty \).

**Proof:**

\( a) \quad \sum_{j=0}^{N} p_{ij}^k F_j \) is non-decreasing in \( i \). So

\[ \sum_{j=0}^{N} p_{Nj}^k F_j = \sum_{\ell=0}^{k+1} p_{Nj}^\ell \left( \sum_{j=0}^{N} p_{0j}^\ell F_j \right) \leq \sum_{j=0}^{N} p_{Nj}^k F_j \quad \text{by Lemma 3.1.3}. \]

The proof of the other part is analogous.

\( b) \quad C^1(N,k) = \min \left\{ \sum_{j=0}^{N} p_{Nj}^k C_j, \sum_{j=0}^{N} p_{Nj}^k L_j \right\} \) is non-increasing in \( k \) by \( a) \) and Lemma 3.1.4. Assume by induction that \( C^{n-1}(N,k) \) is non-increasing in \( k \). Then

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\[ C^N(N, k) = \min\{ \sum_{j=0}^{N} p_{Nj} c_j + \alpha c^{n-1}(0, 0), \sum_{j=0}^{N} p_{Nj} l_j + \alpha c^{n-1}(N, k+1), \sum_{j=0}^{N} p_{Nj} l_j + \alpha \sum_{j=0}^{N} k_{Nj} c^{n-1}(j, 0) \} \]

is non-increasing in \( k \) by induction, a), and Lemma 3.1.4. By Lemma 2.3.1, its limit is \( C^*_n \). The other part is analogous.

c) Let \( n \to \infty \) in part b), and we have the assertion. \( \square \)

We are now ready to prove the results, which provide necessary and sufficient conditions for trivial optimal policies, where a trivial optimal policy is one that specifies no action for all states and all time periods.

**Theorem 3.1.2.** a) No action optimal in all observed states for the 1-period through \((n+1)\)-period problems

\[ \iff C_N - L_N + \sum_{m=1}^{n} \alpha^m \left( \sum_{j=0}^{N} p_{0j}^{m-1} l_j - \sum_{j=0}^{N} p_{Nj}^{m} l_j \right) \geq 0. \]

b) No action optimal in all observed states for the infinite-horizon problem

\[ \iff C_N - L_N + \sum_{m=1}^{\infty} \alpha^m \left( \sum_{j=0}^{N} p_{0j}^{m-1} l_j - \sum_{j=0}^{N} p_{Nj}^{m} l_j \right) \geq 0. \]
Proof: a) The if part is by induction. Note that by Lemma 3.1.3,

\[
\sum_{j=0}^{N} p_{0j}^{m} L_j \leq \sum_{j=0}^{N} p_{0j}^{\ell} \left( \sum_{j=0}^{m-1} p_{\ell j}^{m-1} L_j \right) = \sum_{j=0}^{N} p_{0j}^{m} L_j \leq p_{Nj}^{m} L_j,
\]

so

\[
C_N - L_N + \sum_{m=1}^{q} \alpha^{m} \left( \sum_{j=0}^{N} p_{0j}^{m} L_j - \sum_{j=0}^{N} p_{Nj}^{m} L_j \right) \geq 0 \quad (1)
\]

for \( q = 0, 1, \ldots, n \).

For the one-period problem,

\[
C_N - L_N \geq 0 \Rightarrow \sum_{j=0}^{N} p_{ij}^{k} (C_j - L_j) \geq 0 \quad \text{by Assumption 3}
\]

\[\Rightarrow (\text{cost of repair} \geq \text{cost of no action, all } i, k)\]

so no action is optimal. Now assume by induction that no action is optimal for all observed states in the \( q \)-period problem. For state \((i, k)\) in the \((q+1)\)-period problem, where \( q \leq n \),

\[
\text{cost of repair} - \text{cost of no action}
\]

\[
= \sum_{j=0}^{N} p_{ij}^{k} C_j + \alpha C^{q-1}(0, 0) - \sum_{j=0}^{N} p_{ij}^{k} L_j - \alpha C^{q-1}(i, k+1)
\]

\[
= \sum_{j=0}^{N} p_{ij}^{k} (C_j - L_j) + \alpha(C^{q-1}(0, 0) - C^{q-1}(i, k+1))
\]

\[
\geq \sum_{j=0}^{N} p_{Nj}^{k} (C_j - L_j) + \alpha(C^{q-1}(0, 0) - C^{q-1}(N, k+1))
\]

by Lemma 3.1.3 and Theorem 3.1.1
\[ \geq C_N - L_N + \alpha(C^{q-1}(0,0) - C^{q-1}(N,1)) \quad \text{by Lemma 3.1.6} \]
\[ = C_N - L_N + \sum_{m=1}^{q} \alpha^m \left( \sum_{j=0}^{N} p_{oj}^{m-1} L_j - \sum_{j=0}^{N} p_{nj}^m L_j \right) \]
\[ \geq 0 \quad \text{by Lemma 3.1.5} \]
\[ \geq 0 \quad \text{by (1)} . \]

So repair is not strictly optimal in \((i,k)\) and the \textit{if} part is true by Lemma 3.1.5.

\textbf{Only if part:} We will show that
\[ C_N - L_N + \sum_{l=1}^{n} \alpha^l \left( \sum_{j=0}^{N} p_{oj}^{l-1} L_j - \sum_{j=0}^{N} p_{nj}^l L_j \right) < 0 \]
\[ \Rightarrow \text{no action not always optimal for all states} . \]

If no action is not always optimal in stages \(1, \ldots, n\), there is nothing to prove. So assume it is optimal in all states for the
1-period through \(n\)-periods problems. Consider state \((N,0)\) in the
\((n+1)\)-period problem;

\begin{align*}
\text{cost of repair} - \text{cost of no action} &= C_N - L_N + \alpha(C^0(0,0) - C^0(N,1)) \\
&= C_N - L_N + \sum_{m=1}^{n} \alpha^m \left( \sum_{j=0}^{N} p_{oj}^{m-1} L_j - \sum_{j=0}^{N} p_{nj}^m L_j \right) \quad \text{by assumption} \\
&< 0 \quad \text{by hypothesis} .
\end{align*}
Thus no action is not as good as repair in state \((N,0)\).

b) If part: From part a),

\[
\sum_{j=0}^{N} p_{Oj} L_j - \sum_{j=0}^{N} p_{Nj} L_j \leq 0, \quad \forall m.
\]

So

\[
C_N - L_N + \sum_{l=1}^{n} \alpha^m \left( \sum_{j=0}^{N} p_{Oj} L_j - \sum_{j=0}^{N} p_{Nj} L_j \right) \geq 0, \quad \forall n.
\]

So from part a), no action is optimal in all states for any \(n\)-period problem. So by Lemma 3.1.5,

\[
C^n(i,k) = \sum_{\ell=0}^{n-1} \alpha^\ell \left( \sum_{j=0}^{N} p_{ij} L_j \right),
\]

and by Lemma 2.1.2,

\[
C(i,k) = \sum_{\ell=0}^{\infty} \alpha^\ell \left( \sum_{j=0}^{N} p_{ij} L_j \right).
\]

But no action in all states achieves this cost, so no action is optimal in all states.

Only if part: By Lemma 3.1.5,

\[
C(i,k) = \sum_{\ell=0}^{\infty} \alpha^\ell \left( \sum_{j=0}^{N} p_{ij} L_j \right).
\]

So in state \((N,0)\),

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cost of repair - cost of no action

\[ C_N - L_N + \sum_{1}^{\infty} \alpha^m \left( \sum_{j=0}^{N} p_{o_j}^{m-1} L_j - \sum_{j=0}^{N} p_{n_j}^m L_j \right) . \]

Unless the latter is greater than or equal to zero, then repair is superior to no action, so we have a contradiction.

It should be noted that

(cost of repair - cost of no action)

\[ = \sum_{j=0}^{N} p_{ij}^k (C_j - L_j) + \alpha(C_n^{n-1}(0,0) - C_n^{n-1}(i, k+1)) \]

and

(cost of repair - cost of inspection)

\[ = \sum_{j=0}^{N} p_{ij}^k (C_j - L_j) - M + \alpha(C_n^{n-1}(0,0) - \sum_{j=0}^{N} p_{ij}^{k+1} C_n^{n-1}(j,0)) \]

with analogous expressions for the infinite-horizon problem. By Lemmas 3.1.3 and 3.1.6 and Theorem 3.1.1, these functions are minimized at observed state \((N,0)\). Thus if it is optimal to repair at any state, then it is optimal to repair at state \((N,0)\). We shall develop these functions further in the following sections.

For the average cost case, we can extend Theorem 3.1.2 to get the following:
Corollary 3.1.1. If

\[ C_N - L_N + \sum_{m=1}^{\infty} \left( \sum_{j=0}^{N} p_{Oj}^{m-1} L_j - \sum_{j=0}^{N} p_{Nj}^{m} L_j \right) \geq 0, \]

then no action is optimal in all states for the average-cost case.

Proof: Note that

\[ C_N - L_N + \sum_{m=1}^{\infty} \alpha^m \left( \sum_{j=0}^{N} p_{Oj}^{m-1} L_j - \sum_{j=0}^{N} p_{Nj}^{m} L_j \right) \]

is non-increasing in \( \alpha \) since

\[ \sum_{j=0}^{N} p_{Oj}^{m-1} L_j \leq \sum_{j=0}^{N} p_{Nj}^{m} L_j. \]

So for \( \alpha < 1 \),

\[ C_N - L_N + \sum_{m=1}^{\infty} \alpha^m \left( \sum_{j=0}^{N} p_{Oj}^{m-1} L_j - \sum_{j=0}^{N} p_{Nj}^{m} L_j \right) \geq 0. \] (1)

Now let \( R^* = \) perpetual no action, \( R = \) any other policy. So

\[ \phi(i, k, R^*) = \lim_{\alpha \to 1} (1-\alpha) C(i, k, \alpha, R^*) \]

\[ \leq \lim_{\alpha \to 1} (1-\alpha) C(i, k, \alpha, R) \quad \text{by (1) and the theorem} \]

\[ = \phi(i, k, R). \]

Thus \( R^* \) is optimal. \( \square \)
Thus we have conditions for situations where no action is optimal in all states. In the next section, using the additional Assumption 7, we show that the optimal policy is monotonic when it is not trivial. The remainder of the chapter deals with monotonic and non-trivial policies.

3.2. Optimality of Monotonic Policies Under Introduction of Assumption 7

We now introduce Assumption 7 ($p_{ij} = 0$ for $j < i$) and show how this results in monotonic policies. The significance of this assumption should first be noted. As mentioned in the introduction, a deteriorating system often cannot revert to a better state without repair. We thus feel that Assumption 7 is a quite reasonable assumption (although we shall relax it in Chapter 5). As shown in the Proof of Corollary 2.3.1, the limiting state probability is $\pi_j = \delta_{jN}$ and thus the process tends to wind up in the worst state. Throughout this section we assume Assumptions 1 through 7 hold.

A. The n-period and Infinite-Horizon Problems

We now define two sets of functions that are critical in the analysis in this section. For the n-period problem let

$$ F_{ik}^n = \text{cost of repair} - \text{cost of no action} $$

and

$$ G_{ik}^n = \text{cost of repair} - \text{cost of inspection} $$
when in state \((i, k)\). Let \(F^\infty_{ik}\) and \(G^\infty_{ik}\) denote these same functions for the infinite-horizon problem. The costs refer to the values in the minimization that correspond to the particular choice. For example, for the \(n\)-period problem, cost of repair = \(\sum_{j=0}^{N} p_{ij} C_j + \alpha C_{i}^{n-1}(0,0)\).

The determining factor in whether repair is optimal is whether these functions are positive (or non-negative), and we will be checking for this. In addition, the major idea in showing that monotonic policies are optimal is showing that \(F^n_{ik}, G^n_{ik}, F^\infty_{ik},\) and \(G^\infty_{ik}\) are monotonic in both \(i\) and \(k\).

With two lemmas we can now show that monotonic policies are optimal under these two criteria:

**Lemma 3.2.1.** Assume \(F_j\) is non-increasing (non-decreasing) in \(j\).

Then \(\sum_{j=0}^{N} p_{ij} F_j\) converges down (up) to \(F_N\) as \(k \to \infty\).

**Proof.** The limit follows from the fact that \(p_j = \delta_{jN}\). Also,

\[
\sum_{j=\ell}^{N} p_{ij}^{k+1} = \sum_{j=\ell}^{N} \sum_{m=0}^{N} p_{im}^{k} p_{mj} = \sum_{j=\ell}^{N} \sum_{m=0}^{N} p_{im} \sum_{j=\ell}^{N} p_{mj}^{k} = \sum_{m=0}^{\ell-1} p_{im} \sum_{j=\ell}^{N} p_{mj}^{k} + \sum_{m=\ell}^{N} \sum_{j=\ell}^{N} p_{im} p_{mj}^{k}.
\]

But if \(m \geq \ell\), then \(\sum_{j=\ell}^{N} p_{mj}^{k} = 1\) since \(p_{mn} = 0, n < \ell \leq m\). So

\[
\sum_{j=\ell}^{N} p_{ij}^{k+1} = C + \sum_{m=\ell}^{N} p_{im}^{k} \geq \sum_{m=\ell}^{N} p_{im}^{k} = \sum_{m=\ell}^{\ell} p_{im}^{k} = \sum_{m=\ell}^{\ell} p_{im}^{k} 
\]

where \(C \geq 0\).
So \( \sum_{j=0}^{N} p_{ij}^k \) is non-decreasing in \( k \), and so by Lemma 3.1.1, \( \sum_{j=0}^{N} p_{ij}^k F_j \) is non-increasing (non-decreasing) in \( k \). \( \square \)

**Lemma 3.2.2.** \( C^n(i,k) \) and \( C(i,k) \) are non-decreasing in \( k \).

**Proof:** The proof is almost exactly the same as that of Theorem 3.1.1. The monotonicity of \( \sum_{j=0}^{N} p_{ij}^k L_j \) and \( \sum_{j=0}^{N} p_{ij}^k C_j \) follow from Lemma 3.2.1. \( \square \)

We are now ready to prove the major result of the Chapter.

**Theorem 3.2.1.** For the \( n \)-period problem a monotonic policy is optimal and for the infinite-horizon problem a stationary monotonic policy is optimal.

**Proof:** Let \( F^n_{ik}, G^n_{ik}, F^\infty_{ik} \) and \( G^\infty_{ik} \) be defined as above. We are going to show that \( F^n_{ik} \) and \( G^n_{ik} \) are non-increasing in \( i \) and \( k \). Thus we let

\[
k^*(i,n) = \min\{k : F^n_{ik} < 0, G^n_{ik} < 0\} .
\]

By monotonicity in \( k \), if \( k \geq k^*(i,n) \) it is optimal to repair, and if \( k < k^*(i,n) \), it is optimal not to repair. Also, by definition \( \text{We could also use } \leq \text{ to define } k^*(i,\cdot) \).
of $k^*(i,n)$ and by monotonicity in $i$, we have

$$F^n_{i+1,k^*(i,n)} < 0$$

and

$$G^n_{i+1,k^*(i,n)} < 0 \Rightarrow k^*(i+1,n) \leq k^*(i,n).$$

So by inductively showing that $F^n_{ik}$ is non-increasing and $G^n_{ik}$ is non-increasing in $i$ and $k$, we have the result for the $n$-period problem. Similarly, if we show that $F^\infty_{ik}$ and $G^\infty_{ik}$ are monotonic in $i$ and $k$, we see that a monotonic policy is optimal for the infinite-horizon problem. Since the costs and thus $F^\infty_{ik}$ and $G^\infty_{ik}$ are stationary, such an optimal policy is stationary. Thus we need to show that $F^n_{ik}$, $G^n_{ik}$, $F^\infty_{ik}$, and $G^\infty_{ik}$ are non-increasing in $i$ and $k$. Now

$$F^n_{ik} = \sum_{j=0}^{N} p^{k}_{ij} (C_j - L_j) + \alpha(C^{n-1}(0,0) - C^{n-1}(i, k+1))$$

$$F^\infty_{ik} = \sum_{j=0}^{N} p^{k}_{ij} (C_j - L_j) + \alpha(C(0,0) - C(i, k+1)).$$

By Assumption 3, Lemmas 3.1.3, 3.2.1, and 3.2.2 and Theorem 3.1.1 and from the fact that sums of monotonic functions are monotonic, $F^n_{ik}$ and $F^\infty_{ik}$ are non-increasing in $i$ and $k$. Also

$$G^n_{ik} = \sum_{j=0}^{N} p^{k}_{ij} (C_j - L_j) + \alpha(C^{n-1}(0,0) - \sum_{j=0}^{N} p^{k+1}_{ij} C^{n-1}(j,0)) - M$$

and

$$G^\infty_{ik} = \sum_{j=0}^{N} p^{k}_{ij} (C_j - L_j) + \alpha(C(0,0) - \sum_{j=0}^{N} p^{k+1}_{ij} C(j,0)) - M.$$
By the same results, $G_{ik}^n$ is non-increasing in $i$ and $k$ and $G_{ik}^\infty$ is non-increasing in $i$ and $k$. □

The policy is relatively simple. If one is in $(i,k)$, the optimal action is repair if $k \geq k^*(i,n)$ (or $k^*(i,\infty)$). Otherwise, test the cost functions for either no action or inspection. We now attempt to further describe the optimal policy. At this point note that assumption four ($C_0 > L_0$) is necessary to insure interesting structures. If $L_0 \geq C_0$, then $F_{ik}^n \leq 0$, $G_{ik}^n \leq 0$, $F_{ik}^\infty \leq 0$, and $G_{ik}^\infty \leq 0$, which imply that repair is always optimal under both criteria. The next theorem further characterizes optimal policy. The idea is that one can set the values of $k^*(i,\cdot)$ to $\infty$ or set them at finite values. That is, the critical numbers are either all finite or infinite. The result is summarized in the following theorem, which we precede by a lemma.

**Lemma 3.2.3.** For the n-period and infinite-horizon problems, it cannot be optimal to inspect in state $(N,k)$, all $k$.

**Proof:** For the n-period problem, in state $(N,k)$

\[
\text{cost of no action - cost of inspection} = -M + \alpha(C^{n-1}(N, k+1) - \sum_{j=0}^{N} P_{nj}^{k+1} C^{n-1}(j,0))
\]

\[
= -M + \alpha(C^{n-1}(N,0) - C^{n-1}(N,0)) \quad \text{by Corollary 2.3.1}
\]

\[
= -M < 0 .
\]
So no action is a superior action than inspection. For the infinite-horizon problem, we get the analogous result.

We now prove the theorem:

**Theorem 3.2.2.** a) For the n-period problem, we can set $k^*(N,k) = 0$ or $\infty$. That is, either repair is optimal for state $(N,k)$, all $k$, or no action is optimal in $(N,k)$, all $k$. If the former exists, and repair is strictly optimal, then $k^*(i,n) < \infty$, all $i$. Otherwise repair is not strictly optimal for any state $(i,k)$.

b) For the infinite-horizon problem, we can set $k^*(N,\infty) = 0$ or $\infty$. That is, either repair is optimal for state $(N,k)$, all $k$, or no action is optimal, all $k$. If the former and repair is strictly optimal, then $k^*(i,n) < \infty$, all $i$. Otherwise no action is optimal for all states.

**Proof:** By Lemma 3.2.3, the optimal action in state $(N,k)$ is determined by

$$P^n_{Nk} = \sum_{j=0}^{N} p^k_{Nj}(c_j - L_j) + \alpha(c^{n-1}(0,0) - c^{n-1}(N,k+1))$$

$$= c_N - L_N + \alpha(c^{n-1}(0,0) - c^{n-1}(N,0)) \quad \text{by Corollary 2.3.1}$$

and

$$P^\infty_{Nk} = c_N - L_N + \alpha(c(0,0) - c(N,0))$$
As these are independent of \( k \), we see that the optimal policy can be the same for all \( k \). We now examine two cases for either problem.

Case 1 is repair strictly optimal. Note that if repair is optimal but not strictly optimal then \( F_{Nk}^\infty = 0 \) (or \( F_{Nk}^n = 0 \)). So no action is also optimal so case 2 is no action optimal.

**Case 1.** Repair strictly optimal: We treat the \( n \)-period problem.

For the infinite-horizon problem, just replace \( n \) by \( \infty \). Now

\[
F_{Nk}^n = C_N - L_N + \alpha(\varphi^{n-1}(0,0) - \varphi^{n-1}(N,0)) .
\]

Let \( F_{Nk}^n = -\epsilon, \epsilon > 0 \). By monotonicity we need to show only that \( k^*(0,n) < \infty \). By the monotonicity of \( F_{Ok}^n \) and \( G_{Ok}^n \), we need to find a \( k \) such that \( F_{Ok}^n < 0, G_{Ok}^n < 0 \). It then follows that \( k^*(0,n) \leq k \). So

\[
F_{Ok}^n = \sum_{j=0}^{N} p_{Oj}^k (C_j - L_j) + \alpha(\varphi^{n-1}(0,0) - \varphi^{n-1}(0, k+1)) .
\]

Taking limits as \( k \to \infty \),

\[
F_{Ok}^n \to C_N - L_N + \alpha(\varphi^{n-1}(0,0) - \varphi^{n-1}(N,0)) = F_{NO}^n ,
\]

by Lemma 3.2.1 and Corollary 2.3.1;
\[ C_{0k}^n = \sum_{j=0}^{N} p_{0j}^k (C_j - L_j) + \alpha (C_{n-1}(0,0) - \sum_{j=0}^{N} p_{0j}^{k+1} C_{n-1}(j,0)) - M \]

\[ \to C_N - L_N - M + \alpha (C_{n-1}(0,0) - C_{n-1}(N,0)) \]

So by hypothesis,

\[ F_{0k}^n \to -\epsilon, \quad C_{0k}^n \to -\epsilon - M \]

So pick \( k \) big enough so that \( F_{0k}^n < 0, \ C_{0k}^n < 0 \).

**Case 2.** No action is optimal. In this case \( F_{nk}^n \geq 0 \) (or \( F_{nk}^\infty \geq 0 \)).

By monotonicity of \( F_{ik}^n \) \( F_{ik}^\infty \), \( F_{ik}^n \geq 0, \ \forall \ i, k \) (or \( F_{ik}^\infty \geq 0, \ \forall \ i, k \)).

Thus \( k^*(i,n) = \infty \) by definition (or \( k^*(i,\infty) = \infty \)). So repair can be ruled out for any state, and by Lemma 3.1.5, no action is optimal in any state for the infinite-horizon case.

\[ \square \]

The important point here is that the \( k^*(i,\cdot) \) are all finite if repair is strictly optimal in the states \( (N,k) \) and they can set to be infinity otherwise. This theorem is especially important for the infinite-horizon problem. By this theorem and Theorem 3.1.2 we can state the conditions for the two cases of the theorem:

**Corollary 3.2.1.** If \( C_N - L_N + \sum_{1}^{\infty} \alpha^n (\sum_{j=0}^{N} p_{0j}^{m-1} L_j - L_N) < 0 \) then \( k^*(i,\infty) < \infty, \ \text{all} \ i \). If not, no action is optimal in all states.
Proof: Theorems 3.1.2 and 3.2.2, by direct application. □

For the final result of this section, we consider what happens to \( k^*(i,n) \) as \( n \to \infty \).

**Theorem 3.2.3.** If repair is strictly suboptimal at \( (i, k^*(i,\infty) - 1) \), then \( k^*(i,n) \to k^*(i,\infty) \), where we define \( k^*(i,n) \) as in Theorem 3.2.1.

Proof: From the convergence of \( C^n_i \) to \( C \) it follows that as \( n \to \infty \),

\[
F_i^{n_{ik}} \to F_i^{\infty}, \quad G_i^{n_{ik}} \to G_i^{\infty}. \quad \text{By definition,}
\]

\[
F_i^{\infty}(i,\infty) < 0, \quad G_i^{\infty}(i,\infty) < 0.
\]

So by picking \( \bar{N} \) big enough, for \( n > \bar{N} \),

\[
F_i^{n_{ik}}(i,\infty) < 0, \quad G_i^{n_{ik}}(i,\infty) < 0. \quad (1)
\]

Also, by hypothesis, repair is strictly suboptimal at \( k^*(i,\infty) - 1 \), so either \( F_i^{\infty}(i,\infty)-1 > 0 \) or \( G_i^{\infty}(i,\infty) > 0 \). Assume the former without loss of generality.

So pick \( N' \) big enough so that for \( n > N' \), (1) holds and

\[
F_i^{n_{ik}}(i,\infty)-1 > 0.
\]

Thus, for \( n > N' \), by definition,

\[
k^*(i,n) = k^*(i,\infty). \quad \square
\]
Note that without strict suboptimality, $k^*(i,n)$ oscillates
in a set of possible values. This completes the infinite-horizon
analysis and we go on to the average-cost case.

B. The Average-Cost Case and Conclusions

In this section we show that a monotonic policy is also optimal
in the average-cost case. The technique used is to extend the results
of the infinite-horizon case to the average-cost case. We use equation
(2.2.1). The idea in the proof of the main theorem is that the
$k^*(i,\infty)$ are uniformly bounded in $\alpha$. This enables us to take the
proper limits in extension of the infinite-horizon problem.

**Theorem 3.2.4.** Assume $C_N - L_N + \sum_{k=0}^{\infty} \left( \sum_{j=0}^{N} p_{0j}^k L_j - L_N \right) < 0$. Then
there is an optimal average-cost policy that is a monotonic policy.
Furthermore, there are numbers $k^*(i) < \infty$ such that, if in state
$(i,k)$ and $k \geq k^*(i)$, repair is optimal, and otherwise no action
or inspection is optimal.

**Proof:** By Abel's theorem,

$$\lim_{\alpha \to 1} C_N - L_N + \sum_{k=1}^{\infty} \alpha^k \left( \sum_{j=0}^{N} p_{0j}^{k-1} L_j - L_N \right)$$

$$= C_N - L_N + \sum_{k=0}^{\infty} \left( \sum_{j=0}^{N} p_{0j}^k L_j - L_N \right) .$$

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Thus by monotonicity with respect to $\alpha$ there is an $\alpha^*$ such that for $\alpha > \alpha^*$

$$C_N - L_N + \sum_{k=1}^{\infty} \alpha^k \left( \sum_{j=0}^{N} p_{0j}^{k-1} L_j - L_N \right) < 0.$$

So by Corollary 3.2.1, there are critical numbers $k^*(i, \omega) < \infty$ for all $\alpha > \alpha^*$, and $k^*(i, \omega) \leq k^*(0, \omega)$.

Now

$$F_{0k}^\omega = \sum_{j=0}^{N} p_{0j}^{k} (C_j - L_j) + \alpha(C(0, 0) - C(0, k+1))$$

$$\leq \sum_{j=0}^{N} p_{0j}^{k} (C_j - L_j) + \alpha(C(0, 0) - C(0, k))$$

$$\leq \sum_{j=0}^{N} p_{0j}^{k} [C_j - L_j + \alpha(C(0, 0) - C(j, 0))] = H \quad \text{by Lemma 2.1.4.}$$

Also

$$G_{0k}^\omega \leq \sum_{j=0}^{N} p_{0j}^{k} (C_j - L_j) - M + \alpha(C(0, 0) - \sum_{j=0}^{k} p_{0j}^{k+1} C(j, 0))$$

$$\leq \sum_{j=0}^{N} p_{0j}^{k} (C_j - L_j) - M + \alpha(C(0, 0) - \sum_{j=0}^{N} p_{0j}^{k} C(j, 0))$$

by Lemma 3.2.1 and Theorem 3.1.1.

So $F_{0k}^\omega \leq H$, $G_{0k}^\omega \leq H$.

Now
\[ H \leq \sum_{j=0}^{N-1} p_j^k (C_0 - L_0) + p_{ON}^k (C_N - L_N + \alpha (0,0) - \alpha C(N,0)) \]

by Theorem 3.1.1

\[ \leq (1 - p_{ON}^k) (C_0 - L_0) + p_{ON}^k (C_N - L_N + \alpha (0,0) - \alpha C(N,0)) \]

by Assumption 3.

But for \( \alpha > \alpha^* \), repair is optimal in \((N,0)\). So \( C(N,0) = C_N + \alpha C(0,0) \).

So substituting in,

\[ H \leq (C_0 - L_0) (1 - p_{ON}^k) + p_{ON}^k (C_N - L_N - \alpha C_N + \alpha(1-\alpha) C(0,0)) \]

or

\[ H \leq (C_0 - L_0) (1 - p_{ON}^k) + p_{ON}^k ((1-\alpha) C_N + \alpha(1-\alpha) C(0,0) - L_N) \]

Now take the \( \limsup \) as \( \alpha \to 1 \), and denote the right hand side as \( F(x) \);

\[
\lim_{\alpha \to 1} \sup_{\alpha \geq \alpha_0} F(x) = (C_0 - L_0) (1 - p_{ON}^k) \\
+ \lim_{\alpha \to 1} p_{ON}^k (\lim_{\alpha \to 1} (1-\alpha) C(0,0) - L_N) \\
= F(k). \tag{1}
\]

Now the optimal average cost per period starting in \((0,0)\) is
\[ \text{cost} = \inf_{R^*} \lim_{\alpha \to 1} (1-\alpha) C(0,0,0,0,0) = \inf_{R^*} \lim_{\alpha \to 1} (1-\alpha) C(0,0,0,0) \]

\[ \geq \inf_{R^*} \lim_{\alpha \to 1} (1-\alpha) \inf_s C(0,0,0,0,s) = \lim_{\alpha \to 1} (1-\alpha) C(0,0) . \]

So optimal average cost satisfies

\[ \text{optimal cost} \geq \lim_{\alpha \to 1} (1-\alpha) C(0,0) . \] (2)

Now \[ C_N - L_N + \sum_{k=0}^{\infty} \left( \sum_{j=0}^{N} p_{0j}^k L_j - L_N \right) = -\delta < 0 \] by hypothesis.

Thus

\[ \lim_{k \to \infty} \left( \sum_{j=0}^{N} p_{0j}^k C_j - L_N + \sum_{k=0}^{N} \left( \sum_{j=0}^{N} p_{0j}^k L_j - L_N \right) \right) = -\delta , \ \delta > 0 . \]

Thus by monotonicity, we can find a \( k^* \) such that

\[ \sum_{j=0}^{N} p_{0j}^{k^*} (C_j - L_N) + \sum_{k=0}^{k^*-1} \left( \sum_{j=0}^{N} p_{0j}^k L_j - L_N \right) = -\epsilon < 0 . \]

Now consider the policy of replacing every \( k^* + 1 \) periods. By the above equation,

\[ \text{average cost of this policy} - L_N = -\frac{\epsilon}{k^*+1} , \]

thus

\[ \lim_{\alpha \to 1} (1-\alpha) C(0,0) - L_N \leq -\frac{\epsilon}{k^*+1} \] by (2) .
So by substituting in (1),

\[ F(k) \leq (c_0 - L_0) \left( 1 - \frac{k}{p_{ON}} \right) + p_{ON}\left( -\frac{\epsilon}{k+1} \right). \]

Now pick \( \tilde{\alpha} \) such that

\[ \alpha \geq \tilde{\alpha} \Rightarrow \sup_{\alpha_0 \geq \alpha} F_{\alpha_0}(k) - F(k) \leq \frac{\epsilon}{2(k+1)}. \]

We can do this because \( F(k) \neq \pm \infty \), since \( \lim_{\alpha \to 1} (1-\alpha) C(0,0) \) is bounded above by \( L_N \) and below by zero. Thus for \( \alpha \geq \tilde{\alpha} \),

\[ F_{\alpha}(k) \leq F(k) + \frac{\epsilon}{2(k+1)} \leq (1 - \frac{k}{p_{ON}}) (c_0 - L_0) + p_{ON}\left( -\frac{\epsilon}{k+1} \right) + \frac{\epsilon}{2(k+1)} \]

\[ \leq (c_0 - L_0) (1 - \frac{k}{p_{ON}}) + p_{ON}\left( -\frac{\epsilon}{2(k+1)} \right). \]

(3)

Now \( p_{ON} \to 1 \), so let \( \tilde{k} \) be such that for all \( k \geq \tilde{k} \),

\[ p_{ON}^k > \frac{c_0 - L_0}{c_0 - L_0 + \frac{\epsilon}{2(k+1)}}. \]

(4)

Thus \( F_{\alpha}(k) < 0 \) by combining (3) and (4) for \( k \geq \tilde{k} \). But \( H \leq F_{\alpha}(k) \Rightarrow k*(0, \infty) \leq \tilde{k} \), all \( \alpha \geq \tilde{\alpha} \). Thus we can pick a sequence of \( \alpha_k \geq \tilde{\alpha} \) such that \( \lim \alpha_k = 1 \) and such that one optimal rule repeats infinitely often. As all critical numbers are \( \leq \tilde{k} \), our set of optimal rules is finite. Thus we indeed can find one that repeats infinitely often.
Let $R^*$ be the optimal rule repeated infinitely often and let $R$ be any other rule. So

$$C(i,k,\alpha_k, R^*) \leq C(i,k,\alpha_k, R) , \quad \alpha_k \geq \bar{\alpha}$$

and thus

$$\bar{\phi}(i,k,R^*) = \lim_{\alpha_k \to 1} C(i,k,\alpha_k, R^*) \leq \lim_{\alpha_k \to 1} C(i,k,\alpha_k, R) = \bar{\phi}(i,k,R) .$$

But $R^*$ is a monotonic policy with finite critical numbers.

Similar to the other two optimality criteria, we have two cases. Either no action is optimal in all states (Corollary 3.1.1) or a non-trivial monotonic policy is optimal with finite critical numbers (Theorem 3.2.4). As previously mentioned, a trivial policy is a monotonic one. In conclusion, for all three criteria, we have shown that monotonic policies are optimal with either all finite critical numbers or all infinite critical numbers. It is felt that such a characterization is straightforward. In the next chapter, we use the additional assumption to completely characterize the optimal policy.
CHAPTER 4

COMPLETE POLICY CHARACTERIZATION FOR TP₂ CASE

4.1. Introduction

The question remaining from the analysis of Chapter 3 is the structure of optimal policy for states where repair is not optimal. Note that a monotonic policy stipulates where repair is optimal. For the remainder of the states, we know only that either inspection or no action is optimal. In this chapter we derive, under the additional assumption that $P$ is totally positive of order two ($TP₂$), a complete and straightforward characterization of the optimal policy for the infinite-horizon and average-cost problems. The structure of this optimal policy is analogous to the optimal policy for the Ross model for two states and the Taylor counter-example to the Girshick and Rubin model. For our case however, there are more real states. This policy, which we call a monotonic, four-region policy, is defined as follows:

**Definition.** Consider a monotonic policy with critical numbers $k*(i)$ (or $k*(i,\cdot)$). If there exist numbers $k_1^*(i)$ and $k_2^*(i)$ such that no action is chosen if $0 \leq k < k_1^*(i)$ or $k_2^*(k) < k < k*(i)$, and inspection is chosen if $k_1^*(i) \leq k \leq k_2^*(i)$, then we call such a monotonic policy a monotonic, four-region policy. □

Note the origin of the name "four-region". For any given $i$, there are at most four separate regions of policy for the set of states $(i,k)$. Note also the similarity between this policy and the
optimal policy for the Ross two-state model. In Figure 7, we present an example of a four-region policy. When we refer to a particular region, such as a no-action region, we mean that such region is characterized by the optimality of the indicated action.

We shall show that under the $T_2^p$ assumption the optimal policy in the infinite-horizon and average-cost cases is a monotonic, four-region policy. We shall also show that for those $i$ such that $k_1^*(i) < k^*(i)$ (i.e., some inspection-optimal points $(i,k)$ exist for that $i$), $k_1^*(i)$ is non-increasing in $i$. It is felt that a monotonic, four region policy is a complete and appealing form of an optimal policy. As one varies $k$ in state $(i,k)$ one passes through at most four regions. The first no-action region, the inspection region, and the repair region are all somewhat intuitive. The second

![Figure 7: An Example of a Monotonic, Four-Region Policy](image-url)
no-action region is slightly surprising but its existence was
documented for a similar problem by Taylor and Ross. A possible
intuitive explanation is that it might not be worthwhile to spend for
inspection if one is about to repair shortly anyway. We shall discuss
this further in Section 4.4.

The new assumption that we need replacing Assumption 6 is, as
mentioned, that \( P \) is totally positive of order two (\( TP_2 \)). The
definition of \( TP_2 \) is:

**Definition.** \( P = [p_{ij}] \) is \( TP_2 \) if

\[
\begin{vmatrix}
  p_{i_1 j_1} & p_{i_1 j_2} \\
  p_{i_2 j_1} & p_{i_2 j_2}
\end{vmatrix} \geq 0 \quad \text{for every } i_2 \geq i_1, j_2 \geq j_1.
\]

So Assumption 6' is that \( P \) is \( TP_2 \). In this chapter we assume
Assumptions 1, 2, 3, 4, 5, 6', and 7. It should be noted that
\( TP_2 \Leftrightarrow p_{i_2 k}/p_{i_1 k} \) is non-decreasing in \( k \) for all \( k \) such that
the ratio is defined whenever \( i_2 \geq i_1 \). This latter property is the
property that \( P = [p_{ij}] \) has a monotone likelihood ratio in \( i \). The
equivalence follows directly from the definition. Thus, intuitively,
\( TP_2 \) matrices are similar to IFR matrices in that the mass tends to
the higher-numbered states. In the next section, we discuss some
additional properties of \( TP_2 \) matrices. A reference for total
positivity is Karlin [15] and one for monotone likelihood ratios is
Lehmann [18].

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4.2. Properties and Examples of $TP_2$ Matrices

In addition to the equivalence to monotone likelihood ratios, the $TP_2$ property implies the IFR property.

**Lemma 4.2.1.** $P$ is $TP_2$ $\Rightarrow \sum_{j=2}^{N} p_{ij}$ is non-decreasing in $i$, all $j$.

**Proof:** $TP_2 \Rightarrow p_{i1j1} p_{i2j2} \geq p_{i1j2} p_{i2j1}$ for $i_2 \geq i_1$, $j_2 \geq j_1$.

Let $j_1 < j \leq j_2$ for fixed $j$. So

$$
\sum_{j_1=0}^{j-1} \sum_{j_2=j}^{N} p_{i1j1} p_{i2j2} \geq \sum_{j_2=j}^{N} \sum_{j_1=0}^{j-1} p_{i1j2} p_{i2j1}.
$$

Adding

$$
\sum_{j_2=j}^{N} p_{i1j2} \sum_{j_2=j}^{N} p_{i2j2}
$$

to each side, we get

$$
\sum_{j_2=j}^{N} p_{i2j2} \geq \sum_{j_2=j}^{N} p_{i1j2},
$$

which is the IFR property. \(\square\)

Thus, when we replace Assumption 6 by 6', we have a stronger assumption, and so all of the results of Chapter 3 still hold. In particular, the optimal policy is a monotonic policy.

It is useful to present some examples of classes of $TP_2$ matrices that might represent deteriorating processes:
Example 1. All matrices $P$ such that $p_{ij} = 0$ for $j < i$ or $j > i+1$. This consists of all processes that either stay in the old state or deteriorate exactly one state. This represents a wide class of matrices that might represent real processes.

Example 2. All matrices $P$ such that $p_{ij} = p_{j-1}$ for $j < N$ where $p_k$ is a Polya frequency function of order two, and $p_{iN} = 1 - \sum_{j=0}^{N-1} p_{ij}$.

This class of matrices are often representative of inventory models where state $i$ represents an inventory level of $N-1$ and $p_k$ is the demand function. If the demand exceeds the inventory, the next state is state $N$ (zero inventory), which is why $p_{iN} = 1 - \sum_{j=0}^{N-1} p_{ij}$.

Example 3. All matrices $P$ that represent a structure of $N$ Bernouilli components where the state represents the number of failed components.

For this case the transition function is, for some $q$,

$$p_{ij} = \binom{N-i}{N-j} q^{i-j} (1-q)^{N-j}.$$  

As shown by straightforward calculation, $P = [p_{ij}]$ is $TP_2$. The structures represented by this matrix are an important class of structures in Reliability Theory.

The properties of $TP_2$ matrices are crucial in showing the optimality of a monotonic, four-region policy. We now present some of these properties. In the following results, a sign change of a function $F_i$ occurs when $F_i \cdot F_{i-1} < 0$. When a function has a sign change,
we say that it crosses zero. Sign changes and zero crossings are
intuitive concepts. We note, however, that when sign changes and zero
crossings are determined, zeros of the function are discarded.

\textbf{Lemma 4.2.2.} (Karlin) If \( P \) is \( TP_2 \) and if \( F_i, i = (0, 1, \ldots, N) \)
crosses zero at most once in \( i \), then \( \sum_{j=0}^{N} p_{ij} F_j \) crosses zero at
most once in \( i \). Furthermore, the possible crossing is in the same
direction.

\textbf{Proof:} The proof is based on the proof of Lemma 2, Chapter 3, of
Lehmann [18].

Without loss of generality, we assume

\[
F_j \geq 0 \text{ for } j < j^* \quad \text{and} \quad F_j \leq 0 \text{ for } j \geq j^* . \tag{1}
\]

Then \( F_j \) has at most one sign change. We show that if \( i_1 < i_2, \)

\[
\sum_{j=0}^{N} p_{i_1 j} F_j < 0 \Rightarrow \sum_{j=0}^{N} p_{i_2 j} F_j \leq 0 .
\]

Thus, once \( \sum_{j=0}^{N} p_{ij} F_j \) goes below zero, it stays non-positive, and
we have the result. So assume \( \sum_{j=0}^{N} p_{ij} F_j < 0. \) Recall that \( P \)
is \( TP_2 \) implies that \( p_{i_2 j} / p_{i_1 j} \) is non-decreasing in \( j \) for \( i_2 \geq i_1 \)
for those \( j \) such that the ratio is defined. Now let

\[
j_0 = \min\{ j \geq j^* : p_{i_1 j} \neq 0 \text{ or } p_{i_2 j} \neq 0 \} .
\]
So if \( j_0 \) is defined, then \( \frac{p_{i2j_0}}{p_{i1j_0}} \) is defined. If \( \frac{p_{i2j_0}}{p_{i1j_0}} = \infty \) or \( j_0 \) is not defined, then we must have \( p_{i1j} = 0 \) for \( j \geq j^* \) in which case \( \sum_{j=0}^{N} p_{i1j} F_j \geq 0 \). So since this contradicts the assumption that this sum is strictly less than zero, these cases can be ruled out. Thus, \( j_0 \) is defined and

\[
\frac{p_{i2j_0}}{p_{i1j_0}} = c < \infty ,
\]

and

\( F_j \leq 0 \) on \( S = \{ j : p_{i1j} = 0 \text{ and } p_{i2j} > 0 \} \).

Let

\( T = \{ j : p_{i1j} = 0 \text{ and } p_{i2j} = 0 \} \);

so

\[
\sum_{j=0}^{N} p_{i2j} F_j \leq \sum_{j \in S} p_{i2j} F_j = \sum_{j \in SUT} F_j \frac{p_{i2j}}{p_{i1j}} p_{i1j}
\]

\[
\leq \sum_{j \leq j^* - 1} F_j c p_{i1j} + \sum_{j > j^*} F_j c p_{i1j} \quad \text{by (1) and (2)}
\]

\[
= c \sum_{j=0}^{N} p_{i1j} F_j \leq 0 \quad \text{since } c \geq 0 .
\]

\( \square \)
Lemma 4.2.3. If $P$ is $TP_2$ and if $F_i$, $i = (0, 1, \ldots, N)$ crosses zero at most once in $i$, then $\sum_{j=0}^{N} p_{ij} F_j$ crosses zero at most once in $i$, and the possible zero crossing is in the same direction.

Proof: (induction) The lemma is true for $k = 1$ by Lemma 4.2.2. For case $k+1$, \[
\sum_{j=0}^{N} p_{ij}^{k+1} F_j = \sum_{j=0}^{N} p_{ij} \left( \sum_{\ell=0}^{k} p_{ij}^{\ell} F_j \right),
\]
and so we have the result by induction and Lemma 4.2.2. \[\square\]

Lemma 4.2.4. If $P$ is $TP_2$ and $p_{ij} = 0$ for $j < i$ and if $F_i$, $i = (0, 1, \ldots, N)$, crosses zero at most once in $i$, then $\sum_{j=0}^{N} p_{ij} F_j$ crosses zero at most once in $k$, and the possible crossing is in the same direction.

Proof: Without loss of generality, assume that $F_i$ crosses zero from above. (With no crossing $\sum_{j=0}^{N} p_{ij}^{k} F_j$ has no crossing.) We need to show only that if $\sum_{j=0}^{N} p_{ij} F_j < 0$, then $\sum_{j=0}^{N} p_{ij}^{k+\ell} F_j < 0$. Now let $G_k = \sum_{j=0}^{N} p_{ij} F_j < 0$. By Lemma 4.2.3, $G_m \leq 0$ for $m > i$. So \[
\sum_{j=0}^{N} p_{ij}^{k+\ell} F_j = \sum_{j=0}^{N} p_{ij} G_k^{\ell} = \sum_{j=i}^{N} p_{ij} G_j^{k} \]
since $p_{im}^{\ell} = 0$ for $m < i$. But $\sum_{j=i}^{N} p_{ij} G_j^{k} \leq 0$ since $G_m \leq 0$ for $m > i$ and $G_i^{k} < 0$, so $\sum_{j=0}^{N} p_{ij}^{k+\ell} F_j < 0$. \[\square\]
Using these lemmas, we prove the major result of this chapter in the next section.

4.3. The Monotonic, Four-Region Policy Theorem

The major result of this chapter is that under Assumptions 1-5, 6', and 7, a monotonic four-region policy is optimal for the infinite-horizon and average-cost cases. A proof of this theorem will be presented in this section. Preceding this theorem are some lemmas:

Lemma 4.3.1. Let \( j \in [0, 1, \ldots, N] \). Let \( F_i = \min_{m \in S} \{A_{mi}\} \) and
\[
G_i = \max_{m \in S} \{A_{mi}\},
\]
where \( S \) is a finite set of integers. Assume \( A_{mi} \) crosses zero in \( i \) at most once and if it does, from above. Then both \( F_i \) and \( G_i \) have the same property.

**Proof:** Let \( i^* = \min\{i : F_i < 0\} \) if it exists and let \( m^* = \min\{m : A_{mi^*} < 0\} \). By definition of \( i^* \), \( m^* \) exists. By the crossing property, \( A_{m^*i} \leq 0 \) for \( i \geq i^* \) so \( F_i \leq A_{m^*i} \leq 0 \) for \( i > i^* \). Also, let \( i_* = \min\{i : G_i < 0\} \). So for \( i > i_* \), \( A_{mi} \leq 0 \) since \( A_{mi_*} < 0 \), so \( G_i \leq 0 \). \( \square \)

The next lemma is crucial in the development of the main theorem. We define in the lemma a function \( H_{i,k,l}^{\ell} \) which is equal to \( f \cdot M \) plus the optimal infinite-horizon cost minus the cost of choosing no action for \( \ell \) periods and then inspecting.
Lemma 4.3.2. For the infinite-horizon, discounted problem, for all \( f \geq 1 \),

\[
H_{i,k,f}^\ell = C(i,k) - \sum_{j=0}^{N} p_{ij} L_j - \alpha \sum_{j=0}^{N} p_{ij}^{k+1} L_j - \cdots - \alpha^\ell \sum_{j=0}^{N} p_{ij}^{k+\ell} L_j - \alpha^\ell M
\]

crosses zero at most once in \( i \), and if it does, it does so from above.

(From this point on, we refer to this as the crossing property.)

Proof: The proof is by induction on \( \ell \). We first show the assertion

for \( \ell = 0 \). Now

\[
H_{i,k,f}^0 = C(i,k) - \sum_{j=0}^{N} p_{ij} L_j - \alpha \sum_{j=0}^{N} p_{ij}^{k+1} C(j,0) + M(f-1).
\]

By looking at the terms in the minimization for \( C(i,k) \),

\[
H_{i,k,f}^0 = \min\left\{ \sum_{j=0}^{N} p_{ij}^{k} (C_j - L_j) + \alpha C(0,0) - \alpha \sum_{j=0}^{N} p_{ij}^{k+1} C(j,0) + M(f-1),
\right.
\]

\[
M(f-1) + \alpha C(i, k+1) - \alpha \sum_{j=0}^{N} p_{ij}^{k+1} (C_j, 0), \quad \mathbb{R}\}.
\]

By Lemma 3.1.3 and Theorem 3.1.1, the first term in the minimization

is non-increasing and thus satisfies the crossing property. The

constant \( fM \) satisfies the crossing property. By Lemma 2.1.4 with

\( m = 0 \), the second term is non-negative since \( f \geq 1 \) and thus satisfies

the crossing property. Now assume the assertion is true for case \( \ell-1 \).

For case \( \ell \), by looking at the terms in the minimization of \( C(i,k) \),
\[ H^i_{1,k,f} = C(i,k) - \sum_{j=0}^{N} p^k_{ij} L_j - \alpha \sum_{j=0}^{N} p^{k+1}_{ij} L_j - \cdots - \alpha^l \sum_{j=0}^{N} p^{k+l}_{ij} L_j \]

\[- \alpha^l M - \alpha^{l+1} \sum_{j=0}^{N} p^{k+l+1}_{ij} C(j,0) + fM \]

\[= \min \left( \sum_{j=0}^{N} p^k_{ij} (C_j - L_j) + \alpha C(0,0) - \alpha \sum_{j=0}^{N} p^{k+1}_{ij} L_j \right) \]

\[- \cdots - \alpha^l \sum_{j=0}^{N} p^{k+l}_{ij} L_j - \alpha^l M - \alpha^{l+1} \sum_{j=0}^{N} p^{k+l+1}_{ij} C(j,0) + fM, \]

\[\alpha(C(i,k+1) - \sum_{j=0}^{N} p^{k+1}_{ij} L_j - \cdots - \alpha^{l-1} \sum_{j=0}^{N} p^{(k+1)+(l-1)}_{ij} L_j \]

\[- \alpha^{l-1} M - \alpha^l \sum_{j=0}^{N} p^{(k+1)+(l-1)+1}_{ij} C(j,0) + fM, \]

\[\alpha \sum_{j=0}^{N} p^{k+1}_{ij} (C(j,0) - L_j) - \cdots - \alpha^{l-1} \sum_{j=0}^{N} p^{l-1}_{jm} L_m \]

\[- \alpha^{l-1} M - \alpha^l \sum_{j=0}^{N} p_{jm}^l C(m,0) + \left( \frac{1}{\alpha} + \frac{f}{\alpha} M \right) \].

The first term in the minimization is again non-increasing and thus satisfies the crossing property. The second term is simply \( \alpha H^i_{1,k+1,f/\alpha} \) and so satisfies the crossing property by induction since \( f/\alpha > 1 \). The third term is

\[ \alpha \sum_{j=0}^{N} p^{k+l}_{ij} H^i_{j,0,(l+1)/\alpha} \]

and so by induction and Lemma 4.2.3, the third term satisfies the crossing property. Thus by Lemma 4.3.1, we have the assertion for case \( l \).
This lemma is the main tool that we use in the theorem:

**Theorem 4.3.1.** Under Assumptions 1-5, 6', and 7, where Assumption 6' is that $P$ is $TP_2$, then the infinite-horizon, discounted problem and average-cost problem are each optimized by a monotonic, four-region policy.

**Proof:** We prove the assertion for the infinite-horizon problem first. If a monotonic policy has all infinite critical numbers, then by Lemma 3.1.5, no action is optimal in all states, and so a monotonic, four-region policy is trivially optimal. Otherwise by Theorem 3.2.2, we can set all critical numbers to be finite. We now examine this case. In any observed state $(i,k)$, $k \leq k^*(i,\infty)$, there are $k^*(i,\infty) - k + 1$ options that are possibly optimal: choosing no action for $k^*(i,\infty) - k$ periods and then choosing repair, or choosing no action for $\ell$ periods and then inspecting for $0 \leq \ell \leq k^*(i,\infty) - k - 1$. Let

$$Z_{ik}^\infty = \text{cost of first option} - \min\{\text{cost of any other option}\}.$$ 

Let $k_R(i)$ be the minimum $k \leq k^*(i,\infty)$ such that the first option is optimal. That is, $k_R(i) = \min\{k \leq k^*(i,\infty) : Z_{ik}^\infty \leq 0\}$. Thus if $k_R(i) > 0$, then the strictly optimal action in state $(i, k_R(i) - 1)$ is inspection, and the optimal action in $(i,k)$ for $k \geq k_R(i)$ but $k \leq k^*(i,\infty)$ is no action. (See Figure 8 for a graphical explanation of $k_R(i)$.) So we let $k_{2}^*(i) = k_R(i) - 1$. Thus we need to show that in the interval $[0, k_{2}^*(i)]$, we have a region $[0, k_{1}^*(i) - 1]$ such
Figure 8: Graphical Explanation of Existence of $k^*_R(i)$ and Strategy of Proof

that for $k \in [0, k^*_1(i) - 1]$ no action is optimal at $(i,k)$ and a region $[k^*_1(i), k^*_2(i)]$ such that inspection is optimal. We now assume $k^*_R(i) \geq 2$. (If not, there is nothing left to prove.)

Now let $J^\infty_{ik} =$ (cost of inspection - cost of no action) for the infinite-horizon problem at state $(i,k)$. So
\[ J_{ik}^\infty = \alpha \left( \sum_{j=0}^{N} p_{ij}^{k+1} C(j,0) + \frac{M}{\alpha} - C(i, k+1) \right) . \]

In the region \( k \in [0, k_R(i) - 2] \), by construction \( C(i, k+1) \) can be determined by the minimum of costs from those policies that choose no action for some number of periods and then choose inspection. So

\[ J_{ik}^\infty = \alpha \left( \sum_{j=0}^{N} p_{ij}^{k+1} C(j,0) + \frac{M}{\alpha} - \min_{s \in A} C(i, k+1, \alpha, s) \right) \]

\[ = \alpha \max_{s \in A} \left( \sum_{j=0}^{N} p_{ij}^{k+1} C(j,0) + \frac{M}{\alpha} - C(i, k+1, \alpha, s) \right) \]

where

\[ A = \{ \text{policies that stipulate no action for some number of periods (bounded by } k^*(0,\infty) \text{) and then inspection} \} \]

so

\[ J_{ik}^\infty = \alpha \max_{\ell \leq k^*(0,\infty)} \left( \sum_{j=0}^{N} p_{ij}^{k+1} C(j,0) + \frac{M}{\alpha} - \sum_{j=0}^{\ell} p_{ij}^{k+\ell} L_j \right) \]

\[ - \alpha \sum_{j=0}^{\ell} p_{ij}^{k+\ell} L_j - \ldots - \alpha^\ell \sum_{j=0}^{k+\ell+1} p_{ij} L_j \]

\[ - \alpha^\ell M - \alpha^{\ell+1} \sum_{j=0}^{N} p_{ij}^{k+\ell+2} C(j,0) \]

\[ = \alpha \max_{\ell \leq k^*(0,\infty)} \left( \sum_{j=0}^{N} p_{ij}^{k+1} C(j,0) + \frac{M}{\alpha} - L_j - \sum_{m=0}^{\ell} p_{ij}^{\ell+1} L_m \right) \]

\[ - \ldots - \alpha^\ell \sum_{m=0}^{\ell} p_{ij}^{\ell+1} L_m - \alpha^\ell M \]

\[ - \alpha^{\ell+1} \sum_{m=0}^{N} p_{ij}^{\ell+1} C(m,0) \]

\[ = \alpha \max_{\ell \leq k^*(0,\infty)} \left( \sum_{j=0}^{N} p_{ij}^{k+1} \left( H_{j,0,1/\alpha}^\ell \right) \right) . \]
But by successively applying Lemmas 4.3.2, 4.2.4, and 4.3.1, $J_{1k}^\infty$ satisfies the crossing property in $k \in [0, k_R(i) - 2]$. Also $J_{1k}^\infty, k_2^*(i) < 0$ by construction. Recalling that $k_2^*(i) = k_R(i) - 1$, we can divide $[0, k_2^*(i)]$ into two regions $[0, k_1^*(i) - 1]$ and $[k_1^*(i), k_2^*(i)]$ such that $J_{1k}^\infty \geq 0$ in the former and $J_{1k}^\infty \leq 0$ in the latter, and $J_{1k}^\infty, k_1^*(i) < 0$. Thus in the former region, no action is optimal by definition of $J_{1k}^\infty$, and in the latter, (which may be a singleton) inspection is optimal. Thus we have the assertion for the infinite-horizon problem.

For the average-cost problem, from the proofs of Theorem 3.2.4 and Corollary 3.1.1, the optimal average-cost policy is one that is optimal for an infinite number of infinite-horizon problems for different $\alpha$. Thus the optimal policy is a monotonic, four-region policy. \hfill \square

This theorem is an extremely important result in this paper, for it gives us our most complete characterization of the optimal policy. The structure of this optimal policy is furthermore an appealing one.

4.4. Supplementary Results

To supplement Theorem 4.3.1, we have two additional results. The first result gives us conditions under which the upper no-action region for a given $i$ is vacuous in the infinite-horizon problem.

The second result relates the values of $k_1^*(i)$.
Theorem 4.4.1. A sufficient condition for the top no-action region \((k^*_2(i), k^*(i,\infty))\) for a given \(i\) to be vacuous in the infinite horizon problem is

\[
\sum_{j=0}^{N} p_{ij} \left[ c_j - L_j - M + \alpha(c(0,0) - \sum_{\ell=0}^{N} \beta p_{j\ell} c(\ell,0)) \right] \geq 0
\]

where

\[
m_1(i) = \min\{m : \sum_{j=0}^{N} p_{ij} (c_j - L_j) + \alpha(c(0,0) - c(i, m+1)) < 0\}.
\]

Proof: Note that \(m_1(i) = \min\{m : F_{i1m} < 0\}\). Let \(m_2(i) = \min\{m : G_{i1m} < 0\}\). Note that if \(m_2(i) > m_1(i)\), there are at most three regions, since inspection is optimal for \(m_1(i) \leq k < m_2(i)\) and repair is optimal for \(k \geq m_2(i)\) for states \((i,k)\). But if \(G_{i1m_1(i)} \geq 0\), which is the condition above, then \(m_2(i) > m_1(i)\) by definition.

There is also the question of whether four separate regions are possible. We know that at least for the \(n\)-period problem, four separate regions are possible, as we have generated some examples, and for nearly identical structure, Ross has found an example for the two-state problem. The original existence of such a region was discovered for a similar problem by Taylor [26].

One of the examples that we generated was

\[
P = \begin{bmatrix}
  0.9 & 0.04 & 0.06 \\
  0.0 & 0.9 & 1.0 \\
  0.0 & 0.0 & 1.0
\end{bmatrix}, \quad L = (0,0,1), \quad C = (2.5, 2.5, 2.5), \quad \alpha = 0.9, \quad M = 1
\]
For this case, for the ten-period problem, it was found that as \( k \) varied for state \((0, k)\), no action was optimal for \( k = 0, 7, 8, 10\), inspection was optimal for \( k = 8 \) and repair was optimal for \( k \geq 11 \). Although this example pertains to the \( n \)-period problem, it is conjectured that similar examples can be found for the infinite-horizon and average-cost problems.

The next theorem gives us some insight into the relationship of the four regions in \( k \) for state \((i, k)\) for different \( i \).

**Theorem 4.4.2.** For those \( i \) such that there is some \((i, k)\) where inspection is optimal (i.e., \( k_1^*(i) < k^*(i) \) or \( k^*(i, \infty) \)), we can set \( k_1^*(i) \) such that \( k_1^*(i) \) is non-increasing in \( i \) for both the infinite-horizon and average-cost problems.

**Proof:** We prove the assertion first for the infinite-horizon problem. Assume by contradiction that \( k_1^*(i_2) > k_1^*(i_1) \) for some \( i_2 > i_1 \). Now in state \((i_1, k_1^*(i_1))\) the optimal policy is inspection since \( k_1^*(i_1) < k^*(i_1, \infty) \). Furthermore, from the proof of Theorem 4.3.1, we construct \( k_1^*(i) \) such that inspection is strictly optimal. Now

\[
J_{ik} = \alpha \left( \sum_{j=0}^{N} p_{ij}^{k+1} C(j, 0) + \frac{M}{\alpha} - C(i, k+1) \right)
\]

\[
= \alpha \left( \sum_{j=0}^{N} p_{ij}^{k+1} C(j, 0) + \frac{M}{\alpha} - \min_{s} C(i, k+1, \alpha, s) \right)
\]

\[
= \alpha \max_{s} \left( \sum_{j=0}^{N} p_{ij}^{k+1} C(j, 0) + \frac{M}{\alpha} - C(i, k+1, \alpha, s) \right)
\]

\[
\geq \alpha \max_{\ell \leq k^*(0, \infty)} \left( \sum_{j=0}^{N} p_{ij}^{k+1} H_{j, 0, \ell}^{1/\alpha} \right)
\]
Thus

$$ J_{i_1, k_1^*} \rightarrow 0 \Rightarrow \alpha \max_{\ell \leq k_1^*(0,\infty)} \sum_{j=0}^{N} p_{i_1 j}^1 H_{j,0,1/\alpha}^{\ell} $$

$$ = -\varepsilon < 0 \quad \text{for some } \varepsilon. \quad (1) $$

Now $k_1^*(i_1) < k_1^*(i_2)$, so the optimal policy at observed state $(i_2, k_1^*(i_1))$ is choose no action for some number of periods and then choose inspection, since $k_1^*(i_2) < k_1^*(i_2,\infty)$. In particular, no action is optimal in $(i_2, k_1^*(i_1))$. Therefore, by definition of $k_2^*(i)$,

$$ J_{i_2, k_1^*(i_1)} \geq 0 \Rightarrow \alpha \max_{\ell \leq k_1^*(0,\infty)} \sum_{j=0}^{N} p_{i_2 j}^2 H_{j,0,1/\alpha}^{\ell} \geq 0. \quad (2) $$

Now add $\varepsilon/2$ to both sides of (1) and (2). Recalling the definition of $H_{j,0,1/\alpha}^{\ell}$, we have that

$$ H_{j,0,1/\alpha}^{\ell} + \varepsilon/2 = H_{j,0,1/\alpha+\varepsilon/2M}^{\ell}. $$

Thus

$$ \alpha \max_{\ell \leq k_1^*(0,\infty)} \sum_{j=0}^{N} p_{i_1 j}^1 H_{j,0,1/\alpha+\varepsilon/2M}^{\ell} = -\frac{\varepsilon}{2} < 0 \quad (3) $$

and

$$ \alpha \max_{\ell \leq k_1^*(0,\infty)} \sum_{j=0}^{N} p_{i_2 j}^2 H_{j,0,1/\alpha+\varepsilon/2M}^{\ell} \geq \frac{\varepsilon}{2} > 0. \quad (4) $$

But by successively applying Lemmas 4.3.2, 4.2.3, and 4.3.1, we see that
\[
\alpha \leq \max_{\ell \leq k^*(0,\infty)} \sum_{j=0}^{N} \sum_{\ell} p_{ij}^{*} H_{j,0,1/\alpha+\epsilon/2M}^{\ell}
\]
satisfies the crossing property, which contradicts (3) and (4) since \( i_2 > i_1 \). The assertion for the average-cost case follows trivially as in the proof of Theorem 4.3.1. \( \square \)

Although this result does not give us a complete description of the relationships among the two sets of critical numbers that determine when inspection is optimal, it nevertheless gives us some information about these numbers.
CHAPTER 5
A MORE GENERAL TRANSITION MATRIX

5.1. Introduction

One of the critical assumptions in the past two chapters is Assumption 7, which states that \( p_{ij} = 0 \) for \( j < i \). Although this assumption is reasonable, we would like to find a weaker and more general assumption to replace it. In this chapter we shall see that we can relax Assumption 7 and still obtain the optimality of monotonic policies. In some special cases, which will be outlined in Section 5.5, a four-region, monotonic policy is optimal. The general and weaker condition on the matrix \( P \) that we assume in place of Assumption 7 is:

7') For any \( n \times 1 \) \( F = \{F_j\} \) whose components \( F_j \) are non-increasing in \( j \),

\[
(P \cdot F)_i \leq F_i \vee (\pi \cdot F), \quad i = 0, \ldots, N
\]

where \( P \) is the transition matrix, \( \pi \) is the vector of limiting state probabilities, and \( \vee \) denotes the maximum operator.

Note that if Assumption 7 holds, \( \pi \cdot F = F_N \), and thus Assumption 7 \( \Rightarrow \) Assumption 7'. In general, however, it would seem difficult to determine whether 7') is satisfied because the defining condition must be satisfied for all monotonic \( F \). In Section 5.2, we present examples of matrices that satisfy 7') and present a procedure for
determining whether $7'$ is satisfied by an arbitrary matrix.

Assumption $7'$ is a new condition developed in this paper.

Closely related to Assumption $7'$ is the following property.

**Definition:** Assume $F_k \rightarrow F_\infty$. Then we say that $F_k$ satisfies Property A in $k$ when:

\[
\text{If } F_k \leq F_\infty, \text{ then } F_{k+1} \leq F_\infty,
\]

and

\[
\text{if } F_k \geq F_\infty, \text{ then } F_{k+1} \leq F_k.
\]

We immediately have the following:

**Lemma 5.1.1.** If $P$ satisfies Assumption $7'$ and $F_j$ is non-increasing, then

\[
\sum_{j=0}^{N} p_{ij} F_j
\]

satisfies Property A in $k$.

**Proof:** Let $F_j$ be non-increasing and let

\[
\sum_{j=0}^{N} p_{ij} F_j = G_i.
\]

By Lemma 3.1.3, $G_i$ is non-increasing in $i$. In addition,

\[
\sum_{j=0}^{N} \pi_j G_j = \sum_{j=0}^{N} \pi_j \sum_{\ell=0}^{N} p_{j\ell} F_\ell = \sum_{\ell=0}^{N} \left( \sum_{j=0}^{N} p_{j\ell} \pi_j \right) F_\ell = \sum_{\ell=0}^{N} \pi_\ell F_\ell,
\]

so
\[
\sum_{j=0}^{N} p_{ij} F_j^{k+1} = \sum_{j=0}^{N} p_{ij} G_j \leq G_i \lor \left( \sum_{j=0}^{N} \pi_j G_j \right) \quad \text{since } P \text{ satisfies } 7')
\]

\[
= \sum_{j=0}^{N} p_{ij} F_j \lor \sum_{j=0}^{N} \pi_j F_j .
\]

\[\square\]

We will use the fact that \( \sum_{j=0}^{N} p_{ij} F_j \) satisfies Property A in the analysis of this chapter. The property is precisely what is needed to show that a monotonic policy is optimal. Note that Property A does not directly follow from the other assumptions. An example of a transition matrix that satisfies our other assumptions but does not satisfy Property A is the \( 3 \times 3 \) matrix

\[
\begin{bmatrix}
.5 & .1 & .4 \\
.3 & .5 & .7 \\
.1 & .2 & .7 \\
\end{bmatrix}
\]

for which \( \pi = \left( \frac{62}{273}, \frac{44}{273}, \frac{167}{273} \right) \).

Note that for \( F_0 = 1, F_1 = F_2 = 0 \), Property A is not satisfied. Thus we assume 7'), from which Property A follows.

The class of matrices \( p_{ij} = 0 \) for \( j < i \) is an important subclass of matrices satisfying Assumption 7'). This class of matrices is not exhaustive, however. In the next section we provide some examples of matrices that satisfy 7') and outline how one might determine whether 7') holds for other cases.
5.2. **Testing of the Assumption and Examples**

The only problem with Assumption 7' is that it is expressed in terms of the class of non-increasing functions $F_j$. In this section we describe the class of matrices satisfying 7') in two and three dimensions, give a sample class of such matrices in higher dimensions, and show how one might determine whether 7') holds in the general case. We shall see that several classes of matrices satisfy the assumption.

**Two Dimensions.** For two dimensions, i.e., $\{p_{ij}, i = 0, 1; j = 0, 1\}$, all IFR matrices satisfy Assumption 7':

$$(P \cdot F)_0 = p_{00} F_0 + p_{01} F_1 \leq p_{00} F_0 + p_{01} F_0 = F_0$$

since $F_j$ is non-increasing. Furthermore,

$$(P \cdot F)_1 = p_{10} F_0 + p_{11} F_1 \leq \pi_0 F_0 + \pi_1 F_1 = \pi \cdot F$$

by Lemma 3.1.6. Thus Assumption 7' is satisfied. This is an extremely important class of matrices and we shall look at this class further.

**Three Dimensions.** For three dimensions, i.e., $\{p_{ij}, i = 0, 1, 2; j = 0, 1, 2\}$, if $P$ is IFR, then $P$ satisfies Assumption 7' is equivalent to (where $\wedge$ denotes minimum)

$$p_{10} \leq \pi_0 \wedge \left( \frac{1-p_{11}}{1-\pi_1} \right) \pi_0,$$
which we show as follows:

\[(P\cdot F)_0 = p_{00}F_0 + p_{01}F_1 + p_{02}F_2 \leq p_{00}F_0 + p_{01}F_0 + p_{02}F_0 = F_0;\]

\[(P\cdot F)_2 \leq \pi\cdot F\]

by Lemma 3.1.6.

Without loss of generality let \( F_2 = 0 \). So Assumption 7' holds

\[ \iff (P\cdot F)_1 = p_{10}F_0 + p_{11}F_1 \leq F_1 \vee (\pi_{0}F_0 + \pi_{1}F_1) \quad \forall F_0 \geq F_1 \]

\[ \iff p_{10} \leq (1-p_{11}) F_1 \frac{F_1}{F_0} \text{ or } p_{10} \leq \pi_0 + (p_{1}-p_{11}) F_1 \frac{F_1}{F_0} \quad \forall F_0 \geq F_1 \]

\[ \iff p_{10} \leq \max((1-p_{11}) f, \pi_0 + (p_{1}-p_{11}) f) \quad \forall f \in [0,1]. \]

If \( p_{11} \leq \pi_1 \) then this expression is minimized at \( f = 0 \). If \( p_{11} > \pi_1 \), it is minimized at \( f = \pi_0/(1-\pi_1) \). In the first case,

![Figure 9: Case of Assumption 7' in Three Dimensions](image-url)
we have \( p_{10} \leq \pi_0 \), and in the second case, we have \( p_{10} \leq (1-p_{11}) \pi_0/(1-\pi_1) \).
In either case, \( p_{10} \leq \pi_0 \wedge (1-p_{11})/(1-\pi_1) \pi_0 \). Matrices that satisfy this condition include, for example,

\[
\begin{bmatrix}
.5 & .2 & .3 \\
.3 & .2 & .5 \\
.2 & .2 & .6
\end{bmatrix},
\begin{bmatrix}
.4 & .4 & .2 \\
.3 & .45 & .25 \\
.3 & .3 & .4
\end{bmatrix},
\begin{bmatrix}
.5 & .4 & .1 \\
.1 & .6 & .3 \\
.1 & .4 & .5
\end{bmatrix}
\]
and

\[
\begin{bmatrix}
.6 & .2 & .2 \\
.25 & .4 & .35 \\
.2 & .1 & .7
\end{bmatrix}
\]

**Higher Dimensions.** For higher dimensions, finding equivalent conditions is much more complex. An example of a class of matrices that satisfies Assumption 7' is all \( \mathbf{P} \) such that \( p_{ij} = 0 \) for \( j < i \) and \( j \leq N-3 \),

\[
P_{N-1,N-2} \leq \pi_{N-2} \wedge (\frac{1 - p_{N-1,N-1}}{1 - \pi_{N-1}}) \pi_{N-2},
\]

and \( \mathbf{P} \) IFR. For this class, \( \pi_i = 0 \), \( i \leq N-3 \), and we show the validity of Assumption 7' as follows: For \( i \leq N-2 \),

\[
(P \cdot F)_i = \sum_{j=1}^{N} p_{ij} F_j \leq \sum_{j=1}^{N} p_{ij} F_i = F_i
\]

for \( F_i \) non-increasing. Also \( (P \cdot F)_N \leq \pi \cdot F \) by Lemma 3.1.6, and finally,

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\((P \cdot F)_{N-1} = p_{N-1,N-2} F_{N-2} + p_{N-1,N-1} F_{N-1} + p_{N-1,N} F_N\)

\[\leq F_{N-1} \vee (\pi_{N-2} F_{N-2} + \pi_{N-1} F_{N-1} + \pi_N F_N)\]

by the exact same reasoning as in the three dimensional case. Note that this class includes all \(P\) such that \(p_{ij} = 0\) for \(j < i\).

For an arbitrary IPR matrix we can test whether Assumption 7' is satisfied by using some straightforward linear programming. Let \(F_N \geq 0\) without loss of generality, and let \(\triangle_i = F_i\), \(\triangle_i = F_i - F_{i+1}\), \(i = 0, 1, \ldots, N-1\). Now Assumption 7' does not hold if we can find an \(i\) and an \(F_j\) non-increasing such that \((P \cdot F)_i > F_i\) and \((P \cdot F)_i > \pi \cdot F\). The last two inequalities, however, are equivalent to the linear inequalities

\[A_i = \sum_{\ell=0}^{N} \triangle_i \left( \sum_{j=0}^{\ell} p_{ij} - I_{\{\ell \geq i\}} \right) > 0, \quad (1)\]

\[B_i = \sum_{\ell=0}^{N} \triangle_i \left( \sum_{j=0}^{\ell} (p_{ij} - \pi_j) \right) > 0. \quad (2)\]

Also, note that \(\triangle_i \geq 0\) as \(F_N \geq 0\) and \(F_i\) is non-increasing. We can thus use the simplex method to maximize \(A_i\) subject to the constraint \(B_i = 1\). If an unbounded or positive solution exists for any of these \(N+1\) programs, we see that (1) and (2) holds and thus Assumption 7' is not satisfied. If all programs have a non-positive or infeasible solution then Assumption 7' holds.
5.3. **Optimality of Monotonic Policies for the n-Period and Infinite-Horizon Problems**

Under Assumption 7', it turns out that monotonic policies are optimal only under certain conditions on the costs. In the remainder of this chapter, we show this optimality and derive sufficient conditions for the costs. The sufficient conditions derived are more easily satisfied in the infinite-horizon and average-cost cases, so we shall concentrate more on these cases. These conditions are quite similar to the conditions for the non-trivial case in Chapter 3.

We precede the major results with some lemmas that are important in the analysis.

**Lemma 5.3.1.** Let \( F_k = \max_{j \in A} \{ A_{jk} \} \) where \( A \) is a finite set and \( A_{jk} \) satisfies Property A in \( k \). Then \( F_k \) satisfies Property A in \( k \).

**Proof:** Notice that \( \lim_{k \to \infty} F_k = \max_{j \in A} \{ \lim_{k \to \infty} A_{jk} \} \). We have two cases:

**Case 1.** \( F_k \geq \lim_{k \to \infty} F_k' \). Then \( \max_{j \in A} \{ A_{jk} \} \geq \max_{j \in A} \{ \lim_{k \to \infty} A_{jk} \} \).

Let \( \max_{j \in A} A_{jk} = A_{j^*k} \). Then

\[
A_{j^*k} = \max_{j \in A} A_{jk} \geq \max_{j \in A} \{ \lim_{k \to \infty} A_{jk} \} \geq \lim_{k \to \infty} A_{j^*k}.
\]
Since $A_{j^*k}$ satisfies Property A in $k$,

$$A_{j^*, k+1} \geq A_{j^*k}$$ \hspace{1cm} (1)

**Case 1a.** If $F_{k+1} = A_{j^*, k+1}$ then $F_{k+1} \leq A_{j^*k} = F_k$.

**Case 1b.** Assume $F_{k+1} = A_{j^{**}, k+1}$, $j^{**} \neq j^*$, and

$$A_{j^{**}k} \geq \lim_{k \to \infty} A_{j^{**}k}$$ \hspace{1cm} (2)

Then

$$F_{k+1} = A_{j^{**}, k+1} = \max\{A_{j^{**}, k+1}; A_{j^*, k+1}\}$$

By Property A and (2),

$$A_{j^{**}, k+1} \leq A_{j^{**}k}$$ \hspace{1cm} (3)

By (1) and (3) and Lemma 3.1.4,

$$F_{k+1} = \max\{A_{j^{**}, k+1}; A_{j^*, k+1}\} \leq \max\{A_{j^{**}k}; A_{j^*k}\} = A_{j^*k} = F_k$$

**Case 1c.** Assume $F_{k+1} = A_{j^{**}, k+1}$, $j^{**} \neq j^*$ and

$$A_{j^{**}, k} \leq \lim_{k \to \infty} A_{j^{**}k}$$

So
\[ F_{k+1} = A_{j^{**}, \text{c}, k+1} \leq \lim_{k \to \infty} A_{j^{**}, k} \quad \text{by Property A} \]

\[ \leq \max_j \lim_{k \to \infty} A_{j, k} = \lim_{k \to \infty} F_k \]

\[ \leq F_k \quad \text{by hypothesis.} \]

**Case 2.** \( F_k \leq \lim_{k \to \infty} F_k' \). Pick any \( j \). If \( A_{j, k} \leq \lim_{k \to \infty} A_{j, k} \), then

\[ A_{j, k+1} \leq \lim_{k \to \infty} A_{j, k} \quad \text{by Property A} \]

\[ \leq \max_j \lim_{k \to \infty} A_{j, k} = \lim_{k \to \infty} F_k . \quad (4) \]

If \( A_{j, k} \geq \lim_{k \to \infty} A_{j, k} \), then

\[ A_{j, k+1} \leq A_{j, k} \leq F_k \leq \lim_{k \to \infty} F_k . \quad (5) \]

Thus,

\[ F_{k+1} = \max_j A_{j, k+1} \leq \lim_{k \to \infty} F_k \]

by (4) and (5).

The following lemma deals with sequences of functions that satisfy Property A.
Lemma 5.3.2. Let $F_k^n$ be a sequence (in $n$) of functions that satisfy Property A in $k$. Let

$$F_k^n = \lim_{n \to \infty} F_k^n$$

and assume that the limit exists. Assume also that

$$\lim_{k \to \infty} F_k^\infty = F_\infty^\infty$$

exists, and

$$F_\infty^\infty > \lim_{n \to \infty} F_\infty^n$$

where

$$F_\infty^n = \lim_{k \to \infty} F_k^n$$

has a limit in $n$. Then $F_k^\infty$ satisfies Property A in $k$.

Proof: Let $\Delta_k^n = F_k^n - F_k^\infty$ and $\Delta_k^\infty = F_k^\infty - F_\infty^\infty$, so $\Delta_k^n \to 0$ as $k \to \infty$ and $\Delta_k^\infty \to 0$ as $k \to \infty$. Now by assumption,

$$F_\infty^n + C \to F_\infty^\infty$$

for some $C \geq 0$.

So

$$\Delta_k^n \to \Delta_k^\infty + C$$

as $n \to \infty$. 

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We want to show that \( \Delta_k^n \) satisfies Property A. If \( \Delta_k^\infty > \lim_{k \to \infty} \Delta_k^\infty = 0 \), then for \( n \geq \bar{N} \), some \( \bar{N} \),

\[
\Delta_k^n > 0, \quad \text{since} \quad \lim_{n \to \infty} \Delta_k^n = \Delta_k^\infty + C > 0 .
\]

Thus \( \Delta_k^{n+1} \leq \Delta_k^n \) since \( \Delta_k^n \) satisfies Property A in \( k \). Taking limits of both sides and cancelling \( C \),

\[
\Delta_k^{\infty+1} \leq \Delta_k^\infty .
\]

If \( \Delta_k^\infty \leq 0 \), pick any \( \epsilon \) and then pick \( \bar{N} \) so big that

\[
\Delta_k^n \leq \Delta_k^\infty + C + \epsilon \leq C + \epsilon \quad \text{for} \quad n \geq \bar{N} .
\]

Since \( \Delta_k^n \) satisfies Property A in \( k \),

\[
\frac{\Delta_k^n}{k+1} \leq \max\{\Delta_k^n, 0\} \leq C + \epsilon .
\]

Taking limits as \( n \to \infty \),

\[
\Delta_k^{\infty+1} + C \leq C + \epsilon \Rightarrow \Delta_k^{\infty+1} \leq \epsilon .
\]

Since \( \epsilon \) was arbitrary, \( \Delta_k^{\infty+1} \leq 0 \). Thus \( \Delta_k^\infty \) satisfies Property A in \( k \), and hence so does \( F_k^\infty \). \( \square \)
With one more lemma, we can prove the major results of the section. This lemma shows that we can express $C^n(i,k)$ by a particular structure.

**Lemma 5.3.3.** $C^n(i,k) = \min_{m \in [1,\bar{m}(n)]} \left\{ \sum_{j=0}^{N} p_{ij} A_{mj}^n \right\}$ where $\bar{m}(n) < \infty$

and $A_{mj}^n$ is non-decreasing in $j$.

**Proof:** Induction. The assertion is trivial for $n = 1$. For case $n$,

$$C^n(i,k) = \min\left\{ \sum_{j=0}^{N} p_{ij} C_j + \alpha C^{n-1}(0,0), \sum_{j=0}^{N} p_{ij} L_j + \alpha C^{n-1}(i,k+1), \sum_{j=0}^{N} p_{ij} L_j + M + \alpha \sum_{j=0}^{N} p_{ij} C^{n-1}(j,0) \right\}.$$  

But

$$\sum_{j=0}^{N} p_{ij} L_j + \alpha C^{n-1}(i,k+1)$$

$$= \sum_{j=0}^{N} p_{ij} L_j + \alpha \min_{m \in [1,\bar{m}(n-1)]} \left\{ \sum_{j=0}^{N} p_{ij} A_{mj}^{n-1} \right\}$$

$$= \min_{m \in [1,\bar{m}(n-1)]} \left\{ \sum_{j=0}^{N} p_{ij}(L_j + \alpha \sum_{\ell=0}^{N} p_{j\ell} A_{m\ell}^{n-1}) \right\}.$$  

By substituting this into the expression for $C^n(i,k)$, by noting that

$$\sum_{j=0}^{N} p_{ij}^{k+1} C^{n-1}(j,0) = \sum_{j=0}^{N} p_{ij}^{k} \left( \sum_{\ell=0}^{N} p_{j\ell} C^{n-1}(\ell,0) \right),$$

and by Lemma 3.1.3, the result follows. 

\[\square\]
We now re-examine the functions $F_{ik}^n$, $G_{ik}^n$, $F_{ik}^\infty$, and $G_{ik}^\infty$.

As before,

$$F_{ik}^\infty = \text{cost of repair - cost of no action}$$

$$G_{ik}^\infty = \text{cost of repair - cost of inspection}$$

for the infinite-horizon problem, and $F_{ik}^n$ and $G_{ik}^n$ represent the same functions for the n-period problem. Note that the possible optimality of repair is determined by the signs of these functions, that $G_{ik}^n$ and $G_{ik}^\infty$ have limits in $k$, and that by Lemma 2.3.1 and Theorem 2.3.1, $F_{ik}^n$ and $F_{ik}^\infty$ also have limits in $k$. We define these limits below. Recall that $\pi = (\pi_0, \pi_1, \ldots, \pi_N)$ is the vector of limiting state probabilities,

$$F_{ik}^n = \lim_{k \to \infty} F_{ik}^n = \sum_{j=0}^{N} \pi_j (C_{j} - L_{j}) + \alpha C^{n-1}(0,0) - \alpha C^{n-1}_*$$

$$G_{ik}^n = \lim_{k \to \infty} G_{ik}^n = \sum_{j=0}^{N} \pi_j (C_{j} - L_{j}) - M + \alpha C^{n-1}(0,0) - \alpha \sum_{j=0}^{N} \pi_j C^{n-1}(j,0)$$

$$F_{ik}^\infty = \lim_{k \to \infty} F_{ik}^\infty = \sum_{j=0}^{N} \pi_j (C_{j} - L_{j}) + \alpha C(0,0) - \alpha C_*$$

$$G_{ik}^\infty = \lim_{k \to \infty} G_{ik}^\infty = \sum_{j=0}^{N} \pi_j (C_{j} - L_{j}) - M + \alpha C(0,0) - \alpha \sum_{j=0}^{N} \pi_j C(j,0) .$$

Given these limits, we can now prove the optimality of monotonic policies for both the n-period and infinite-horizon problems. As mentioned before, we require additional conditions, which are stated in the theorem:
Theorem 5.3.1. a) Assume $F_n^* \leq 0$ and $G_n^* \leq 0$. Then a monotonic policy is optimal for the n-period problem. Furthermore, it is optimal to repair in state $(N, k)$, all $k$.

b) Assume $F_\infty^* \leq 0$ and $G_\infty^* \leq 0$. Then a stationary monotonic policy is optimal for the infinite-horizon problem. Furthermore, it is optimal to repair in state $(N, k)$, all $k$.

Proof:

$$G_{ik}^n = \sum_{j=0}^{N} p_{ij} (C_j - L_j) - M + \alpha C^{n-1}(0,0) - \alpha \sum_{j=0}^{N} p_{ij} n_{j}^{n-1}(j,0)$$

$$= \sum_{j=0}^{N} p_{ij} [C_j - L_j - M + \alpha (C^{n-1}(0,0) - \sum_{j=0}^{N} p_{j\ell} n_{j}^{n-1}(\ell,0))] .$$

By Lemma 5.1.1, $G_{ik}^n$ satisfies Property A in $k$;

$$F_{ik}^n = \sum_{j=0}^{N} p_{ij} (C_j - L_j) + \alpha (C^{n-1}(0,0) - C^{n-1}(i, k+1))$$

$$= \max_{m \in [1, \ldots, m(n-1)]} \left[ \sum_{j=0}^{N} p_{ij} [C_j - L_j + \alpha C^{n-1}(0,0) - \alpha \sum_{\ell=0}^{N} p_{j\ell} A_{m\ell}^{n-1}] \right] \quad \text{by Lemma 5.3.3.}$$

Note that the term in brackets is non-increasing by Lemma 3.1.1. So by Lemmas 5.1.1 and 5.3.1, $F_{ik}^n$ satisfies Property A in $k$. Furthermore, by Lemmas 2.1.2 and 2.3.2,

$$F_\infty^* = \lim_{n \to \infty} F_n^* \quad \text{and} \quad G_\infty^* = \lim_{n \to \infty} G_n^* .$$

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Since $F^n_{ik} \to F^\infty_{ik}$ and $G^n_{ik} \to G^\infty_{ik}$ as $n \to \infty$, we have by Lemma 5.3.2 that $F^\infty_{ik}$ and $G^\infty_{ik}$ also satisfy Property A in $k$.

We now complete the proof of the theorem for the $n$-period problem. The proof for the infinite-horizon is completed in exactly the same manner, and the existence of a stationary optimal policy follows trivially as in Chapter 3. Pick any $i$. If $F^n_{i0} \leq F^n_\ast$ then $F^n_{i0} \leq 0$. Since $F^n_{ik}$ satisfies Property A in $k$, then

$$F^n_{ik} \leq F^n_\ast \leq 0$$

for any $k$.

Also by Property A, if $F^n_{i0} > F^n_\ast$, then $F^n_{ik}$ is non-increasing in $k$ until $F^n_{ik}$ decreases to or below $F^n_\ast$. Beyond that point $F^n_{ik} \leq F^n_\ast \leq 0$. There is thus a

$$k = \min \{k : F^n_{ik} \leq 0\}$$

such that for $k \geq k$, $F^n_{ik} \leq 0$, and thus repair is just as good or better than no action. An analogous value can be found for $G^n_{ik}$.

Thus we can define

$$k^\ast(i,n) = \min \{k : F^n_{ik} \leq 0, G^n_{ik} \leq 0\}$$

such that $F^n_{ik} \leq 0$ and $G^n_{ik} \leq 0$ for $k \geq k^\ast(i,n)$. That is, repair is optimal for $k \geq k^\ast(i,n)$, and no action or inspection is optimal otherwise. Thus the $k^\ast(i,n)$ are critical numbers in an optimal monotonic policy. To show that $k^\ast(i,n)$ is non-increasing in $i,$

\[ \text{We use } \leq \text{ in this chapter to allow the optimality of monotonic policies when } F^\ast = 0 \text{ or } G^\ast = 0. \]
note that $F_{ik}^n$ and $G_{ik}^n$ are non-increasing in $i$ by the same reasoning given in the proof of Theorem 3.2.1. Furthermore,

$$F_{NO}^n = C_N - L_N + \alpha C^{n-1}(0,0) - \alpha C^{n-1}(N,1)$$

$$\leq \sum_{j=0}^{N} \pi_j (C_j - L_j) + \alpha C^{n-1}(0,0) - \alpha C_{*}^{n-1}$$

by Lemma 3.1.6

$$= F_*^n \leq 0 ;$$

$$G_{NO}^n = C_N - L_N - M + \alpha C^{n-1}(0,0) - \alpha \sum_{j=0}^{N} p_{Nj} C^{n-1}(j,0)$$

$$\leq \sum_{j=0}^{N} \pi_j (C_j - L_j) - M + \alpha C^{n-1}(0,0) - \alpha \sum_{j=0}^{N} \pi_j C^{n-1}(j,0)$$

$$= G_*^n \leq 0$$

also by Lemma 3.1.6;

since $F_{NO}^n \leq F_*^n$ and $G_{NO}^n \leq G_*^n$,

$$F_{Nk}^n \leq F_*^n \leq 0 \quad \text{and} \quad G_{Nk}^n \leq G_*^n \leq 0$$

by Property A.

Thus repair is optimal in $(N,k)$, $\forall k$. As mentioned, the infinite-horizon case is exactly the same. $\square$

The only problem with this result is that it is difficult to show whether $F_*^n \leq 0$, $G_*^n \leq 0$, $F_*^\infty \leq 0$ or $G_*^\infty \leq 0$. The remainder of this section is mainly intended to derive sufficient conditions for these inequalities. The infinite-horizon (and average-cost) cases
are easier to analyze than the n-period case. Consequently, the sufficient conditions derived for the former two cases are more often satisfied than the sufficient conditions for the latter case.

Note that under the new Assumption 7', we do not have the two cases of Theorem 3.2.2 (that is, either all infinite or all finite critical numbers), and sometimes no monotonic policy is optimal. Instead of two cases, there are now four cases. We describe these cases for the infinite-horizon problem. By replacing $\infty$ by $n$, we get the same cases for the n-period problem.

**Case 1.** $F_\infty^* < 0$, $G_\infty^* < 0$. A monotonic policy is optimal. Furthermore, since $F_{0k}^\infty \rightarrow F^*_{\infty} < 0$ as $k \rightarrow \infty$, and $G_{0k}^\infty \rightarrow G_{\infty}^* < 0$, we can pick $k$ big enough such that $F_{0k}^\infty < 0$, $G_{0k}^\infty < 0$. Thus we see that $k^*(0,n) \leq k < \infty \Rightarrow k^*(i,n) < \infty, \forall i$. We could define $k^*(i,n) = \min\{k : F_{ik}^\infty < 0, G_{ik}^\infty < 0\}$ in this case and the proof of Theorem 5.3.1 would work with $<$ required for repair to be optimal.

**Case 2.** $F_\infty^* \leq 0$, $G_\infty^* \leq 0$. In this case a monotonic policy is optimal, but the critical numbers may not all be finite.

**Case 3.** Not Case 1 or 2, but not both $F_{NO}^\infty < 0$, $G_{NO}^\infty < 0$. Repair is not optimal in any state, and in the infinite-horizon case, no action is optimal in all states.

**Case 4.** Not 1 or 2, and $F_{NO}^\infty < 0$, $G_{NO}^\infty < 0$. This is the only case where a monotonic policy is not optimal. As $k$ increases in any
observed state \((N,k)\), regions where repair is optimal give way to regions where repair is not optimal.

Under the original Assumption 7, we have that \(F^*_n = F^n_{NO}\), \(F^* = F^0_{NO}\), \(G^*_n = G^n_{NO}\), and \(G^* = G^0_{NO}\), and thus Cases 1 and 3 are collectively exhaustive. For the infinite-horizon problem, these two alternatives are the two possibilities of Corollary 3.2.1.

The remainder of the section is devoted to finding sufficient conditions for Cases 1 and 2. For \(n = 1\), we see that necessary and sufficient conditions for \(F^*_n \leq 0\) and \(G^*_n \leq 0\) is

\[
\sum_{j=0}^{N} \pi_j(C_j - L_j) \leq 0.
\]

For \(n > 1\), we have the following result:

**Theorem 5.3.2.** For the \(n\)-period problem with \(n > 1\), a sufficient condition for Case 1 or Case 2 is

\[
\sum_{j=0}^{N} \pi_j((1-\alpha)C_j - L_j) + \alpha C_0 \leq 0.
\]

**Proof:** First note that

\[
c^n(j,0) - c^n(0,0) = \min(C_j + \alpha c^{n-1}(0,0) - c^n(0,0), L_j
\]

\[
+ \alpha c^{n-1}(1,j) - c^n(0,0), L_j + M
\]

\[
+ \alpha \sum_{\ell=0}^{N} p_{j\ell} c^{n-1}(\ell,0) - c^n(0,0)\]

\[
\geq \min(C_j - C_0, L_j - L_0 + \alpha c^{n-1}(1,j) - \alpha c^{n-1}(0,1),
\]

\[
L_j - L_0 + \alpha \sum_{\ell=0}^{N} (p_{j\ell} - p_0\ell) c^{n-1}(\ell,0))
\]

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since $c^n(0,0) \leq \text{costs using any one of the three options}$

$\geq \min\{c_j-c_0, l_j-l_0\} \geq c_j - c_0$ \quad \text{by Assumption 3.}$

So

$c^n(j,0) - c^n(0,0) \geq c_j - c_0$; \quad (1)

$F^n_x = \sum_{j=0}^{N} \pi_j(c_j_l_j) + \alpha(c^{n-1}(0,0) - c^n_x)$

$\leq \sum_{j=0}^{N} \pi_j(c_j_l_j) + \alpha \sum_{j=0}^{N} \pi_j(c^{n-1}(0,0) - c^{n-1}(j,0))$

by letting $k \to \infty$ in Lemma 2.1.4

$\leq \sum_{j=0}^{N} \pi_j((1-\alpha) c_j - l_j + \alpha c_0)$ \quad \text{by (1)}

$\leq 0$;

$G^n_x \leq \sum_{j=0}^{N} \pi_j(c_j_l_j) - M + \alpha(c^{n-1}(0,0) - \sum_{j=0}^{N} \pi_j c^{n-1}(j,0))$

$\leq 0$ \quad \text{as above.}$

It should be noted that the condition in Theorem 5.3.2 does not have any special meaning and is not related to any conditions of Derman or others. The bound in the theorem is somewhat crude and thus does not provide us conditions that are too easily satisfied. For the infinite-horizon case, we derive conditions that are more
easily satisfied and that provide some intuitive meaning. We precede the result by two lemmas.

**Lemma 5.3.4.** Necessary and sufficient conditions for Case 1 or Case 2 in the infinite-horizon cases are:

\[ \sum_{j=0}^{N} \pi_j((1-\alpha)C_j - L_j) + \alpha(1-\alpha)(C_{0,0}) \leq 0 \]  

(1)

and

\[ -M \sum_{j=0}^{N} \pi_j(C_j - L_j) + \alpha \sum_{j=0}^{N} \pi_j(C_{0,0} - C(j,0)) \leq 0 \]  

(2)

**Proof:** Necessity: Since \( F_{\ast}^{\infty} \leq 0 \),

\[ \sum_{j=0}^{N} \pi_j(C_j - L_j) + \alpha(C_{0,0} - C_{\ast}) \leq 0 \]  

(3)

From the recursive equation for \( C_{\ast} \), it follows that

\[ C_{\ast} = \sum_{j=0}^{N} \pi_j C_j + \alpha C(0,0) \]

Substituting this into (3) gives (1), and (2) is a restatement of

\( C_{\ast}^{\infty} \leq 0 \).

**Sufficiency:** By (2), \( C_{\ast}^{\infty} \leq 0 \), and we eliminate one of the terms in the minimization of \( C_{\ast} \), so

\[ C_{\ast} = \min \{ \sum_{j=0}^{N} \pi_j C_j + \alpha C(0,0), \sum_{j=0}^{N} \pi_j L_j + \alpha C_{\ast} \} \]  

(4)
Assume it is not true that $F_*^\infty \leq 0$. Then

$$
\sum_{j=0}^{N} \pi_j (C_j - L_j) + \alpha(C(0,0) - C_*) > 0, \quad (5)
$$

and thus by (4),

$$
C_* = \sum_{j=0}^{N} \pi_j L_j + \alpha C_* \Rightarrow (1-\alpha)C_* = \sum_{j=0}^{N} \pi_j L_j. \quad (6)
$$

Combining (5) and (6),

$$
\sum_{j=0}^{N} \pi_j C_j + \alpha C(0,0) - C_* > 0. \quad (7)
$$

However,

$$
(1-\alpha) \left[ \sum_{j=0}^{N} \pi_j C_j + \alpha C(0,0) - C_* \right]
$$

$$
= \sum_{j=0}^{N} \pi_j ((1-\alpha) C_j - L_j) + \alpha(1-\alpha) C(0,0) \quad \text{by (6)}
$$

$$
\leq 0 \quad \text{by (1)}
$$

which contradicts (7). So it must be that $F_*^\infty < 0$ also. \hfill \square

We also use the following lemma:

**Lemma 5.3.2.** Assume that $(A_1, A_2, \ldots)$ and $(B_1, B_2, \ldots)$ have minima. Then

$$
\min_{i} (A_i) - \min_{i} (B_i) \geq \min_{i} (A_i - B_i)
$$
\[ |\min_i (A_i) - \min_i (B_i)| \leq \sup_i \{|A_i - B_i|\} \, . \]

**Proof:**

\[ \min_i (A_i) - \min_i (B_i) = \min_i (A_i - B_i), \quad \text{some } i^* \]

\[ \geq \min_i (A_i - B_i) \quad \text{since } B_i^* \leq B_i. \]

Also

\[ \min_i (A_i) - B_i^* \leq A_i - B_i^* \leq \sup_i \{|A_i - B_i|\}. \]

Similarly

\[ \min_i (B_i) - \min_i (A_i) \leq \sup_i \{|A_i - B_i|\}. \]

We now use these lemmas to prove the theorem.

**Theorem 5.3.3.** Let

\[ F_{i^*j} = \begin{cases} 
\min(0, (1-\alpha) \sum_{i=0}^\infty \sum_{\ell=0}^N p_{i\ell}^i L_\ell - C_j) & \text{if } j \geq i^* \\
L_0 - L_j + \alpha \sum_{\ell=0}^N (p_{0\ell} - p_{j\ell}) C_\ell & \text{if } j < i^* 
\end{cases} \]

Then sufficient conditions for Case 1 or Case 2 in the infinite-horizon case (and thus for the optimality of a monotonic policy) are
\[ \sum_{j=0}^{N} \pi_j (c_j - L_j) + \sum_{i=1}^{\infty} \alpha^i \left( \sum_{j=0}^{N} p_{0j}^{i-1} L_j - \sum_{j=0}^{N} \pi_j L_j \right) \leq 0, \quad (1) \]

and

\[ -M + \sum_{j=0}^{N} \pi_j (c_j - L_j) + \alpha \max_{i* \leq N} \sum_{j=0}^{N} \pi_j F_{i*} \leq 0. \quad (2) \]

Before the theorem is proven, one should note several important observations relating to these conditions. First, for the original case, that is under Assumption 7, these conditions reduce to

\[ C_N - L_N + \sum_{i=1}^{\infty} \alpha^i \left( \sum_{j=0}^{N} p_{0j}^{i-1} L_j - L_N \right) \leq 0. \]

Comparing this with Theorem 3.1.2 and Corollary 3.2.1, we see that this condition is also a necessary condition for \( F_{*}^{\infty} \leq 0 \) and \( C_{*}^{\infty} \leq 0. \) For this case, we see that our conditions are as tight as possible. Second, all the terms in (1) and (2), except possibly \( \sum_{j=0}^{N} \pi_j (c_j - L_j) \) contribute non-positive amounts to the left sides of the inequalities. For example,

\[ \left[ \sum_{j=0}^{N} p_{0j}^{i-1} L_j - \sum_{j=0}^{N} \pi_j L_j \right] \leq 0, \quad L_0 - L_j \leq 0, \]

and

\[ \sum_{\ell=0}^{N} \left( p_{0\ell} - p_{j\ell} \right) c_{\ell} \leq 0. \]
Thus if \[ \sum_{j=0}^{N} \pi_j(C_j-L_j) \leq 0, \] the conditions are immediately satisfied.

If not, the theorem gives other quantities to add to \[ \sum_{j=0}^{N} \pi_j(C_j-L_j) \]
to test for Cases 1 or 2. The conditions are thus readily satisfied.

In addition, by successively adding negative terms until the sum
becomes (hopefully) non-positive, one does not usually have to calculate
infinite summations to check the conditions. Intuitively, we see

that \[ \sum_{j=0}^{N} \pi_j(C_j-L_j) \]
represents one-period cost differences inherent
in \( F_\infty \) and \( G_\infty \), and the other terms represent bounds on future cost
differences. Further intuitive ideas are in the proof. Finally,
except for the case where Assumption 7 holds, these conditions are
not related to any other conditions on costs.

We now proceed with the proof:

**Proof of Theorem 5.3.3:** Condition (1) implies condition (1) of
Lemma 5.3.4 as follows: By dividing (1) of Lemma 5.3.4 by \((1-\alpha)\),
we obtain

\[
\sum_{j=0}^{N} \pi_j(C_j-L_j) + \alpha \left( C(0,0) - \sum_{j=0}^{\infty} \frac{\pi_j L_j}{1-\alpha} \right) \leq 0. \tag{3}
\]

A possible policy in state \((0,0)\) is continual no action. Thus
Another bound exists on \( C(0,0) \) that is tighter than this one here. That is,

\[
C(0,0) \leq \min_{k^* \leq \infty} \frac{1}{1-\alpha^{k^*+1}} \left\{ \sum_{i=0}^{k^*-1} \alpha^i \sum_{j=0}^{N} p_{0j} \pi_j L_j + \alpha^{k^*} \sum_{j=0}^{N} p_{0j} \pi_j \right\} = H,
\]

where the right side of (5) is the cost obtained by repair every \( k^*+1 \) periods. However, it is easy to see that equation (3) substituting (4) is always true when equation (3) substituting (5) is true. In other words, for the purposes of this theorem, bound (4) is just as effective as bound (5), or

\[
\sum_{j=0}^{N} \pi_j (C_j - L_j) + \alpha (H - \sum_{j=0}^{N} \pi_j L_j (1-\alpha)) \leq 0 \Rightarrow (1).
\]

To show this, let \( M = \infty \) and assume the opposite of (1). Thus,

\[
\sum_{j=0}^{N} p_{0j}^k (C_j - L_j) + \alpha \sum_{i=0}^{N} \pi_j (C_j - L_j) + \alpha \sum_{i=0}^{\infty} \sum_{j=0}^{N} p_{0j}^i L_j - \alpha \sum_{j=0}^{N} \sum_{i=0}^{\infty} \pi_j L_j \\
\geq \sum_{j=0}^{N} \pi_j (C_j - L_j) + \alpha \sum_{i=0}^{\infty} \sum_{j=0}^{N} p_{0j}^i L_j - \alpha \sum_{j=0}^{N} \sum_{i=0}^{\infty} \pi_j L_j
\]

by Lemma 3.1.6

\[
\geq 0,
\]

\( \forall n \) since the opposite of (1) holds. Thus by induction, as \( M = \infty \), this implies that

\[
C^n(0,k) = \sum_{i=0}^{n-1} \alpha^i \sum_{j=0}^{N} p_{0j}^{i+k} L_j \text{ and thus } C(0,0) = \sum_{i=0}^{\infty} \alpha^i \sum_{j=0}^{N} p_{0j}^i L_j.
\]

Thus

\[
\sum_{j=0}^{N} \pi_j (C_j - L_j) + \sum_{i=1}^{\infty} \alpha^i \sum_{j=0}^{N} p_{0j}^{i-1} L_j - \sum_{j=0}^{N} \pi_j L_j > 0
\]

\[
\Rightarrow \sum_{j=0}^{N} \pi_j (C_j - L_j) + \alpha (C(0,0) - \sum_{j=0}^{N} \pi_j L_j (1-\alpha)) > 0
\]

\[
\Rightarrow \sum_{j=0}^{N} \pi_j (C_j - L_j) + \alpha (H - \sum_{j=0}^{N} \pi_j L_j (1-\alpha)) > 0 \quad \text{by (5)}.
\]

Since the last implication does not depend on \( M \), it must be true for any \( M \). The desired result is simply the contrapositive of this.
and combining (4) and (1), we obtain (3) and thus (1) of Lemma 5.3.4. To show that (2) of Lemma 5.3.4 holds and consequently complete the proof of the theorem, we need a lower bound on $C(j,0) - C(0,0)$. Now

$$
C(j,0) - C(0,0) = \min(C_j - (1-\alpha) C(0,0), L_j - C(0,0)
+ \alpha C(j,1), L_j - C(0,0) + M
+ \sum_{\ell=0}^{N} p_j \ell \ C(\ell,0))
$$

where the first term corresponds to repair, etc. Now from the monotonicity of $F_{ik}^\alpha$ and $G_{ik}^\alpha$, there exists an $i$ such that if in state $(i,0)$ and $i \geq i$, repair is optimal. Furthermore, condition (1), Lemma 3.1.6 and the discussion following Theorem 3.1.2 imply that $i \leq N$. Thus,

$$
C(j,0) - C(0,0) = \min_{i* \leq N} H_{i* j}
$$

where

$$
H_{i* j} = \begin{cases}
L_j - C(0,0) + \alpha \min(C(j,1), M + \sum_{\ell=0}^{N} p_j \ell \ C(\ell,0)) & \text{for } j < i* \\
C_j - (1-\alpha) C(0,0) & \text{for } j \geq i*
\end{cases}
$$

Thus if we can show that $-F_{i* j} \leq H_{i* j}$, then we have a lower bound on $C(j,0) - C(0,0)$ which when combined with (2) yields (2) of Lemma 5.3.4. For $j \geq i*$,
\[-F_{i^*j} \leq \max\{0, C_j - (1-\alpha) C(0,0)\} \quad \text{by (4)}\]

\[\leq C_j - (1-\alpha) C(0,0) \quad \text{since } C_j + \alpha C(0,0) \geq C(j,0) \geq C(0,0)\]

\[= H_{i^*j}.\]

For $j < i^*$,

\[-F_{i^*j} = L_j - L_0 + \alpha \sum_{\ell=0}^N (p_{j\ell} - p_{O\ell}) C_\ell\]

\[= L_j - L_0 + \alpha \min\left( \sum_{\ell=0}^N (p_{j\ell} - p_{O\ell}) C_\ell, \sum_{\ell=0}^N (p_{j\ell} - p_{O\ell}) L_\ell \right)\]

since $\sum_{\ell=0}^N p_{j\ell}(C_\ell - L_\ell)$ is non-increasing in $j$

\[\leq L_j - L_0 + \alpha \min\left( \sum_{\ell=0}^N p_{j\ell} C_\ell - C(0,1) + \alpha C(0,0), \sum_{\ell=0}^N p_{j\ell} L_\ell - C(0,1) + \alpha C(0,2), \sum_{\ell=0}^N p_{j\ell} L_\ell \right)\]

- $C(0,1) + \alpha C(0,2), \sum_{\ell=0}^N p_{j\ell} L_\ell$

- $C(0,1) + M + \alpha \sum_{\ell=0}^N p_{O\ell} C(\ell,0))$

since $C(0,1) \leq \text{cost using any single option}$

\[\leq L_j - L_0 + \alpha C(j,1) - \alpha C(0,1) \quad \text{by definition, Lemma 3.1.3, and Theorem 3.1.1}\]

\[\leq L_j + \alpha C(j,1) - C(0,0) \quad \text{since } C(0,0) \leq L_0 + \alpha C(0,1).\]

(7)
In addition, for \( j < i^* \), defining \( C(-1,0) = L_{-1} = C_{-1} = 0 \),

\[-F_{i^*j} = L_j - L_0 + \alpha \sum_{\ell=0}^{N} (P_{j\ell} - P_{0\ell}) C_{\ell},\]

\[= L_j - L_0 + \alpha \sum_{m=0}^{N} (C_m - C_{m-1}) \sum_{\ell=m}^{N} (P_{j\ell} - P_{0\ell})\]

\[= L_j - L_0 + \alpha \sum_{m=0}^{N} \min(C_m - C_{m-1}, L_m - L_{m-1}) \sum_{\ell=m}^{N} (P_{j\ell} - P_{0\ell})\]

by Assumption 3

\[\leq L_j - L_0 + \alpha \sum_{m=0}^{N} (C(m,0) - C(m-1,0)) \sum_{\ell=m}^{N} (P_{j\ell} - P_{0\ell})\]

by the IFR property and by Lemmas 5.3.5 and 3.1.5 and Theorem 3.1.1

\[= L_j - L_0 + \alpha \sum_{\ell=0}^{N} (P_{j\ell} - P_{0\ell}) C(\ell,0)\]

\[\leq L_j - C(0,0) + \alpha \sum_{\ell=0}^{N} P_{j\ell} C(\ell,0) + M. \quad (8)\]

Combining (7) and (8) gives

\[-F_{i^*j} \leq H_{i^*j} \quad \text{for } j < i^*\]

which completes the proof. \( \square \)

Since it also follows that \( C(0,0) \geq L_0 / (1-\alpha) \), since the minimum cost per period is \( \min(C_0, L_0) = L_0 \), it follows from (1) of Lemma 5.3.4 that a necessary condition for Case 1 or Case 2 is
\[ \sum_{j=0}^{N} \pi_j ((1-\alpha) C_j - L_j) + \alpha L_0 \leq 0. \]

5.4. The Average-Cost Problem

We can extend the results of the infinite-horizon case to the average-cost case by an analysis similar to that of Chapter 3. The major result, as in the infinite-horizon case, is the existence of an optimal monotonic policy under certain cost conditions. The conditions are similar to those of the last section.

Theorem 5.4.1. Assume that

\[ \lim_{\alpha_0 \to 1} \sup_{\alpha \geq \alpha_0} (1-\alpha) C(0,0) - \sum_{j=0}^{N} \pi_j L_j < 0, \quad (1) \]

\[ -M + \lim_{\alpha_0 \to 1} \sup_{\alpha \geq \alpha_0} \left[ \sum_{j=0}^{N} (C_j - L_j) + C(0,0) - \sum_{j=0}^{N} \pi_j C(j,0) \right] < 0. \quad (2) \]

Then a monotonic policy is optimal for the average-cost problem. Furthermore, there are non-increasing critical numbers \( k^*(i) \).

Proof: Note that (1) and (2) imply that for some \( \tilde{\alpha} \), for all

\[ 1 > \alpha \geq \tilde{\alpha}, \]

\[ \sum_{j=0}^{N} \pi_j ((1-\alpha) C_j - L_j) + \alpha (1-\alpha) C(0,0) < 0, \quad (3) \]

\[ -M + \sum_{j=0}^{N} \pi_j ((C_j - L_j) + \alpha C(0,0) - \alpha \sum_{j=0}^{N} \pi_j C(j,0) < 0 \right) . \quad (4) \]

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Thus by Lemma 5.3.4, we have that monotonic policies are optimal for the infinite-horizon case for all $1 > \alpha \geq \bar{\alpha}$. If we can show that $k^*(i,\infty) \leq k_0$, some $k_0$ for all $1 > \alpha \geq \underline{\alpha}$, some $\underline{\alpha}$, then by the same method used in the proof of Theorem 3.2.4, we have the result. The rest of the proof shows this. Note that

$$G_{0k^*}^\infty = \sum_{j=0}^{N} p_{0j}(c_j - L_j) - M + \alpha(c(0,0) - \sum_{j=0}^{N} p_{0j}^k c(j,0)).$$

By (2), we can find an $\alpha_0 > \bar{\alpha}$ such that

$$G_{\infty} = -M + \sup_{\alpha \geq \alpha_0} \left[ \sum_{j=0}^{N} p_{0j}(c_j - L_j) + \alpha(c(0,0) - \sum_{j=0}^{N} \pi_j c(j,0)) \right]$$

$$= -\varepsilon < 0.$$  

The claim is that if

$$G_k = -M + \sup_{\alpha \geq \alpha_0} \left[ \sum_{j=0}^{N} p_{0j}(c_j - L_j) + \alpha(c(0,0) - \sum_{j=0}^{N} p_{0j}^k c(j,0)) \right],$$

Then $G_k \to G_{\infty}$. This follows directly if

$$\sum_{j=0}^{N} |p_{0j}^k - \pi_j| (c(j,0) - c(0,0)) \to 0 \quad \text{as} \quad k \to \infty \quad (5)$$

uniformly in $\alpha_0 \leq \alpha < 1$. Now

$$0 \leq c(j,0) - c(0,0) \leq c_j - (1-\alpha) c(0,0)$$

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with the last inequality following since \(C(j,0) \leq \text{cost using repair.}\)

Since \(C(j,0) - C(0,0)\) is bounded, we have the uniform convergence desired in (5). So \(G_K \to G_\infty\) and for \(\tilde{k}\) big enough, for any \(k \geq \tilde{k}\),

\[
G_k \leq G_\infty + \frac{\varepsilon}{2} = -\frac{\varepsilon}{2} .
\]

So for \(k \geq \tilde{k}\) and \(1 > \alpha \geq \alpha_0\),

\[
G^\infty_{ik} \leq C^\infty_{0k} \leq G_k \leq -\frac{\varepsilon}{2} ,
\]

so repair is better than inspection.

Now by (1),

\[
\lim_{\alpha \to 1} \sup_{1 > \alpha \geq \alpha_0} \left[ \sum_{j=0}^{N} \pi_j ((1-\alpha) C_j - L_j) + \alpha (1-\alpha) C(0,0) \right] < 0 .
\]

Thus for \(\alpha \geq \alpha_0 \geq \tilde{\alpha}\),

\[
F^\infty = \sup_{1 > \alpha \geq \alpha_0} \left[ \sum_{j=0}^{N} \pi_j ((1-\alpha) C_j - L_j) + \alpha (1-\alpha) C(0,0) \right] = -\varepsilon < 0 .
\]

Recall that for \(\alpha \geq \tilde{\alpha}\), from the proof of Lemma 5.3.4,

\[
C_* = \sum_{j=0}^{N} \pi_j C_j + \alpha C(0,0) ,
\]

so

\[
F^\infty = \sup_{1 > \alpha \geq \alpha_0} \left[ \sum_{j=0}^{N} \pi_j (C_j - L_j) + \alpha (C(0,0) - C_*) \right] .
\]
If

$$F_k = \sup_{1 > \alpha > \alpha'} \left[ \sum_{j=0}^{N} p_{o,j}^k (C_j - L_j) + \alpha (C(0,0) - C(0,k+1)) \right],$$

then the claim is that $F_k \to F_\infty$ as $k \to \infty$, which follows if

$$C(0,k) \to C_* \quad \text{as} \quad k \to \infty \quad (7)$$

uniformly in $1 > \alpha > \alpha'$. We show this as follows:

$$C_* = \sum_{j=0}^{N} p_{o,j} C_j + \alpha C(0,0)$$

$$= \min \left\{ \sum_{j=0}^{N} p_{o,j} C_j + \alpha C(0,0), \sum_{j=0}^{N} p_{o,j} L_j + \alpha \sum_{j=0}^{N} p_{o,j} C_j + \alpha^2 C(0,0), \ldots, \frac{1-\alpha^{n+1}}{1-\alpha} \sum_{j=0}^{N} p_{o,j} L_j + \alpha^{n+1} \sum_{j=0}^{N} p_{o,j} C_j + \alpha^{n+2} C(0,0), \ldots \right\}.$$ 

Now let $k \geq \tilde{k}, \alpha > \alpha'$; since inspection cannot be optimal,

$$C(0,k) = \min \left\{ \sum_{j=0}^{N} p_{o,j}^k C_j + \alpha C(0,0), \sum_{j=0}^{N} p_{o,j}^k L_j + \alpha \sum_{j=0}^{N} p_{o,j}^k C_j \right\}$$

$$= \min \left\{ \sum_{j=0}^{N} p_{o,j}^k C_j + \alpha C(0,0), \sum_{j=0}^{N} p_{o,j}^k L_j + \alpha \sum_{j=0}^{N} p_{o,j}^k C_j \right\}$$

$$= \min \left\{ \sum_{j=0}^{N} p_{o,j}^k C_j + \alpha C(0,0), \sum_{j=0}^{N} p_{o,j}^k L_j + \alpha \sum_{j=0}^{N} p_{o,j}^k C_j \right\}$$

by recursion.
In the above minimization, \( n \) denotes how many periods to wait before repair. By (3) and (4) we have Case 1 of the infinite-horizon case and the minimization will occur for some \( n \) for each \( \alpha \). Thus by Lemma 5.3.5,

\[
|C(0,k) - C_\star| \leq \sup \{ |\sum_{j=0}^{N} (p_{0j}^k - \pi_j)C_j|, |\sum_{j=0}^{N} (p_{0j}^k - \pi_j)L_j + \alpha \sum_{j=0}^{N} (p_{0j}^{k+1} - \pi_j)C_j|, \ldots, |\sum_{i=0}^{m} \alpha^i \sum_{j=0}^{N} (p_{0j}^{k+i} - \pi_j)L_j + \alpha^{m+1} \sum_{j=0}^{N} (p_{0j}^{k+m+1} - \pi_j)C_j| \}
\]

\[
\leq \sum_{i=0}^{\infty} \sum_{j=0}^{N} |p_{0j}^{k+i} - \pi_j|L_j + \sup_{m \geq 0} \sum_{j=0}^{N} |p_{0j}^{k+m} - \pi_j|C_j.
\]  \( (8) \)

Recalling Assumption 5 and the ergodic theorem (see Fisz [8], for example),

\[
|p_{0j}^{k+i} - \pi_j| \leq (1-\delta)^{(k+i)/r} - 1
\]

for \( j \) recurrent

for some \( r > 0, \delta < 1 \). For \( j \) not recurrent,

\[
|p_{0j}^{k+i} - \pi_j| = p_{0j}^{k+i} \leq 1 - \sum_{\text{recurrent class}} p_{0j}^{k+i}
\]

\[
\leq \sum_{\text{recurrent class}} |\pi_j - p_{0j}^{k+i}| \leq (N+1) (1-\delta)^{(k+i)/r} - 1.
\]

So in any case,
\[ |p_{ij}^{k+1} - \pi_j| \leq \left( \frac{N+1}{1-\delta} \right)^{k+1}/r. \]

Substituting this into (8),

\[ |C(0,k) - c| \leq \sum_{i=0}^{\infty} \sum_{j=0}^{N} \left( \frac{N+1}{1-\delta} \right)^{k+m}/r \]

\[ + \sup_{m \geq 0, j=0} \sum_{m} \left( \frac{N+1}{1-\delta} \right)^{(k+m)/r} \]

\[ \leq \text{constant} \cdot (1-\delta)^k \rightarrow 0. \]

Thus we have the uniform convergence in (7), so \( F_k \rightarrow F_\infty \). Since \( F_\infty < 0 \), we can let \( k_0 \geq \hat{k} \) be big enough so that \( F_k < 0 \) for \( k \geq k_0 \).

Thus for \( \alpha > \alpha_0 \), \( k \geq k_0 \),

\[ F_{ik}^\infty \leq F_{0k}^\infty \leq F_k < 0 \quad (9) \]

So repair is better than no action. By combining (9) and (6), we see that for \( 1 > \alpha > \alpha_0 \),

\[ k^*(i, \infty) \leq k_0, \]

which completes the proof. \( \square \)

We finish this section by finding sufficient conditions for the hypothesis of Theorem 5.4.1. These conditions are analogous to those of Theorem 5.3.3. The comments that apply to Theorem 5.3.3
apply to this theorem in an analogous manner. It is felt, that is, that these conditions are tight, easily satisfied and not difficult to check.

**Theorem 5.4.2.** Let

\[
G_{i^*j} = \begin{cases} 
\min(O, \sum_{\ell=0}^{N} \pi_{i\ell} L_{i\ell} - C_j) & j \geq i^* \\
L_0 - L_j + \sum_{\ell=0}^{N} (p_{i\ell} - p_{j\ell})C_{i\ell} & j < i^*
\end{cases}
\]

Then sufficient conditions for the hypothesis of Theorem 5.4.1 to hold (and thus for the optimality of a monotonic policy in the average-cost case) are

\[
\sum_{j=0}^{N} \pi_j (C_j - L_j) + \sum_{k=0}^{\infty} \left( \sum_{j=0}^{N} p_{0j} L_j - \sum_{j=0}^{N} \pi_j L_j \right) < 0 , \quad (1)
\]

and

\[
-M + \sum_{j=0}^{N} \pi_j (C_j - L_j) + \max_{i^* \leq N} \sum_{j=0}^{N} \pi_j G_{i^*j} < 0 . \quad (2)
\]

**Proof:** By (1); since \(\sum_{j=0}^{N} p_{0j} L_j \leq \sum_{j=0}^{N} \pi_j L_j\) by the first part of Lemma 3.1.6; and by the geometric convergence in the proof of Theorem 5.4.1, we can find a \(k^*\) such that

\[
\sum_{j=0}^{N} p_{0j} C_j - \sum_{j=0}^{N} \pi_j L_j + \sum_{k=0}^{k^*-1} \left( \sum_{j=0}^{N} p_{0j} L_j - \sum_{j=0}^{N} \pi_j L_j \right) < 0 .
\]
In other words,

\[ H = (k^* + 1) \left( \text{average-cost of repair every } k^* + 1 \text{ periods} - \sum_{j=0}^{N} \pi_j L_j \right) < 0. \]

By (2) of Theorem 3.2.4,

\[ \lim_{\alpha_0 \to 1} \sup_{1 > \alpha \geq \alpha_0} (1-\alpha) C(0,0) - \sum_{j=0}^{N} L_j \leq \frac{H}{k^* + 1} < 0. \]

Thus we have (1) of Theorem 5.4.1. Now by equation (2.2.1),

\[ \lim_{\alpha \to 1} (1-\alpha) \sum_{i=0}^{\infty} \alpha^i \sum_{p_0 \leq L_0} \frac{L}{l^*} = \sum_{j=0}^{N} \pi_j L_j. \]

Combining this with (2), we see that for some \( \alpha \) and \( \varepsilon \) and

\[ 1 > \alpha \geq \alpha, \]

\[ -M + \sum_{j=0}^{N} \pi_j (C_j - L_j) + \alpha \max_{i^* \leq N} \sum_{j=0}^{N} \pi_j F_{i^* j} \leq -\varepsilon < 0, \]

where \( F_{i^* j} \) is the same as in Theorem 5.3.3. Thus from the proof of Theorem 5.3.3, we have

\[ -M + \sum_{j=0}^{N} \pi_j (C_j - L_j) + \alpha \sum_{j=0}^{N} \pi_j (C(0,0) - C(j,0)) \leq -\varepsilon \quad (3) \]

for \( 1 > \alpha \geq \alpha \). Equation (2) of Theorem 5.4.1 follows from (3) above.
5.5. **Cases Where Monotonic, Four-Region Policies are Optimal**

In addition to deriving the optimality of monotonic policies, we would also like to determine, under Assumption 7' and the TP₂ assumption, when four-region policies are optimal. As Lemma 4.2.4 does not generally hold unless Assumption 7 holds, the techniques of Chapter 4 do not generally insure the optimality of four-region policies. However, as seen in Chapter 4, one needs to show that $J_{ik}^\infty$ satisfies the crossing property in $k$ only for those real states $i$ such that repair is not optimal in observed state $(i,0)$. Thus, we need to prove the property in Lemma 4.2.4 only for those $i$ such that repair is not optimal in $(i,0)$. Lemma 4.2.4, however, holds for any given $i$ such that $p_{ij} = 0$ for $j$ less than that particular $i$. It follows that we need $p_{ij} = 0$ for $j < i$ only for $(i,0)$ not a repair-optimal state. So if Assumption 7' holds, if we have the hypotheses of Theorem 5.3.3, (5.4.2), if $p_{ij} = 0$ for $j < i$ for $(i,0)$ not repair-optimal, and if $P$ is TP₂, then a monotonic, four-region policy is optimal for the infinite-horizon (average-cost) problem. Checking whether $p_{ij} = 0$ for $j < i$ and $(i,0)$ not repair-optimal might be difficult. It therefore might be worthwhile to further explore this condition in future work. The major point is that we can characterize other cases where we know monotonic, four-region policies are optimal. One such case is all $P$ such that $P$ is TP₂ and $p_{ij} = 0$ for $j < i$ and $j < N-1$. This case, of course, includes all $2 \times 2$ IFR matrices, and is a subclass of one of our examples of matrix classes satisfying Assumption 7'.

1With strict inequality so that critical numbers are finite.
CHAPTER 6

SPECIAL CASES AND OTHER TOPICS

In this chapter, we briefly examine one very important special case, cases that involve variation of the cost structure, and ramifications of the theory on computational procedures. Relevant references are cited in each section.

6.1. The $2 \times 2$ Case

The $2 \times 2$ case, in which the process is in a good or a bad state, is important for several reasons. First, it represents our most simple model; second, it has been looked at the most by others; third, it represents a case where we can develop additional results; and finally, it represents a case where all the matrix assumptions reduce to one simple inequality. For this case, state zero is the good state and state one is the bad state. The transition matrix is

$$
P = \begin{bmatrix}
    a & 1-a \\
    1-b & b
\end{bmatrix}.
$$

From past results, we know that for two states $TP_2 \Leftrightarrow IFR \Rightarrow Assumption 7'$ satisfied. But $P$ is $TP_2 \Leftrightarrow a+b \geq 1$. Hence all matrix assumptions reduce to this single inequality. This paper was largely motivated by the two-state case where inspection information did not give the precise state, as studied by Girshick and Rubin. Using a more general state space, Ross [23] examined the case where $b = 1$ and where

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inspection information does give the precise state. Ross found a four-region structure of the optimal policy and found cases for the n-period problem where four non-vacuous regions existed. Our results for the $2 \times 2$ case are consequently new only for the case $b \neq 1$.

The existence of an optimal monotonic four-region policy for the infinite-horizon and average-cost cases follows from Section 5.5. Under the proper assumptions of cost functions, such as Theorem 5.4.2 for the average-cost case, we see that the optimal policy is repair in state $(1, k)$, all $k$, and that the set $\{(0, k)\}$ can be broken up into at most four regions. An analogous result holds for the n-period problem also, and we can combine these results with a relatively simple proof.

**Theorem 6.1.1.** Assume that the proper cost function assumptions hold for the appropriate optimization criterion (Theorem 5.3.2, 5.3.3 or 5.4.2). Then under any of the three criteria, if $a+b \geq 1$, a monotonic four-region policy is optimal for the 2-state case. Specifically, repair is optimal in state $(1, k)$, all $k$, and the set $\{(0, k)\}$ can be broken up into at most four regions: a no-action region, an inspection region, another no-action region, and a repair region.

**Proof:** The optimality of a monotonic policy with repair optimal in state $(1, k)$, all $k$, follows from Theorem 5.3.2, 5.3.3, or 5.4.2. For the n-period problem, let $J^n_{ik} = (\text{cost of inspection} - \text{cost of no action})$. So
\[ J_{ik}^n = \alpha \left( \sum_{j=0}^{k+1} p_{ij} c^{n-1}(j,0) - c^{n-1}(i, k+1) \right) + M \]

\[ = M + \alpha \max_{m \in [0, m(n-1)]} \left\{ \sum_{j=0}^{k+1} p_{ij} [c^{n-1}(j,0) - A_{mj}^{n-1}] \right\} \]

by Lemma 5.3.3.

But the function in brackets is either non-increasing or non-decreasing since \( j = 0 \) or 1. Thus the function in braces is either non-increasing or non-decreasing. The maximum is consequently unimodal with one minimum. Since \( J_{ik}^n \) is unimodal with one minimum, the region \([0, k^*(0,n) - 1]\) can be broken up into at most three regions. The assertion follows for the infinite-horizon case since

\[ \lim_{n \to \infty} J_{ik}^n = J_{ik}^{\infty} \]

is unimodal, and the assertion follows for the average-cost case as in Theorem 5.4.1.

\[ \square \]

6.2. **Special Cases Involving Variations in the Inspection Structure**

It should be noted that the model can be varied to incorporate slightly different structures for inspection. For example, inspection information could be made available at the beginning of a time period, in which case the process will be in state \((j,1)\) at the beginning of the next time period with probability \( p_{ij}^k \). Similar results can be derived in this case. It is simply a matter of convention. One could also let the inspection cost vary with the real state. Again, analogous results can be derived, although special assumptions on such variations would have to be made. It was felt that a constant inspection cost would be quite realistic.

\[ ^1 \text{By Lemma 3.1.6.} \]
There is one special case where changing the inspection cost structure might provide a more realistic model. This case was given as an example of a $TP_2$ matrix and consists of a machine with $N$ equivalent structures or components that are subject to failure. The state of the process is the number of failed components. Presumably, an inspection act tells the operator which components have failed. If a component that has failed stays failed, the operator has to inspect only those components that he knew were working at the last act of inspection or repair. For such a structure, it might be more realistic to let the inspection cost be a non-increasing cost $M_i$ of the integer $i$, last real state known, of the observed state $(i,k)$. Under this assumption, we replace $M$ by $M_i$ in the expressions for $F_{ik}^n, G_{ik}^n, F_{ik}^\infty, G_{ik}^\infty,$ and $J_{ik}^\infty$ and obtain the following result.

**Theorem 6.2.1.** Assume $C_j - L_j - \sum_{\ell=0}^{N} p_j^\ell M_\ell$ is non-increasing, all $k$.

For a machine with $N$ equivalent failing structures with an inspection cost that is a non-increasing function of the last known real state, then for any given $i$, the set $\{(i,k)\}$ can be broken up into at most four-regions that describe optimal policy for the infinite-horizon and average-cost cases. The order (for increasing $k$) of the regions is no-action, inspection, no-action, and repair.

**Proof:** We first show that, by induction,

$$C_j - L_j - \sum_{\ell=0}^{N} p_j^\ell C^{(\ell,k)}$$

is non-increasing in $j$ for (1) any $h, m, n,$ and $k$. 

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For \( n = 0 \), we have the result trivially. Now assume the assertion for case \( n-1 \), so

\[
C_j - L_j - \alpha^h \sum_{\ell=0}^N p_{j\ell}^m C^n(\ell,k)
\]

\[
= (C_j - L_j - \sum_{\ell=0}^N p_{j\ell}^m M_\ell) + (1-\alpha^h) \sum_{\ell=0}^N p_{j\ell}^m M_\ell
\]

\[
\quad - \alpha^h \sum_{\ell=0}^N p_{j\ell}^m [C^n(\ell,k) - M_\ell].
\]

By hypothesis, the first two terms are non-increasing. Furthermore,

\[
C^n(\ell,k) - M_\ell = \min\left( \sum_{m=0}^N p_{\ell m}^k C_m + \alpha C^{n-1}(0,0) - M_\ell, \sum_{m=0}^N p_{\ell m}^k L_m \right)
\]

\[
\quad + \alpha C^{n-1}(\ell, k+1) - M_\ell, \sum_{m=0}^N p_{\ell m}^k L_m
\]

\[
\quad + \alpha \sum_{m=0}^N p_{\ell m}^{k+1} C^{n-1}(m,0))
\]

is non-decreasing in \( \ell \) by induction and since \( M_\ell \) is non-increasing. So \( C_j - L_j - \alpha^h \sum_{\ell=0}^N p_{j\ell}^m C^n(\ell,k) \) is non-increasing in \( j \), and by taking limits,

\[
C_j - L_j - \alpha^h \sum_{\ell=0}^N p_{j\ell}^m C(\ell,k)
\]

is non-increasing in \( j \).

Thus

\[
G_{ik}^\infty = \sum_{j=0}^N p_{ij}^k (C_j - L_j) + \alpha C(0,0) - M_1 \alpha \sum_{j=0}^N p_{ij}^{k+1} C(j,0)
\]

is non-increasing in \( k \).
Also,

\[ h^{1}_{ik} = \sum_{j=0}^{N} p_{ij} (c_{j} - l_{j}) \]

is non-increasing in \( k \),

and

\[ h^{n}_{ik} = \sum_{j=0}^{N} p_{ij} (c_{j} - l_{j}) + \alpha c^{n-1}(0,0) \]

\[ - \alpha^{h} \min \{ \sum_{j=0}^{N} k+1 \} \]

\[ + \alpha c^{n-2}(0,0), \sum_{j=0}^{N} k+1 l_{j} + M \]

\[ + \alpha \sum_{j=0}^{N} p_{ij} c^{n-2}(j,0) \]

is non-increasing in \( k \) by induction and (1).

By letting \( n \to \infty \), we see that \( F_{ik}^{\infty} \) is non-increasing in \( k \). Thus we can define critical numbers for repair. Furthermore, by reasoning similar to that of Chapter 3, either all critical numbers are finite or a trivial optimal policy is optimal. In the former case, since

\[ c_{j} - l_{j} - \alpha^{n} \sum_{\ell=0}^{N} \sum_{j=0}^{N} p_{j\ell} c(\ell,k) \]

is non-increasing in \( j \),

the proof of Theorem 4.3.1 is still valid and we have four-regions. The average-cost case follows as in Chapter 4. \( \Box \)
Notice that this result does not say anything about monotonicity of critical numbers. These numbers, in fact, may not be monotonic for this special case.

6.3. **Computational Procedures**

A typical dynamic programming problem has several methods available for solution, including successive approximations, which iterates $C^n$ until it gets arbitrarily close to $C$, policy improvement method, which uses recursive equations, and the linear program ([3] or [19]) for the average-cost problem. All of these techniques are usable for our model, and the results obtained in this paper help in two separate ways. First, because the form of the optimal policy is known, use of the policy improvement method is simplified. Second, by use of results such as Theorem 3.2.4 and their proofs, we can find bounds on the critical numbers. This enables us to reduce the problem to a finite-state problem which allows us to use the linear program method and simplifies policy improvement procedures.

So the theoretical results help in computational procedures in addition to adding insight to the general problem.
CHAPTER 7

CONCLUSIONS

The major result of this paper is that for deteriorating processes under uncertainty that are represented by $TP_2$ matrices $P$ such that $p_{ij} = 0$ for $j < i$, an optimal infinite-horizon or average-cost policy is a monotonic, four-region policy. It is felt that such a policy is a straightforward one. It was also found that if $P$ is IFR and $p_{ij} = 0$ for $j < i$, then an optimal policy is monotonic under all three criteria; and that if $P$ is IFR and satisfies Assumption 7', then an optimal policy is monotonic under certain cost conditions and is four-region in certain cases if $P$ is also $TP_2$. It is felt that the results add to the understanding and insight of such deteriorating processes and furthermore help computational efforts in particular problems.

In future work, we shall look at several topics. It is conjectured, of course, that a four-region policy is optimal for the $n$-period problem and this possibility will be examined. Other topics include the relationship of critical numbers, total policy characterization for IFR matrices, more examples and special cases, and structures where inspection information does not give the precise state.
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Deteriorating Markov Processes Under Uncertainty

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Report Date:
May 6, 1974

Report Number:
N00014-67-A-0112-0052

Program Element, Project, Task Area & Work Unit Numbers:
(NR-042-002)

Distribution Statement (of this Report):
Distribution of this document is unlimited.

Keywords:
MARKOV CHAINS
PARTIALLY OBSERVABLE PROCESSES
REPAIR
DYNAMIC PROGRAMMING
MAINTENANCE MODELS
INSPECTION
DETERIORATING PROCESSES
COSTLY INSPECTION

Abstract:
A model is developed that represents a deteriorating Markov process with imperfect or costly information. The process might be, for example, a deteriorating machine or inventory system with several states. In the context of the machine example, at each time period, the machine operator has three possible actions to choose from. If he chooses repair, an
expected repair cost is incurred and the system reverts to the best state. If he chooses inspection, an inspection cost and an expected operating cost are incurred, and the operator determines exactly which state the system will be in at the beginning of the next time period. Finally, if he chooses no action, an expected operating cost is incurred, and he obtains no new information about the state of the process. Such processes have been studied by others, but under the imperfect information assumption, results are incomplete.

Under the structure assumed, one can characterize a state space of the process as observed by the operator. In observed state \((i, k)\), the operator knows that \(k\) time units ago, the underlying Markov process was in state \(i\), and that no new information has been gathered in \(k\) time units. Under straightforward assumptions on costs and under the assumptions that the Markov matrix \(P\) is IFR or stochastically increasing and \(p_{ij} = 0\) for \(j < i\) (upper triangular), it is shown that there are numbers \(k*\) \((i)\) non-increasing in \(i\) such that it is optimal to repair if \(k \geq k*(i)\) in state \((i, k)\) and optimal not to repair otherwise. This optimality holds for the \(n\)-period, infinite-horizon (discounted), and average-cost criteria. Under the stronger assumption that \(P\) is totally positive of order two (TP2), it is further shown that, under the latter two criteria, that the interval \(k \in [0, k*(i)-1]\) for state \((i, k)\) can be broken into at most three additional regions: a no-action-optimal region, an inspection-optimal region and a second no-action-optimal region. Thus for the TP2, upper-triangular case, we have a "four-region" policy. The same results are developed for matrices satisfying more general conditions than \(p_{ij} = 0\) for \(j < i\), and some special cases are examined.

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