SAFETY STOCK SENSITIVITY ANALYSIS IN DYNAMIC ECONOMIC LOT SIZE MODELS: A PARAMETRIC APPROACH

BY

R. SCOTT SHIPLEY

TECHNICAL REPORT NO. 173
FEBRUARY 20, 1976

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Abstract

The question of safety stock levels is considered in a dynamic economic lot size model where: (1) the cumulative total requirement over a fixed number of future time periods is known, but the individual by-period requirements can only be point-estimated; and (2) stockout costs are significant, but unknown. A dynamic programming procedure is described which details the minimum total cost of production and inventory activities (and the corresponding optimal production plan) parametrically as a function of all possible values of a uniform safety stock parameter.

1. INTRODUCTION

Dynamic economic lot size models address the questions of when and how much of a single product to provide in order to satisfy time-varying requirements over a discrete number of future time periods with minimal total cost. Existing models typically make two rather restrictive assumptions that:

(1) the by-period requirements either

(a) are known with certainty, or

(b) behave according to some specified

(usually, normal) random distributions; and

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1The author is currently a Ph.D. student in Operations Research at Stanford under the auspices of Bell Telephone Laboratories.
(2) stockout costs (i.e., costs associated with lost sales, damaged goodwill, backordering, etc.) are known.

For instance, the traditional Wagner-Whitin model [10] assumes (1a) and, with regard to (2), assumes that stockout costs are infinite. The more recent work of Kaiman [4,5] typifies models assuming (1b) and (2). Unfortunately, in some situations, these assumptions are not reasonable approximations to real-world applications.

To illustrate, consider an automobile firm which must develop a procurement plan for satisfying all future service requirements of a particular component which is no longer used in the current product line. Based on the actual total sales of all past car models which utilized the component, a prediction of the total future service requirements (to obsolescence) can often be made with sufficient accuracy to warrant considering the sum forecast as constant. However, any forecast made with regard to the way in which this total sum will be distributed over the individual future months (or years) could be subject to substantial inaccuracies, since actual requirements may be related to such things as future weather patterns and gasoline prices (which affect component deterioration rates). Often, the only information available is some sort of "best guess", in the form of a point-estimate, as to what the expected requirement in each future month might be. To complicate matters, true stockout costs are extremely difficult to ascertain. In
particular, consider the ire of a motorist with an inoperative vehicle who is told by his local dealer that the service part needed to repair his car is out-of-stock in Detroit! Such occurrences can cause irreparable goodwill damage in the form of irritated customers switching allegiances to other brands of automobiles. Just how does one quantify the loss of future revenues because of inability to provide immediate service? One can perhaps only say that actual stockout costs are substantially more than just the additional expediting expenses that might be incurred to clear backorders.

Situations similar to that described above can undoubtedly be found in the service parts organization of many manufacturing firms. In such cases, neither assumptions (1) or (2) are really appropriate. Other authors [6] have previously noted the problems associated with (2) and, considering the inherent difficulties in making accurate forecasts, it should come as no surprise that, in some cases, (1a) and (1b) are simply not applicable. Therefore, the subject addressed here is the matter of guidance, with regard to providing safety stock, in dynamic economic lot size models where:

(1) the sum of all requirements over a fixed number of future time periods is known, but only point-estimates are available for the individual by-period requirements; and
(2) stockout costs are significant, but unknown.
Under conditions (1) and (2), it would be impossible to declare any single plan as "best". However, guidance can be provided by detailing what the minimum (known) costs of production and inventory activities will be for various levels of safety stock. Specifically, we will show that by modifying the standard Wagner-Whitin model and algorithm, it is possible (with little additional work) to detail the minimum costs of production and inventory activities (and the corresponding production plans) parametrically as a function of all possible values of a uniform safety stock parameter. Using the parametric solutions, a number of significant sensitivity questions can be answered, such as:

What are the marginal costs of additional units of safety stock? (Beware, the marginal cost is not constant!)

For purposes of continuity, the first section of this paper is devoted to a review of the work of Wagner and Whitin; most of the necessary symbols and terms are also introduced there. The second section consists of a detailed description of our parametric model and the algorithm which solves it. An illustrative example problem and attempts at intuitive explanation supplement the mathematical descriptions of both sections.

2. REVIEW OF WAGNER-WHITIN [10]

The Wagner-Whitin Model

Suppose the requirements \( d_1, d_2, \ldots, d_N \) for a single product in time periods \( 1, 2, \ldots, N \) are given nonnegative numbers. Let \( x_n \) be the amount produced (or ordered) in period \( n \) and let \( y_n \) be the remaining inventory on hand at the end of period \( n, n = 1, \ldots, N. \)
Let $a_n$, $b_n$ and $h_n$ be the nonnegative setup cost, unit production cost and unit inventory carrying cost, respectively, in period $n$, $n = 1, \ldots, N$. Denote by $\delta(\cdot)$ the indicator function:

$$
\delta(x) = \begin{cases} 
0, & \text{if } x = 0 \\
1, & \text{if } x \neq 0.
\end{cases}
$$

Then the Wagner-Whitin model can be stated as

minimize

$$
\sum_{n=1}^{N} a_n \cdot \delta(x_n) + b_n x_n + h_n y_n
$$

subject to ($n = 1, \ldots, N$):

$$
x_n + y_{n-1} - y_n = d_n
$$

(2)

$$
x_n \geq 0
$$

(3)

$$
y_n \geq 0
$$

(4)

$$
y_0 = y_N = 0.
$$

(5)

The objective (1) represents the total production and inventory costs over the planning interval. Constraints (2) are simple inventory balance equations. Constraints (3) restrict the model to purely production (or ordering) activities (i.e., disposal activities, which would correspond to negative values of $x_n$, are not allowed). Constraints (4) dictate that back-ordering is not allowed. With regard to (5), Wagner and Whitin demonstrate that there is no loss of generality in assuming that the initial inventory ($y_0$) is zero. Similarly, the absence of a product disposal activity and the assumption of known requirements dictate that only exact total requirements need be produced ($y_N = 0$).
The vector \( x = (x_1, \ldots, x_N) \) of production quantities will be called a **schedule**, with the corresponding inventories \( (y_1, \ldots, y_N) \) being implicitly given by (2). A schedule \( x \) will be called **feasible** if it satisfies (2)-(5) and **optimal** if it is both feasible and minimizes (1) over all feasible schedules.

For simplicity of presentation, we will henceforth always assume that \( d_1 > 0 \).\(^1\) It will also prove convenient to adopt the following notation for cumulative requirements \( (D_0 = 0) \):

\[
D_n = \sum_{i=1}^{n} d_i = D_{n-1} + d_n, \quad n = 1, \ldots, N.
\]

The Wagner-Whitin Algorithm

Given a schedule \( x \), period \( n \) will be called a **production point** if \( x_n > 0 \) and similarly, an **inventory point** if \( y_n = 0 \). The Wagner-Whitin algorithm is a dynamic programming recursion derived from the fundamental fact, called the **regenerative point property**, that there is always an optimal schedule \( \bar{x} \) which satisfies

\[
\bar{x}_n \cdot \bar{y}_{n-1} = 0, \quad n = 1, \ldots, N. \tag{6}
\]

That is, period \( n \) is a production point only if period \( (n-1) \) is an inventory point.

The above property allows the search for an optimal schedule to be restricted to only those schedules satisfying (6) and accounts for the efficiencies of the recursive technique. In particular, suppose we are interested in a feasible schedule having period \( n \) as an inventory point and last previous production

\(^1\)Modifications necessary in the case \( d_1 = 0 \) are outlined in the Appendix.
point at period $i$ ($i \leq n$). Then, by definition, $y_n = 0$, $x_i > 0$ and $x_{i+1} = x_{i+2} = \ldots = x_n = 0$. Also, by (6), $y_{i-1} = 0$. Hence, the production in period $i$ must satisfy exactly the requirements in periods $i, i+1, \ldots, n$:

$$x_i = \sum_{j=1}^{n} d_j = D_n - D_{i-1}.$$ 

Similarly, the inventories over this interval are given by

$$y_j = D_n - D_j, \quad j = 1, \ldots, n.$$ 

Hence, the costs associated with this schedule in periods $i, i+1, \ldots, n$ will be

$$a_i + b_i (D_n - D_{i-1}) + \sum_{j=1}^{n} h_j (D_n - D_j). \quad (7)$$

It should be noted that in the above discussion, we have implicitly assumed that $(D_n - D_{i-1}) > 0$, so that period $i$ was indeed a production point. However, in order to generalize (7) to cover the possible case of $(D_n - D_{i-1}) = 0$, we simply replace the first term of (7) by $a_i \cdot \delta(D_n - D_{i-1})$. The entire Wagner-Whitin recursion is based simply on this generalized version of (7).

For $n = 1, \ldots, N$, let $f_n$ denote the minimum cost of satisfying the requirements in periods $1, 2, \ldots, n$ with $y_n = 0$; define $f_0 = 0$. Obviously, we seek $f_N$ and an associated optimal schedule having cost $f_N$. Similarly, let $g_n(1)$ denote the minimum cost of satisfying the requirements in periods $1, 2, \ldots, n$ with $y_n = 0$ and with last previous production point\(^1\) at period $i$ ($1 \leq i \leq n < N$).

---

\(^1\)Strictly speaking, this interpretation of $g_n(i)$ is only accurate when $D_n - D_{i-1} > 0$, as explained in the text following Equation (7); the special case $D_n - D_{i-1} = 0$ will remain implicit.
Since any partial schedule yielding minimum cost \( f_n \) must have a last production point somewhere in the interval \([1, \ldots, n]\), we have

\[
f_n = \min_{1 \leq i \leq n} \{ g_n(i) \}, \quad n = 1, \ldots, N. \tag{8}
\]

On the other hand, if period \( i \) is a production point, then \( y_{i-1} = 0 \) by (6). By definition, the best we can do in periods 1 through \( i-1 \) (when \( y_{i-1} = 0 \)) is to use the partial schedule having cost \( f_{i-1} \); this fact, coupled with our previous observation (7), leads immediately to

\[
g_n(i) = [\min \text{ cost of satisfying } 1, \ldots, i-1 \text{ with } y_{i-1} = 0] + [\min \text{ cost of satisfying } i, \ldots, n \text{ with } y_{i-1} = y_n = 0 \text{ and } x_{i+1} = x_{i+2} = \ldots = x_n = 0]
\]

\[
= f_{i-1} + a_i \cdot \delta(D_n - D_{i-1}) + b_i (D_n - D_i) + \sum_{j=1}^{n} h_j (D_n - D_j),
\]

\[
i = 1, \ldots, n; \quad n = 1, \ldots, N. \tag{9}
\]

The recursive procedure alternates between (8) and (9), successively generating \( f_0, g_1(1), f_1, g_2(1), g_2(2), f_2, g_3(1), g_3(2), g_3(3), \ldots, f_N \). This is commonly referred to as the forward recursion and, as presented here, requires \( N(N+1)/2 \) computations of (9) and a total of \( N(N+1)/2 \) comparisons in the \( N \) evaluations of (8). However, as Wagner and Whitin demonstrate in their Planning Horizon Theorem [10], the above computation estimates are upper bounds which may be significantly reduced in many cases.
Once the forward recursion is completed, we have the minimum N-period cost \( f_N \); the task which remains is that of finding an optimal schedule \( \bar{x} \) which actually realizes the minimum cost \( f_N \). This part of the algorithm is commonly referred to as the backward lookup and proceeds as follows. Consider (8) for the case \( n = N \). The equality must be satisfied for some period \( i^* \):

\[
f_N = g_N(i^*) = \min_{1 \leq i \leq N} \{ g_N(i) \}.
\]

Hence, period \( i^* \) is the last production point in the optimal schedule, so \( \bar{x}_{i^*} = D_N - D_{i^*-1} \) and \( \bar{x}_j = 0, j = i^*+1, \ldots, N \). Now, from (9) with \( n = N \) and \( i = i^* \), \( g_N(i^*) = f_{i^*-1} + \ldots \), so the remaining undetermined portion of the optimal schedule \( \bar{x} \) must be the partial schedule having cost \( f_{i^*-1} \) in periods \( 1, 2, \ldots, i^*-1 \). Hence, proceeding in a manner analogous to that described above, we now consider (8) with \( n = i^*-1 \) to find the second-to-last production point of \( \bar{x} \), etc. In practice, the lookup is performed by keeping a set of pointers to indicate where equality is realized in expression (8) for each period \( n, n = 1, \ldots, N \).

An Example Problem

While the literature is abundant with excellent numerical illustrations and explanations of the Wagner-Whitin algorithm [1,3,9,10], we shall present a small \( (N = 4) \) example here which will later facilitate the illustration of our own results. Table 1 depicts the costs and requirements data for our 4-period example, with each row of the table detailing the data pertinent to a specific time period.
**TABLE 1**

<table>
<thead>
<tr>
<th>n</th>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$h_n$</th>
<th>$d_n$</th>
<th>$D_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40</td>
<td>4</td>
<td>2</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>35</td>
<td>4</td>
<td>1</td>
<td>25</td>
<td>55</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
<td>3</td>
<td>1</td>
<td>20</td>
<td>75</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>1</td>
<td>-</td>
<td>10</td>
<td>85</td>
</tr>
</tbody>
</table>

Example Problem: Data ($N = 4$)

Table 2 depicts the results from applying the recursions (8) and (9) to the example data. The $n^{th}$ row of Table 2 details the costs $g_n(i)$ for $i = 1, \ldots, n$; $f_n$, being the minimum element of the $n^{th}$ row, is detailed by underlining. From the bottom row of Table 2, we see that the minimum total cost is $f_4 = 435$. Backtracking through the table, we see that the associated optimal schedule is $\bar{x}_4 = D_4 - D_3 = 10$, $\bar{x}_3 = 0$, $\bar{x}_2 = D_3 - D_1 = 45$, and $\bar{x}_1 = D_1 = 30$. The corresponding optimal inventory levels are $\bar{y}_1 = 0$, $\bar{y}_2 = 20$, $\bar{y}_3 = 0$, and $\bar{y}_4 = 0$. As the recursion calculations for this example are few in number and quite simple, any reader unfamiliar with the Wagner-Whitin approach is strongly
encouraged to duplicate our results at this point.

**TABLE 2**

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>160</td>
<td></td>
<td></td>
<td>$g_n(i)$</td>
</tr>
<tr>
<td>2</td>
<td>310</td>
<td>295</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>450</td>
<td>395</td>
<td>400</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>530</td>
<td>455</td>
<td>440</td>
<td>435</td>
</tr>
</tbody>
</table>

Example Problem: Recursion Results

Notice in the optimal Wagner-Whitin schedule for the example problem that inventory levels drop to zero in three of the four time periods. While this characteristic of Wagner-Whitin schedules is obviously optimal with regard to known requirements $d_1, ..., d_N$, it is certainly not appealing in cases where $d_1, ..., d_N$ are only point-estimates which may deviate from the requirements that will actually be incurred. To illustrate, if either $D_1$ or $D_3$
of the example problem underestimate the corresponding actual (partial) cumulative requirements, then lost sales and/or damaged goodwill will result if the Wagner-Whitin schedule is followed (as some of the actual requirements will not be satisfied on time). Our results, presented in the next section, revise the Wagner-Whitin model and algorithm to remedy this type of conflict in the cases where $d_1, \ldots, d_N$ are estimates, but $D_N$ is sufficiently accurate to be treated as constant.

3. SAFETY STOCK PARAMETERIZATION

The Parametric Model

We now suppose that $d_1, \ldots, d_N$ are estimated requirements for a single product in periods $1, \ldots, N$. Even though the by-period quantities are estimates, we assume that $N$ is a known, finite duration and that the total cumulative requirements projection $D_N$ is sufficiently accurate to be treated as a constant. Thus, the question that is critical here is the way in which the total cumulative requirements $D_N$ will actually be distributed over the individual time periods. Examples where this situation might arise were previously cited in the introductory section.

Since the total cumulative requirements $D_N$ are known, we still desire a schedule $x$ which produces total exact requirements (i.e., $y_N = 0$). However, as we have seen in the preceding section, application of the Wagner-Whitin algorithm to the projections $d_1, \ldots, d_N$ will lead to a schedule with zero inventories in all time periods which immediately precede periods of production.
Thus, if the actual (partial) cumulative requirements through any period \( n \) ever exceed the corresponding projection \( D_n \), there is the substantial possibility of incurring lost sales and/or damaged goodwill if the Wagner-Whitin schedule is followed. In order to protect against such possibilities, we therefore propose to alter the model by imposing a uniform lower bound on all projected inventory levels until all of the required \( D_N \) units have been produced; this lower bound will be called the safety stock parameter and its value will be denoted by \( s \).

An immediate consequence of the safety stock restriction is that no production can occur after the first period \( n \) where remaining requirements \( (D_N - D_n) \) either equal or fall below the value \( s \). Denote by \( \alpha(s) \) the last period where production can occur when the safety stock parameter has value \( s \):

\[
\alpha(s) = \min_{1 \leq n \leq N} \{ n | D_N - D_n \leq s \}.
\tag{10}
\]

Then our safety stock restriction can be stated as

\[
y_n \geq \begin{cases} 
  s, & n = 1, \ldots, \alpha(s) \\
  D_N - D_n, & n = \alpha(s) + 1, \ldots, N.
\end{cases}
\tag{11}
\]

Obviously, the second set of inequalities in (11) are in fact equalities (since \( y_N = 0 \)) and furthermore, are sufficient to insure that \( x_n = 0, n = \alpha(s)+1, \ldots, N \). From the definition of \( \alpha(s) \), it is easy to see that (11) is equivalent to the following unified expression:

\[
y_n \geq \min(s, D_N - D_n), \quad n = 1, \ldots, N.
\tag{12}
\]
The revised parametric model thus takes the form of minimizing (1) subject to constraints (2), (3), (5) and (12). A schedule \( x \) will be called an \( s \)-schedule if it satisfies these constraints when the safety stock parameter has value \( s \). It should be noted that the special case of the revised model with \( s = 0 \) is precisely the Wagner-Whitin model, since (12) reduces to (4) in this instance (i.e., an optimal Wagner-Whitin schedule is an optimal 0-schedule).

To those familiar with dynamic programming, it may be fairly obvious how to modify the Wagner-Whitin algorithm to find optimal \( s \)-schedules for any fixed value of \( s \). However, the results of our study show that a single application of the Wagner-Whitin algorithm, suitably modified, actually solves the revised model parametrically for all values of \( s \geq 0 \)! That is, with our algorithm, it is possible to specify optimal \( s \)-schedules as a function of \( s \geq 0 \). Furthermore, the modified algorithm involves little additional computation over that of the standard Wagner-Whitin recursion. We now proceed to detail the parametric algorithm, with the proof of its correctness relegated to the Appendix.

The Parametric Algorithm with Illustration

Define \( v(1) = 0 \) and let

\[
v(n) = b_1 - b_n + \sum_{i=1}^{n-1} h_i, \quad n = 2, \ldots, N. \tag{13}
\]

Our algorithm is based on the following two facts (proven in the Appendix):
(1) there is always an optimal s-schedule $\bar{x}$ which satisfies
\[ \bar{x}_n \cdot (\bar{y}_{n-1} - s) = 0, \quad n = 2, \ldots, N; \text{ and} \quad (14) \]

(2) the least cost of an s-schedule satisfying (14) and having last production point in period $i$ ($i \leq \alpha(s)$) is given by
\[ g_N(i) + v(i) \cdot s, \quad i = 1, \ldots, \alpha(s), \quad (15) \]

where the $g_N(i)$ are computed in the standard recursions (8) and (9).

To provide some intuitive interpretation, note that (14) is simply a generalization of the regenerative point property (6), while (15) represents the cost of an altered schedule obtained by taking the standard Wagner-Whitin schedule $x$ which realizes cost $g_N(i)$, adding $s$ to $x_1$ and subtracting $s$ from $x_i$. (The fact that this simple type of transformation actually preserves the sub-optimal character between Wagner-Whitin 0-schedules and general s-schedules is the primary proof contained in the Appendix.)

Denote by $f_N(s)$ the minimum total cost, over all s-schedules, when the value of the safety stock parameter is $s$. Since, as we have noted earlier, production is not feasible in periods $\alpha(s)+1, \ldots, N$, we have from (15) that
\[ f_N(s) = \min_{1 \leq i \leq \alpha(s)} \{ g_N(i) + v(i) \cdot s \}, \quad s \geq 0. \quad (16) \]

Period $i$ is a potential production point if and only if $i \leq \alpha(s)$. Thus, from (10) we have that period $i > 1$ is a potential production
point if and only if \( s < D_N - D_{i-1} \), while period 1 is always a potential production point. Keeping this observation in mind, we can now state the forward recursion of our modified algorithm:

(F1) apply the standard recursions (8) and (9) to the projections \( d_1, \ldots, d_N \) until \( g_N(i) \), \( i = 1, \ldots, N \), have been computed;

(F2) on a single graph, plot the \( N \) linear functions (15) (having intercepts \( g_N(i) \) and slopes \( v(i) \), \( i = 1, \ldots, N \)) versus \( s \geq 0 \) over their respective ranges \( (0 \leq s < D_N - D_{i-1}, \text{ for } i > 1, \text{ and } 0 \leq s < \infty, \text{ for } i = 1) \);

(F3) \( f_N(s) \), versus \( s \geq 0 \), is then given by the lower envelope of the \( N \) functions plotted in (F2).

To illustrate, consider the example problem of the previous section. Using (13), we have from Table 1 that \( v(1) = 0 \), \( v(2) = 2 \), \( v(3) = 4 \), and \( v(4) = 7 \). From the bottom row of Table 2, we have \( g_4(1) = 530 \), \( g_4(2) = 455 \), \( g_4(3) = 440 \), and \( g_4(4) = 435 \). The four \((N = 4)\) linear functions to be plotted and their respective ranges (as given in (F2) above) thus appear as in Table 3.

Figure 1 depicts the plots of the four functions versus \( s \); the lower envelope, delineated by the heavier line, is the minimum total cost \( f_4(s) \) as a function of the safety stock parameter value \( s \). In particular, we see that the minimum cost is given
TABLE 3

<table>
<thead>
<tr>
<th></th>
<th>( g_4(i) + v(i) \cdot s )</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>530 + 0s</td>
<td>( 0 \leq s &lt; \infty )</td>
</tr>
<tr>
<td>2</td>
<td>455 + 2s</td>
<td>( 0 \leq s &lt; 55 )</td>
</tr>
<tr>
<td>3</td>
<td>440 + 4s</td>
<td>( 0 \leq s &lt; 30 )</td>
</tr>
<tr>
<td>4</td>
<td>435 + 7s</td>
<td>( 0 \leq s &lt; 10 )</td>
</tr>
</tbody>
</table>

Example Problem: Linear Functions to be Plotted

parametrically by:¹,²

\[
\begin{align*}
    f_4(s) &= \begin{cases} 
        435 + 7s, & 0 \leq s < 5/3 \\
        440 + 4s, & 5/3 \leq s < 15/2 \\
        455 + 2s, & 15/2 \leq s < 75/2 \\
        530, & 75/2 \leq s < \infty.
    \end{cases}
\end{align*}
\]

¹With regard to the fractions, we remind the reader that the model is equally valid for continuous production measures (e.g., tons of fertilizer); if an integer measure fits the particular application (e.g., number of bulldozers), then the fractions should be rounded appropriately.

²Notice that we have defined the upper limit of each interval to be "open" (e.g., \( s < 5/3 \)). Usually, the alternative "closed" representation (e.g., \( s \leq 5/3 \)) will be equally valid. However, consider what the lower envelope of Figure 1 would look like if the intercepts for \( i = 1, 2 \) and 3 were each incremented by 500; in this event, the lower envelope would "jump" at \( s = 10 \) (the range extent of the linear function for \( i = 4 \)) and only the open representation (e.g., \( s \leq 10 \)) would be valid.
Example Problem: Minimum Cost vs. Safety Stock Parameter (not to scale)

Minimum cost J^*(s)

(heavy line) is lower envelope.

Note:

Figure 1
In general, $f_4(\cdot)$ will be piecewise linear (but not necessarily continuous -- see previous footnote), as depicted in the example above. That is, there will be numbers $0 = s_0 < s_1 < s_2 < \ldots < s_M = \infty$ and indices $i_1, i_2, \ldots, i_m = 1 (M \leq N)$ such that

$$f_N(s) = g_N(i_m) + v(i_m) \cdot s, \text{ for } s_{m-1} \leq s < s_m, \ m = 1, \ldots, M. \ (16)$$

(To illustrate, in the example problem, $M = 4$, $s_0 = 0$, $s_1 = 5/3$, $s_2 = 15/2$, $s_3 = 75/2$, $s_4 = \infty$, $i_1 = 4$, $i_2 = 3$, $i_3 = 2$ and $i_4 = 1$.) Specifically, the index $i_m$ is the last production point period of an optimal s-schedule, whenever $s_{m-1} \leq s < s_m, \ m = 1, \ldots, M$. With this representation of $f_N(\cdot)$ in mind, we can state the modified backward lookup procedure for finding the parametric optimal s-schedule $\bar{x}$ when $s_{m-1} \leq s < s_m$:

\begin{itemize}
  \item [(B1)] the last period of production is $i^* = i_m$ with 
  $$\bar{x}_{i^*} = D_N - D_{i^*-1} - s \ (\text{and } \bar{x}_n = 0, \ n = i^*+1, \ldots, N);$$

  \item [(B2)] to determine the remainder of the optimal s-schedule, 
  use the standard Wagner-Whitin lookup procedure 
  starting with $n = i^*-1$ in (8); the period 1 production quantity ($x_1$) determined in this manner should 
  then be incremented by s to get $\bar{x}_1$.
\end{itemize}

To find optimal s-schedules for all $s \geq 0$, the above lookup procedure must be employed M times (once for each interval $s_{m-1} \leq s < s_m, \ m = 1, \ldots, M$).

To illustrate, we will now find the optimal s-schedule in the example problem for $s_1 = 5/3 \leq s < 15/2 = s_2$ (i.e., $m = 2$). Since $i_2 = 3$, we have from (B1) that production last occurs is period $i^* = 3$ with $\bar{x}_3 = D_4 - D_2 - s = 30 - s$ (and $\bar{x}_4 = 0$). We now refer...
to Table 2 and employ the standard Wagner-Whitin backtracking scheme starting from row $1^*-1 = 2$ of the table. We see that $\bar{x}_2 = D_2 - D_1 = 25$ and (making the period 1 correction stated in (B2)) that $\bar{x}_1 = D_1 + s = 30 + s$. The optimal s-schedule is thus given parametrically by $\bar{x}_1 = 30 + s$, $\bar{x}_2 = 25$, $\bar{x}_3 = 30 - s$ and $\bar{x}_4 = 0$, for $5/3 \leq s < 15/2$. The corresponding inventories are easily seen to be $\bar{y}_1 = s$, $\bar{y}_2 = s$, $\bar{y}_3 = 10$, and $\bar{y}_4 = 0$. Calculations of the parametric optimal s-schedules for the other intervals follow similarly and Table 4 summarizes the complete set of parametric solutions to the example problem for all safety stock parameter values $s \geq 0$.

**TABLE 4**

<table>
<thead>
<tr>
<th></th>
<th>Min. Cost</th>
<th>$\bar{x}_1$</th>
<th>$\bar{x}_2$</th>
<th>$\bar{x}_3$</th>
<th>$\bar{x}_4$</th>
<th>$\bar{y}_1$</th>
<th>$\bar{y}_2$</th>
<th>$\bar{y}_3$</th>
<th>$\bar{y}_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq s &lt; 5/3$</td>
<td>435+7s</td>
<td>20+s</td>
<td>45</td>
<td>0</td>
<td>10-s</td>
<td>s</td>
<td>s</td>
<td>s</td>
<td>0</td>
</tr>
<tr>
<td>$5/3 \leq s &lt; 15/2$</td>
<td>440+4s</td>
<td>30+s</td>
<td>25</td>
<td>30-s</td>
<td>0</td>
<td>s</td>
<td>s</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$15/2 \leq s &lt; 75/2$</td>
<td>455+2s</td>
<td>30+s</td>
<td>55+s</td>
<td>0</td>
<td>0</td>
<td>s</td>
<td>30</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$75/2 \leq s &lt; \infty$</td>
<td>530</td>
<td>85</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>55</td>
<td>30</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

Example Problem: Optimal Parametric Solutions
It can be shown that a complete lookup to generate all parametric solutions takes at most \( N(N+1)/2 \) steps, where each step involves only a pointer-access and a single subtraction or addition.

**Sensitivity Analysis**

Having solved the revised model for all values \( s \geq 0 \), we are now in a position to answer a variety of sensitivity questions with regard to providing safeguards against lost sales and/or damaged goodwill. We shall illustrate using the example problem.

**Q1.** How much insurance against lost sales will an extra $50 allocation to production and inventory expenses buy?

**A1.** From Figure 1, \( f_4(15) = 485 = 435 + 50 \), so $50 will allow the safety stock parameter to be set at \( s = 15 \). The corresponding optimal 15-schedule is given by the third row of Table 4: \( \bar{x}_1 = 45, \bar{x}_2 = 40, \bar{x}_3 = 0, \bar{x}_4 = 0 \). Thus, the extra $50 will insure that no lost sales will be incurred whenever the actual requirements in period 1 do not exceed 45.

**Q2.** What are the marginal costs of providing extra insurance against lost sales?

**A2.** From Figure 1, we see that the marginal cost (per unit increase in \( s \)) is $7 for \( 0 \leq s < 5/3 \), $4 for \( 5/3 \leq s < 15/2 \), and $2 for \( 15/2 \leq s < 75/2 \). For \( s \geq 75/2 \), it is optimal to produce all requirements in period 1 (guaranteeing no lost sales) at total cost $530.
Q3. What is the minimum total cost (and associated optimal schedule) required to guarantee avoidance of lost sales when it is known that the actual by-period requirements will never fluctuate more than 2 units over the estimates?

A3. The maximum possible fluctuation over $D_1$ is 2, over $D_2$ is 4, and over $D_3$ is 6, so we want to set $s = 6$. Referring to row 2 of Table 4, we see that the minimum total cost is $464 and the optimal 6-schedule is: $\bar{x}_1 = 36, \bar{x}_2 = 25, \bar{x}_3 = 24, \bar{x}_4 = 0$.

Q4. Same question as Q3 when it is known that actual by-period requirements will never exceed more than 20% of the estimates?

A4. The maximum possible fluctuation over $D_1$ is $0.2D_1 = 6$ and over $D_2$ is $0.2D_2 = 11$. However (since $D_4 = 85$ is constant), the maximum possible fluctuation over $D_3$ is given by

$$\min\{0.2D_3, D_4 - D_3\} = \min\{15, 10\} = 10.$$  

Thus, we want to set $s = 10$ and the optimal 11-schedule (with cost $477$) should be used: $\bar{x}_1 = 41, \bar{x}_2 = 14, \bar{x}_3 = 0, \bar{x}_4 = 0$.

Q5. Find the schedule which maximizes $s$, yet has cost within 15% of the cost of an optimal Wagner-Whitin 0-schedule.

A5. The Wagner-Whitin schedule has total cost $435$, so the schedule we seek can cost no more than $1.15(435) = 500.25$. From Figure 1, $f_4(22.5) = 500.25$, so $s = 22.5$ is the maximum value that may be attained with the additional 15% allocation. The optimal 22.5-schedule is given by: $\bar{x}_1 = 52.5, \bar{x}_2 = 32.5, \bar{x}_3 = 0, \bar{x}_4 = 0$.  

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Obviously, the answers to a host of additional sensitivity questions are within easy reach once a parametric solution has been calculated.

5. CONCLUDING REMARKS

The principal advantage of the parametric model is that its application does not require the sometimes-restrictive assumptions of known stockout cost and known distributions (or constants) for the by-period requirements. Restated, to apply the parametric approach, a production planner does not need the quantity of input information necessary for other techniques. Furthermore, the amount of computation is on the same order as other dynamic programming approaches. In particular, if the accuracy of the projections $d_1, \ldots, d_N$ is suspect, then the parametric approach should certainly be preferred over the Wagner-Whitin technique [10], as the former addresses a problem much more in line with the real world, providing substantially more relevant output with comparable computational aspects.

With regard to possible shortcomings, it should be pointed out that a price is indeed paid by the parametric approach for requiring less input information. Specifically, the parametric model works with a uniform safety stock parameter, while other techniques [4,5] allow safety stock levels to vary over the planning horizon. Also, we do not advocate any particular "best" criterion for choosing a final value for the safety stock parameter (although a number of budgetary and statistical
possibilities are implied in the preceding subsection concerning sensitivity analysis).

In summary, as with all planning techniques, the practitioner will have to weigh the relevancy and costs of the parametric approach in his specific application against those of all other alternatives. As Gorenstein [2, p. 44] so aptly puts it: "..., if the model fits, use it; if it doesn't, derive one that does. If that is impossible or impractical, use the one that fits best by comparing the costs incurred by each under your operating conditions."

APPENDIX

The Appendix consists of two subsections; the first establishes the correctness of the parametric algorithm, while the second details necessary modifications for the troublesome case \( d_1 = 0 \).

On the Correctness of the Parametric Algorithm

In this subsection, we establish via two theorems the properties (14) and (15), from which the parametric algorithm follows. Theorem 1 (Regenerative Point Property). For \( s \geq 0 \) there exists at least one optimal \( s \)-schedule \( \bar{x} \) satisfying:

\[
\bar{x}_n (\bar{y}_{n-1} - s) = 0, \quad n = 2, \ldots, N.
\]

(17)

Proof. Fix \( s \geq 0 \) and let \( \lambda = \alpha(s) \). Then for any \( s \)-schedule, as demonstrated in the text, \( x_n = 0 \) and \( y_{n-1} = D_N - D_{n-1} \) for
n = \ell + 1, \ldots, N. Thus, it suffices to establish (17) for \n = 2, \ldots, \ell. If \ell = 1, there is nothing to prove, so assume \ell \geq 2. Define translated variables \( u_n = y_n - s, \ n = 1, \ldots, \ell - 1 \). Then (by substituting the known values of \( x_n \) and \( y_{n-1} \) for \( n = \ell + 1, \ldots, N \) and using \( u_n \) in place to \( y_n, \ n = 1, \ldots, \ell - 1 \)), the revised model can be equivalently rewritten as:

\[
\sum_{n=1}^{\ell} (a_n \delta(x_n) + b_n x_n) + \sum_{n=1}^{\ell - 1} h_n u_n + s \sum_{n=1}^{\ell - 1} h_n + \sum_{n=\ell}^{N} h_n(D_N - D_n) \quad (18)
\]

subject to:

\[
\begin{align*}
\begin{cases}
\begin{align*}
x_n + u_{n-1} - u_n &= d_n, & n = 1, \ldots, \ell \\
u_n &= 0 \\
u_{\ell} &= D_N - D_\ell \\
x_n &\geq 0, \ u_n \geq 0, & n = 1, \ldots, \ell.
\end{align*}
\end{cases}
\end{align*}
\]

(19)

The objective (18) is a concave function of the variables on the nonnegative orthant (since \( a_n \geq 0 \)). Thus, (18) is minimized at an extreme point of the convex polyhedron defined by (19). But (19) defines a bounded Leontief Substitution System with extreme points characterized by the property [7, 8]: \( x_n u_{n-1} = 0, \ n = 1, \ldots, \ell \). That is, there exists an \( s \)-schedule \( \bar{x} \) minimizing (18) with the property \( \bar{x}_n \bar{u}_{n-1} = 0, \ n = 1, \ldots, \ell \), or equivalently, \( \bar{x}_n (y_{n-1} - s) = 0, \ n = 2, \ldots, \ell \).

By Theorem 1, it suffices to restrict our attention to only those \( s \)-schedules satisfying (17). Let \( W_1(s) \) denote the set of
s-schedules satisfying (17) and having last production point in period i. As noted in the preceding proof, \( W_i(s) \) is null for \( i = \alpha(s)+1, \ldots, N \). For \( i = 1, \ldots, \alpha(s) \), we have from Theorem 1 that \( x \in W_i(s) \) implies that \( y_{i-1} = s \) and hence, \( x_i = D_N - D_{i-1} - s \).

For any s-schedule \( x \), let \( c(x) \) be its total cost. Denote by \( C_i(s) \) the minimum cost over all s-schedules in \( W_i(s) \):

\[
C_i(s) = \min_{x \in W_i(s)} c(x), \quad i = 1, \ldots, \alpha(s)
\]

For \( s = 0 \), (17) reduces to the standard Wagner-Whitin regenerative point result, so by definition, \( C_i(0) = \mathcal{E}_N(i), \quad i = 1, \ldots, N \).

**Theorem 2.** If \( d_1 > 0 \), then for \( s \geq 0 \):

\[
C_i(s) = \mathcal{E}_N(i) + v(i) \cdot s, \quad i = 1, \ldots, \alpha(s).
\]  

(20)

**Proof.** Fix \( s \geq 0 \) and fix \( i, 1 \leq i \leq \alpha(s) \). Define the linear transformation \( T : \mathbb{R}^N \rightarrow \mathbb{R}^N \) by

\[
T(x)_n = \begin{cases} x_1 + s, & n = 1 \\ x_n, & n \neq 1,1 \\ x_i - s, & n = i. \end{cases}
\]  

(21)

It is easily verified that

\[
W_i(s) = T(W_i(0))
\]  

(22)

(that is, \( W_i(s) \) is the image of \( W_i(0) \) under the linear transformation \( T \)) by successively establishing the set relations

\[
W_i(s) \subseteq T(W_i(0)) \quad \text{and} \quad W_i(0) \subseteq T^{-1}(W_i(s)).
\]

Since \( d_1 > 0 \) and \( y_0 = 0 \) together imply that \( x_1 > 0 \), for all \( x \in W_i(0) \cup W_i(s) \), it follows immediately from (21) and the definition of \( v(i) \) that
\[ c(T(x)) = c(x) + v(i) \cdot s, \quad \text{for all } x \in W_1(0). \quad (23) \]

Thus, from (22) and (23),

\[
C_1(s) = \min_{x \in W_1(s)} \{ c(x) \} \\
= \min_{x \in W_1(0)} \{ c(T(x)) \} \\
= \min_{x \in W_1(0)} \{ c(x) + v(i) \cdot s \} \\
= \min_{x \in W_1(0)} \{ c(x) \} + v(i) \cdot s \\
= C_1(0) + v(i) \cdot s \\
= g_N(i) + v(i) \cdot s
\]

Theorems 1 and 2 thus establish properties (14) and (15).

The Special Case: \( d_1 = 0 \)

When \( d_1 = 0 \), (23) fails to hold and Theorem 2 cannot be established directly, in which case the parametric procedure given in the text may break down. To remedy this problem, it may be necessary to apply the recursions (8) and (9) twice whenever \( d_1 = 0 \).

The standard Wagner-Whitin algorithm is first used to solve for the optimal 0-schedule (i.e., \( s = 0 \)). To find optimal s-schedules for \( s > 0 \), the parametric algorithm is then applied to a perturbed problem by changing only step (F1) of the forward recursion to:
apply the standard recursions (8) and (9), replacing the second term of (9) by \(a_1 \cdot (D_n - D_0)\) whenever \(i = 1\), to the projections \(d_1, \ldots, d_N\) until \(g_N(i), i = 1, \ldots, N\) have been computed.

Making the above exception to (9) whenever \(i = 1\) corresponds to solving a perturbed problem with \(d_1 > 0\), but arbitrarily close to zero. In this case, Theorem 2 can be established for \(s > 0\), with the corresponding \(g_N(i) (i = 1, \ldots, N)\) in (20) coming from the perturbed problem, as calculated in (Fl') above. We note that if 0-schedules are not of interest when \(d_1 = 0\), then it suffices to only apply the parametric algorithm (with perturbed step (Fl')) to find all optimal s-schedules for \(s > 0\).
REFERENCES


**SAFETY STOCK SENSITIVITY ANALYSIS IN DYNAMIC ECONOMIC LOT SIZE MODELS: A PARAMETRIC APPROACH**

**R. Scott SHIPLEY**

**Department of Operations Research and Department of Statistics**
Stanford University, Stanford, Ca. 94305

**Statistics and Probability Program**
Office of Naval Research Code 436
Arlington, Virginia 22217

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**lot size models**

**production models**

**inventory models**

**safety stock**

**dynamic programming**
TR 173, SAFETY STOCK SENSITIVITY ANALYSIS IN DYNAMIC ECONOMIC LOT SIZE MODELS: A PARAMETRIC APPROACH

The question of safety stock levels is considered in a dynamic economic lot size model where: (1) the cumulative total requirement over a fixed number of future time periods is known, but the individual by-period requirements can only be point-estimated; and (2) stockout costs are significant, but unknown. A dynamic programming procedure is described which details the minimum total cost of production and inventory activities (and the corresponding optimal production plan) parametrically as a function of all possible values of a uniform safety stock parameter.