A TANDEM STORAGE SYSTEM AND ITS DIFFUSION LIMIT

BY

J. MICHAEL HARRISON
and
L. A. SHEPP

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A Tandem Storage System and Its Diffusion Limit

J. Michael Harrison
Stanford University

L.A. Shepp
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Abstract. We consider a two-dimensional diffusion process \( Z(t) = [Z_1(t), Z_2(t)] \) that lives in the half strip \( \{0 \leq Z_1 \leq 1, 0 \leq Z_2 < \infty \} \). On the interior of this state space, \( Z \) behaves like a standard Brownian motion (independent components with zero drift and unit variance), and there is instantaneous reflection at the boundary. The reflection is in a direction normal to the boundary at \( Z_1 = 1 \) and \( Z_2 = 0 \), but at \( Z_1 = 0 \) the reflection is at an angle \( \theta \) below the normal \( (0 < \theta < \pi/2) \).

This process \( Z \) is shown to arise as the diffusion limit of a certain tandem storage or queueing system. It is shown that \( Z(t) \) has a non-defective limit distribution \( F \) as \( t \to \infty \), and the marginal distributions of \( F \) are computed explicitly. The marginal limit distribution for \( Z_1 \) is uniform (this result is essentially trivial), but that for \( Z_2 \) is much more complicated.

A Tandem Storage System and Its Diffusion Limit

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1. Introduction and Summary

This paper is concerned with approximate analysis of a certain discrete-time storage process. The physical system giving rise to this two-dimensional process is pictured in Figure 1 below. It consists of a buffer (or dam or reservoir) having finite capacity $b$, followed by another buffer having infinite capacity. We denote by $S^1_t$ and $S^2_t$ the contents of the finite buffer and the infinite buffer respectively at time $t = 0, 1, \ldots$. Imposing very particular assumptions on the input process to the first buffer, the transfer process between buffers, and the output process from the second buffer, we study here the approximate behavior of the storage process $S_t = (S^1_t, S^2_t)$ as $b \rightarrow \infty$. To be more specific, let us define a continuous-time process $S^*(t) = [S^*_1(t), S^*_2(t)]$ by setting

\begin{equation}
S^*_1(t) = \frac{1}{b} S^1_t \quad \text{and} \quad S^*_2(t) = \frac{1}{ob} S^2_t
\end{equation}

for $t \geq 0$, where $[x]$ denotes the integer part of $x$ (the largest integer less than or equal to $x$). There is a two-dimensional diffusion process $Z$ such that $S^*$ converges weakly to $Z$ as $b \rightarrow \infty$, and we would like to determine the steady-state behavior of $Z$. 

1
mean potential input per period = \( \lambda \)
variance of potential input per period = 1

buffer or reservoir of finite capacity \( b \)

mean potential transfer per period = \( \lambda \)
variance of potential transfer per period = 0

buffer or reservoir of infinite capacity

mean potential output per period = \( \lambda \)
variance of potential output per period = \( \sigma^2 \)

Figure 1. The Storage System under Study

In §2 we define precisely the discrete-time storage process \( S \), explaining how it is constructed from a given input sequence, transfer sequence and output sequence. The diffusion limit \( Z \) is then defined and characterized in §3. Weak convergence of \( S^* \) to \( Z \) is implied by a general result of Wenocur's [7] to appear shortly, so we shall not give a formal proof. By emphasizing parallels in the definitions of \( S \) and \( Z \), however, we hope to make the convergence result more than plausible.

Our diffusion limit \( Z(t) = [Z_1(t), Z_2(t)] \) lives in the semi-infinite strip \( \Sigma \) pictured in Figure 2 below. We use the letters \( x \) and \( y \) to denote elements of the strip \( \Sigma \). When it is necessary to speak of individual components, we write \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \). On the interior of its state space, \( Z \) behaves like a two-dimensional standard Brownian motion (independent components with zero mean and unit variance).
At the boundary, $Z$ reflects instantaneously. On the sides $Z_1 = 1$ and $Z_2 = 0$ we have what is called normal reflection. On the side $Z_1 = 0$ there is oblique reflection in the direction pictured in Figure 2, where $a = 1/\sigma$ and $\sigma^2$ is the variance parameter introduced in Figure 1. The precise meaning of this boundary behavior will be explained in §3. In §4 it will be shown that $P(Z(t) \leq y) \rightarrow F(y)$ as $t \rightarrow \infty$, where $F$ is a non-defective distribution on the strip $\Sigma$, and this limit distribution (or steady-state distribution) will be shown to satisfy a

![Figure 2. State Space $\Sigma$ and Angles of Reflection for the Diffusion Limit $Z$](image)

certain adjoint relation. Using this adjoint relation and Wiener-Hopf techniques, we calculate the marginal distributions of $F$ in §5.

The marginal steady-state distribution of $Z_1$ is uniform, which follows immediately from the fact that $Z_1$ is a one-dimensional standard
Brownian motion constrained to $[0,1]$ by reflecting barriers. But the marginal steady-state distribution of $Z_2$, 

$$G(\xi) = F(1,\xi), \quad u \geq 0,$$

is much more complex. Let 

$$c \equiv \pi \frac{\Gamma(1 - \frac{\theta}{\pi})}{\Gamma(1 - \frac{2\theta}{\pi})} \Gamma\left(\frac{\theta}{\pi}\right),$$

(1.2) 

$$p(u) \equiv e^{-\theta u}(1 - e^{-\pi u})^{-2\theta/\pi}, \quad u \geq 0,$$

(1.3) 

where $\theta = \tan^{-1} a$ as in Figure 2, and $\Gamma(\cdot)$ is the standard gamma function 

$$\Gamma(s) \equiv \int_0^\infty e^{-t} t^{s-1} dt, \quad s > 0.$$

It will be shown that $cp(u)$ is a probability density function on $(0,\infty)$ with mean $1/a$, and that this distribution describes the asymptotic behavior of a certain increasing process associated with the boundary $Z_1 = 0$. Our main result is the following.

(1.4) **Theorem.** $G(d\xi) = g(\xi) d\xi$ where 

$$g(\xi) = \int_0^\infty cp(u)du.$$ 

From (1.4) it follows that the steady-state marginal distribution $G$ has Laplace transform
(1.5) \[ \int_0^\infty e^{-\alpha \xi} G(d\xi) = \left( a/\alpha \right) \left[ 1 - c\Gamma\left( \frac{\pi - 2\theta}{\pi} \right) \Gamma\left( \frac{\theta + \alpha}{\pi} \right) / \pi \Gamma\left( \frac{\pi - \theta + \alpha}{\pi} \right) \right]. \]

Using (1.5), one can calculate the mean of \( G \) in terms of the tabulated digamma and trigamma functions, cf. Chapter 6 of Abramowitz and Stegun [1]. Also, from (1.3), we see that \( p(u) \sim \exp(-\theta u) \) as \( u \to \infty \), and hence from (1.4)

(1.6) \[ g(\xi) \sim \frac{ae}{\theta} e^{-\theta \xi}, \quad \text{as } \xi \to \infty. \]

An intriguing aspect of (1.6) is that the tail of \( G \) remains relatively fat even as \( \theta \to \pi/2 \), which presumably corresponds to the case where \( Z \) is instantaneously displaced to the origin whenever the boundary \( Z_t = 0 \) is hit (see Figure 2 and the construction of \( Z \) in §3).

2. Construction of the Storage Process

Taken as primitive are three mutually independent IID sequences denoted by \( \{u^0_t\}, \{u^1_t\} \) and \( \{u^2_t\} \) for \( t = 1, 2, \ldots \). We call \( u^0_t \) the input variable, \( u^1_t \) the transfer variable and \( u^2_t \) the output variable respectively for period \( t \). (Period \( t \) extends from time \( t-1 \) to time \( t \).) As shown in Figure 1, we assume throughout that

(2.1) \[ E(u^0_t) = E(u^1_t) = E(u^2_t) \equiv \lambda, \]

and

(2.2) \[ \text{Var}(u^1_t) = 0, \]

while \( u^0_t \) and \( u^2_t \) both have positive variance. Paraphrasing (2.1) and (2.2), we say that ours is a balanced storage system with deterministic
transfer between the two buffers. As a convenient normalization we can assume that \( \text{Var}(u^0_t) = 1 \), and then we set \( \sigma^2 = \text{Var}(u^2_t) \). These notational conventions are also displayed in Figure 1. For simplicity we take \( S_0^1 = S_0^2 = 0 \), meaning that both buffers are initially empty.

The content process for the finite buffer is now defined inductively by

\[
S_t^1 = \begin{cases} 
0, & \text{if } 0 \leq S_{t-1}^1 + u_t^0 - u_t^1 \\
S_{t-1}^1 + u_t^0 - u_t^1, & \text{if } 0 < S_{t-1}^1 + u_t^0 - u_t^1 \leq b \\
b, & \text{if } S_{t-1}^1 + u_t^0 - u_t^1 \geq b
\end{cases}
\]

(2.3)

for \( t = 1, 2, \ldots \). This is of course the standard construction for the content process of a finite dam in discrete time [6]. Actual input to the dam during period \( t \) may be less than the full potential input \( u_t^0 \) because of the capacity restriction, and actual output may similarly be less than the full potential output \( u_t^1 \) because of the restriction \( S_2^1 \geq 0 \). Defining

\[
x_t^1 = u_t^0 - u_t^1,
\]

(2.4)

\[
y_t^0 = [S_{t-1}^1 + x_t^1 - b]^+,
\]

(2.5)

\[
y_t^1 = [S_{t-1}^1 + x_t^1]^-
\]

(2.6)

we observe that \( y_t^0 \) represents lost potential input to the finite buffer during period \( t \), and \( y_t^1 \) represents lost potential output from the finite buffer, which amounts to lost potential transfer between the two buffers.
buffer, which amounts to lost potential transfer between the two buffers pictured in Figure 1. That is, actual input to the finite buffer during period $t$ is $u^0_t - y^0_t$, actual transfer between the two buffers during period $t$ is $u^1_t - y^1_t$, and (2.3) is equivalent to (2.4)-(2.6) plus

$$S^1_t = S^1_{t-1} + x^1_t + y^1_t - y^0_t,$$

for $t = 1, 2, \ldots$. Having established that the actual input to the infinite buffer during period $t$ is $u^1_t - y^1_t$, we can define the content process for the infinite buffer inductively by setting

$$S^2_t = \begin{cases} S^2_{t-1} + u^1_t - y^1_t - u^2_t & \text{if } S^2_{t-1} + u^1_t - y^1_t - u^2_t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for $t = 1, 2, \ldots$. Defining

$$x^2_t \equiv u^1_t - u^2_t,$$

we see that $y^2_t$ represents lost potential output from the infinite buffer during period $t$. That is, actual output during period $t$ is $u^2_t - y^2_t$, and (2.8) is equivalent to (2.9)-(2.10) plus

$$S^2_t = S^2_{t-1} + x^2_t - y^1_t + y^2_t,$$

for $t = 1, 2, \ldots$.

To derive an equivalent and very useful characterization of the storage process $S$, let us define the cumulative quantities
\[ x_t^k = x_1^k + \cdots + x_t^k, \quad \text{for } t = 0, 1, \ldots \text{ and } k = 1, 2, \]

\[ y_t^k = y_1^k + \cdots + y_t^k, \quad \text{for } t = 1, 2, \ldots \text{ and } k = 0, 1, 2, \]

with \( x_0^k = y_0^k = 0 \). By summing (2.7) and (2.11) over periods 1, \( \ldots, t \) we have

\[ s_t^1 = x_t^1 + y_t^1 - y_t^0, \]

\[ s_t^2 = x_t^2 + y_t^2 - y_t^1 \]

for \( t = 0, 1, \ldots \). By construction we have

\[ 0 \leq s_t^1 \leq b \quad \text{and} \quad s_t^2 \geq 0 \quad \text{for } t = 0, 1, \ldots \]

and one can easily verify that

\[ s_t^1 \Delta y_t^1 = (b - s_t^1) \Delta y_t^0 = s_t^2 \Delta y_t^2 = 0 \quad \text{for } t = 1, 2, \ldots, \]

where \( \Delta y_t^k = y_t^k - y_{t-1}^k = y_t^k \). Obviously (2.16) and (2.17) together require that \( \Delta y_t^1 = 0 \) except when \( s_t^1 = 0 \), that \( \Delta y_t^0 = 0 \) except when \( s_t^1 = b \), and that \( \Delta y_t^2 = 0 \) except when \( s_t^2 = 0 \).

We now define the normalized continuous-time storage process

\[ S^*(t) = [S_1^*(t), S_2^*] \] by applying to \( S \) the transformation (1.1). In a similar vein, let \( X^*(t) = [X_1^*(t), X_2^*(t)] \) and \( Y^*(t) = [Y_0^*(t), Y_1^*(t), Y_2^*(t)] \) be defined by

\[ X_1^*(t) = \frac{1}{b} x_1^1 \left[ b^2 t \right], \quad X_2^* = \frac{1}{bd} x_2^2 \left[ b^2 t \right], \]
\[(2.19)\quad Y_0^*(t) = \frac{1}{b^2} Y_0^0 \quad [b^2 t] , \quad Y_1^*(t) = \frac{1}{b^2} Y_1^1 \quad [b^2 t] , \quad Y_2^*(t) = \frac{1}{b^2} Y_2^2 \quad [b^2 t] \]

for \( t \geq 0 \). Obviously \( X^* \) is right-continuous with left-hand limits (RCLL), and from (2.7), (2.11), and (2.14)-(2.17) it follows that \( Y^* \) and \( S^* \) have the following properties:

\[(2.20)\quad Y_k^* \text{ is RCLL and non-decreasing with } Y_k^*(0) = 0 \quad (k = 0, 1, 2),\]

\[(2.21)\quad S_1^*(t) = X_1^*(t) + Y_1^*(t) - Y_0^*(t) ,\]

\[(2.22)\quad S_2^*(t) = X_2^*(t) + Y_2^*(t) - a Y_1^*(t) \quad \text{where} \quad a = \frac{1}{c} ,\]

\[(2.23)\quad 0 \leq S_1^*(t) \leq 1 \quad \text{and} \quad S_2^*(t) \geq 0 ,\]

\[(2.24)\quad \int_0^t S_1^*(u) \, dY_1^*(u) = \int_0^t [1 - S_1^*(u)] \, dY_0^*(u) = \int_0^t S_2^*(u) \, dY_2^*(u) = 0 \]

for all \( t \geq 0 \), where the integrals in (2.24) are defined path-by-path in the Lebesgue-Stieltjes sense. (These integrals do not exist in the Riemann-Stieltjes sense because, for example, \( S_1^* \) and \( Y_1^* \) jump simultaneously.)

We have arrived at (2.20)-(2.24) as characterizations of processes constructed previously, but these relationships can actually be used to define \( Y^* \) and \( S^* \) in terms of \( X^* \). That is, with \( X^* \) defined in terms of the primitive sequences \( \{ u_k^* \} \) via (2.4), (2.9), (2.12) and (2.18), there exists a unique pair of processes \( (Y^*, S^*) \) that jointly satisfy (2.20)-(2.24). Thus (2.20)-(2.24) implicitly define an operator which maps \( X^* \) into \( (Y^*, S^*) \). In the next section we use relationships analogous to (2.20)-(2.24) to define a pair of processes \( (U, Z) \) in terms of a
two-dimensional standard Brownian motion \( W \). Donsker's Theorem [2] tells us that \( X^* \) converges weakly to \( W \) as \( b \to \infty \), and then the continuous mapping theorem [2] can be used to show that

\[
(X^*, Y^*, S^*) \Rightarrow (W, U, Z) \quad \text{as} \quad b \to \infty,
\]

where \( \Rightarrow \) denotes weak convergence in a seven-fold product space. We shall not go further into the convergence argument here, referring the interested reader to Wenocur [7]. The remainder of this paper is devoted to construction and characterization of the diffusion limit \( Z \).

3. The Limiting Diffusion Process

Let \( W(t) = [W_1(t), W_2(t)] \) be a two-dimensional standard Brownian motion (independent components with zero shift and unit variance). Defining \( a \equiv 1/\sigma \) as in (2.22), we wish to construct processes \( U(t) = [U_0(t), U_1(t), U_2(t)] \) and \( Z(t) = [Z_1(t), Z_2(t)] \) which jointly satisfy

\[
(3.1) \quad U_k \text{ is continuous and non-decreasing with } U_k(0) = 0 \quad (k = 0, 1, 2),
\]

\[
(3.2) \quad Z_1(t) = W_1(t) + U_1(t) - U_0(t),
\]

\[
(3.3) \quad Z_2(t) = W_2(t) + U_2(t) - aU_1(t),
\]

\[
(3.4) \quad 0 \leq Z_1(t) \leq 1 \quad \text{and} \quad Z_2(t) \geq 0,
\]
\[ (3.5) \quad \int_0^t Z_1(s) \, dU_1(s) = \int_0^t [1 - Z_1(s)] \, dU_0(s) = \int_0^t Z_2(s) \, dU_2(s) = 0 \]

for \( t \geq 0 \). Condition (3.4) says that \( Z \) lives within the semi-infinite
strip \( \Sigma \) pictured in Figure 2. Obviously (3.5) requires that \( U_1 \)
increase only when \( Z_1 = 0 \), that \( U_0 \) increase only when \( Z_1 = 1 \), and that
\( U_2 \) increase only when \( Z_2 = 0 \). Conditions (3.2) and (3.3) determine the
direction in which \( Z \) is driven upon hitting the boundary of \( \Sigma \). On each
of the three boundary surfaces we have reflection in the (constant)
direction pictured in Figure 2 above.

For each (continuous) sample path \( W \) there is in fact a unique pair
of sample paths \((U, Z)\) satisfying (3.1)-(3.5). The argument goes as
follows. If we delete all mention of \( U_2 \) and \( Z_2 \) in (3.1)-(3.5), the
remaining conditions are those which define \( Z_1 \) as a one-dimensional
standard Brownian motion restricted to \([0, 1] \) by reflecting barriers; \( U_1 \)
and \( U_0 \) are increasing processes (local time processes) associated with
the lower boundary and upper boundary respectively. For a construction of
\( U_0 \) and \( U_1 \) in terms of \( W_1 \) see §5 of [5]. With \( U_0 \) and \( U_1 \)
determined, one may argue as in §2 of [3] that the remaining requirements
of (3.1)-(3.5) are uniquely satisfied by taking

\[ (3.6) \quad U_2(t) = \sup_{0 \leq s \leq t} \left[ -W_2(s) + aU_1(s) \right]^+, \quad t \geq 0 . \]

This construction of \((U, Z)\) from \( W \) is valid for any starting state \( W(0) = (x_1, x_2) \in \Sigma \). Holding \( \{W(t) - W(0), t \geq 0\} \) fixed, it can be verified
that
(3.7) \( Z_1(t) \) and \( Z_2(t) \) are non-decreasing functions of both \( x_1 \) and \( x_2 \) for all \( t \geq 0 \).

Using an argument like that in §2 of [3], it follows that \( Z \) is a Markov process, and we shall denote by \( P_x(\cdot) \) the distribution on the path space of \( Z \) corresponding to initial state \( Z(0) = W(0) = x \). From (3.6) we have that

\[
(3.8) \quad F_t(x,y) \equiv P_x\{Z(t) \leq y\} \text{ is non-increasing in both } x_1 \text{ and } x_2 \text{ for all } t \geq 0 \text{ and } y \in \Sigma.
\]

The process \( U \) satisfying (3.1)-(3.5) is non-anticipating with respect to \( W \). Thus \( Z \) is a continuous semimartingale, using the filtration generated by \( W \), and we can develop its analytical theory using the multidimensional Itô Formula. For functions \( f : \mathbb{R}^2 \to \mathbb{R} \) that are twice continuously differentiable, we define the differential operators

\[
\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2},
\]

\[
D_0 \equiv -\frac{\partial}{\partial x_1}, \quad D_1 \equiv \frac{\partial}{\partial x_1} - a \frac{\partial}{\partial x_2} \quad \text{and} \quad D_2 \equiv \frac{\partial}{\partial x_2}.
\]

Defining the boundary surfaces

\[
\Sigma_0 \equiv \{(x_1,x_2) \in \Sigma : x_1 = 1\},
\]

\[
\Sigma_1 \equiv \{(x_1,x_2) \in \Sigma : x_1 = 0\},
\]

and

\[
\Sigma_2 \equiv \{(x_1,x_2) \in \Sigma : x_2 = 0\},
\]

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we observe that \( D_k f \) is the directional derivative of \( f \) in the direction of reflection associated with \( \Sigma_k \) (see Figure 2). The following proposition is virtually identical to Theorem 2 of [3], so we shall not prove it.

\[(3.9) \text{ Proposition. If } f : \mathbb{R}^2 \to \mathbb{R} \text{ is twice continuously differentiable, then for all } \lambda \text{ and } t \geq 0 \]

\[
e^{\lambda t} f(Z(t)) = f(Z(0)) + \sum_{k=1}^{2} \int_0^t e^{\lambda s} \frac{\partial}{\partial x_k} f(Z(s)) \, dW_k(s) \]

\[+ \int_0^t e^{\lambda s}(\lambda + \frac{1}{2} \Delta) f(Z(s)) \, ds \]

\[+ \sum_{k=0}^{2} \int_0^t e^{\lambda s} D_k f(Z(s)) \, dU_k(s). \]

Here the integrals involving \( dW_k \) are of Ito type, while those involving \( dU_k \) are defined path by path in the Riemann-Stieltjes sense.

\[(3.10) \text{ Proposition. } E_x[U_0(t)] \sim \frac{1}{2} t, E_x[U_1(t)] \sim \frac{1}{2} t, \text{ and } E_x[U_2(t)] \sim \frac{3}{2} t \text{ as } t \to \infty (x \in \Sigma).\]

**Proof.** Set \( \lambda = 0 \) and \( f(x_1, x_2) = x_1 \) in (3.9). Then \((\lambda + \Delta)f = 0, -D_0 f = D_1 f = 1, \text{ and } D_2 f = 0. \) Moreover, the Ito integrals have zero expectation, because their integrands are bounded. Thus, taking \( E_x \) of both sides, we obtain

\[(3.11) \quad E_x[Z_1(t)] = x_1 + E_x[U_1(t)] - E_x[U_0(t)]. \]

Similarly, by taking \( \lambda = 0 \) and \( f(x) = x_2 \) we have
(3.12) \[ E_x[Z_2(t)] = x_2 - aE_x[U_1(t)] + E_x[U_2(t)], \]

and the choice \( \lambda = 0 \) and \( f(x) = (x_1)^2 \) yields

(3.13) \[ E_x[(Z_1(t))^2] = (x_1)^2 + t - 2E_x[\int_0^t Z_1(s) \, dU_0(s)] \]

\[ + 2E_x[\int_0^t Z_1(s) \, dU_1(s)] = (x_1)^2 + t - 2E_x[U_0(t)]. \]

For the latter equality in (3.12), we use the fact that \( U_0 \) increases only when \( Z_1 = 1 \), and \( U_1 \) increases only when \( Z_1 = 0 \), by (3.5).

Dividing both (3.11) and (3.13) by \( t \) and letting \( t \to \infty \), we obtain \( E_x[U_0(t)] \sim \frac{1}{2} t \) and \( E_x[U_1(t)] \sim \frac{1}{2} t \) as desired, because \( Z_1(t) \) is bounded. If it can be shown that

(3.14) \[ \frac{1}{t} E_x[Z_2(t)] \to 0 \quad \text{as} \quad t \to \infty, \]

the proof will be completed by dividing (3.11) by \( t \) and letting \( t \to \infty \).

To establish (3.14), first observe that (3.6) implies

(3.15) \[ U_2(t) \leq \sup_{0 \leq s \leq t} [W_2(s)]^+ + \sup_{0 \leq s \leq t} [aU_1(s)] \]

\[ = aU_1(t) + \sup_{0 \leq s \leq t} [W_2(s)]^+. \]

and hence (3.3) gives

(3.16) \[ Z_2(t) \leq W_2(t) + \sup_{0 \leq s \leq t} [W_2(s)]^+. \]
Taking $E_X$ of both sides in (3.16), dividing by $t$ and letting $t \to \infty$ gives (3.14), thus completing the proof.

(3.17) **Proposition.** Let $\lambda > 0$, $\alpha \in (0, \pi/2)$ and $\beta > 0$ be constants, let $f(x) = \cos(\alpha(1-x_1)) \exp(\beta x_2)$, and define $M(t) = \exp(\lambda t) f(Z(t))$, $t \geq 0$. If the constants $\lambda$, $\alpha$, $\beta$ simultaneously satisfy

(3.18) \[ \lambda + \frac{1}{2} (\beta^2 - \alpha^2) = 0 , \]

(3.19) \[ \alpha \sin \alpha - a \beta \cos \alpha = 0 , \]

then the stopped process $\{M(t \wedge T), t \geq 0\}$ is a supermartingale, where

(3.20) \[ T = \inf\{t \geq 0 : Z_2(t) = 0\} . \]

**Remark.** The stopped process is in fact a martingale, but we shall only need the weaker property, and it is slightly easier to prove.

**Proof.** First observe that

(3.21) \[ D_0 f(1, x_2) = 0 , \]

regardless of how $\lambda$, $\alpha$ and $\beta$ are chosen, and that (3.18) and (3.19) imply

(3.22) \[ (\lambda + \frac{1}{2} \lambda) f(x_1, x_2) = 0 , \]

and

(3.23) \[ D_1 f(0, x_2) = 0 \]

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respectively. Combining (3.21)-(3.23) with the fact that \( U_2(T) = 0 \), we see that all terms on the right side of (3.9) except for the Ito integrals are zero, provided \( t \leq T \). Thus \( \{ M(t \wedge T) \} \) is a local martingale. But \( f \) is a positive function, hence \( M \) is a positive process, and a positive local martingale is a supermartingale (Fatou's Lemma). This completes the proof.

(3.24) Proposition. \( E_X(T) < \infty \) for all \( x \in \Sigma \), where (3.20) defines \( T \).

Proof. First we show that there exist constants \( \lambda > 0, \alpha \in (0, \pi/2) \) and \( \beta > 0 \) satisfying (3.18) and (3.19). Obviously (3.19) is equivalent to

(3.25) \[ \beta = \frac{\alpha}{a} \tan \alpha. \]

Substituting this into (3.19), we arrive at the requirement that

(3.26) \[ \phi(\alpha) \equiv \alpha^2 (1 - a^{-2} \tan^2 \alpha) = 2\lambda. \]

It can be verified that \( \phi(0) = 0, \phi'(0) > 0 \), and \( \phi(\alpha) \to -\infty \) as \( \alpha \to \pi/2 \), so there exists \( \alpha \in (0, \pi/2) \) satisfying (3.26) if \( \lambda > 0 \) is chosen small enough. Choosing \( \beta \) according to (3.25), we then have a triple \( (\lambda, \alpha, \beta) \) satisfying the hypotheses of Proposition (3.17). Thus \( M(t \wedge T) \) is a supermartingale, meaning that \( E_X[M(0)] \geq E[M(t \wedge T)] \), which is equivalent to

(3.27) \[ f(x) \geq E_X[e^{\lambda(t \wedge T)} f(Z(t \wedge T))] \geq E_X[e^{\lambda(t \wedge T)}] \cos \alpha. \]

Letting \( t \to \infty \) in (3.27) gives
(3.28) \[ E_x(e^{\lambda T}) \leq f(x)/\cos\alpha \]

which of course implies \( E_x(T) < \infty \) as desired.

4. The Analytical Problem

Hereafter it will be assumed that \( W(0) = Z(0) = 0 \), and the symbols \( P(\cdot) \) and \( E(\cdot) \) will be understood to mean \( P_0(\cdot) \) and \( E_0(\cdot) \) respectively. Similarly, recalling the notation \( F_t(x,y) \equiv P_x(Z(t) \leq y) \), let us agree to write

\[ (4.1) \quad F_t(y) = P(Z(t) \leq y) = F_t(0,y). \]

Because the origin is the least element of our strip \( \Sigma \) (under the usual partial ordering), it follows from the monotonicity property (3.8) that \( Z(t) \) is stochastically increasing in \( t \), meaning that \( F_t(y) \) is a non-increasing function of \( t \) for each \( y \in \Sigma \). Consequently,

\[ (4.2) \quad F(y) = \lim_{t \to \infty} F_t(y) \]

exists, and it follows from Proposition (3.24) that \( F \) must furthermore be a non-defective distribution function. Our objective is to compute the marginal distribution

\[ (4.3) \quad G(\xi) \equiv F(1,\xi) = \lim_{t \to \infty} P(Z_2(t) \leq \xi), \quad \xi \geq 0. \]

Incidentally, there is an elegant coupling argument which shows that \( F_t(x,y) \to F(y) \) as \( t \to \infty \) for all starting states \( x \in \Sigma \), but we shall have no need for this fact. (The key observation is that if two sample
paths of $W$ have different starting states but identical increments, then the corresponding paths of $Z$ will intersect at some finite time $T$ and be identical thereafter.

We now want to derive an analytical characterization of the steady-state distribution $F$. A useful first step is to define the functions

\begin{equation}
H_t(u) = \frac{2}{t} E[\int_0^t 1_{Z_2(s) \leq u} \, dU_1(s)] , \quad t \geq 0 .
\end{equation}

Obviously, $H_t(*)$ is increasing with $H_t(\infty) = 2E[U_1(t)]/t$, which one can easily show to be finite. Proposition (3.10) shows that $H_t(\infty) \to 1$ as $t \to \infty$.

\begin{equation}
\text{(4.5) Proposition.} \quad H_t \Rightarrow H \text{ as } t \to \infty, \text{ where } H \text{ is a distribution function on } [0,\infty) \text{ with}
\end{equation}

\begin{equation}
\int_0^\infty u \, H(du) = \int_0^\infty [1 - H(u)] \, du = \frac{1}{a} .
\end{equation}

Furthermore,

\begin{equation}
G(\xi) = a \int_0^\xi [1 - H(u)] du .
\end{equation}

**Proof.** It will be useful to define the distribution functions

\begin{equation}
G_t(\xi) \equiv P(Z_2(t) \leq \xi) = F_t(1, \xi) , \quad \xi \geq 0 .
\end{equation}
Let $\alpha > 0$ be arbitrary, put into (3.9) the test function $f(x_1, x_2) = \exp(-\alpha x_2)$, and take $E(\cdot)$ of both sides. The Ito integrals have zero expectation (integrands are bounded), $D_0f = 0$, $D_1f = a\alpha f$, $D_2f(x_1, 0) = -\alpha$, and $\Delta f = \alpha^2 f$, so we have

\begin{align*}
(4.9) \quad E[e^{-\alpha Z_2(t)}] &= 1 + \frac{1}{2} \alpha^2 E[\int_0^t e^{-\alpha Z_2(s)} ds] \\
&\quad + \alpha E[\int_0^t e^{-\alpha Z_2(s)} dU_1(s)] - \alpha E[U_2(t)].
\end{align*}

Set $G^*_t(\alpha) \equiv \int_{[0, \infty]} e^{-\alpha \xi} G_t(d\xi)$, and let other Laplace transforms be denoted similarly. Since $G_t \Rightarrow G$, the continuity theorem gives

\begin{align*}
(4.10) \quad \frac{1}{t} E[\int_0^t e^{-\alpha Z_2(s)} ds] &= \frac{1}{t} \int_0^t E[e^{-\alpha Z_2(s)}] ds \\
&= \frac{1}{t} \int_0^t G^*_s(\alpha) ds \to G^*(\alpha) \quad \text{as } t \to \infty.
\end{align*}

Of course

\begin{align*}
(4.11) \quad \frac{1}{t} E[U_2(t)] + \frac{1}{2} a &= \text{as } t \to \infty
\end{align*}

by (3.10), and a monotone class argument gives

\begin{align*}
(4.12) \quad \frac{1}{t} E[\int_0^t e^{-\alpha Z_2(s)} dU_1(s)] &= \frac{1}{2} \int_{[0, \infty]} e^{-\alpha \xi} H_t(d\xi) = \frac{1}{2} H^*_t(\alpha).
\end{align*}

Dividing (4.9) through by $t$, letting $t \to \infty$, and using (4.10)-(4.13), we obtain $H^*_t(\alpha) + 1 - \alpha G^*(\alpha)/a$ as $t \to \infty$. Thus $H_t \Rightarrow H$, where $H$ is an increasing function whose Laplace transform is related to that of $G$ via
\[(4.13) \quad G^*(\alpha) = \frac{3}{\alpha} \left[ 1 - H^*(\alpha) \right].\]

Since \(G\) is a distribution function it follows from (4.13) that \(H\) is also a distribution function with mean given by (4.6). Finally, (4.13) is the transform version of (4.7), so the proof is complete.

For the next proposition, we want to insert in the Ito Formula (3.9) test functions \(f : \Sigma \to \mathbb{R}\) such that

\[(4.14) \quad f \text{ is twice continuously differentiable, with bounded first and second-order partials on } \Sigma, \text{ and } D_0f(1,x_2) = D_2f(x_1,0) = 0.\]

\[(4.15) \quad \text{Proposition. If } f \text{ satisfies (4.14) then}\]

\[(4.16) \quad 0 = \int_{\Sigma} \Delta f(x) F(dx) + \int_{[0,\infty)} D_1 f(0,u) H(du).\]

\textbf{Proof.} Set \(\lambda = 0\) in (3.9) and take \(E(\cdot)\) of both sides. The Ito integrals have zero expectation (bounded integrands), and \(D_0f(1,x_2) = D_2f(x_1,0) = 0\) by hypothesis, so we obtain
(4.17) \[ E[f(Z(t))] = f(0) + E\left[ \int_0^t \frac{1}{2} \Delta f(Z(s)) \, ds \right] \]
\[ + E\left[ \int_0^t D_1 f(Z(s)) \, dU_1(s) \right] \]
\[ = f(0) + \frac{1}{2} \int_0^t \left[ \int_\Sigma \Delta f(x) \, F_t(dx) \right] \]
\[ + \frac{1}{2} t \int_{[0,\infty)} D_1 f(0,u) \, H_t(du) . \]

Recall that \( F_t \rightarrow F \) and \( H_t \rightarrow H \) as \( t \to \infty \). Using the boundedness and continuity assumptions of (4.14), we can then divide (4.17) by \( t \) and let \( t \to \infty \) to obtain (4.16), completing the proof.

It will be seen shortly that Proposition (4.15) is sufficient for the determination of \( H \), so we will not prove more. There may be some value, however, in stating a more general form of (4.15) that treats the three boundary surfaces of \( \Sigma \) symmetrically. Let \( f \) be a test function satisfying all the requirements of (4.14) except that its normal derivatives along \( \Sigma_0 \) and \( \Sigma_2 \) are arbitrary. It can be shown that

(4.18) \[ 0 = \int_\Sigma \frac{1}{2} \Delta f(x) \, F(dx) + \sum_{k=0}^2 \int_\Sigma D_k f(x) \, v_k(dx) , \]

where \( v_0, v_1 \) and \( v_2 \) are finite, positive measures satisfying (defined by)

(4.19) \[ \frac{1}{t} E\left[ \int_0^t 1_A(Z(s)) \, dU_k(s) \right] + v_k(A) \quad \text{as } t \to \infty . \]
Since $\nu_k$ concentrates all of its mass on $\Sigma_k$ by (3.5), we can rewrite (4.18) as

\begin{equation}
0 = \int \frac{1}{2} \Delta f(x) F(dx) + \sum_{k=0}^{2} \int_{\Sigma_k} D_k f(\sigma) \nu_k(d\sigma),
\end{equation}

and (3.10) shows that $\nu_0(\Sigma_0) = \nu_1(\Sigma_1) = 1/2$ and $\nu_2(\Sigma_2) = a/2$. Obviously, (4.15) specializes (4.20) to test functions $f$ of the class (4.14), with $H(u) \equiv 2\nu_1[0,u]$. For more on adjoint relations like (4.20) see §9 of [4].

5. The Steady-State Distribution

By putting a well chosen test function $f$ into the adjoint relation (4.15), we can determine the steady-state boundary distribution $H$ defined by (4.4)-(4.5). Let

\begin{equation}
\phi^+(z) \equiv \int_{[0,\infty)} e^{izu} H(du), \quad \text{Im}(z) \geq 0,
\end{equation}

\begin{equation}
\phi^-(z) \equiv \int_{[0,\infty)} e^{-izu} H(du), \quad \text{Im}(z) \geq 0,
\end{equation}

(5.3) **Proposition.** $\phi^+$ and $\phi^-$ are bounded, analytic in the upper half-plane and lower-half plane respectively, and satisfy
\begin{equation}
\frac{\phi^-(z)}{\phi^+(z)} = \frac{a \cosh z - i \sinh z}{a \cosh z + i \sinh z} \quad \text{for } z \text{ real}.
\end{equation}

**Proof.** The boundedness and analyticity follow immediately from the definitions (5.1), (5.2). To prove (5.4), let $z \in \mathbb{R}$ be arbitrary and define

$$f(x) \cosh ix_2 \cosh z(1-x_1), \quad \text{for } x \in \Sigma.$$ 

It is easy to verify that $f$ satisfies all the restrictions of (4.14), plus $\Delta f = 0$, so (4.15) gives us

\begin{equation}
0 = \int_{[0, \tau)} D_1 f(0, u) H(du).
\end{equation}

(This is actually a pair of identities, one for the real part of $D_1 f$ and one for the imaginary part, but we shall continue to use the efficient notation of complex variables.) Now direct computation gives

\begin{equation}
D_1 f(0, u) = \left( \frac{\partial}{\partial x_1} - a \frac{\partial}{\partial x_2} \right) f(0, u)
\end{equation}

$$= - z \sinh z \cosh iz u - aiz \cosh z \sinh iz u$$

$$= - \frac{1}{2} z[(\sinh z + a \cosh z) e^{iz u} + (\sinh z - a \cosh z) e^{-iz u}]$$

$$= \frac{1}{2} iz[(a \cosh z + i \sinh z) e^{-iz u} - (a \cosh z - i \sinh z) e^{iz u}].$$

Substituting (5.6) into (5.5) and dividing in the obvious way gives the desired identity (5.4).
It will be shown shortly that Proposition (5.3) actually gives a unique characterization of \( \phi^+ \) and \( \phi^- \). Anticipating the eventual solution, let

\[
(5.7) \quad \phi^+(z) = \int_0^\infty e^{izu} p(u) \, du, \quad \text{Im}(z) \geq 0,
\]

\[
(5.8) \quad \phi^-(z) = \int_0^\infty e^{-izu} p(u) \, du, \quad \text{Im}(z) \leq 0,
\]

where \( p \) is defined by (1.3). Using the change of variable \( v = \exp(-\pi u) \), one finds that

\[
(5.9) \quad \psi^+(z) = \int_0^\infty e^{izu} e^{-\theta u} (1 - e^{-\pi u})^{-2\theta/\pi} \, du
\]

\[
= \int_0^1 (1-v)^{\alpha-1} v^\beta \frac{1}{\pi v} \, dv
\]

\[
= \frac{1}{\pi} \int_0^1 (1-v)^{\alpha-1} v^{\beta-1} \, dv \equiv \frac{1}{\pi} B(\alpha, \beta),
\]

where \( \alpha \equiv (\pi-2\theta)/\pi \), \( \beta \equiv (\theta-iz)/\pi \) and \( B(\cdot, \cdot) \) is the beta function.

Using the identity \( B(\alpha, \beta) = \Gamma(\alpha) \Gamma(\beta)/\Gamma(\alpha+\beta) \), we can express (5.9) in the final form

\[
(5.10) \quad \psi^+(z) = \Gamma(1 - \frac{2\theta}{\pi}) \Gamma(\frac{\theta-iz}{\pi})/\pi \Gamma(1 - \frac{\theta+iz}{\pi}).
\]

Similar calculations give

\[
(5.11) \quad \psi^-(z) = \Gamma(1 - \frac{2\theta}{\pi}) \Gamma(\frac{\theta+iz}{\pi})/\pi \Gamma(1 - \frac{-\theta-iz}{\pi}).
\]
Using the identity \( \Gamma(\alpha) \Gamma(1-\alpha) = \pi/\sin(\pi\alpha) \), we divide (5.11) by (5.10) to obtain

\[
(5.12) \quad \frac{\phi^-(z)}{\phi^+(z)} = \frac{\sin(\theta-iz)}{\sin(\theta+iz)}.
\]

Expanding the sines on the right side of (5.12), we find that (5.4) holds with \((\phi^+, \phi^-)\) in place of \((\phi^+, \phi^-)\). Furthermore, \(p\) is positive and integrable on \((0, \infty)\), so \((\phi^+, \phi^-)\) have all the properties ascribed to \((\phi^+, \phi^-)\) in Proposition (5.3).

\[
(5.13) \quad \text{Proposition. } \phi^+(z) = c\phi^+(z), \text{ where } c \text{ is defined by (1.2). Thus } H(du) = cp(u)du.
\]

Proof. Let \(k(z) = \phi^+(z)/\psi^+(z)\) for \(\text{Im}(z) \geq 0\), \(k(z) = \phi^-(z)/\psi^-(z)\) for \(\text{Im}(z) \leq 0\). These two definitions agree for real \(z\) because \((\phi^+, \phi^-)\) and \((\phi^+, \phi^-)\) both satisfy (5.4). Recall that the \(\Gamma\) function has no zeros, and its singularities are poles on the negative real axis, cf. 6.1.3 of Abramowitz and Stegun [1]. From this and the analyticity of \(\phi^+, \phi^-, \phi^+\), \(\phi^-\) on their respective half planes, it follows that \(k(z)\) is an entire function of \(z\). Further,

\[
(5.14) \quad \Gamma(z+a) z^{b-a}/\Gamma(z+b) \to 1
\]

as \(|z| \to \infty, z \neq -a-n\) or \(-b-n\) \((n = 0, 1, \ldots)\), cf. 6.1.4 of Abramowitz and Stegun [1]. From (5.10) and (5.11) we then have

\[
(5.15) \quad |\phi^+(z)| \geq \gamma |z|^{1-2\theta}/\pi \quad \text{as } |z| \to \infty, \text{Im}(z) \geq 0,
\]
(5.16) \[ |\phi^-(z)| \geq \gamma |z|^{1-2\theta/\pi} \quad \text{as} \quad |z| \to \infty, \text{Im}(z) \leq 0, \]

for some constant \( \gamma > 0 \). Since \( \phi^+ \) and \( \phi^- \) are bounded in their respective half-planes, it follows that

(5.17) \[ |k(z)| \leq A |z|^{-\left(1-2\theta/\pi\right)} \quad \text{as} \quad |z| \to \infty, \]

for some constant \( A > 0 \). Since \( 0 < 1-2\theta/\pi < 1 \), it follows from (5.17) and Liouville's Theorem that such an entire function \( k(z) \) is a constant. So \( \phi^+(z) = c\psi^+(z) \) for some \( c > 0 \). Of course \( \phi^+(0) = 1 \) by definition (\( \psi^+ \) is the characteristic function of a probability distribution), so it must be that \( c = 1/\phi^+(0) \), so (5.10) shows that \( c \) is given by (1.2) as claimed. This proves the first statement of the proposition, and the second follows from the continuity theorem for characteristic functions.

The preceding argument gives no hint of the means originally used to calculate \( p \), which was roughly the following. First the right side of (5.4) was factored into products of zeros, using the Hadamard product formulas for the numerator and denominator individually. The individual factors were allocated to \( \phi^+ \) and \( \phi^- \) so as to insure analyticity in the upper half-plane and lower half-plane respectively, and then the Euler product for \( \Gamma \) was used to obtain (5.10) and (5.11). Finally, a partial fraction decomposition and sum revealed that \( \phi^+ \) is the characteristic function of the density \( p \) defined by (1.3).

To obtain the final formula for \( G \) stated earlier as Theorem (1.4), one simply combines (5.13) and (4.7). The equivalent Laplace transform solution (1.5) is obtained by combining (4.13), (5.13) and (5.10).
6. Another Approach to the Key Transform Relation

Our starting point in the previous section was equation (5.4), connecting \( \phi^+(z) \) and \( \phi^-(z) \). To conclude the paper, we give an alternative, heuristic derivation of the key transform relation (5.4). This derivation would be difficult to make rigorous, but it helps one develop an intuition for the diffusion process under study.

Consider a two-dimensional process \( \{X(t), Y(t); t \geq 0\} \) that lives in the infinite strip \(|Y| \leq 1\), pictured in Figure 3 below. On the interior of this strip, the process \((X,Y)\) behaves like standard Brownian motion,

![Diagram](image)

Figure 3. Boundary Behavior of the Process \((X,Y)\)
and it jumps upon hitting the boundary. If the boundary is hit at a point \((x, 1)\), then there is an instantaneous jump to the point \((x - \alpha \text{ sgn } x, 1 - \varepsilon)\), after which the Brownian movement resumes. If the boundary is hit at \((x, -1)\), then the jump is to \((x - \alpha \text{ sgn } x, -1 + \varepsilon)\). Let \(T_n\) be the nth time at which the boundary is hit, and set \(X_n \equiv X(T_n)\). We claim that \(\{X_n\}\) is a Markov chain, that it has a unique stationary distribution, and that the stationary distribution of \(|X_n|\) converges weakly to the boundary distribution \(H\) of §4 as \(\varepsilon \to 0\). We shall make no attempt to justify any of these claims, hoping that the symmetry apparent in Figure 3 makes them at least plausible.

Hereafter assume that the initial state of \((X, Y)\) is distributed randomly over the boundary of the strip according to the stationary distribution of \(\{X_n\}\). Then the transition mechanism for our stationary Markov chain \(\{X_n\}\) can be expressed as

\[
X_{n+1} = X_n - \alpha \text{ sgn } X_n + \xi_{n+1},
\]

where \(\xi_{n+1}\) is the increment of Brownian movement (in the \(X\) direction) between consecutive hitings of the boundary. Defining

\[
\phi_\varepsilon(z) \equiv E[\exp(izX_n)],
\]
(6.3) \[ \phi_\varepsilon(z) \equiv E[\exp(iz\xi_{n+1})], \]

we have from (6.1) that

(6.4) \[ \phi_\varepsilon(z) = \phi_\varepsilon(z) E[\exp[iz(X_n - a\varepsilon \text{ sgn } X_n)]] , \]

because \( \{X_n\} \) is stationary and \( \xi_{n+1} \) is independent of \( X_n \) (using the spatial homogeneity of Brownian motion). Next, using the fact that

\[ M(t) \equiv \exp\{izX(t) + zY(t)\}, \quad t \geq 0, \]

is a martingale for arbitrary complex \( z \), one can easily derive the formula

(6.5) \[ \phi_\varepsilon(z) = \cosh(z(1-\varepsilon))/\cosh z. \]

Now defining

(6.6) \[ \phi_\varepsilon^+(z) = E[\exp(iz|X_n|)] , \]

(6.7) \[ \phi_\varepsilon^-(z) = E[\exp(iz|X_n|)] , \]

symmetry and (6.2) give us

(6.8) \[ \phi_\varepsilon(z) = \frac{1}{2} \phi_\varepsilon^+(z) + \frac{1}{2} \phi_\varepsilon^-(z) \]

(6.9) \[ E[\exp[iz(X_n - a\varepsilon \text{ sgn } X_n)]] \]

\[ = \frac{1}{2} \phi_\varepsilon^+(z) e^{-iza\varepsilon} + \frac{1}{2} \phi_\varepsilon^-(z) e^{iza\varepsilon} . \]

Substituting (6.5), (6.8) and (6.9) into (6.4) and simplifying, we obtain
\begin{equation}
\frac{\phi^-_\varepsilon(z)}{\phi^+_\varepsilon(z)} = \frac{\cosh z - \exp(-i\varepsilon z) \cosh(z(1-\varepsilon))}{\cosh z - \exp(i\varepsilon z) \cosh(z(1-\varepsilon))}.
\end{equation}

Let \( \phi^+(z) \) and \( \phi^-(z) \) be defined in terms of \( H \) as in §5. If the stationary distribution of \( |X_n| \) converges weakly to \( H \) as \( \varepsilon \to 0 \), as claimed earlier, then of course \( \phi^+_\varepsilon(z) \to \phi^+(z) \) and \( \phi^-_\varepsilon(z) \to \phi^-(z) \) as \( \varepsilon \to 0 \). Assuming the weak convergence, we may then let \( \varepsilon \to 0 \) in (6.10) to obtain (5.4).
References


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**Authors:**
J. Michael Harrison and L. A. Shepp

**Performing Organization:**
DEPT'S OF OPERATIONS RESEARCH AND STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305

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**ABSTRACT:**
See next page.
We consider a two-dimensional diffusion process \( Z(t) = [Z_1(t), Z_2(t)] \) that lives in the half strip \( \{0 \leq Z_1 \leq 1, 0 \leq Z_2 < \infty\} \). On the interior of this state space, \( Z \) behaves like a standard Brownian motion (independent components with zero drift and unit variance), and there is instantaneous reflection at the boundary. The reflection is in a direction normal to the boundary at \( Z_1 = 1 \) and \( Z_2 = 0 \), but at \( Z_1 = 0 \) the reflection is at an angle \( \theta \) below the normal \( (0 < \theta < \pi/2) \).

This process \( Z \) is shown to arise as the diffusion limit of a certain tandem storage or queueing system. It is shown that \( Z(t) \) has a non-defective limit distribution \( F \) as \( t \to \infty \), and the marginal distributions of \( F \) are computed explicitly. The marginal limit distribution for \( Z_1 \) is uniform (this result is essentially trivial), but that for \( Z_2 \) is much more complicated.