APPLIED MATHEMATICS AND STATISTICS LABORATORY

STANFORD UNIVERSITY
CALIFORNIA

STATISTICAL PRODUCTION MODELS AND CONTINUOUS SAMPLING PLANS

By
I. RICHARD SAVAGE

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GERALD J. LIEBERMAN, PROJECT DIRECTOR
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I. RICHARD SAVAGE

Abstract: Several plausible models for a production process are presented. Under these models it is possible to evaluate the operating characteristics of a certain type of continuous sampling plan. The algebraic forms of the operating characteristics although complicated are amenable to numerical analysis. Several detailed examples are given.

0. Introduction and Summary: The purpose of this report is to consider the operating characteristics of a particular type of continuous sampling plan for a particular type of production model. The approach differs from most work in this field in the following respects:

1. The continuous sampling plan incorporates an explicit rule for the stopping of production and looking for trouble.

2. The plan is assessed not through its AOQ function but through the income that results by application of the plan.

3. The production process is not considered to be in a state of statistical control. Instead it is assumed that quality will degenerate in a specified manner unless corrective action is taken.

In this section we will give an outline of the contents of the remainder of the report. Rather than summarizing the report we present one of the more simple cases.

Consider then the following set-up:

Sampling Plan: Every $i^{th}$ item produced is inspected. If the item is found good, production is continued, and if it is found bad, production is
stopped and trouble if present is removed. The time required for trouble shooting is $A$ units and the cost of trouble shooting is $B$ units.

**Production Process States:** If units of time after trouble has been removed, the state of the production process is given by $z(t)$, $z(t)$ is a chance quantity and it is assumed to have a Poisson distribution with parameter $\Delta t$. Large values of $z(t)$ will tend to produce poor items.

**Quality Process:** If the production process state is $z(t)$, then the probability that the item produced at time $t$ is good is $R^{z(t)}$. The constant $R$ measures the rate of decay of the quality. It is assumed that $0 < R < 1$. Thus soon after trouble is removed the items produced will have a high probability of being good and as time goes on this probability will decrease.

**Costs:** The value of a good item is one unit and the value of a bad item is zero. The inspection process is destructive and the cost of inspecting a single item is $c$ units.

Under the above assumptions the average income per unit of time, $L$, is given by the formula

$$L = \frac{e^{FA(1-R)} [E(i*) - 1] - B - cE(i*)}{FE(i*) + A}$$

where

$$E(i*) = \sum_{j=0}^{\infty} e^{-jAF} e^{AF(1-R^j)/(1-R)}$$

and the average outgoing quality, AOQ, is given by the formula
\[ \text{AOQ} = 1 - \left( \frac{\text{FA}(1-R)}{\text{FA}(1-R)} - 1 \right) \left( 1 - \frac{1}{\text{E}(1^*)} \right) \]

The two parameters that must be estimated before choosing a plan, \( F \) value, are \( \Delta \) and \( R \). When \( \Delta \) and \( R \) have been evaluated we can choose \( F \) in order to maximize \( L \). If \( \Delta \) and \( R \) cannot be found exactly we can explore the consequences of using various possible values for \( F \) by plotting \( L \) as a function of \( F \) for several combinations of the parameters. In this way a reasonable plan can be arrived at.

In the course of this report the above assumptions are given in a more general form. Consideration is given as to the reasonableness of the assumptions. In particular it is pointed out that although the plans considered are simple they have much in common with most of the plans now in use. It is also shown that the assumptions for the production process states and the quality states have many of the properties needed to describe these phenomenon adequately. In Section 6 several examples are given in detail.

1. Preliminary Remarks: The number of continuous sampling plans that have been proposed is large [see references]. This wealth of plans makes it possible for the quality control engineer to select a plan that will have many of the properties desired in a particular situation.

There are several reasons for using continuous sampling plans. All of the proposed plans insure a desired AOQL under specified conditions. In part this is accomplished by the screening effects of the plans. While using the plans the inspector is kept informed on the state of the production process. This allows him to search for trouble either by examining the process in operation, or in extreme cases, to shut down the process
and remove sources of trouble. Most of the work that has been done on continuous sampling plans has been restricted to the case where the production process is in a state of control. Also most of the plans have not been formulated with a clear rule as to when trouble shooting should be done. In this paper we consider several models (more general than being in a state of control) with explicit rules for the inspection process as well as when to look for trouble.

Lieberman [9] has considered the results of lack of control on the operating characteristics of a plan proposed by Dodge in [2]. The production process that he considers is the one that leads to the poorest results, i.e., gives the largest AOQL. His results are concerned only with the screening effects of the plans and do not touch on the trouble shooting aspects.

In [7] Girshick and Rubin consider a particular type of production process and then formulate a plan that will give the best possible operating characteristics among all possible plans. Their plans have a screening effect. Further, their plan has a trouble shooting rule so that the process will tend to turn out better material when the plan is used than when it is not used.

A difficulty with the Girshick-Rubin plan is that it is very hard to arrive at the numerical results required to put this plan into operation with known operating characteristics. So far very limited results have been obtained and hence these plans have not been used.

In [13] Savage discusses a plan that explicitly takes into account the trouble hunting nature of continuous sampling plans. For this plan the operating characteristics have been found only when the production process
is in a state of statistical control.

Currently, Mr. Geoffrey Gregory of the Applied Mathematics and Statistics Laboratory, Stanford University, is working on sampling plans that are much like some that have already been proposed, but with the additional feature that at some point in the cycle of the inspection plan there is an explicit order for the process to be stopped, examined, and brought into the proper state of statistical control. The model that Mr. Gregory is using for this work is that proposed by Girshick and Rubin [7]. Explicitly the model is:
if the process is in the good state at a particular time, then the probability that the next item will be produced while the process is still in the good state is $1 - g$ and the probability that the next item will be produced in the bad state is $g$. Further, the process will stay in the bad state forever unless corrective action is taken.

Mr. Gregory's study treats then a very simple production process, but on the other hand it is possible to analyze fairly complicated continuous sampling plans under this model. In this paper we shall consider only simple continuous sampling plans, but the production models that we use will be substantially more realistic than the Girshick-Rubin model. Their model is not a special case of the one considered here. Although we only consider simple sampling inspection plans, an evaluation of the operating characteristics of more complicated plans might be done under these models if that were desired.

In Section 2, Stochastic Processes, we point out how the production model and the sampling inspection plan may be formulated as stochastic processes and discuss several possibilities for these choices. Then in Section 3, Realizations, we discuss how these models might arise in practice, and thus give a justification for their consideration. In Section 4, Fundamental
Quantities, the important quantities related to these stochastic processes are defined and formulae for their evaluation are presented. The derivations of these algebraic quantities are in Section 7. In Section 5, Income, we show how the expected values can be used for finding the costs and incomes from different sampling plans when different models are assumed. In Section 6, Examples, material is presented on the line of thought useful for suggesting which of the proposed plans would be best and the numerical analysis of the models for particular examples.

2. Stochastic Processes: In all of the models about to be considered there are three stochastic processes required for the description of the combined production and inspection gestalt. Each of these stochastic processes will be a function of time.

I. The inspection process tells what the inspector (manager) is doing at each time. The states in which this process may be are:

   A. Produce an item and do not inspect it.
   B. Produce an item and inspect it.
   C. Do not produce an item.

II. At each instant of time there is a number $z(t)$ which describes the condition of the production process. $z(t)$ is a stochastic process. In general large values of $z(t)$ will be considered indicative of poor quality and small values will be associated with good quality. We assume that $z(t)$ is a non-decreasing function of $t$ unless the inspection process is in state (I.C.), i.e., the only way the quality can improve is to stop the production process.
III. The random variable $x(t)$ will equal one if the item produced at time $t$ is good and will equal zero if the item produced at time $t$ is bad. We assume the probability distribution of $x(t)$ at time $t$ depends only on the value of $z(t)$ at time $t$. Further, given $z(t)$, the joint distribution of $x(t)$ for several values of $t$ will be mutually independent.

In the above we have thought of the production process as producing items at regularly spaced intervals of time. For many applications this is a realistic assumption. But for theoretical purposes, also because it will fit many other production situations, we can think of the production process putting out a continuous stream of goods such as a machine producing wire or linoleum. In that case by saying that the item produced at a particular time is good we mean that the cross section at that particular time contains no defects. Depending on the particular model used, we will shift back and forth between these two possible frameworks of reference.

Now let us become more specific and describe in detail some of the basic assumptions we will make about the various stochastic processes involved in the gestalt.

We are going to restrict ourselves to a narrow class of sampling inspection plans. In these plans we stop production at the first sign of trouble, i.e., stop the first time that a defective item is inspected. In order to describe this process in more detail we need the technical term "cycle." By a cycle is meant the length of time, and all that happens during that time, between the starting of the production until the production (1) has been stopped by the sampling inspection plan, (2) has been repaired and (3) is ready to reconvene production. All of the stochastic processes are described in terms of cycles and we will now begin with the ideas for the
sampling inspection plan. We make the following basic assumption.

Assumption: The length of time between the beginning of production until the first item is inspected is a positive random variable, $I_1$. If the first item inspected is found to be bad the production process is stopped and the time to repair the process is a function only of $z(I_1)$. If the first item inspected is found to be good the length of time before the next item is inspected is a random variable, $I_2$, independent of the first waiting time but with the same distribution as the first waiting time. If the second item inspected is found to be bad the process is stopped and the time required for repair is a function of $z(I_1 + I_2)$. If the item inspected is found to be good the inspection process is continued in the obvious manner. We shall use $I_1, I_2, \ldots$, to denote the lengths of time between inspections, and $S_1 = \sum_{j=1}^{I_1} I_j$. The basic assumption is that these random variables are mutually independent and identically distributed. Further the time required to repair the process is a function only of $z(S)$ where $S$ is the time the process is stopped. It should be noted that the number of random variables $I_1$ which will appear in a cycle is a random variable.

We will now consider several sampling plans or alternatively several formulations for the distribution of $I_1$.

**Sampling Plan 1** (SP$_1$)

$$
Pr(I_1 = F) = 1
$$

In the case where we are thinking of the items being produced at regularly spaced intervals we must choose $F$ as a positive integer.
case $F = 1$ corresponds to one hundred percent inspection. In the case where we are thinking of a continuous stream of production, the only requirement is $F > 0$.

**Sampling Plan II.** (SP$_{II}$)

In the case of discrete production let $\Pr(I_1 = k) = (F-1)^{k-1}/F^k$ where it is assumed that $F$ is a number larger than 1. In this case the average distance between inspections is $F$. In the case of a continuous stream of production let $\Pr(I_1 < x) = 1 - e^{-x/F}$. Again, the average distance between inspections is $F$.

**Sampling Plan General.** (SP$_G$)

$$\Pr(I_1 \leq x) = D(x)$$

Where $D(x)$ is an arbitrary cumulative distribution function which satisfies $D(0^-) = 0$ and $\int_0^\infty xD(x) = F$.

The choice of the function of $z(S)$ describing the length of time to repair the machine and make it as good as new when the process is stopped in state $z(S)$ is not critical and we shall, for convenience, assume it has the form $A_1 + A_2 z(S)$ where $A_1$ and $A_2$ are non negative constants. The formula arises from the fixed time ($A_1$) to shut the process down plus a time proportional ($A_2$) to the wear $[z(S)]$ of the machine to do the repair work.

Regarding the stochastic process $z(t)$ which describes the state of the machine, we will make the following assumption:

**Assumption:** If the production process were allowed to go on uninterrupted, the process $z(t)$ would be a stationary process with independent, non-negative
increments. Or more specifically, if $t_1, t_2, \ldots, t_n$ is an increasing sequence of times, then the random variables $z(t_i) - z(t_{i-1})$ for $i = 1, 2, \ldots, n$ are mutually independent, assume only non-negative values and further the distribution of $z(t_i) - z(t_{i-1})$ depends on $t_i$ and $t_{i-1}$ only through $t_i - t_{i-1}$ . As a convention choose $t_0 = 0$ and $z(t_0) = 0$.

We will now consider several specific stochastic processes for the production process states.

**Production Process States I (PPS_I)**

The production process states is a Poisson stochastic process with parameter $\Delta$ . More specifically,

$$\Pr[z(t_i) - z(t_{i-1}) = k] = [\exp - \Delta(t_i - t_{i-1})][\Delta(t_i - t_{i-1})]^k/k!$$

The values of the differences of the $z$'s in disjoint time intervals are independently distributed.

**Production Process States General (PPS_G)**

The process $z(t)$ is an arbitrary stationary stochastic process with non-negative, independent increments. There are other examples besides the Poisson of such a process, i.e., gamma variables with the number of degrees of freedom depending on the time interval length. But these examples are rather awkward to deal with and not likely to occur in practice so we will not consider them in the following.

Finally, we must consider the stochastic process for quality of the item produced at time $t$. We consider only the following special case.
Quality Process:

For a fixed realization of the process \( z(t) \) the qualities (good or bad) of items produced at different times are mutually independent and

\[
\Pr[x(t) = 1] = \Pr^z(t) = 1 - \Pr[x(t) = 0]
\]

where \( x(t) = 1 \) means that the item produced at time \( t \) is good and \( x(t) = 0 \) means that the item produced at time \( t \) is bad.

3. Realizations: The stochastic processes introduced in the preceding section to describe various phases of the production process and sampling plans might at first appear artificial. Hence, it is desirable to give motivation for their consideration. First of all it should be made clear that the general forms that have been given for the production process states and for the quality process are such as to lead to simple mathematical results. But in the following notes we hope to make it clear that there were further, more important, reasons for making the particular choices.

Sampling inspection plans that call for a drastic change in behavior when the first sign of trouble occurs might appear to be not at all what is done in practice. A little thought will show that in fact nearly all of the plans proposed so far actually have this feature. In particular, in the use of quality control charts the standard procedure at the present time is to take radical steps whenever a point falls outside of control limits. This radical procedure involves the looking for trouble in the production process. For control charts the point out of control can either act as warning that there is trouble or actually be used as an indication that there is trouble present. If the latter interpretation is used by the quality control engineer, then the plan being proposed here is actually the one being used. Control charts and continuous sampling plans are very closely related in that
their use has the same effects on the production process and the output. For a detailed discussion of this aspect of control charts see Duncan [5].

In continuous sampling plans such as that of Dodge [2] radical action is taken whenever a defective is found. In particular, one hundred percent inspection is begun whenever a defect is found. One hundred percent inspection will in many cases call for trouble shooting and an increased inspection effort that might almost stop production.

The multi-level plans of Lieberman and Solomon [10] were designed to soften some of the radical changes caused by finding the first defect, but they still make abrupt changes whenever a defect has been found. The plans described in [1, 6, 11] have been further modifications of the original Dodge plan to smooth abrupt changes from the appearance of one defect but it still remains that the first defect causes some abrupt behavioral changes on the part of the inspectors. The plan of Savage [13] is another example of a plan where the first sign of trouble causes a radical change. Even under the Girshick-Rubin [7] plans one defect causes abrupt changes. Thus a survey of the plans being used indicates that all of them take notice of the first sign of trouble and hence the proposed plan, although apparently simple, might really be a close approximation to what is going on in the other plans.

Another reason why this plan is a close approximation to the Dodge plan is that the period of one hundred percent inspection for the Dodge plan may be thought of as a checking out time for the machine to see if it is back into the proper state. If the inspection is destructive, then the one hundred percent inspection period of the Dodge plan can hardly be considered as a weeding out period and the above interpretation becomes the only applicable one.
Now consider the method of finding the times to make inspections. In Dodge's paper [2] it is not clear just when the observations are made and in much of the subsequent writing this is not completely clarified. However, Lieberman [9] makes it clear that the production is broken into lots of size $F$ and then that one item selected at random from each lot is inspected. The plans that we propose are similar to this interpretation of the Dodge plan. Resnikoff [11] discusses the sampling methods in detail.

It is not desirable to inspect at regular intervals those situations where the unscrupulous workers on the production line could make sure that the work that was to be inspected would be good. Another advantage of the Dodge plan is that it insures there will never be an interval of length greater than $2F-2$ with no inspected units in it. In this manner the process will be under surveillance with no large gap in which it could go out of control without chance of detection.

In $SP_1$ we suggest that every $F^{th}$ item be inspected. It might just so happen that the process has some natural periodicities in it, in which case this is not a good inspection plan. Thus if $F$ is an even number and the produce is alternately coming from two different machines, then it would happen that only the production from one of the machines is inspected. Clearly, we could get into trouble if the percent defectives from these machines differ substantially. An advantage of the regular spacing of inspection is that there will never be more than $F-1$ consecutive items that go uninspected. In studies made for the Girshick-Rubin model to be reported by Mr. Gregory, it was shown that for some situations the sampling at regular intervals insures better results than the block sampling plan of Dodge. As long as the process is in a state of statistical control both the block and regular sampling plan give the same results. Under the block
sampling plan it is assumed all of the material in a block is passed on to the consumer even if the item inspected in the block was found to be defective. In all of the plans treated here all uninspected items that are produced are passed on to the consumers. This is in accordance with the original Dodge plan. It is assumed, however, that the inspection results are obtained instantaneously and are used in making decisions regarding the production of the next item or stopping the process.

Sampling Plan II has the desirable property that the times at which the observations are made are random. It does have the disadvantage that there might be substantial distances between observations, and hence an increase in the hazards from spotty production.

The choices of stochastic processes for describing the production process states has been made with the idea of giving processes that would be more flexible and more realistic than those considered by Girshick and Rubin. In their model the production process is at one of two levels and once it gets to the bad level it remains there indefinitely or until the process is stopped and has been returned to the good state. In the processes that we are considering here there are an infinite number of states and the process keeps deteriorating as time goes on. The process does not make abrupt changes from good to bad but instead change can be relatively slow. (The process under study and the Girshick-Rubin process coincide when the bad state of the latter consists of only producing bad items.) It is more realistic for things to keep getting worse as time goes on. It is a general phenomenon that in manufacturing the quality will change slowly and keep going towards a lower level if no corrective action is taken. Of course, if there is no corrective action, eventually the process will break down and fail to produce. This need not be a matter of concern for in every
plan used in practice the production would be stopped a long time before this has happened.

The choice of stationary processes with positive independent increments is of course a mathematical simplification and allows for easy computation of the necessary algebraic results. However, the proposed processes have a rational basis when thinking of models for the production process.

Positive increments, or more exactly non-negative increments, mean that the process goes in only one direction, i.e., from good to bad. We are not permitting the process to improve itself.

The assumption of stationary and independent increments is closely related to the facts: (1) the changes in short time intervals are small compared to what can happen in the long run and (2) the process will not be allowed to go to the bitter end, but will be stopped a long time before that occurs. The assumption that the increments are Poisson distributed is used since this is the simplest (conceptually and algebraically) process that has all of the properties postulated.

In the function for the quality process the parameter $P$ is the probability of producing a good item when the production process is working at its best level. (In many situations the parameter $P$ will be close to 1.) The function $R^z(t)$ is a decreasing function of the state of the process implying that as the process gets worse the quality decreases. The apparent generalization of the quality process function to $PR^A + Bz(t)$ is actually a special case of the function already chosen. The quantity $PR^A$ plays the role of $P$ in the original function. And $R^B$ plays the role of $R$ in the original function. The particular function chosen has the desirable properties: (1) at the beginning of time it is at its maximum, (2) if time goes on long enough it will approach zero, (3) if $R$ is chosen near one,
the probability of producing a good item will change slowly when considered as a function of \( z(t) \) or of \( t \).

4. **Fundamental Quantities:** In this section we present some of the fundamental random variables and their expected values required in finding the operating characteristics of the plans under the proposed models. The presentation of the derivations of the results of this section is in Section 7.

We define the random variable \( S_i = \sum_{j=1}^{i} I_j \). \( S_i \) is the elapsed time up to and including the time at which the \( i^{th} \) inspection is made. A fundamental random variable is \( S \) defined as the time at which the first defective item is found by inspection. We first give a general formula for \( \Pr(S > S_i) \) and then special cases of this formula for the particular modes of sampling and models under discussion. Nearly all of the other parameters can be found in terms of these probabilities. Thus we give only general results for the other expected values since we may substitute the results for these probabilities in studying special cases.

**Definition:** \( S \) is the time at which the first defective is found.

I. **FPS\(_G\)** and **SP\(_G\)**

\[
\Pr(S > S_i) = p^i \pi \sum_{j=0}^{\infty} \left( E[R \times (I_1^j)] \right) \text{ for } i = 0, 1, \ldots
\]

II. **FPS\(_I\)** and **SP\(_G\)**

\[
\Pr(S > S_i) = p^i \pi \sum_{j=0}^{\infty} \left( E[e^{-\Delta I_1(1-R_j)}] \right) \text{ for } i = 0, 1, \ldots
\]
III. $\text{PPS}_I$ and $\text{SP}_I$

$$\Pr(S > S_0) = 1$$

$$\Pr(S > S_i) = (P e^{-\Delta F})^i e^{\Delta R(1-R^i)/(1-R)} \quad \text{for } i = 1, 2, \ldots.$$ 

IV. $\text{PPS}_I$ and $\text{SP}_I$ (Discrete)

$$\Pr(S > S_0) = 1$$

$$\Pr(S > S_i) = \frac{(\frac{P}{F} e^{-\Delta})^i e^{\Delta R(1-R^i)/(1-R)}}{\pi \left[1-e^{-\Delta(1-R^i)}(1 - \frac{1}{F})\right]} \quad \text{for } i = 1, 2, \ldots.$$

V. $\text{PPS}_I$ and $\text{SP}_I$ (Continuous)

$$\Pr(S > S_i) = \frac{P^i}{\pi} \frac{1}{[1 + \Delta F(1-R^j)]^{-1}} \quad \text{for } i = 0, 1, \ldots.$$ 

VI. $\text{PPS}_G$ and $\text{SP}_G$

$$E(S) = F \sum_{i=0}^{\infty} \Pr(S > S_i)$$

**Definition:** $i^*$ is the number of items inspected in a cycle.

VII. $\text{PPS}_G$ and $\text{SP}_G$

$$E(i^*) = \sum_{i=0}^{\infty} \Pr(S > S_i) = E(S)/F$$
Definition: $x(\alpha)$ is the quality of the item produced at time $\alpha$. In particular, if the item at time $\alpha$ is good, then $x(\alpha) = 1$ and $x(\alpha) = 0$ if the item is bad.

VIII. $PPS_G$ and $SP_G$

$$E[x(\alpha)] = E[PR^z(\alpha)]$$

IX. $PPS_I$ and $SP_G$

$$E[x(\alpha)] = PE[e^{-\alpha} \Delta(1-R)]$$

Definition: $X_0$ is the amount of good production before the first inspection.

X. $PPS_G$ and $SP_G$ (discrete)

$$E(X_0) = PE[\sum_{\alpha=1}^{I-1} R^z(\alpha)]$$

XI. $PPS_G$ and $SP_G$ (continuous)

$$E(X_0) = PE[\int_0^{I-1} R^z(\alpha) \; d\alpha]$$

XII. $PPS_I$ and $SP_G$ (discrete)

$$E(X_0) = \frac{[P - e^{\Delta(1-R)} Pr(S > S_1)]}{[e^{\Delta(1-R)} - 1]}$$

XIII. $PPS_I$ and $SP_G$ (continuous)

$$E(X_0) = \frac{[P - Pr(S > S_1)]}{\Delta(1-R)}$$
Definition: $X^*$ is the amount of good uninspected production before the process is stopped.

XIV. $PPS_G$ and $SP_G$

$$E(X^*) = E(X_0)[\frac{E(i^*) - 1}{Pr(S > S_1)}]$$

Definition: $Z^*$ is the state of the production process at the time that is stopped.

XV. $PPS_G$ and $SP_G$

$$E(Z^*) = \Delta E(S)$$

5. Income: The only way in which a sampling plan can be evaluated is in terms of the income assumed under the model. There are many ways of defining income. We consider the average income per unit of time (in the long run).

The income will be measured by:

$$[Income \ up \ to \ time \ T]/T$$

This quantity is a random variable. We center our interest on

$$\lim_{T \to \infty} [Income \ up \ to \ time \ T]/T$$

Girshick and Rubin [7] have shown for models like the ones under consideration that this limit exists with probability 1 and that in fact this limit is given by
\[ L = E(\text{Income per cycle}) / E(\text{Length of a cycle}) \]

The proof of Girshick and Rubin applies in this case with minor modifications. (See page 7 for definition of cycle.)

In short lengths of production the average income per unit of time may be quite different from \( L \). However, we are thinking of using these sampling plans in situations where the total length of time of the production process will be large and hence these numbers will be indicative of the adequacy of the plan being used.

Before showing how one can compute \( L \), we must first define several economic parameters of the model. In particular assume a good uninspected item is worth 1 unit and a bad item is worth 0 units. In any cost problem it is always possible to choose two of the costs in an arbitrary manner as we have just done and then to determine the other costs in terms of these. For the particular choice just made the value of the production at time \( t \) is \( x(t) \). Let \( c_0(c_1) \) be the value of a bad (good) inspected item less the cost of inspection. Thus, if the inspection were destructive, \( c_0(c_1) \) would be the negative of the cost of inspecting a bad (good) item. In the case of non-destructive testing, it might be that \( c_1 > 1 \) due to the fact that a tested good item would then be known to be good and hence could be sold at a premium. Assume the time required to fix the process when it is stopped in state \( z^* \) is \( A_1 + A_2 z^* \) and that the cost of doing this adjustment is \( B_1 + B_2 z^* \).

We are now able to evaluate \( L \) in terms of the expected values defined in the previous section and the constants which have just been introduced.
XVII. \[ L = \frac{[E(X*) - B_1 - B_2 E(Z*) + (c_0 - c_1) + c_1 E(i*)]}{[E(S) + A_1 + A_2 E(Z*)]} \]

XVIII. Proportion of time devoted to production = \[ \frac{1}{1 + \frac{1}{E(S)}} \]

XIX. Proportion of good items produced (discrete case) = \[ \frac{[E(X*) + E(i*) - 1]}{E(S)} \]  
= 1 - AOG

XX. Proportion of production time giving good quality (continuous case) 
= \[ \frac{E(X*)}{E(S)} \]

XXI. Proportion of items produced that are inspected = \[ \frac{E(i*)}{E(S)} = \frac{1}{F} = API \]

Formulas XVIII through XXI are a few of the special cases of XVII that are of interest. The important thing to note is that these formulas can all be obtained from the few important expected values given in the preceding section by judicious choices of the cost parameters of this section.

6. Examples: In this section we consider two examples in detail in order to illustrate the meaning of the preceding formulas and to emphasize that in spite of the fact the formulas are not so simple as to allow the computation
of tables they are in such a form that particular problems can be analyzed. In the course of the analysis of these examples we find only the best plan among those considered. It is unlikely the class under consideration actually contains the best possible plan.

The example that we are about to consider is highly idealized, but affords a situation illustrating all of the theory. Further in this example we are able from a priori reasons to assure ourselves that the conditions of the model are satisfied.

The hypothetical production process that we wish to consider is a counting process that arises in the following manner. The census bureau has recorded for each of the hundred and sixty million individuals in the country data on a large sheet (one for each individual). The sheet contains 1000 positions and each position is either filled or blank. The probability that a position is filled is $\frac{1}{2}$ for each individual in each position. The quantity that is desired is the total number of positions filled in a sheet. The method of finding this total is to place a scoring screen over each sheet which makes contact with each of the positions and records the total number of filled contacts. Unfortunately, the screen, after it has been in use for some time, will make mistakes in certain positions due to the contact points breaking. This breaking, for instance, might be caused by getting grease on the points or physical breaking. The machine will make a mistake if a filled position occurs under a broken contact point.

It seems reasonable that the number of broken contact points can be represented by a Poisson stochastic process which will correspond to our $z(t)$ process. This will be a good approximation under the assumptions: (1) the failure times for the individual points are independently distributed,
(2) we are only interested in the case where there are few failures. The failure rate for the contact points can be easily found from previous experience. Let, \( \Delta \), the number of failures per unit of time (the time to process a sheet), be given by .0001. When the process begins with a new screen there will be no failures and thus \( P = 1 \). Finally \( R = 1/2 \) since the probability of no mistakes occurring when there are \( k \) failed points will be the probability that in the sheet at hand those \( k \) positions are not filled.

The inspection procedure consists in taking a sheet and examining it to see if the total number of filled positions on the sheet agrees with that of the machine. We will assume that the cost of the inspection by hand is 11 units where it should be remembered that the value of a properly recorded sheet is one and of a sheet with the wrong total is zero. A difficulty arises, since the time to do the inspection process is probably equivalent to the time of totaling several sheets. Nevertheless, for the purposes of this example it will be assumed that the inspection process is instantaneous and as a result of the inspection process we make the proper decision as to whether the next item should be allowed to go through the machine or whether the machine should be stopped.

Assume the way the machine is repaired is to stop the process and remove the old screen and replace it with a new one. Let the number of time units required for this operation be 25 and the cost of carrying out the operation 50 units. We are making the assumption that the time and costs for repairs do not depend on the state of the screen. The situation being that if a mistake has been found then we know that the screen is defective and must be replaced so that having found a mistake there are no alternatives but to stop the machine and do something.
First use sampling at rate $F$ with equally spaced time intervals, i.e., $SP_I$.

$$E(i^*) = \sum_{i=0}^{\infty} [(e^{-0.0001 F})^i (0.0001 F(1 - 2^{-i}))]$$

$$\Pr(S > S_1) = e^{-0.0005 F}$$

$$E(X_0) = \frac{1 - e^{-0.0005 (F - 1)}}{e^{0.0005} - 1}$$

To complete the analysis the following quantities must be computed for selected values of $F$.

$$L = \frac{E(X_0)[E(i^*) - 1]}{\Pr(S > S_1)} - 50 - 1 - 10 E(i^*) - \frac{F E(i^*) + 25}{F E(i^*)}$$

$$f = \frac{1}{1 + 25/F E(i^*)} \quad \text{(proportion of time devoted to production)}$$

$$AOQ = 1 - \frac{E(X_0)[E(i^*) - 1]}{\Pr(S > S_1) + E(i^*) - 1}$$
TABLE I

<table>
<thead>
<tr>
<th>$F$</th>
<th>$E(i^*)$</th>
<th>$\Pr(S &gt; S_0)$</th>
<th>$E(S)$</th>
<th>$E(x^*)$</th>
<th>$L$</th>
<th>$f$</th>
<th>$AOQ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.0 \times 10^2$</td>
<td>101.491</td>
<td>.99501</td>
<td>$1.0149 \times 10^4$</td>
<td>9.9736 $\times 10^3$</td>
<td>.8855</td>
<td>.99754</td>
<td>.00738</td>
</tr>
<tr>
<td>$3.8 \times 10^2$</td>
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<td>.98118</td>
<td>$1.0557 \times 10^4$</td>
<td>1.0248 $\times 10^4$</td>
<td>.94689</td>
<td>.99764</td>
<td>.02673</td>
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<tr>
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<td>.98069</td>
<td>$1.0572 \times 10^4$</td>
<td>1.0255 $\times 10^4$</td>
<td>.94701</td>
<td>.99764</td>
<td>.02746</td>
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<td>.98020</td>
<td>$1.0586 \times 10^4$</td>
<td>1.0263 $\times 10^4$</td>
<td>.94706</td>
<td>.99764</td>
<td>.02811</td>
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<tr>
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<td>$1.0600 \times 10^4$</td>
<td>1.0270 $\times 10^4$</td>
<td>.94706</td>
<td>.99765</td>
<td>.02879</td>
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<td>.97922</td>
<td>$1.0615 \times 10^4$</td>
<td>1.0278 $\times 10^4$</td>
<td>.94704</td>
<td>.99765</td>
<td>.02946</td>
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<td>$1.0728 \times 10^4$</td>
<td>1.0337 $\times 10^4$</td>
<td>.94603</td>
<td>.99768</td>
<td>.03454</td>
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<tr>
<td>$1.0 \times 10^3$</td>
<td>11.419</td>
<td>.95123</td>
<td>$1.1419 \times 10^4$</td>
<td>1.0673 $\times 10^4$</td>
<td>.9182</td>
<td>.99782</td>
<td>.06445</td>
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<tr>
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<td>.77880</td>
<td>$1.6084 \times 10^4$</td>
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<td>$1.0 \times 10^4$</td>
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<td>.60653</td>
<td>$2.0875 \times 10^4$</td>
<td>1.4108 $\times 10^4$</td>
<td>.6716</td>
<td>.99880</td>
<td>.32412</td>
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</tbody>
</table>

Max $L = .947$ at $F \approx 410$

Thus from the numerical analysis the best value for $F$ is about 410. At this point the average income per unit of time is .94706. The shape of the income function is very flat in the neighborhood of the maximum. If for administrative reasons it was more convenient to use $F = 400$, there would be no appreciable loss and the AOQ would be slightly better. The AOQ can be made as small as desired by picking $F$ small, but this becomes costly. The total expected amount of production before stopping using $F = 410$ is 10,600 which is slightly more than 10,000 the expected waiting time for the first contact point to break. The proportion of time devoted to production is very insensitive to the value of $F$ used. On the other hand, the number of inspections
made depends quite heavily on $F$ being about 26 for the optimum.

Since the numbers of items between inspections are so large, it is reasonable to think of this as the continuous case and use $SP_{II}$ (continuous).

We then have

$$E(i^*) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \left[ 1 + .0001 F(1 - 2^{-j}) \right]^{-1}$$

$$\Pr(S > S_1) = [1 + .00005 F]^{-1}$$

$$E(X_o) = \frac{1 - [1 + .00005 F]^{-1}}{.00005}$$

$$E(X_o) \left[ \frac{E(i^*) - 1}{\Pr(S > S_1)} \right] - 10 E(i^*) - 50$$

$$L = \frac{E(X_o) \left[ \frac{E(i^*) - 1}{\Pr(S > S_1)} \right] - 10 E(i^*) - 50}{F E(i^*) + 25}$$

$$f = [1 + 25/F E(i^*)]^{-1}$$

$$\Lambda_{AQ} = 1 - \frac{E(X_o)[E(i) - 1]}{F E(i^*)\Pr(S > S_1)}$$
TABLE II

<table>
<thead>
<tr>
<th>F</th>
<th>E(1*)</th>
<th>Pr(S&gt;S₁)</th>
<th>E(S)</th>
<th>E(x*)</th>
<th>L</th>
<th>f</th>
<th>AOQ</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>35.295</td>
<td>.98522</td>
<td>1.05885 × 10⁴</td>
<td>1.0289 × 10⁴</td>
<td>.93141</td>
<td>.9976⁴</td>
<td>.02829</td>
</tr>
<tr>
<td>340</td>
<td>31.369</td>
<td>.98328</td>
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<td>1.0325 × 10⁴</td>
<td>.93184</td>
<td>.9976⁴</td>
<td>.03188</td>
</tr>
<tr>
<td>350</td>
<td>30.527</td>
<td>.98280</td>
<td>1.0684 × 10⁴</td>
<td>1.0335 × 10⁴</td>
<td>.93182</td>
<td>.9976⁴</td>
<td>.03267</td>
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<tr>
<td>400</td>
<td>26.950-</td>
<td>.98039</td>
<td>1.0780 × 10⁴</td>
<td>1.0380 × 10⁴</td>
<td>.93110</td>
<td>.9976⁴</td>
<td>.03711</td>
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<tr>
<td>10⁴</td>
<td>3.4627</td>
<td>.66667</td>
<td>2.4627 × 10⁴</td>
<td>1.4627 × 10⁴</td>
<td>.5903</td>
<td>.9989⁹</td>
<td>.40606</td>
</tr>
</tbody>
</table>

Max L = .9318⁴ at F = 340

The best sampling rate now turns out to be about 340 in contrast to 410 for the regularly spaced inspections. The maximum rate of income has been reduced about one percent until it now is .9318⁴. And in general this plan does not seem as satisfactory for the AOQ is larger, and the E(1*) is larger. However, it should be remembered that the advantage of random spacing lies in the fact that the consumer is protected against periodicities, etc.

As the second example consider the following production process: Items are being produced one at a time by an automatic machine that is rather complicated in its operation. It has been noticed that as time goes on, if the machine is left unattended, the quality will degenerate slowly. The explanation here is that there are many parts of the machine that have a chance to malfunction, and as time goes on some of these parts become imperfect. For instance, cutting edges become chipped and adjustments lose their calibration. The machine is so constructed that even if one of these minor flaws exist it does not necessarily happen that the output at that time will contain
a flaw. Thus, even if a cutting edge has a chip in it, it does not necessarily follow that the cutting edge will mar the finished product. The mar might only occur if the chip hits at the beginning of a cutting operation. Having collected a large quantity of data on the operation of this machine it has been found that the quality of the items produced at time \( t \) is approximately of the form \( \Pr[z(t)] \) where \( P = .99 \), \( R = .95 \) and \( z(t) \) is a Poisson process with parameter \( .1 \). The fact \( P = .99 \) means that even when the machine is in the best condition possible one percent of the output will be defective; hence it is clear that the AOQ is less than \( .99 \). The choice of \( R = .95 \) means that even when a defect appears in the machine the probability that it will be effective in hurting the output is only \( .95 \). The origin of this function for the quality of the output was presumably found empirically in the research that went into the development of the production process. Presumably it is a close approximation to the true production process.

Again, let the value of a good item be \( 1 \) unit and the value of a bad item be \( 0 \) units. Presume the inspection cost is negligible, but that the inspection process is destructive. Also assume that the cost of repair is of the form \( 5 + z(t) \). Finally, let the length of time to repair the machine be of the form \( 3 + z(t) \).

Let us analyze this process using \( S_{PI} \) (discrete). A reason for using \( S_{PI} \) is that there is some possibility that there are periodicities in the quality of the items produced, and the discreteness is used because the number of items between inspections will not be large. This being the case it is necessary to compute the following functions for selected values of \( F \).
\[
\text{Pr}(S > S_1) = \frac{.99 e^{-.1}}{F} \frac{e^{.1(19)(.05)}}{[1-e^{-.1(.05)}(1-1/F)]}
\]

\[
E(X_o) = \frac{[.99 - e^{.1(.05)}\text{Pr}(S > S_1)]}{(.1)(.05)}
\]

\[
E(i*) = 1 + \sum_{i=1}^{\infty} \left\{ \frac{.99 e^{-.01 i}}{F} \frac{e^{.01(19)(1-.95^i)}}{1 \pi [1-e^{-.1(1-.95^i)}(1-1/F)]} \right\}
\]

\[
E(S) = F E(i*)
\]

\[
E(X^*) = E(X_o) \left[ \frac{E(i*) - 1}{\text{Pr}(S > S_1)} \right]
\]

\[
E(Z^*) = .1 E(S)
\]

\[
L = \left\{ \frac{E(X^*) - 5 - E(Z^*) - E(i*)}{E(S) + 3 + E(Z^*)} \right\}
\]

\[
f = [1 - \frac{3 + E(Z^*)}{E(S)}]^{-1}
\]

\[
\text{AOQ} = 1 - \frac{E(X^*)}{E(S)}
\]
TABLE III

<table>
<thead>
<tr>
<th>F</th>
<th>E(1*)</th>
<th>Pr(S &gt; S_1)</th>
<th>E(S)</th>
<th>E(X*)</th>
<th>L</th>
<th>f</th>
<th>AOQ</th>
</tr>
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<tr>
<td>15</td>
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<td>.4502</td>
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<td>.364645</td>
<td>.897900</td>
<td>.461041</td>
</tr>
</tbody>
</table>

Maximizing F is about 25 in which case L = .465

For this problem the AOQ corresponding to the optimum F is .28 which for many products of course would not be tolerable. Also the optimum income per unit of time is less than half of what would be obtained if production was all good. Further, for the optimum F repair work must be done on the average once for about every hundred items produced. Even cutting F down to fifteen does not make an appreciable improvement in the AOQ, although it would not cause much of an economic loss. With the information supplied in Table III it might be considered desirable not to start production until the process can be improved substantially.

7. Proofs: Define x(0) = 1 , and remember that S_0 = 0 and z(0) = 0 .

Then by definition

\[ \Pr(S > S_1) = \Pr[ \prod_{j=0}^{1} x(S_j) = 1] . \]
\[
= \mathbb{E}\left[ \prod_{j=0}^{i} x(S_j) \right]
= \mathbb{E}\left\{ \mathbb{E}\left[ \prod_{j=0}^{i} x(S_j) | S_0, \ldots, S_i ; z(S_0), \ldots, z(S_i) \right] \right\}
= \mathbb{E}\left\{ \mathbb{E}\left[ \prod_{j=0}^{i} \left( \mathbb{P}\left( R_{S_j} \right) \right) | S_0, \ldots, S_i \right] \right\}.
\]

Since \( z(t) \) is stationary with independent increments, we may for fixed \( S_0, \ldots, S_i \) use
\[
z(S_k) = \sum_{j=1}^{k} z'(I_j)
\]
where \( I_j = S_j - S_{j-1} \) and the \( z'(I_j) \) are mutually independent with the same distribution as \( z(I_j) \) and obtain
\[
\mathbb{P}(S > S_i) = \mathbb{E}\left\{ \mathbb{E}\left[ \prod_{j=1}^{i} (i-j+1)z'(I_j) \right] | I_1, \ldots, I_i \right\}
= \mathbb{E}\left\{ \prod_{j=1}^{i} \left[ \mathbb{E}(\mathbb{P}(R_{I_j}) | I_j) \right] \right\}
= \prod_{j=1}^{i} \mathbb{E}\left[ \mathbb{E}(\mathbb{P}(R_{I_j}) | I_j) \right]
\]
\[
= \frac{1}{\pi} \sum_{j=1}^{\infty} \left\{ \mathbb{E}[PR_{I_j}] \right\}
\]

\[
= \frac{1}{\pi} \sum_{j=1}^{\infty} \left\{ \mathbb{E}[PR_{I_1}] \right\}
\]

\[
= p^i \frac{1}{\pi} \sum_{j=0}^{\infty} \left\{ \mathbb{E}[R^{j}(I_1)] \right\}
\]

which is \( P \). To obtain II we note under \( P_{I} \) that \( z(I_1) \) has for
fixed \( I_1 \) a Poisson distribution with parameter \( I_1 \Delta \).

Starting with \( \text{Pr}(S > S_i) = p^i \frac{1}{\pi} \sum_{j=0}^{\infty} \mathbb{E}[R^{j}(I_1)] \)

we need to do the following computation:

\[
\mathbb{E}[R^{j}(I_1)|I_1] = \sum_{k=0}^{\infty} \left[ \frac{(-\Delta I_1)^k}{k!} R^{jk} \right]
\]

\[
= e^{-\Delta I_1}(1-R^j)
\]

When the condition \( SP_1 \) is added, \( I_1 = F \). Hence III is obtained from
II by replacing \( I_1 \) by \( F \) thusly:

\[
\text{Pr}(S > S_i) = p^i \frac{1}{\pi} \sum_{j=0}^{\infty} \left[ e^{-\Delta F(1-R^j)} \right]
\]

\[
= p^i \left[ e^{-\Delta F[(i+1) - \sum_{j=0}^{\infty} R^j]} \right]
\]

\[
= p^i e
\]
\[ \frac{1}{\pi} e^{-\Delta} F[(i+1) - (1-R_i+1)/(1-R)] \]

which reduces to the desired form.

Using \( SP_{II} \) (discrete) we have from \( II \) and the distribution of \( I_1 \)

\[
\Pr(S > S_i) = \frac{1}{\pi} \sum_{j=0}^{\infty} \left( \sum_{k=1}^{\infty} \left( e^{-\Delta} k(1-R^j) \right) \left[ 1 - \frac{1}{F} k \frac{1}{F} \right] \right)
\]

\[
= \frac{1}{\pi} \sum_{j=0}^{\infty} \left( \frac{e^{-\Delta(1-R^j)}}{F[1 - (1 - \frac{1}{F}) e^{-\Delta(1-R^j)}]} \right)
\]

and then \( IV \) is a simplification. Similarly, using \( SP_{II} \) (continuous) we have

\[
\Pr(S > S_i) = \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-\Delta} t(1-R^j) - t/F}{\int_{0}^{\infty} F[\Delta(1-R^j) + \frac{1}{F}] \frac{e^{-t[\Delta(1-R^j) + \frac{1}{F}]}}{F[\Delta(1-R^j) + \frac{1}{F}]}} dt
\]

\[
= \frac{1}{\pi} \sum_{j=0}^{\infty} \left( \frac{F[\Delta(1-R^j) + \frac{1}{F}]}{F[\Delta(1-R^j) + \frac{1}{F}]} \right)^{-1}
\]

and hence \( V \).
The proof of VI begins with

\[ E(S) = E\left[ \sum_{i=1}^{\infty} I_i \prod_{j=0}^{i-1} x(S_j) \right] \]

Now note that \( I_i \) is independent of \( S_0, \ldots, S_{i-1}, x(S_0), \ldots, x(S_{i-1}), \) and \( z(S_0), \ldots, z(S_{i-1}) \). Hence

\[ E(S) = \sum_{i=1}^{\infty} \left( E(I_i) E\left[ \prod_{j=0}^{i-1} x(S_j) \right] \right) \]

and thus VI is obtained.

To prove VII we need only note that \( i^* = \sum_{i=0}^{\infty} \prod_{j=0}^{i} x(S_j) \).

Results VIII and IX are the same as I and II for the case of \( i=1 \) and \( \alpha \) playing the role of \( S_1 \).

To obtain X note that

\[ x_o = \sum_{\alpha=1}^{I_1-1} [x(\alpha)] \]

and for XI we have

\[ x_o = \int_0^{I_1} x(\alpha) d\alpha \]
and thus

\[ E(X_0) = E[\int_0^{I_1} x(\alpha) d\alpha] \]

\[ = E \left\{ E[\int_0^{I_1} x(\alpha) d\alpha | I_1 \text{ and } z(\alpha) \text{ for } 0 \leq \alpha \leq I_1] \right\} \]

\[ = E \left\{ \int_0^{I_1} E[x(\alpha) | I_1 \text{ and } z(\alpha) \text{ for } 0 \leq \alpha \leq I_1] d\alpha \right\} \]

\[ = E[\int_0^{I_1} PZ(\alpha) d\alpha] \]

Under PPS_{\perp} (discrete) using VIII, IX and X we have

\[ E(X_0) = \sum_{\alpha=1}^{I_1-1} e^{-\alpha \Delta(1-R)} \]

\[ = \frac{P \sum_{\alpha=1}^{I_1-1} e^{-\alpha \Delta(1-R)}}{[1-e^{-\Delta(1-R)}]} \left[ E[1 - e^{-(I_1-1)\Delta(1-R)}] \right] \]

and now using II with i=1 we obtain XII.

To prove XIII we proceed in the same manner as follows:

\[ E(X_0) = FE[\int_0^{I_1} e^{-\alpha(1-R)\Delta} d\alpha] \]

\[ = FE[\frac{1-e^{-I_1(1-R)\Delta}}{\Delta(1-R)}] \]
To prove XIV we note that in the discrete case

\[
E(X^*) = E \left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{i+1} \pi X(S_j) \left[ \sum_{\alpha=1}^{S_{i+1}} X(\alpha) \right] \right\}
\]

\[
= E \left\{ \sum_{i=0}^{\infty} [P^i \pi R] z(S_i) \left[ R \sum_{\alpha=1}^{I'-1} PR' z'(\alpha) \right] \right\}
\]

where \( I' \) is distributed like \( I_{i+1} \) and is independent of all random variables occurring up to and including time \( S_i \) and \( z'(1), \ldots, z'(I-1) \).

Hence

\[
E(X^*) = E(X_0) E \left\{ \sum_{i=0}^{\infty} \sum_{j=1}^{i+1} \pi j z'(I_j) \right\}
\]

where \( \sum_{j=2}^{1} = 0 \) and where the \( z'(I_j)'s \) are independently distributed like the \( z(I_j)'s \). Whence

\[
E(X^*) = E(X_0) \left\{ \sum_{i=0}^{\infty} P^i \pi \sum_{j=1}^{i+1} j z(I_j) \right\}
\]

where \( \sum_{j=2}^{1} = 1 \). Now dividing and multiplying by \( PE[R_{j=2}^{z(I_1)}] \) we obtain
and now XIV follows from I and VII. The continuous case of XIV is done analogously.

To prove XV we note (1)

\[ Z^* = \sum_{i=1}^{\infty} [z(S_i) - z(S_{i-1})][ \prod_{j=0}^{i-1} x(S_j)] \]

and (2) \([z(S_i) - z(S_{i-1})]\) is independent of everything up to and including \(S_{i-1}\).
REFERENCES


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