AN ECONOMIC APPROACH TO THE CHOICE OF CONTINUOUS SAMPLING PLANS

By
GEOFFREY GREGORY

TECHNICAL REPORT NO. 30
September 20, 1956

PREPARED UNDER CONTRACT N6onr-25126 (NR-042-002)
FOR
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GERALD J. LIEBERMAN, PROJECT DIRECTOR
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<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Summary and Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. Model and Costs</td>
<td></td>
</tr>
<tr>
<td>2.1 The Discrete Case</td>
<td>6</td>
</tr>
<tr>
<td>2.2 The Continuous Case</td>
<td>7</td>
</tr>
<tr>
<td>3. The Discrete Case - General Procedure and Remarks</td>
<td></td>
</tr>
<tr>
<td>3.1 Method A</td>
<td>8</td>
</tr>
<tr>
<td>3.2 Method B</td>
<td>29</td>
</tr>
<tr>
<td>4. The Basic Sampling Plan</td>
<td></td>
</tr>
<tr>
<td>4.1 Systematic Sampling</td>
<td>32</td>
</tr>
<tr>
<td>4.2 Stratified Sampling</td>
<td>33</td>
</tr>
<tr>
<td>4.3 Random Sampling</td>
<td>34</td>
</tr>
<tr>
<td>5. Simplified Dodge Plan</td>
<td></td>
</tr>
<tr>
<td>5.1 Systematic Sampling</td>
<td>39</td>
</tr>
<tr>
<td>5.2 Stratified Sampling</td>
<td>41</td>
</tr>
<tr>
<td>5.3 Random Sampling</td>
<td>43</td>
</tr>
<tr>
<td>6. An Extended Version of the Dodge Plan</td>
<td>45</td>
</tr>
<tr>
<td>7. The Savage Three Decision Plan (Attributes)</td>
<td>48</td>
</tr>
<tr>
<td>8. The Savage Three Decision Plan (Attributes-Variables)</td>
<td>52</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (Cont.)

9. An Extended Version of the Basic Plan
   9.1 Systematic Sampling .................. 56
   9.2 Random Sampling ................... 63

10. The Derman-Littauer-Solomon Multi-Level Plan MLP-T ........ 71

11. The Continuous Case - General Procedure and Remarks ............. 76

12. A Minimax Method of Choosing the Plan. ...................... 77

13. A Comparison of Methods of Sampling ........................ 80

14. A Numerical Example for the Discrete Case .................... 83

Acknowledgments ........................................ 85

Bibliography ........................................... 86-87
<table>
<thead>
<tr>
<th>Table/Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 3.1.1</td>
<td>12-13</td>
</tr>
<tr>
<td>Table 3.1.2</td>
<td>14</td>
</tr>
<tr>
<td>Figure 4.3.1</td>
<td>37</td>
</tr>
<tr>
<td>Figure 12.1.1</td>
<td>79</td>
</tr>
<tr>
<td>Figure 12.1.2</td>
<td>79</td>
</tr>
</tbody>
</table>
0. Application of the Method of Choosing a Plan.

In this report an indication is given of a method by which a continuous sampling plan can be chosen. In the following sections an account of the theory involved is given, but in this section the proposed procedure to be used in practice is given.

The criterion by which a plan is selected is the maximization of the average income. By income is meant the actual income derived from the use of the machine and it consists of the costs and values listed below. It is assumed that these cost parameters are known. In the discrete case where attributes sampling is used these are:

1. The value of a good item.

2. The value of a defective item which is passed.

3. The cost per unit of inspection. This may be different for different sampling rates.

4. The cost per unit time of examining and repairing the machine.

It is assumed further that this can have at most two values, one corresponding to the case where the machine was actually in good working order, and the other to the case where the machine was in need of repair. The full meaning of the two cases will become apparent when the model is considered.

Non-destructive testing is used in the analysis. Destructive testing would involve some slight modification of the computations.

For any analysis of a continuous sampling plan, a model for the
process must be assumed. The recent work of I. R. Savage* discusses various statistical production models. Here, however, a two-state model is assumed, one state corresponding to good quality, the other to bad. The full assumption is that the quality characteristic under consideration can have one of two distributions. When the machine is started up, the characteristic has the distribution corresponding to the good state (state I) and there is a probability $g$ that before each item is produced, the distribution of characteristic changes to that corresponding to the inferior state (state II). The only way in which the process can return to state I is by corrective action at the machine. The plan has to specify not only the sampling procedure but also the point at which the corrective action is taken of stopping and, if necessary, repairing the machine.

The assumptions made in the following sections are:-

1. In state I the machine is producing a proportion $1-p$ of defectives.

2. In state II the machine is producing a proportion $1-q$ of defectives. $p > q$.

3. When the machine is being "repaired" in state I, it is shut down for a period of $d_1$ time units. A time unit is the time to produce a single item.

4. When the machine is being repaired in state II, it is shut down for a period of $d_2$ time units.

To apply this model, values must be given to the parameters $p$, $q$, $g$, $d_1$, $d_2$.

$p$ is the proportion non-defective when the machine is in good running order and there should be little difficulty in obtaining an estimate of this quantity. $q$ should be taken as the proportion defective when the machine is in need of repair. It is closely related to the lot tolerance percent defective of lot-by-lot sampling. $g$ is the probability that the process changes state before the next item is produced. Estimation of $g$ may present difficulties, particularly if no data on the process is available. If this is the case, it is suggested that initially a pessimistic view is taken and $g$ is overestimated. Then, after the process has been running for some time, the estimate can be reviewed. For any plan the expected length of time between shut down periods is a function of $g$ and hence this affords a method of checking the estimate. $d_1$, the length of time that the machine is shut down when it is in fact in good working order, will presumably be constant and easily estimable. $d_2$, the corresponding length when the process is in state II is assumed constant, although it may actually depend on the type of fault in the machine. However, provided that there is not too much difference between the shut down times, $d_2$ can be taken as the average time, weighted, if possible, by the frequency of occurrence of the fault.

If, now, all these factors can be determined, it is possible to find a sampling plan, which is at least an approximation to the optimal plan, for any of the classes considered in sections 4 through 10. A description of each plan, the results obtained and the application of the results to a particular example is given in this section. The example is described
by the following parameters

\[ p = 0.99 \quad b_1 = 10 \]
\[ q = 0.90 \quad b_2 = 0 \]
\[ g = 0.0005 \quad c = 1 \]
\[ d_1 = 10 \quad e = 1 \]
\[ d_2 = 10 \quad n_1 = 5 \]
\[ n_2 = 10 \]

In the statements of the plans reference will be made to sampling at the rate \( F \). This can be interpreted as inspecting every \( F \)th item (regular sampling), as dividing the production into blocks of size \( F \) and inspecting one at random from each block (systematic sampling), or as inspecting each item with probability \( 1/F \) (random sampling). In the results and the example only regular sampling will be considered, although the results using stratified or random sampling will not differ appreciably.

1. **The Basic Plan.**

Sample at the rate \( F \). Stop the process and examine the machine as soon as a defective occurs.

**Results.**

Approximation to the optimal \( F \),

\[
F^* = \sqrt{\frac{2(1-q)[c + n_1(1-p)[pb_1 + (1-p)b_2 + d_1]]}{g(b_1-b_2)(p-q)(1+q)}}
\]

Approximate optimal income,

\[
V^* = pb_1 + (1-p)b_2 - 2\sqrt{\left[c + n_1(1-p)[pb_1 + (1-p)b_2 + d_1]\right] \left[\frac{(b_1-b_2)(p-q)(1+q)}{2(1-q)}\right]}
\]
In the numerical example
\[ F^* = 22, \quad V^* = 9.715 \]

2. The Simplified Dodge Plan.

Sample at the rate \( F \). When a defective is found, change to one hundred percent inspection. If the next \( j \) consecutive items are non-defective, return to sampling at the rate \( F \). If, however, a defective is found within the next \( j \) items, stop the machine immediately and examine the process. When the machine starts up again, the plan begins with inspection at the rate \( F \).

Results.

Approximation to the optimal \( F \),
\[
F^* = \sqrt{\frac{n(1-p)(1-p^q)(pb + (1-p)b + d) + c + e(1-p^q)}}
\]

Approximate optimal income,
\[ V^* = pb + (1-p)b \]

\[-2\sqrt{g(b_1-b_2)(p-q) \left[ \frac{1}{(1-q)(1-q^j)} - \frac{1}{2} \right] [n(1-p)(1-p^q)(pb + (1-p)b + d) + c + e(1-p^q)]]\]

In the numerical example
\[ F^* = 17, \quad j^* = 23, \quad V^* = 9.737 \]

The optimal value of \( j, j^* \), is found by a trial and error method of maximizing \( V^* \). \( F^* \) can then be determined directly.

Sample at the rate F. When a defective is found, change to one hundred percent inspection and continue thereon until either

(a) \( j \) consecutive non-defective items are found

or (b) \( k \) defectives are found.

In case (a) return to sampling every \( F^{th} \) item and in case (b) stop the process and examine the machine. When the machine starts up again, the plan begins with sampling at the rate F.

Results.

Approximation to the optimal \( F \),

\[
F^* = \sqrt{c + \frac{e(1-p^j)[1-(1-p^j)^k]}{p^j} + n_1(1-p)(1-p^j)^k \left[ d_1 + pb_1 + (1-p)b_2 \right]} \\
\quad \cdot g(p-q)(b_1-b_2) \left[ \frac{1}{(1-q)(1-q^j)^k} - \frac{1}{2} \right]
\]

Approximate optimal income,

\[
V^* = pb_1 + (1-p)b_2 - 2\sqrt{g} \left\{ (b_1-b_2)(p-q) \left[ \frac{1}{(1-q)(1-q^j)^k} - \frac{1}{2} \right] \right\} \cdot \left[ n_1(1-p)(1-p^j)^k \left( pb_1+(1-p)b_2+d_1 \right) + c + \frac{e(1-p^j)[1-(1-p^j)^k]}{p^j} \right]
\]

In the numerical example

\[
F^* = 15, \quad j^* = 29, \quad k^* = 2, \quad V^* = 9.738
\]

The optimal values of \( j \) and \( k \) (\( j^* \) and \( k^* \)) are the maximizing
values of \( V^* \) and are found by trial and error methods. Substitution of these values in the formula for \( F^* \) yields the approximate optimal sampling rate.

4. The Savage Three-Decision Plan (Attributes).

Inspect at the rate \( F \). When a defective is found, change to a sequential sampling plan by attributes whose terminal decisions are:

i) Return to sampling at the rate \( F \).

ii) Stop the process and examine the machine.

The procedure reverts to sampling at the rate \( F \) when the machine starts up again after a shut-down period.

Results.

Suppose the sequential sampling plan has probabilities \( \alpha \) and \( \beta \) of errors of types I and II respectively. Here they are the probabilities of making the wrong decisions during the period of probation.

Approximation to the optimal \( F \).

\[
F^* = \sqrt{\frac{e(1-p)[h_1(1-\alpha) - h_2\alpha]}{c + \frac{s - 1 + p}{s - 1 + p}} + \frac{n_1(1-p)\alpha[d_1 + pb_1 + (1-p)b_2]}{g(p-q)(b_1-b_2)} \left(\frac{1}{(1-q)(1-\beta)} - \frac{1}{2}\right)}
\]

Approximate optimal income

\[
V^* = pb_1 + (1-p)b_2 - 2g \left(\frac{1}{(p-q)(b_1-b_2)} \left[\frac{1}{(1-q)(1-\beta)} - \frac{1}{2}\right] - \sqrt{\frac{e(1-p)[h_1(1-\alpha) - h_2\alpha]}{c + \frac{s - 1 + p}{s - 1 + p}} + \frac{n_1(1-p)\alpha[d_1 + pb_1 + (1-p)b_2]}{g(p-q)(b_1-b_2)} \left(\frac{1}{(1-q)(1-\beta)} - \frac{1}{2}\right)}\right)
\]
where

\[ h_1 = \frac{\log \frac{1-\alpha}{\beta}}{\log \frac{p(1-q)}{q(1-p)}} \]

\[ h_2 = \frac{\log \frac{1-\beta}{\alpha}}{\log \frac{p(1-q)}{q(1-p)}} \]

\[ s = \frac{\log \frac{p/q}{q/(1-p)}}{\log \frac{p(1-q)}{q(1-p)}} \]

\( h_1, h_2 \) and \( s \) define the sequential sampling plan.

For the numerical example, reasonable values of \( \alpha \) and \( \beta \) are \( \alpha = 0.01, \beta = 0.05 \), since shutting down the process when it is in the good state will be more serious than allowing it to run for a further period of time when it is in state II. Maximization over \( \alpha \) and \( \beta \) would be very tedious.

\[ h_1 = 1.2451 \quad h_2 = 1.8991 \quad s = 0.039745 \]

Then \( F^* = 18, \quad V^* = 9.740 \)

5. The Savage Three-Decision Plan (Attributes-Variables).

The procedure for this plan is identical to that of the previous plan, except that here the sequential sampling is done by variables.

Suppose that an item is considered defective if a certain quality characteristic \( x \) is greater than some specification \( \bar{U} \).
Define $\mu_1$ and $\mu_2$ by

\[
p = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2\sigma^2}} \, dx
\]

\[
q = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu_2)^2}{2\sigma^2}} \, dx
\]

or $K_p = \frac{U - \mu_1}{\sigma}$, $K_q = \frac{U - \mu_2}{\sigma}$,

where $t = \int_{-\infty}^{K_t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$

Results.

Approximation to the optimal $F$,

\[
F^* = \sqrt{\frac{c + \frac{e(1-p)[h_1(1-\alpha) - h_2\alpha]}{s - \mu_1} + n_1(1-p) \alpha [d_1 + pb_1 + (1-p)b_2]}{g(p-q)(b_1-b_2)[(1-q)(1-\beta) - \frac{1}{2}]}}
\]

Approximate optimal income

\[
V^* = pb_1 + (1-p)b_2 - 2\sqrt{\left\{ \left(\frac{1}{(p-q)(b_1-b_2)} \left[\frac{1}{(1-q)(1-\beta)} - \frac{1}{2}\right]\right)^2 \right\}}
\]

\[
\sqrt{\left[ c + \frac{e(1-p)[h_1(1-\alpha) - h_2\alpha]}{s - \mu_1} + n_1(1-p)\alpha [d_1 + pb_1 + (1-p)b_2] \right]}\}
\]
where

\[ s = \frac{\mu_1 + \mu_2}{2} \]

\[ h_1 = -\sigma^2 \log \frac{\beta}{1-\alpha} \]

\[ h_2 = \frac{\sigma^2 \log \frac{1-\beta}{\alpha}}{\mu_2 - \mu_1} \]

\[ \mu_1 = \bar{U} - \sigma K_p \quad \mu_2 = \bar{U} - \sigma K_q \]

\[ s = \bar{U} - \sigma \left( \frac{K_p + K_q}{2} \right) \]

\[ s - \mu_1 = \sigma \left( \frac{K_p - K_q}{2} \right) \]

\[ \mu_2 - \mu_1 = \sigma (K_p - K_q) \]

\[ \frac{h_1(1-\alpha) - h_2\alpha}{s - \mu_1} = -2 \left[ \frac{(1-\alpha) \log \beta/1-\alpha + \alpha \log 1-\beta/\alpha}{(K_p - K_q)^2} \right] \]

independent of \( \sigma \).

In the numerical example

\[ \frac{h_1(1-\alpha) - h_2\alpha}{s - \mu_1} = 2.2928 \]

\[ F^* = 15 \quad V^* = 9.764 \]

**Note.** The reason for the high value of \( V^* \) is because the cost of inspection by variables has been taken the same as that by attributes.
6. **An Extended Version of the Basic Plan.**

Sample at the rate \( F \). When a defective item is found, if a further defective is found within \( k \) inspected items of the defective item found, stop the process and examine the machine.

**Results.**

Approximation to the optimal \( F \)

\[
F^* = \sqrt{\frac{c(2-p^k) + n_1(1-p)(1-p^k)(pb_1 + (1-p)b_2 + d_1)}{g(p-q)(b_1-b_2) \left[ \frac{q}{p-q} \left( 1 - \frac{(1-p)(1-p^k)}{(1-q)(1-q^k)} \right) + \frac{1}{(1-q)(1-q^k)} + (1-p^k) \left( \frac{1}{2} + \frac{q}{1-q} \right) \right]}}
\]

Approximate optimal income

\[
V^* = pb_1 + (1-p)b_2 - 2\sqrt{g} \left\{ \left\{ c + \frac{\frac{n_1(1-p)(1-p^k)}{2-p^k}}{pb_1 + (1-p)b_2 + d_1} \right\} \frac{(p-q)(b_1-b_2)}{2-p^k} \cdot \sqrt{\frac{q}{p-q} \left( 1 - \frac{(1-p)(1-p^k)}{(1-q)(1-q^k)} \right) + \frac{1}{(1-q)(1-q^k)} + (1-p^k) \left( \frac{1}{2} + \frac{q}{1-q} \right) \right]}
\]

In the numerical example, the approximations are \( F^* = 15 \), \( k^* = \infty \), \( V^* = 9.704 \)

The optimal value of \( k \), \( k^* \), is the value which maximizes \( V^* \) and is found by trial and error. Substitution of this value in the formula for \( F^* \) yields the optimal sampling rate.

The approximation \( k^* = \infty \) indicates that a large value of \( k \) is to be used. \( k \) is the inspector's "memory" and for practical considerations it should not be too large. Presumably a good plan would be to take \( k \) as large as possible and sample every fifteenth item.
7. The Derman-Littauer-Solomon Multi-Level Plan MLP-T.

The plan consists of \( k+1 \) levels of inspection.

**Level 0.** Inspect every item. If \( i \) consecutive non-defectives are found, proceed to level 1. If a defective is found before this happens, stop the process and examine the machine.

**Level \( j \), \( 1 \leq j \leq k-1 \).** Inspect at the rate \( p_j^j \). If the next \( i \) inspected items are non-defective, proceed to level \( j+1 \). If a defective occurs before this happens, return to level 0.

**Level \( k \).** Inspect at the rate \( p_k^k \). If a defective occurs, return to level 0.

When the machine is started up after a period of inspection, the plan begins again at level 0.

Results.

The income function for this plan is lengthy. Approximations to the optimal sampling rate and other parameters of the plan have not been made. Reference should be made to Section 10 for the full analysis.

The approximations quoted may be used to define the plan to be used, but a better procedure would be to use them as a guide to the exact optimal plan, which could be determined by further trial and error methods.
AN ECONOMIC APPROACH TO THE CHOICE
OF CONTINUOUS SAMPLING PLANS

By
Geoffrey Gregory

1. Summary and Introduction.

The production of a machine consists of either a sequence of discrete items or a continuous flow of material, which is directed past the inspector’s station. If it is now decided to use a continuous sampling plan on this production, the first question which arises concerns the purpose of this sampling. A continuous sampling plan can achieve either:-

1. Improvement in quality by the elimination of defective production.
2. Detection of any change in the quality of production and specification of the appropriate corrective action.

Plans designed to satisfy the first purpose achieved this by increasing the amount of sampling whenever poor quality is suspected. However, to improve the quality significantly in this manner requires a good deal of inspection, and it is felt that if a process is not producing sufficiently good quality it is better to attack the problem at its source, namely the machine itself, rather than in the final production stage. In cases where the machine may be subject to tool-wear, quality will never improve unless some corrective action is taken at the machine. Were plans of the former type followed to the letter, the inspector would find himself on one hundred percent inspection almost all the time. In practice he would soon realize that something was wrong with the process and then take appropriate action. The problem is to obtain a precise rule for when to take this action. To achieve the second purpose with the existing plans a
non-objective decision rule is normally used. The purpose here is to formulate an objective procedure.

To examine the effect of a plan upon the process, some assumption must be made about the nature of the process, and it is here that the second major question arises. Obviously the assumption that the process is in a state of control is meaningless for a plan designed for the purpose indicated. There are many possible models to choose from, each of which could possibly fit a situation found in practice, but here the simple case where the process can only be in one of two states is considered.

The different states correspond to different probability distributions of the quality measurement under consideration. One of the states will be the state where the machine is producing material of satisfactory quality, the other state being that of inferior quality. A further refinement is that although the process may pass from the satisfactory state to the inferior whilst in production, the only way in which it may reverse this transition is for the machine to be shut down and repaired.

The final step which must be taken before any consideration of the plans can be made is the evaluation of the economic factors involved.

One particular set of cost parameters is defined in the next section. Alternative factors such as the difference between destructive and non-destructive sampling, the replacement or non-replacement of defective material by non-defective, whether or not good material has its value increased by virtue of being inspected, etc., all may be included by the appropriate interpretation of the cost parameters.

It will be seen therefore that there are many situations for which
it might be desired to design a continuous sampling plan. The results obtained here will only apply to a number of these situations and should not be applied generally. It is felt, however, that the methods used in obtaining the results may be adapted to use in many situations found in practice.

The criterion used for selection of a sampling plan will be that of maximizing the average income. Girshick and Rubin [5] considered this very general criterion under the same model, but their answer consisted of integral equations, which, except for one or two special cases, have so far remained unsolved. It was, therefore, decided that, instead of attempting to maximize the income over every possible continuous sampling plan, only certain classes of plans would be considered and the maximization would be done within each class. For the case where the machine is producing a series of discrete items, seven classes of plans have been considered. Each plan is defined by several parameters and the object is to find the parameters whose plan yields the maximum income. It is in most cases impossible to present the maximization in an explicit form, but it is proved that if the plans are of a certain general form then an approximation to the optimal principal sampling rate can be obtained.

If the production of the machine consists of a continuous flow of material, the model and the costs can be adapted from the discrete case without difficulty. Similarly the plans may be adapted, and an indication is given of the way in which an approximation to the optimal principal sampling rate may be obtained.

Next a brief discussion of methods of sampling is included. When a
plan indicates sampling one item in $F$ this could be interpreted in the following ways.

1. Sample every $F^{th}$ item.
2. Divide the production into blocks of size $F$. Select an item at random from each block.
3. Sample every item with probability $1/F$.

The relative merits of these methods is considered when the object is maximization of average income.

Finally, one particular set of values will be given to the states and costs for the discrete case, and it will be demonstrated how a plan of the basic class could be selected for this situation.

The Plans Considered.

Most of the existing plans contain no provision for shutting down the process, and hence where a plan considered is referred to by name it will have been appropriately modified. The plans will be described in the form suitable for the discrete case, and in all but the last plan attributes sampling is being used.

1. Inspect one in every $F$ items. When a defective is found, stop the process and examine the machine. This is the simplest form of plan and is similar to the procedure used in control chart theory.

2. Inspect one in every $F$ items. When a defective is found, inspect every item until either a defective is found, or $j$ consecutive non-defectives have been found. In the former case stop the process and examine the machine, and in the latter return to inspection of one in every $F$ items. This is a
simplified version of the Dodge Plan [3].

3. Inspect one in every $F$ items. When a defective is found, inspect every item until either $k$ more defectives are found, or $j$ consecutive non-defectives have been found. In the former case stop the process and examine the machine, and in the latter return to inspection of one in every $F$ items. This is a more complete version of the Dodge Plan [3].

4. Inspect one in every $F$ items. When a defective is found, use a sequential sampling plan for attributes where the terminal decisions are to return to sampling one in every $F$ items and to shut down the process and examine the machine. When the process starts up again, the plan is recommenced at the beginning of the sequential sampling. This plan is essentially the Savage Three-Decision Plan [9].

5. The procedure is the same as plan 4 except that the sequential sampling is done by variables. This enables the inspector to come to his decision with the minimum of production time elapsing.

6. Inspect one in every $F$ items. If two defectives occur within $k$ inspected items stop the process and examine the machine. This plan has the property that the amount of inspection remains constant.

7. The plan consists of $k+1$ levels of inspection. Whenever the machine has been shut down, the plan begins again at level 0.

**Level 0.** Inspect every item. If $i$ consecutive non-defectives are found, proceed to level 1. If a defective is found before this happens, stop the process and examine the machine.

**Level j, 1 \leq j \leq k-1.** Inspect one in every $F^j$ items. If the next $i$
inspected items are non-defective, proceed to level \( j+1 \). If a defective
occurs before this happens, return to level 0.

**Level \( k \).** Inspect one in every \( F^k \) items. If a defective occurs, return to
level 0.

This plan is based on the tightened multi-level plan MLP-T [2], [7].

2. **Model and Costs.**

   The model used is essentially a special case of that described by
Girshick and Rubin [5].

2.1 **The discrete case.**

   - **State I.** The probability that an item is non-defective is \( p \).
   - **State II.** The probability that an item is non-defective is \( q \).

   \[ p > q. \]

   If the process is in state I, there is a constant probability \( g \) that
before the next item is produced the process will go into state II. Once
the process is in state II it will not return to state I until the process
is shut down and the machine repaired.

   The time taken to produce an item will be taken as the unit of time.
If the decision is made to stop the process, it will be assumed that the
length of time that the machine is shut down depends only on the state of
the process. If the process were in state I, the time shut down is \( n_1 \)
units, and \( n_2 \) units if the process were in state II.
When the process is started up again, it will always be in state I, unless otherwise specified.

**Costs and Values.**

- Value of a non-defective item $b_1$
- Value of a defective item $b_2$
- (if $b_1 > b_2$)
- Cost per unit of inspection (sampling) $c$
- Cost per unit of inspection (one hundred percent inspection) $e$
- Cost per unit time of examining the machine - state I $d_1$
- Cost per unit time of examining the machine - state II $d_2$

2.2 The continuous case.

The production consists of a continuous flow of material with defects occurring at instants of time. Take some arbitrary unit of time and define the unit length of production as the length produced in this time.

**State I.** The distance between consecutive defects has an exponential distribution with mean $\frac{1}{\mu}$.

**State II.** The distance between consecutive defects has an exponential distribution with mean $\frac{1}{\nu}$.

$\nu > \mu$

If the process is in state I, the time before it goes into state II has an exponential distribution with mean $\frac{1}{\beta}$. Once the process is in state II it will not return to state I until the process is shut down and the machine repaired. As before, the length of time that the machine is shut
down is \( n_1 \) units if the process is in state I, \( n_2 \) units in state II, and the process starts up again in state I.

Costs and values.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value per unit length of production</td>
<td>( b_1 )</td>
</tr>
<tr>
<td>Loss when a defect occurs</td>
<td>( b_2 )</td>
</tr>
<tr>
<td>Cost per unit length of inspection (sampling)</td>
<td>( c )</td>
</tr>
<tr>
<td>Cost per unit length of inspection (one hundred percent inspection)</td>
<td>( e )</td>
</tr>
<tr>
<td>Cost per unit time of examining the machine - state I.</td>
<td>( d_1 )</td>
</tr>
<tr>
<td>Cost per unit time of examining the machine - state II.</td>
<td>( d_2 )</td>
</tr>
</tbody>
</table>

3. **The Discrete Case - General Procedure and Remarks.**

Two methods of computing the average income function are used, which, although equivalent, differ in their ease of computation for the various plans.

3.1 The first, which will be referred to as method A, is essentially that used in the paper by Girshick and Rubin [5]. Define as a cycle the production between consecutive times when the machine leaves the repair shop. Both the length of a cycle \( L \) and the income from a cycle \( I \) are random variables. Suppose \( V \) is the long run average income per unit of time, using a particular sampling plan. Then Girshick and Rubin [5, p. 118] use the following theorem to compute \( V \).
Theorem 3.1  If, for any plan,

(i)  \( E(L) < \infty \)

(ii) the income or cost for any unit is finite,

then \( V = \frac{E(I)}{E(L)} \), where \( E \) denotes expectation.

The proof of this theorem is outlined in the above-mentioned paper and will not be repeated here.

The procedure followed is to compute first \( E(I) \). From this \( E(L) \) may be deduced by an appropriate choice of the cost parameters \([b_1 = 1, b_2 = 1, c = 0, e = 0, d_1 = -1, d_2 = -1]\). Thus \( V \) is found. The problem is solved by finding the values of the parameters which maximize \( V \). Should the usual method of maximization fail to yield an explicit form for the maximizing parameters, it is suggested that a method of trial and error be used. As an alternative to this, some approximation to the optimal parameters may be found. In the following discussion the assumption is made that the transition probability \( g \) is small. This assumption is plausible both from a practical point of view and also since if \( g \) were not small, a relatively small length of production could be determined, at the end of which the process would be in state II with high probability. In most cases it would be more economical not to sample at all, but instead to stop the machine at the end of these periods of production. However, in the case of small \( g \) the sampling plan has to be determined. To this end it will be proved that for all plans of a certain general class, the optimal sampling rate is always of a given order of \( g \).
Definition. Suppose a sampling plan consists of three stages.

Stage 1 (Sampling). Sample an average of one in $F$ items. (This may be interpreted in any of the three ways described in section 1 and considered in section 12). When a defective item is found, proceed to stage 2.

Stage 2 (Action). After each item is produced, one of three actions is taken

1. Return to stage 1
2. Remain in stage 2
3. Proceed to stage 3

Stage 3 (Shut Down). In stage 3 the process is stopped and the machine is being examined. After this the plan returns to stage 1. Then, if the parameter $F$ of stage 1 is independent of the parameters of stages 2 and 3, the plan will be called regular.

Theorem 3.2 For any regular plan used on a process which is assumed to be subject to a two-state model of the type described in section 2, provided

(i) the process does not go from state I to state II when the plan is in stage 2.

(ii) the expected length of a cycle is finite,

then the value of $F$ which maximizes the long-run average income $V$ is equal to $c \sqrt{g} + o\left(\frac{1}{\sqrt{g}}\right)$ where $c$ is a constant.

Proof. Suppose the process is in state I. Let $\pi_1$ be the probability that, given that the plan is in stage 2, the process returns to stage 1 directly.
Let \( \pi_2 \) be the corresponding probability when the process is in state II.

The length of production, \( l \), and the income from the production, \( i \), in stage 2 are random variables.

Let \( E_1(l|j) \) = expected length in stage 2 given that the process is in state I and the plan proceeds to stage \( j \). \( j=1, 3 \).

\( E_2(l|j) \) = expected length in stage 2 given that the process is in state II and the plan proceeds to stage \( j \). \( j=1, 3 \).

Let \( E_1(i|j), E_2(i|j) \) be the corresponding incomes in stage 2 for the two states \( j=1, 3 \).

Let \( E_1(l) = \pi_1 E_1(l|1) + (1-\pi_1) E_1(l|3) \)
\( E_2(l) = \pi_2 E_2(l|1) + (1-\pi_2) E_2(l|3) \)
\( E_1(i) = \pi_1 E_1(i|1) + (1-\pi_1) E_1(i|3) \)
\( E_2(i) = \pi_2 E_2(i|1) + (1-\pi_2) E_2(i|3) \)

the expected lengths and incomes in stage 2.

**Lemma 3.1** For any regular plan satisfying the conditions of the theorem where either systematic or stratified sampling is used in stage 1, the long run average income is of the form

\[
V = \frac{\alpha_1 F + \beta_1 F \lambda + \gamma_1 \lambda + \delta_1 + \epsilon_1 \frac{(1-\lambda)(1-g)}{g} + \xi_1 \frac{(1-\lambda)(1-g)}{gF}}{\alpha_2 F + \beta_2 F \lambda + \gamma_2 \lambda + \delta_2 + \epsilon_2 \frac{(1-\lambda)(1-g)}{g}} \quad \text{...(3.1.1)}
\]

where \( \alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i, \xi_i \) \( i=1, 2 \) represent quantities independent of \( F \) and \( g \). In particular
(1) $\alpha_1 + \beta_1 + \epsilon_1 = pb_1 + (1-p)b_2$
(2) $\alpha_2 + \beta_2 + \epsilon_2 = 1$
(3) $(\gamma_2 + \delta_2) (\alpha_1 + \beta_1 + \epsilon_1) > \gamma_1 + \delta_1 + \epsilon_1$
(4) $\left(\beta_1 + \frac{\epsilon_1}{2}\right) > (\beta_2 + \frac{\epsilon_2}{2}) (\alpha_1 + \beta_1 + \epsilon_1)$

Proof (a) Systematic Sampling.

When the plan is in stage 1, the production is divided into blocks of size $F$, the last item in each block being inspected. Let $I_1$ be the expected income from the remainder of the cycle, given that a block is about to be inspected and the process is in state I, $I_2$ the corresponding income if the process is in state II.

$I_1$ is the sum of products of the quantities tabulated in Table 3.1.1.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Conditional Expected Income</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Process remains in state I and the inspected item is non-defective.</td>
<td>$(F-1)(pb_1 + (1-p)b_2) + b_1 - c + I_1$</td>
<td>$(1-g)^F p$</td>
</tr>
<tr>
<td>Process remains in state I, the inspected item is defective, and the plan goes to stage 3.</td>
<td>$(F-1)(pb_1 + (1-p)b_2) + b_2 - c \quad + E_1(i</td>
<td>3) - n_1 d_1$</td>
</tr>
<tr>
<td>Process remains in state I, the inspected item is defective, and the plan returns to stage 1 directly.</td>
<td>$(F-1)(pb_1 + (1-p)b_2) + b_2 - c \quad + E_1(i</td>
<td>1) + I_1$</td>
</tr>
</tbody>
</table>

(table cont. on next page)
(continued)

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Conditional Expected Income</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Process shifts to state II after s items (0 ≤ s &lt; F) and the inspected item is non-defective.</td>
<td>[ s(pb_1 + (1-p)b_2) + (F-1-s)(qb_1 + (1-q)b_2) ] ( (1-g)^s g_q ) + ( b_1 - c + I_2 )</td>
<td>[ \frac{b_2 - c + E_2(i</td>
</tr>
<tr>
<td>Process shifts to state II after s items (0 ≤ s &lt; F), the inspected item is defective and the plan goes to stage 3.</td>
<td>[ s(pb_1 + (1-p)b_2) + (F-1-s)(qb_1 + (1-q)b_2) ] ( (1-g)^s g(1-q)(1-\pi_2) ) + ( b_2 - c + E_2(i</td>
<td>1) + I_2 )</td>
</tr>
<tr>
<td>Process shifts to state II after s items (0 ≤ s &lt; F), the inspected item is defective and the plan returns to stage 1 directly.</td>
<td>[ s(pb_1 + (1-p)b_2) + (F-1-s)(qb_1 + (1-q)b_2) ] ( (1-g)^s g(1-q) \pi_2 ) + ( b_2 - c + E_2(i</td>
<td>1) + I_2 )</td>
</tr>
</tbody>
</table>

Table 3.1.1

\( I_1 \) is the sum of the products of the conditional expected incomes and the corresponding probabilities, where for the last three tabulated, the sum is also taken over s from 0 to F-1. \( I_2 \) is the sum of products of the quantities tabulated in table 3.1.2.
The inspected item is non-defective. 
\[(F-1)(q b_1 + (1-q) b_2) + b_1 - c + I_2\] 
\[q\]

The inspected item is defective and the plan goes to stage 3. 
\[(F-1)(q b_1 + (1-q) b_2) + b_2 - c + E_2(i|3) - n_2 d_2\] 
\[(1-q)(1-\pi_2)\]

The inspected item is defective and the plan returns to stage 1. 
\[(F-1)(q b_1 + (1-q) b_2) + b_2 - c + E_2(i|1) + I_2\] 
\[(1-q)\pi_2\]

Table 3.1.2

\[I_2 = \frac{\left[(F-1)(q b_1 + (1-q) b_2) + b_2 - c\right]}{(1-q)(1-\pi_2)} + \frac{(b_1-b_2)q}{(1-q)(1-\pi_2)}\]

\[+ E_2(i|3) + \frac{\pi_2 E_2(i|1)}{1-\pi_2} - n_2 d_2.\]

and, writing \(\lambda = (1-q)^F\)

\[1 - \lambda p - \pi_1 \lambda (1-p)] I_1 = \frac{(b_1 F-c)[1 - \lambda q - \pi_2 \lambda (1-q)]}{(1-q)(1-\pi_2)}\]

\[+ (b_1-b_2) \left[ \frac{1}{1-q} - 1 \right] (1-\lambda)(1-p-q) - F \frac{[1 - \lambda q - \pi_2 \lambda (1-q)]}{1-\pi_2}\]

\[+ \lambda(1-p) E_1(i) + \frac{(1-\lambda)}{1-\pi_2} E_2(i) - n_1 d_1 \lambda(1-p)(1-\pi_1) - n_2 d_2(1-\lambda)\]

Putting \(b_1 = b_2 = 1, c = 0, d_1 = d_2 = -1, E_1(i) = E_1(\ell), E_2(i) = E_2(\ell)\)

\[1 - \lambda p - \pi_1 \lambda (1-p)] L_1 = \frac{F[1 - \lambda q - \pi_2 \lambda (1-q)]}{(1-q)(1-\pi_2)} + \lambda(1-p)E_1(\ell)\]

\[+ \frac{(1-\lambda)}{1-\pi_2} E_2(\ell) + n_1 \lambda(1-p)(1-\pi_1) + n_2(1-\lambda)\]
Then \( V = \frac{I_1}{L_1} \) (by theorem 3.1), which is of the required form.

For later reference the following equations, denoted as equations (3.1.2), are given

\[
\alpha_1 = \frac{q b_1 + (1-q) b_2}{(1-q)(1-\pi_2)}
\]

\[
\beta_1 = -\frac{[q + \pi_2(1-q)][q b_1 + (1-q) b_2]}{(1-q)(1-\pi_2)}
\]

\[
\gamma_1 = \frac{c[q + \pi_2(1-q)]}{(1-q)(1-\pi_2)} + (1-p) E_1(l) - \frac{E_2(l)}{1-\pi_2} - n_1 d_1 (1-p)(1-\pi_1) + n_2 d_2
\]

\[
\delta_1 = \frac{-c}{(1-q)(1-\pi_2)} + \frac{E_2(l)}{1-\pi_2} - n_2 d_2
\]

\[
\epsilon_1 = (b_1 - b_2)(p-q)
\]

\[
\zeta_1 = 0
\]

\[
\alpha_2 = \frac{1}{(1-q)(1-\pi_2)}
\]

\[
\beta_2 = -\frac{[q + \pi_2(1-q)]}{(1-q)(1-\pi_2)}
\]

\[
\gamma_2 = (1-p) E_1(l) - \frac{E_2(l)}{1-\pi_2} + n_1 (1-p)(1-\pi_1) - n_2
\]

\[
\delta_2 = \frac{E_2(l)}{1-\pi_2} + n_2
\]

\[
\epsilon_2 = 0
\]
(b) **Stratified Sampling.**

When the plan is in stage 1, the production is divided into blocks of size $F$ and one item is selected at random from each block and inspected. If a defective is found, it is assumed that stage 2 begins at the end of the block in which the defective is found and not immediately after the defective item.

A similar computation to that used in the proof of lemma 3.1.(a) yields the required result. The only difference is that a distinction has to be made, when the process changes state, between the case where the inspection is made before the process changes state and vice versa.

If $I_1$ and $I_2$ are defined as before,

$$I_2 = \frac{[F(qb_1 + (1-q)b_2) - c]}{(1-q)(1-\pi_2)} + \frac{\pi_2 E_2(i|1)}{1-\pi_2} + E_2(i|3) - n_2 \bar{d}_2$$

and

$$[1-\lambda p - \pi_1 \lambda (1-p)] I_1 = (b_1 F - c) \left[ \frac{1 - \lambda p - \pi_2 \lambda (1-p)}{(1-q)(1-\pi_2)} + \frac{(1-\lambda)(1-q)(p-q)}{qF(1-q)} \right]$$

$$- F(b_1 - b_2) \left[ \lambda (1-p) + \frac{(1-\lambda)}{(1-\pi_2)} \right]$$

$$+ \lambda (1-p) \pi_1 E_1(i|1) + \lambda (1-p)(1-\pi_1) E_1(i|3)$$

$$+ \frac{(1-\lambda) \pi_2}{(1-\pi_2)} E_2(i|1) + (1-\lambda) E_2(i|3)$$

$$- n_1 \bar{d}_1 \lambda (1-p)(1-\pi_1) - n_2 \bar{d}_2 (1-\lambda)$$
Hence

\[
[1 - \lambda p - \pi_1 \lambda (1-p)]I_1 = F \left[ \frac{1 - \lambda p - \pi_2 \lambda (1-p)}{(1-q)(1-\pi_2)} + \frac{(1-\lambda)(1-q)(p-q)}{gF(1-q)} \right]
+ \lambda (1-p) \pi_1 E_1(\ell|1) + \lambda (1-p)(1-\pi_1) E_1(\ell|3)
+ \frac{(1-\lambda)\pi_2}{1-\pi_2} E_2(\ell|1) + (1-\lambda) E_2(\ell|3)
+ n_1 \lambda (1-p)(1-\pi_1) + n_2 (1-\lambda).
\]

Thus the lemma is proved for stratified sampling. The following equations give the income coefficients.

Equations (3.1.3)

\[
\alpha_1 = \frac{q b_1 + (1-q) b_2}{(1-q)(1-\pi_2)}
\]

\[
\beta_1 = \frac{[p + \pi_2 (1-p)] (q b_1 + (1-q) b_2)}{(1-q)(1-\pi_2)}
\]

\[
\gamma_1 = \frac{c [p + \pi_2 (1-p)]}{(1-q)(1-\pi_2)} + (1-p) E_1(\ell) - \frac{E_2(\ell)}{1-\pi_2} - n_1 \frac{d_1}{(1-p)(1-\pi_1)} + n_2 \frac{d_2}{1-\pi_2}
\]

\[
\delta_1 = \frac{c}{(1-q)(1-\pi_2)} + \frac{E_2(\ell)}{1-\pi_2} - n_2 \frac{d_2}{1-\pi_2}
\]

\[
\epsilon_1 = \frac{b_1 (p-q)}{1-q}
\]

\[
\zeta_1 = \frac{-c (p-q)}{1-q}
\]

\[
\alpha_2 = \frac{1}{(1-q)(1-\pi_2)}
\]

\[
\beta_2 = \frac{[p + \pi_2 (1-p)]}{(1-q)(1-\pi_2)}
\]

\[
\gamma_2 = (1-p) E_1(\ell) - \frac{E_2(\ell)}{1-\pi_2} + n_1 (1-p)(1-\pi_1) - n_2
\]
\[ \delta_2 = \frac{E_2(l)}{1-\pi_2} + n_2 \]

\[ \epsilon_2 = \frac{p-q}{1-q} \]

Results (1) and (2) can be verified without difficulty.

In both cases

\[
(\gamma_2 + \delta_2)(\alpha_1 + \beta_1 + \epsilon_1) - (\gamma_1 + \delta_1 + \epsilon_1) = c + n_1(1-p)(1-\pi_1)(pb_1 + (1-p)b_2 + d_1) \\
+ (1-p)[(pb_1 + (1-p)b_2)E_1(l) - E_1(1)]
\]

\[ E_1(1) = (pb_1 + (1-p)b_2)E_1(l) - \text{inspection costs of stage 2} \]

\[ E_1(1) < (pb_1 + (1-p)b_2)E_1(l) \]

Hence result (3).

For systematic sampling

\[
(\beta_1 + \frac{\epsilon_1}{2}) - \beta_2(\alpha_1 + \beta_1 + \epsilon_1) = (p-q)(b_1-b_2) \left[ \frac{q + \pi_2(l-q)}{(1-q)(1-\pi_2)} + \frac{1}{2} \right]
\]

and for stratified sampling

\[
(\beta_1 + \frac{\epsilon_1}{2}) - \beta_2(\alpha_1 + \beta_1 + \epsilon_1) = (p-q)(b_1-b_2) \left[ \frac{p + \pi_2(l-p)}{(1-q)(1-\pi_2)} + \frac{(1-p)}{2(1-q)} \right]
\]

Hence result (4)

The proof of the theorem for systematic and stratified sampling now follows by considering the effect on \( V \) of six forms of behaviour of \( F \) as \( g \) becomes small. Arranged in order of magnitude they are:
Case 1. $F \to F_0$, a finite positive constant.

Case 2. $F \to \infty$, but $F = \frac{\omega(g)}{\sqrt{g}}$ where $\omega(g) \to 0$

Case 3. $F \to \infty$, but $F = c/\sqrt{g}$, $c$ a constant to the order $o(\sqrt{g})$.

Case 4. $F \to \infty$, but $F = \frac{h(g)}{\sqrt{g}}$ where $h(g) \to \infty$ and $\sqrt{g} h(g) \to 0$

Case 5. $F \to \infty$, but $F = \frac{\tilde{h}(g)}{\sqrt{g}}$ where $\tilde{h}(g) \to \infty$ and $\sqrt{g} \tilde{h}(g) \to 0$, $0 < g_0 < \infty$.

Case 6. $F \to \infty$, but $F = \frac{\tilde{h}(g)}{\sqrt{g}}$ where $\tilde{h}(g) \to \infty$ and $\sqrt{g} \tilde{h}(g) \to \infty$.

For the purpose of maximization of $V$, $F$ is treated as a continuous variable.

It will be shown that for all $F$ of the form of case 3, there is a $c$ which maximizes $V$. Let $V^*$ be the income corresponding to this $c$. Then the proof is completed by showing that for any $F$ of any of the other cases, there is some positive $\varepsilon$ such that $V^* - V = \varepsilon A + o(\varepsilon)$, where $A$ is a positive constant.

The six cases exhaust all forms of behaviour of $F$.

Because of the method of proof case 3 will be the first case considered.

**Case 3.** $F = c/\sqrt{g} + o(\frac{1}{\sqrt{g}})$

$$F = \frac{c}{\sqrt{g}} (1 + o(1))$$

Hence $\frac{1}{F} = \frac{\sqrt{g}}{c} (1 + o(1)) = \frac{\sqrt{g}}{c} + o(\sqrt{g})$

$$\lambda = (1-g)^{\frac{1}{F}} = 1 - c\sqrt{g} + o(\sqrt{g})$$

$$\lambda \frac{1}{F} = \frac{\sqrt{g}}{c} + o(\sqrt{g})$$
\[
\frac{(1-g)(1-\lambda)}{g^p} = 1 - \frac{c\sqrt{g}}{2} + o(\sqrt{g})
\]

\[
\frac{(1-g)(1-\lambda)}{g^{p^2}} = \frac{\sqrt{g}}{c} + o(\sqrt{g})
\]

Write

\[
V = \frac{\alpha_1 + \beta_1 \lambda + \gamma_1 \frac{\lambda}{F} + \delta_1 \frac{1}{F} + \varepsilon_1 \frac{(1-\lambda)(1-g)}{g^p} + \xi_1 \frac{(1-\lambda)(1-g)}{g^{p^2}}}{\alpha_2 + \beta_2 \lambda + \gamma_2 \frac{\lambda}{F} + \delta_2 \frac{1}{F} + \varepsilon_2 \frac{(1-\lambda)(1-g)}{g^p}}
\]

Then

\[
V = \frac{\alpha_1 + \beta_1 (1-c\sqrt{g}) + \gamma_1 \frac{\sqrt{g}}{c} + \delta_1 \frac{\sqrt{g}}{c} + \varepsilon_1 (1 - \frac{c\sqrt{g}}{2}) + \xi_1 \frac{\sqrt{g}}{c} + o(\sqrt{g})}{\alpha_2 + \beta_2 (1-c\sqrt{g}) + \gamma_2 \frac{\sqrt{g}}{c} + \delta_2 \frac{\sqrt{g}}{c} + \varepsilon_2 (1 - \frac{c\sqrt{g}}{2}) + o(\sqrt{g})}
\]

\[
= \frac{(\alpha_1 + \beta_1 + \varepsilon_1) - (\beta_1 + \frac{\varepsilon_1}{2})c\sqrt{g} + (\gamma_1 + \delta_1 + \xi_1) \frac{\sqrt{g}}{c} + o(\sqrt{g})}{(\alpha_2 + \beta_2 + \varepsilon_2) - (\beta_2 + \frac{\varepsilon_2}{2})c\sqrt{g} + (\gamma_2 + \delta_2) \frac{\sqrt{g}}{c} + o(\sqrt{g})}
\]

\[
= \frac{(\alpha_1 + \beta_1 + \varepsilon_1) - \sqrt{g} \left\{ [ (\beta_1 + \frac{\varepsilon_1}{2}) - (\beta_2 + \frac{\varepsilon_2}{2})(\alpha_1 + \beta_1 + \varepsilon_1)]c \\
+ [ (\gamma_2 + \delta_2)(\alpha_1 + \beta_1 + \varepsilon_1) - (\gamma_1 + \delta_1 + \xi_1) \frac{1}{c}] \right\} + o(\sqrt{g})}{(\alpha_2 + \beta_2 + \varepsilon_2) - (\beta_2 + \frac{\varepsilon_2}{2})c\sqrt{g} + (\gamma_2 + \delta_2) \frac{\sqrt{g}}{c} + o(\sqrt{g})}
\]

Write \( \Gamma = (\beta_1 + \frac{\varepsilon_1}{2}) - (\beta_2 + \frac{\varepsilon_2}{2})(\alpha_1 + \beta_1 + \varepsilon_1) \)

\( \Delta = (\gamma_2 + \delta_2)(\alpha_1 + \beta_1 + \varepsilon_1) - (\gamma_1 + \delta_1 + \xi_1) \)

By (3) and (4) of the lemma \( \Gamma > 0, \Delta > 0 \).

\[
V = pb_1 + (1-p)b_2 - \sqrt{g} \left[ \Gamma c + \frac{\Delta}{c} \right] + o(\sqrt{g})
\]

\[
= pb_1 + (1-p)b_2 - \sqrt{g} \left[ (\sqrt{\Gamma c} - \frac{\sqrt{\Delta}}{c})^2 + 2\sqrt{\Gamma \Delta} \right] + o(\sqrt{g})
\]
Hence $V$ is maximized by

$$c^* = \sqrt{\frac{\Delta}{1^2}} + o(1)$$

i.e. the optimal $F = \frac{\sqrt{\Delta}}{\sqrt{1^2}} (1 + o(1))$ ....(3.1.4)

The maximum $V$ is

$$V^* = pb_1 + (1-p)b_2 - 2\sqrt{g} \sqrt{\Delta} + o(\sqrt{g}) ....(3.1.5)$$

**Case 1.**

$$\lim_{g \to 0} F = F_o, \quad 0 < F_o < \infty$$

$$\lim_{g \to 0} \lambda = \lim_{g \to 0} (1-g)^F = 1 \text{ and } \lim_{g \to 0} \frac{(1-\lambda)}{g} = F_o$$

Hence

$$\lim_{g \to 0} V = \frac{\alpha_1 F_0 + \beta_1 F_0 + \gamma_1 + \delta_1 + \epsilon_1 F_0 + \zeta_1}{\alpha_2 F_0 + \beta_2 F_0 + \gamma_2 + \delta_2 + \epsilon_2 F_0}$$

$$= \frac{F_0(\alpha_1 + \beta_1 + \epsilon_1) + (\gamma_1 + \delta_1 + \zeta_1)}{F_0 + \gamma_2 + \delta_2}$$

$$\lim_{g \to 0} V^* - \lim_{g \to 0} V = \frac{(F_0 + \gamma_2 + \delta_2)(\alpha_1 + \beta_1 + \epsilon_1) - F_0(\alpha_1 + \beta_1 + \epsilon_1) - (\gamma_1 + \delta_1 + \zeta_1)}{F_0 + \gamma_2 + \delta_2}$$

$$= \frac{\Delta}{F_0 + \gamma_2 + \delta_2} > 0$$

Hence

$$V^* - V = \frac{\Delta}{F_0 + \gamma_2 + \delta_2} + o(1).$$
Case 2. $F \to \infty$, $F = \frac{\omega(g)}{\sqrt{g}}$ where $\omega(g) \to 0$.

\[
\lambda = (1-g)^F = 1 - Fg + \frac{F(F-1)}{2} g^2 - \ldots
\]

\[= 1 + o(\sqrt{g}) \text{ since } F/\sqrt{g} = \omega(g) \to 0\]

\[
\frac{\lambda}{F} = \frac{1}{F} + o(\sqrt{g})
\]

\[
\frac{(1-\lambda)(1-g)}{gF} = (1-g)(1 + o(\sqrt{g})) = 1 + o(\sqrt{g})
\]

\[
\frac{(1-\lambda)(1-g)}{gF^2} = \frac{1}{F} + o(\sqrt{g})
\]

Dividing the numerator and denominator of $V$ by $F$, we have

\[
V = \frac{(\alpha + \beta + \epsilon) + \frac{1}{F}(\gamma + \delta + \xi) + o(\sqrt{g})}{1 + \frac{1}{F}(\gamma_2 + \delta_2) + o(\sqrt{g})}
\]

In fact,

\[
V = \frac{(\alpha + \beta + \epsilon) + \frac{1}{F}(\gamma + \delta + \xi) + o(\sqrt{g})}{1 + \frac{1}{F}(\gamma_2 + \delta_2) + o\left(\frac{1}{F}\right)}
\]

\[= (\alpha + \beta + \epsilon) - \frac{\Delta}{F} + o\left(\frac{1}{F}\right)\]

\[
V^* = (\alpha + \beta + \epsilon) + o\left(\frac{1}{F}\right)
\]

\[
V^* - V = \frac{\Delta}{F} + o\left(\frac{1}{F}\right)
\]

Case 4. $F \to \infty$, $F = \frac{\Omega(g)}{\sqrt{g}}$ where $\Omega(g) \to \infty$, $\sqrt{g} \Omega(g) \to 0$

Let $\eta = gF = \sqrt{g} \Omega(g)$

$\eta \to 0$ as $g \to 0$.

\[
\lambda = (1-g)^F = 1 - Fg + \frac{F(F-1)}{2} g^2 - \ldots
\]

\[= 1 - \eta + o(\eta)\]
\[
\frac{1}{F} = o(\eta) \quad \text{since} \quad \frac{1}{gF^2} = \frac{1}{[\Omega(g)]^2} \to 0 \quad \text{as} \quad g \to 0
\]

\[
\frac{\lambda}{F} = o(\eta)
\]

\[
\frac{(1 - \lambda)(1 - g)}{gF} = 1 - \frac{\eta}{2} + o(\eta)
\]

\[
\frac{(1 - \lambda)(1 - g)}{gF^2} = o(\eta)
\]

Hence

\[
V = \frac{\alpha_1 + \beta_1 (1 - \eta) + \epsilon_1 (1 - \frac{\eta}{2}) + o(\eta)}{\alpha_2 + \beta_2 (1 - \eta) + \epsilon_2 (1 - \frac{\eta}{2}) + o(\eta)}
\]

\[
= (\alpha_1 + \beta_1 + \epsilon_1) - \eta \left[ (\beta_1 + \frac{\epsilon_1}{2}) - (\alpha_1 + \beta_1 + \epsilon_1)(\beta_2 + \frac{\epsilon_2}{2}) \right] + o(\eta)
\]

\[
= (\alpha_1 + \beta_1 + \epsilon_1) - \eta \Gamma + o(\eta)
\]

Now \( \sqrt{g} = o(\eta) \)

Hence \( V^* = (\alpha_1 + \beta_1 + \epsilon_1) + o(\eta) \)

\[
V^* - V = \eta \Gamma + o(\eta)
\]

Case 5. \( F \to \infty, \frac{\Omega(g)}{\sqrt{g}} = \infty, \sqrt{g}, \Omega(g) \to g_0, 0 < g_0 < \infty \)

Hence

\[
\lim_{g \to 0} \lambda = \lim_{g \to 0} (1 - g)^F = e^{-g_0}
\]

\[
\lim_{g \to 0} V = \frac{\alpha_1 + \beta_1 e^{-g_0} + \epsilon_1 \left(1 - e^{-g_0}\right)}{\alpha_2 + \beta_2 e^{-g_0} + \epsilon_2 \left(1 - e^{-g_0}\right)}
\]
\[
\lim_{g \to 0} V^* - \lim_{g \to 0} V = (\alpha_1 + \beta_1 + \epsilon_1) - \frac{(\alpha_1 + \beta_1 + \epsilon_1) - \beta_1 \left(1 - e^{-g_0}\right) - \epsilon_1 \left(\frac{g_0 - 1 + e^{-g_0}}{g_0}\right)}{1 - \beta_2 \left(1 - e^{-g_0}\right) - \epsilon_2 \left(\frac{g_0 - 1 + e^{-g_0}}{g_0}\right)} > 0 \text{ if }
\]

\[
\left(1 - e^{-g_0}\right) (\beta_1 - \beta_2 (\alpha_1 + \beta_1 + \epsilon_1)) + \left(\frac{g_0 - 1 + e^{-g_0}}{g_0}\right) (\epsilon_1 - \epsilon_2 (\alpha_1 + \beta_1 + \epsilon_1)) > 0
\]

Since \( g_0 \) is positive \( 1 > e^{-g_0} \)

Also \( g_0 - 1 + e^{-g_0} > 0 \)

\[
\beta_1 - \beta_2 (\alpha_1 + \beta_1 + \epsilon_1) = \frac{[q + \pi_2(1-q)] (p-q) (b_1 - b_2)}{(1-q)(1-\pi_2)} \text{ for systematic sampling}
\]

\[
= \frac{[p + \pi_2(1-p)] (p-q) (b_1 - b_2)}{(1-q)(1-\pi_2)} \text{ for stratified sampling}
\]

> 0 in both cases

and

\[
\epsilon_1 - \epsilon_2 (\alpha_1 + \beta_1 + \epsilon_1) = (b_1 - b_2) (p-q) \text{ for systematic sampling}
\]

\[
= \frac{(b_1 - b_2) (p-q) (1-p)}{(1-q)} \text{ for stratified sampling}
\]

> 0 in both cases

Hence \( \lim_{g \to 0} V^* - \lim_{g \to 0} V = A > 0 \)

\[
V^* - V = A + o(1)
\]
Case 6. $F \to \infty, \quad F = \frac{\Omega(g)}{\sqrt{g}}, \quad \Omega(g) \to \infty, \quad \Omega(g) \to \infty$

$$\lim_{g \to 0} \lambda = \lim_{g \to 0} (1-g)^F = \lim_{g \to 0} e^{-gF} = 0$$

$$\lim_{g \to 0} V = \frac{\alpha_1}{\alpha_2} = q\beta_1 + (1-q)\beta_2$$

$$\lim_{g \to 0} V^* - \lim_{g \to 0} V = (p-q)(\beta_1 - \beta_2)$$

$$V^* - V = (p-q)(\beta_1 - \beta_2) + o(1).$$

Thus, for all possible values of $F$, provided $g$ is small, the income is maximized for a value of $F$ equal to $\frac{c}{\sqrt{g}}(1 + o(1))$ for the cases of systematic and stratified sampling.

Random Sampling.

There remains the consideration of the cases where random sampling is employed in stage 1. Each item is sampled with probability $1/F$. Writing $f = 1/F$, it will be proved that the maximizing $f$ can be found explicitly, and that it is of order $O(\sqrt{g})$.

Lemma 3.2 For any regular plan satisfying the conditions of the theorem where random sampling is used in stage 1, the long run average income is of the form

$$V = \frac{a_1 f^2 + b_1 f + c_1}{a_2 f^2 + b_2 f + c_2}$$

...(3.1.6)

where $a_1 = O(1), b_1 = O(1), c_1 = O(g)$

$a_2 = O(1), b_2 = O(1), c_2 = O(g)$
Proof. Let $I_1$, $I_2$ be the expected incomes from the remainder of the cycle, given that an item is about to be produced and the process is in states I, II respectively.

Then, considering the possible outcomes,

$$I_1 = (1-g)(1-f)(pb_1 + (1-p)b_2 + I_1)$$
+ $(1-g)f(p)(b_1 - c + I_1)$
+ $(1-g)f(1-p)(b_2 - c + E_1(i|1) + I_1)$
+ $(1-g)f(1-p)(1-\pi_1)(b_2 - c + E_1(i|3) - n_1 \cdot d_1)$
+ g(1-f)(qb_1 + (1-q)b_2 + I_2)
+ gf(qb_1 - c + I_2)$
+ gf(1-q)\pi_2(b_2 - c + E_2(i|1) + I_2)$
+ gf(1-q)(1-\pi_2)(b_2 - c + E_2(i|3) - n_2 \cdot d_2)$

and

$$I_2 = (1-f)(qb_1 + (1-q)b_2 + I_2)$$
+ f(qb_1 - c + I_2)$
+ f(1-q)\pi_2(b_2 - c + E_2(i|1) + I_2)$
+ f(1-q)(1-\pi_2)(b_2 - c + E_2(i|3) - n_2 \cdot d_2)$

$I_1 = E(I)$ and so

$[g + (1-g)f(1-p)(1-\pi_1)]E(I)$

$$= (1-g)(pb_1 + (1-p)b_2 - cf) + \frac{g[qb_1 + (1-q)b_2 - cf]}{f(1-q)(1-\pi_2)}$$
+ $(1-g)f(1-p)E_1(i) + \frac{gE_2(i)}{1-\pi_2} - n_1 \cdot d_1(1-g)f(1-p)(1-\pi_1) - n_2 \cdot d_2 \cdot g$
and hence

\[ [g + (1-g)f(1-p)(1-\pi_1)]E(L) = 1-g + \frac{g}{f(1-q)(1-\pi_2)} + (1-g)f(1-p)E_1(\ell) \]
\[ + \frac{gE_2(\ell)}{1-\pi_2} + n_1(1-g)f(1-p)(1-\pi_1) + n_2 g. \]

\[ V = \frac{E(I)}{E(L)} = \frac{a_1 f^2 + b_1 f + c_1}{a_2 f^2 + b_2 f + c_2} \]

and hence equations (3.1.7) follow:

\[ a_1 = -c(1-g) + (1-g)(1-p)E_1(i) - n_1 d_1 (1-g)(1-p)(1-\pi_1) = 0(1) \]
\[ b_1 = (1-g)(p b_1 + (1-p)b_2) - \frac{c g}{(1-q)(1-\pi_2)} + \frac{gE_2(i)}{1-\pi_2} - n_2 d_2 g = 0(1) \]
\[ c_1 = \frac{g[q b_1 + (1-q)b_2]}{(1-q)(1-\pi_2)} = 0(g) \]

\[ a_2 = (1-g)(1-p)E_1(\ell) + n_1(1-g)(1-p)(1-\pi_1) = 0(1) \]
\[ b_2 = (1-g) + \frac{gE_2(\ell)}{1-\pi_2} + n_2 g = 0(1) \]
\[ c_2 = \frac{g}{(1-q)(1-\pi_2)} = 0(g) \]

Hence the lemma.

Returning to the proof of the theorem

\[ \frac{\partial V}{\partial f} = \frac{[a_1 b_2 - a_2 b_1] f^2 + 2 [a_1 c_2 - a_2 c_1] f + [b_1 c_2 - b_2 c_1]}{[a_2 f^2 + b_2 f + c_2]^2} \]
\[ \frac{\partial V}{\partial f} = 0 \text{ if} \]
\[
(a_1 b_2 - a_2 b_1) f^2 + 2(a_1 c_2 - a_2 c_1) f + (b_1 c_2 - b_2 c_1) = 0 \quad \ldots (3.1.6)
\]

\[
f = \frac{a_2 c_1 - a_1 c_2 + \sqrt{(a_1 c_2 - a_2 c_1)^2 - (a_1 b_2 - a_2 b_1)(b_1 c_2 - b_2 c_1)}}{(a_1 b_2 - a_2 b_1)}
\]

\[
= g(1-g) A + \sqrt{g^2 (1-g)^2 A^2 - g(1-g)(B + Cg + Dg^2)}
\]

\[
(1-g)(E + Fg)
\]

where \( A, B, C, D, E, F \) do not depend on \( g \).

Hence a first approximation to \( f \) is

\[
f = \frac{\sqrt{-Bg}}{E}, \text{ provided } E \neq 0, B < 0
\]

\[
E = -c + (1-p) E_1(1) - n_1 d_1 (1-p)(1-\kappa_1) - (pb_1 + (1-p)b_2)(l-p)E_1(l) + n_1 (1-p)(1-\kappa_1)
\]

\[
= -[c + n_1 (1-p)(1-\kappa_1)(d_1 + pb_1 + (1-p)b_2) + (1-p)(p b_1 + (1-p)b_2)E_1(l) - (1-p)E_1(l)]
\]

< 0, since \( E_1(l) < (p b_1 + (1-p)b_2)E_1(l) \)

Hence \( E \neq 0 \)

and

\[
B = E \frac{(p-q)(b_1-b_2)}{(1-q)(1-\kappa_2)} < 0 \text{ since } E < 0
\]

Thus the approximation is

\[
f = \frac{g(p-q)(b_1-b_2)}{(1-q)(1-\kappa_2)\{c+(1-p)[p b_1 + (1-p)b_2]E_1(l) - (1-p)E_1(l) + n_1 (1-p)(1-\kappa_1)[d_1 + p b_1 + (1-p)b_2]\}}
\]

\[
\ldots (3.1.8)
\]
and the approximate optimal income

\[ V^* = p b_1 + (1-p) b_2 \]

\[ -2\sqrt{g} \frac{1}{(1-q)(1-\pi_2)} \left\{ c + (1-p) \left[ (p b_1 + (1-p) b_2) \bar{E}_1(\varepsilon) - \bar{E}_1(1) \right] + \pi_1 (1-\pi_1) (1-p) (d_1 + p b_1 + (1-p) b_2) \right\} \]

\[ \ldots (3.1.9). \]

Hence the maximizing \( f \) is of order \( \sqrt{g} \), and the theorem is proved for all cases.

Note. The assumption of the theorem that the process can not change state during stage 2 can be justified by the fact that stage 2 is a period of probation when the process is under close surveillance and highly unlikely to deteriorate appreciably. Also for most plans the time spent in stage 2 will usually be small, and hence the probability of a change of state would under any circumstances be very small. The assumption will be made for all the regular type plans considered.

3.2 The second method - method B - was used by Lieberman and Solomon [7], Resnikoff [8] and Derman, Littauer and Solomon [2], in their analyses of the multi-level continuous sampling plan for a one-state model. Villegas [10] adapted the method to a two-state model for a version of the Doige Plan.

If a sampling plan is being used on a production process, the "state of nature" can be described by the following:

1. The state of the machine (I or II)

2. The action taken (e.g. produce an item and inspect, produce and
pass without inspection, or do not produce since the machine is shut down).

3. The relative position within the plan.

These states form a stochastic process, each state of which corresponds to a unit of production time. The probability of passing from one state to another may depend on the transition probability, the previous state, the result of inspection at the previous state, or the sampling plan in use.

The process is stationary and also the process is so defined in all cases illustrated that the transition probabilities depend only on the previous state. The process is, therefore, a Markov chain.

Applying a theorem of Doob [4, p. 182] it can be stated that if

(i) the number of states is finite,

(ii) there are no transient states,

(iii) there is only one ergodic class, then the probability that at a time \( t \) the Markov chain is in a state \( s \) tends to a limit \( P(s) \) as \( t \to \infty \). This limit forms a probability distribution over the states of the Markov chain.

Also if conditions (i), (ii), (iii) hold and if \( I(s) \) is the expected income for state \( s \) of the chain, then by the Strong Law of Large Numbers for Markov chains [4, p. 219], the long run average income

\[
V = \sum_s I(s)P(s).
\]

Suppose the Markov chain has \( n \) states \( (E_1, E_2, \ldots, E_n) \). Let \( p_{ij} \) be the transition probability from state \( E_i \) to \( E_j \). Let \( P \) be the
stochastic matrix

\[
\begin{pmatrix}
  p_{11}, p_{12}, \cdots p_{1n} \\
p_{21}, p_{22}, \cdots p_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
p_{n1}, p_{n2}, \cdots p_{nn}
\end{pmatrix}
\]

Then if \( P(E_i) \) is the limiting probability of the state \( E_i \),

\[
(P(E_1), P(E_2), \ldots P(E_n)) \begin{pmatrix}
  p_{11}, p_{12}, \cdots p_{1n} \\
p_{21}, p_{22}, \cdots p_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
p_{n1}, p_{n2}, \cdots p_{nn}
\end{pmatrix} = (P(E_1), P(E_2), \ldots P(E_n))
\]

\[\ldots(3.2.1)\]

Thus \((P(E_1), \ldots P(E_n))\) is the eigenvector, the sum of whose components is unity, corresponding to the unit eigenvalue of the stochastic matrix. (Every stochastic matrix has a unit eigenvalue).

This is a general method of obtaining the probabilities of the states. In practice equation (3.2.1) is equivalent to \( n \) linear equations in the \( P(E_i) \), most of which are of a very simple form. These will be indicated whenever this method is used.

It was the author's experience that, when a plan called for the use of either systematic or stratified sampling, the first method using the theorem of Girshick and Rubin (3.1.1) yielded the simpler computation of \( V \), but that when random sampling is indicated the above method was simpler, particularly when the plan was not of the regular type.
4. The Basic Sampling Plan.

Three versions of this plan are considered, the versions differing in their interpretation of the method of sampling.

4.1 Systematic Sampling.

Procedure. Inspect every $F^{th}$ item. Stop the process and examine the machine as soon as a defective occurs.

This is a regular plan where stage 2 is empty and $\pi_1 = \pi_2 = 0$. It satisfies the conditions of theorem 3.2.

Results.

Income coefficients (equation 3.1.1, page 11)

$$
\alpha_1 = \frac{qb_1 + (1-q)b_2}{1-q}
$$

$$
\alpha_2 = \frac{1}{1-q}
$$

$$
\beta_1 = -\frac{q(qb_1 + (1-q)b_2)}{1-q}
$$

$$
\beta_2 = \frac{-q}{1-q}
$$

$$
\gamma_1 = \frac{c_2}{1-q} - n_1 d_1 (1-p) + n_2 d_2
$$

$$
\gamma_2 = n_1 (1-p) - n_2
$$

$$
\delta_1 = \frac{-c}{1-q} - n_2 d_2
$$

$$
\delta_2 = n_2
$$

$$
\epsilon_1 = (b_1 - b_2)(p-q)
$$

$$
\epsilon_2 = 0
$$

$$
\zeta_1 = 0
$$

Approximation to optimal $F$ for small $g$

$$
F = \sqrt{\frac{2(1-q) \{ c + n_1 (1-p)[pb_1 + (1-p)b_2 + d_1] \}}{g(b_1 - b_2)(p-q)(1+q)}}
$$
Approximate optimal income

$$V^* = pb_1 + (1-p)b_2 - 2\sqrt{g} \sqrt{c + n_1(l-p)[pb_1 + (1-p)b_2 + d_1] - \frac{(b_1-b_2)(p-q)(l-q)}{2(l-q)}}$$

**Proof**

Stage 2 is empty, and hence $E_1(i) = E_2(i) = E_1(\lambda) = E_2(\lambda) = 0$.

Also $\pi_1 = \pi_2 = 0$.

Thus the income parameters may be deduced directly from equations (3.1.2). Also the approximation to $F$ is given by equation (3.1.4), $F = \sqrt{\frac{\Delta}{g}}$, the approximation $V^*$ by equation (3.1.5).

4.2 **Stratified Sampling.**

**Procedure.** Divide the production into blocks of size $F$ and inspect one item drawn at random from each block. Stop the process and examine the machine if a defective occurs.

Again this is a regular plan satisfying the conditions of theorem 3.2.

**Results.**

**Income coefficients (equation 3.1.1, page 11)**

- $\alpha_1 = \frac{qb_1 + (1-q)b_2}{1-q}$
- $\alpha_2 = \frac{1}{1-q}$
- $\beta_1 = \frac{-p(qb_1 + (1-q)b_2)}{1-q}$
- $\beta_2 = \frac{-p}{1-q}$
- $\gamma_1 = \frac{cp}{1-q} - n_1 d_1(l-p) + n_2 d_2$
- $\gamma_2 = n_1(1-p) - n_2$
- $\delta_1 = \frac{-c}{1-q} - n_2 d_2$
- $\delta_2 = n_2$
- $\epsilon_1 = \frac{b_1(p-q)}{1-q}$
- $\epsilon_2 = \frac{p-q}{1-q}$
- $\zeta_1 = \frac{-c(p-q)}{1-q}$
Approximation to optimal \( F \) for small \( g \)

\[
F = \sqrt{\frac{2(1-q)\{c + n_1(1-p)[pb_1 + (1-p)b_2 + d_1]\}}{g(b_1-b_2)(p-q)(1+p)}}
\]

Approximate optimal income

\[
V^* = pb_1 + (1-p)b_2 - 2\sqrt{g\{c + n_1(1-p)[pb_1 + (1-p)b_2 + d_1]\}} \left(\frac{(b_1-b_2)(p-q)(1+p)}{2(1-q)}\right)
\]

Proof. The proof for the case where stratified sampling is used is exactly the same as that for the case of systematic sampling, except that equations (3.1.3) are used for the income function.

4.3 Random Sampling.

Procedure. Inspect each item with probability \( f \). Stop the process and examine the machine as soon as a defective is found.

This is a regular plan where stage 2 is empty, \( \pi_1 = \pi_2 = 0 \).

Results.

**Income coefficients** (equation 3.1.6, page 25)

\[
a_1 = -c(1-g) - n_1 d_1(1-g)(1-p)
\]

\[
b_1 = (1-g)(pb_1 + (1-p)b_2) - \frac{cg}{1-q} - n_2 d_2 g
\]

\[
c_1 = \frac{g[qb_1 + (1-q)b_2]}{(1-q)}
\]

\[
a_2 = n_1(1-g)(1-p)
\]

\[
b_2 = (1-g) + n_2 g
\]

\[
c_2 = \frac{g}{1-q}
\]
Optimal parameter $f$. The optimal $f$ is a solution of the quadratic

$$(a_1 b_2 - a_2 b_1) x^2 + 2(a_1 c_2 - a_2 c_1) f + (b_1 c_2 - b_2 c_1) = 0.$$  

Approximation to the solution for small $g$

$$f = \frac{(p-q)(b_1 - b_2) g}{\sqrt{(1-q) \left\{ n_1 (1-p) [pb_1 + (1-p) b_2 + d_1] + c \right\}}}$$  

Approximate optimal income

$$V^* = pb_1 + (1-p)b_2 - 2\sqrt{g} \sqrt{\frac{(p-q)(b_1 - b_2) \left\{ c + n_1 (1-p) [pb_1 + (1-p) b_2 + d_1] \right\}}{1-q}}$$  

Proof.

$$E_1(1) = E_2(1) = E_1(\varepsilon) = E_2(\varepsilon) = 0$$  

Hence the equations for the income may be deduced directly from equations (3.1.7). The equation giving the optimal $f$ is equation (3.1.6). Equation (3.1.8) gives the approximate $f$.

Alternative Procedure. As an illustration of the use of method B of computing the average income, this method will now be used to find the above results for the random sampling case.

Consider the triple $(s, a, n)$ where

- $s = 1$ if the machine is in state I.
- $s = 2$ if the machine is in state II.
- $a = 0$ if an item is produced and passed without inspection.
- $a = 1$ if an item is produced and inspected.
- $a = 2$ if the machine is shut down.
\( n = 0 \) if \( a = 0, 1 \)  

number of time units remaining before the machine goes back into production if \( a = 2 \).

Then the sequence of random variables taking the values \( (s, a, n) \) is a stationary Markov chain with \( 4 + n_1 + n_2 \) states. There are no transient states and there is only one ergodic class.

Figure 4.3.1 is the stochastic matrix \( P \) for this plan. The row vector \( E \) such that \( EP = E \) is to be found.

Let \( A(1-f) \) be the first column of \( P \). Then \( Af \) is the second column of \( P \). Also if \( B(1-f) \) is the third column of \( P \), \( Bf \) is the fourth column.

Then  
\[
\begin{align*}
P(1, 0, 0) &= EA(1-f) \\
P(1, 1, 0) &= EAf \\
P(2, 0, 0) &= EB(1-f) \\
P(2, 1, 0) &= EBf
\end{align*}
\]

Also  
\[
\begin{align*}
P(1, 2, n_1) &= (1-p) P(1, 1, 0) \\
P(1, 2, h) &= P(1, 2, h-1), h = n_1, \ldots, 2. \\
P(2, 2, h_2) &= (1-q) P(2, 1, 0) \\
P(2, 2, k) &= P(2, 2, k-1), k = n_2, \ldots, 2
\end{align*}
\]

Thus all the stationary probabilities can be expressed in terms of \( EA \) and \( EB \).

\[
EA = (1-g)P(1,0,0) + (1-g)p P(1,1,0) + (1-g)P(1,2,1) + (1-g)P(2,2,1) \\
= (1-g)EA(1-f) + (1-g)p EAf + (1-g)(1-p)EAF + (1-g)(1-q)EBf \\
EA = \frac{(1-g)(1-q)f}{g} EB
\]
\[
\begin{array}{ccccccccccccccccccc}
(1,0,0) & (1,1,0) & (2,0,0) & (2,1,0) & (1,2,n_1) & (1,2,n_1-1) & \cdots & (1,2,1) & (2,2,n_2) & (2,2,n_2-1) & \cdots & (2,2,1) \\
\hline
(1,0,0) & (1-g)(1-f) & (1-g)f & g(1-f) & g & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
(1,1,0) & (1-g)(1-f)p & (1-g)f_p & g(1-f)p & gfp & 1-p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
(2,0,0) & 0 & 0 & 1-f & f & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
(2,1,0) & 0 & 0 & (1-f)q & f_q & 0 & 0 & \cdots & 0 & 1-q & 0 & \cdots & 0 \\
(1,2,n_1) & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
(1,2,n_1-1) & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(1,2,2) & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & c \\
(1,2,1) & (1-g)(1-f) & (1-g)f & g(1-f) & g & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & c \\
(2,2,n_2) & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
(2,2,n_2-1) & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(2,2,2) & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
(2,2,1) & (1-g)(1-f) & (1-g)f & g(1-f) & g & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\end{array}
\]
Also the sum of the probabilities of the states is \( 1 \). Using this, \( EA \) and \( EB \) may be determined explicitly and hence \( P(s, a, n) \) may be determined for all \( s, a, n \).

Let \( V(s, a, n) \) be the expected income from state \( (s, a, n) \)

\[
V(1, 0, 0) = pb_1 + (1-p)b_2
\]
\[
V(1, 1, 0) = pb_1 + (1-p)b_2 - c
\]
\[
V(1, 2, h) = -d_1, \ h = n_1, \ldots, 1
\]
\[
V(2, 0, 0) = qb_1 + (1-q)b_2
\]
\[
V(2, 1, 0) = qb_1 + (1-q)b_2 - c
\]
\[
V(2, 2, k) = -d_2, \ k = n_2, \ldots, 1.
\]

Then

\[
V = (1-f)EA[pb_1 + (1-p)b_2] + fEA[pb_1 + (1-p)b_2 - c]
\]
\[
- n_1(1-p)EA[f d_1 + (1-f)EB[qb_1 + (1-q)b_2]]
\]
\[
+ fEB[qb_1 + (1-q)b_2 - c] - n_2(1-q)EB d_2
\]

\[
= EA \left\{ pb_1 + (1-p)b_2 - cf - n_1 d_1 (1-p)f + \frac{g[qb_1 + (1-q)b_2]}{f(1-g)(1-q)} - \frac{gc}{(1-g)(1-q)} \right\}
\]

\[
- \frac{n_2 d_2 g}{1-g}
\]

and

\[
l = EA \left\{ 1 + \frac{g}{f(1-g)(1-q)} + n_1 f(1-p) + \frac{n_2 g}{1-g} \right\}.
\]

Hence \( V \) is found as before.

Finding the optimal \( f \) now follows exactly as in the case where method \( A \) is used.
5. The Simplified Dodge Plan.

5.1 Systematic sampling.

Procedure. Inspect every Fth item. When a defective is found, change to one hundred percent inspection. If the next \(j\) consecutive items are non-defective, return to sampling every Fth item. If, however, a defective is found within the next \(j\) items, stop the machine immediately and examine the process. When the machine starts up again, the plan begins with inspection of every Fth item.

This is a regular plan satisfying the conditions of the theorem.

Results.

Income coefficients (equation 3.1.1, page 11).

\[
\begin{align*}
\alpha_1 &= \frac{qb_1 + (1-q)b_2}{(1-q)(1-q^j)} \\
\beta_1 &= -\frac{[q + q^j(1-q)][qb_1 + (1-q)b_2]}{(1-q)(1-q^j)} \\
\gamma_1 &= \frac{c[q + q^j(1-q)]}{(1-q)(1-q^j)} + (1-p^j)[pb_1 + (1-p)b_2 - e] - \frac{[qb_1 + (1-q)b_2 - e]}{(1-q)} \\
&\quad - n_1d_1(1-p)(1-p^j) + n_2d_2 \\
\delta_1 &= -\frac{c}{(1-q)(1-q^j)} + \frac{[qb_1 + (1-q)b_2 - e]}{1-q} - n_2d_2 \\
\epsilon_1 &= (b_1 - b_2)(p-q) \\
\xi_1 &= 0 \\
\alpha_2 &= \frac{1}{(1-q)(1-q^j)} \\
\beta_2 &= -\frac{[q + q^j(1-q)]}{(1-q)(1-q^j)}
\end{align*}
\]

\[ \gamma_2 = 1 - p^j - \frac{1}{1-q} + n_1(1-p)(1-p^j) - n_2 \]

\[ \delta_2 = \frac{1}{1-q} + n_2 \]

\[ \epsilon_2 = 0 \]

Approximation to optimal \( F \) for small \( g \).

\[ F = \sqrt{\frac{[n_1(1-p)(1-p^j)(pb_1 + (1-p)b_2 + d_1) + c + e(1-p^j)]}{g(b_1 - b_2)(p-q) \left[ \frac{1}{(1-q)(1-q^j)} - \frac{1}{2} \right]}} \]

Approximate optimal income.

\[ y^* = pb_1 + (1-p)b_2 \]

\[ -2\sqrt{g} \sqrt{b_1 - b_2)(p-q) \left[ \frac{1}{(1-q)(1-q^j)} - \frac{1}{2} \right] [n_1(1-p)(1-p^j)(pb_1 + (1-p)b_2 + d_1) + c + e(1-p^j)]} \]

Proof. \( \pi_1 = \) Probability of no defectives in \( j \) items

\[ \pi_1 = p^j \]

\[ \pi_2 = q^j \]

\[ E_1(i) = \sum_{r=0}^{j=1} (1-p)^r (pb_1 + b_2 - (r+1)e) + j(b_1 - e)p^j \]

\[ = \frac{(1-p^j)}{(1-p)} [pb_1 + (1-p)b_2 - e] \]

Similarly \( E_2(i) = \frac{(1-q^j)}{(1-q)} [qb_1 + (1-q)b_2 - e] \)

Putting \( b_1 = b_2 = 1, \ e = 0 \)
\[ E_1(\alpha) = \frac{(1-p^j)}{(1-p)} \]

and \[ E_2(\alpha) = \frac{(1-q^j)}{(1-q)} \]

Substitution in equations (3.1.2) now yields the income coefficients and in equations (3.1.4) and (3.1.5) the approximate optimal \( F \) and the corresponding approximate income respectively.

5.2 Stratified Sampling.

Procedure. Divide the production into blocks of size \( F \) and inspect one item drawn at random from each block. If a defective item is found in some block, inspect one hundred percent the production following this block. If the next \( j \) items inspected are non-defective, revert to sampling one in every \( F \). If, however, a defective is found, stop the process and examine the machine. When the machine starts up again, the plan begins with inspection of one in every \( F \) items.

Again this is a regular plan satisfying the conditions of the theorem.

Results.

Income coefficients (equations 3.1.1, page 11)

\[ \alpha_1 = \frac{qb_1 + (1-q)b_2}{(1-q)(1-q^j)} \]

\[ \beta_1 = -\frac{[p + q^j(1-p)](qb_1 + (1-q)b_2)}{(1-q)(1-q^j)} \]
\[
\gamma_1 = \frac{c[p + q^j(p-1)]}{(1-q)(1-q^j)} + (1-p^j)(pb_1 + (1-p)b_2 - e) - \frac{(qb_1 + (1-q)b_2 - e)}{1-q} \\
- n_1 d_1 (1-p)(1-p^j) + n_2 d_2 \\
\delta_1 = - \frac{c}{(1-q)(1-q^j)} + \frac{(qb_1 + (1-q)b_2 - e)}{1-q} - n_2 d_2 \\
\epsilon_1 = \frac{b_1(p-q)}{1-q} \\
\theta_1 = - \frac{c(p-q)}{1-q} \\
\alpha_2 = \frac{1}{(1-q)(1-q^j)} \\
\beta_2 = - \frac{[p + q^j(p-1)]}{(1-q)(1-q^j)} \\
\gamma_2 = (1-p^j) - \frac{1}{1-q} + n_1 (1-p)(1-p^j) - n_2 \\
\delta_2 = \frac{1}{1-q} + n_2 \\
\epsilon_2 = \frac{p-q}{1-q}
\]

Approximation to optimal $F$ for small $g$.

\[
F = \frac{(1-q)[c + e(1-p^j) + n_1 (1-p)(1-p^j)(d_1 + p_{b_1} + (1-p)b_2)]}{g(p-q)(b_1-b_2) \left[ \frac{1}{1-q^j} - \frac{1-p}{2} \right]} 
\]
Approximate optimal income

\[ V^* = pb_1 + (1-p)b_2 \]

\[ -2 \sqrt{ \frac{(p-q)(b_1-b_2)}{1-q} } \left[ \frac{1}{1-q^j} - \frac{1-p}{2} \right] \left[ c+e(1-p^d)+n_1(1-p)(1-p^d)(d_1+pb_1+(1-p)b_2) \right] \]

**Proof.** Stage 2 is unaffected by the method of sampling used in stage 1. Hence the values of \( \pi_1, \pi_2, E_1(\bar{1}), E_2(\bar{1}), E_1(\bar{2}), E_2(\bar{2}) \) are the same as in section 5.1. To find the values of the income coefficients these values are now substituted in equations (3.1.3) and the approximate optimal \( F \) and \( V^* \) are found from equations (3.1.4) and (3.1.5) respectively as before.

5.3 Random Sampling.

**Procedure.** Inspect every item with probability \( f \). When a defective is found, change to one hundred percent inspection. If the next \( j \) consecutive items are non-defective, return to sampling with probability \( f \). If, instead, a defective item is found within the next \( j \) items inspected, stop the machine and examine the process. When the machine starts up again, the plan begins with sampling each item with probability \( f \).

This is, of course, still a regular plan satisfying the conditions of the theorem.

**Results.**

Income coefficients (equation 3.1.6, page 25)

\[ a_1 = -c(1-g)(1-g)(1-p^d)[pb_1+(1-p)b_2-e]-n_1d_1(1-g)(1-p)(1-p^d) \]

\[ b_1 = (1-g)[pb_1+(1-p)b_2] - \frac{cg}{(1-q)(1-q^d)} + \frac{g[qb_1+(1-q)b_2-e]}{1-q} - n_2d_2g \]
\[ c_1 = \frac{g(qb_1 + (1-q)b_2)}{(1-q)(1-q^j)} \]

\[ a_2 = (1-g)(1-p^j) + n_1(1-g)(1-p)(1-p^j) \]

\[ b_2 = (1-g) + \frac{g}{1-q} + n_2 \cdot g \]

\[ c_2 = \frac{g}{(1-q)(1-q^j)} \]

**Optimal parameter \( f \).** The optimal parameter \( f \) is a solution of the quadratic

\[ (a_1 b_2 - a_2 b_1) f^2 + 2(a_1 c_2 - a_2 c_1) f + (b_1 c_2 - b_2 c_1) = 0 \]

**Approximation to the solution for small \( g \).**

\[ f = \sqrt{\frac{g(p-q)(b_1-b_2)}{(1-q)(1-q^j)}} \left\{ c + e(1-p^j) + n_1(1-p)(1-p^j)[d_1 + pb_1 + (1-p)b_2] \right\} \]

**Approximate optimal income**

\[ v^* = pb_1 + (1-p)b_2 \]

\[-2\sqrt{g} \sqrt{(p-q)(b_1-b_2)} \left\{ c + e(1-p^j) + n_1(1-p)(1-p^j)[d_1 + pb_1 + (1-p)b_2] \right\} \]

**Proof.**

\( \pi_1, \pi_2, E_1(i), E_2(i), E_1(\ell), E_2(\ell) \) are the same as in section 5.1. The income coefficients may now be determined from equation (3.1.7), and hence the quadratic equation leading to the evaluation of the optimal \( f \) is found. The approximation is found from equation (3.1.8).
Note. It will be observed that for any regular plan the values of \( \pi_1, \pi_2, E_1(i), E_2(i), E_1(\delta), E_2(\delta) \) are the same no matter what method of sampling is used in stage 1. Once these quantities have been determined, it is merely a matter of substitution in the appropriate formulae, and the income coefficients, the approximation to the optimal sampling rate and the approximate average income can be found directly. Because of this, in the future, for any regular plan, only the case of systematic sampling will be considered.


Procedure. Inspect every \( F^{th} \) item. When a defective item is found, change to one hundred percent inspection and continue thereon until either

(a) \( j \) consecutive non-defective items are found

or (b) \( k \) defectives are found.

In case (a) return to sampling every \( F^{th} \) item and in case (b) stop the process and examine the machine. When the machine starts up again, the plan begins with sampling every \( F^{th} \) item.

This is a regular plan and because of this, only systematic sampling will be considered.

Results.

Income coefficients (equation 3.1.1, page 11)

\[
\alpha_1 = \frac{q b_1 + (1-q) b_2}{k} \frac{1}{(1-q)(1-q^j)}
\]
\begin{align*}
\beta_1 &= \frac{- (1-(1-q)(1-q^j)^k)(qb_1+(1-q)b_2)}{(1-q)(1-q^j)^k} \\
\gamma_1 &= \frac{c[1-(1-q)(1-q^j)^k]}{(1-q)(1-q^j)^j} + \frac{(1-p)^j[1-(1-p^j)^k][pb_1+(1-p)b_2-e]}{p^j} \\
&\quad - \frac{[1-(1-q^j)^k]}{q^j(1-q)(1-q^j)^k-1} - n_1d_1(1-p)(1-p^j)^k + n_2d_2 \\
\delta_1 &= - \frac{c}{(1-q)(1-q^j)^j} + \frac{[1-(1-q^j)^k]}{q^j(1-q)(1-q^j)^k-1} - n_2d_2 \\
\epsilon_1 &= (b_1-b_2)(p-q) \\
\xi_1 &= 0 \\
\alpha_2 &= \frac{1}{(1-q)(1-q^j)^k} \\
\beta_2 &= 1 - \frac{1}{(1-q)(1-q^j)^k} \\
\gamma_2 &= \frac{(1-p)^j[1-(1-p^j)^k]}{p^j} - \frac{[1-(1-q^j)^k]}{q^j(1-q)(1-q^j)^k-1} + n_1(1-p)(1-p^j)^k - n_2 \\
\delta_2 &= \frac{[1-(1-q^j)^k]}{q^j(1-q)(1-q^j)^k-1} + n_2 \\
\epsilon_2 &= 0 \\
\end{align*}

Approximation to the optimal \( F \) for small \( q \):

\[
F = \sqrt{\frac{c + \frac{e(1-p)^j[1-(1-p^j)^k]}{p^j} + n_1(1-p)(1-p^j)^k[d_1 + pb_1 + (1-p)b_2]}{g(p-q)(b_1-b_2) \left[ \frac{1}{(1-q)(1-q^j)^k} - \frac{1}{2} \right]}}
\]
Approximate optimal income

\[ V^* = pb_1 + (1-p)b_2 - 2\sqrt{\frac{1}{(b_1-b_2)(p-q)}} \left( \frac{1}{(1-q)(1-q^j)^k} - \frac{1}{2} \right) \]

\[ \cdot \sqrt{\frac{n_1(1-p)(1-p^j)^k(pb_1 + (1-p)b_2 + a_1) + c + e(1-p^d)[1-(1-p^d)^k]}{p^j}} \]

Proof.

\( \pi_1 = \) probability of \( j \) consecutive non-defectives before \( k \) defectives are found

\[ = p^j + (1-p^j)p^j + \ldots + (1-p^j)^{k-1}p^j \]

\[ = \frac{p^j[1-(1-p^j)^k]}{p^j} = 1-(1-p^j)^k \]

\( \pi_2 = 1-(1-q^j)^k \)

Let \( I_r \) be the expected income remaining in stage 2 given that \( r \) defectives have been found in this stage.

\[ I_k = 0 \]

\[ I_r = \sum_{s=0}^{j-1} (1-p)p^s(sb_1 + b_2 - (s+1)e + I_{r+1}) + p^j(1-e) \]

\[ r = 0, \ldots, k-1 \]

\[ = (1-p^j)I_{r+1} + \frac{(1-p^j)}{1-p}[pb_1 + (1-p)b_2 - e] \]

Hence \( E_1(i) = I_0 = \frac{(1-p^j)[1-(1-p^j)^k][pb_1 + (1-p)b_2 - e]}{p^j(1-p)} \)

Similarly \( E_2(i) = \frac{(1-q^j)[1-(1-q^j)^k][qb_1 + (1-q)b_2 - e]}{q^j(1-q)} \)
and from these

\[ E_1(I) = \frac{(1-p^d)[1-(1-p^d)^k]}{p^d(1-p)} \]

\[ E_2(I) = \frac{(1-q^d)[1-(1-q^d)^k]}{q^d(1-q)} \]

Substitution in equations (3.1.2), (3.1.4) and (3.1.5) yields the income coefficients, the approximate optimal \( F \) and the corresponding approximate income respectively. It may be observed that the simplified Dodge plan is the special case of this plan when \( k = 1 \).

7. The Savage Three Decision Plan (Attributes).

Procedure. Inspect every \( F \)th item. When a defective is found, change to a sequential sampling plan by attributes whose terminal decisions are:

i) Return to inspecting every \( F \)th item.

ii) Stop the process and examine the machine.

The procedure reverts to the inspection of every \( F \)th item when the machine starts up again after a shut-down period.

The plan is regular and when the assumption is made that the process does not change state during the sequential sampling, it also satisfies the conditions of theorem 3.1.2.

Results.

Suppose the sequential sampling stage (i.e. stage 2) is designed to give prescribed \( \pi_1, \pi_2 \). Let \( \pi_1 = 1-\alpha, \pi_2 = \beta \).
Then, for the income coefficients (equation 3.1.1, page 11)

\[\alpha_1 = \frac{qb_1 + (1-q)b_2}{(1-q)(1-\beta)}\]

\[\beta_1 = \frac{[q + \beta(1-q)](qb_1 + (1-q)b_2)}{(1-q)(1-\beta)}\]

\[\gamma_1 = \frac{c[q + \beta(1-q)]}{(1-q)(1-\beta)} + \frac{(1-p)[qb_1 + (1-p)b_2 - e][h_1(1-\alpha) - \alpha h_2]}{s - 1 + p} - \frac{[qb_1 + (1-q)b_2 - e][h_2(1-\beta) - \beta h_1]}{(1-\beta)(1-q-s)} - n_1 d_1(1-p)\alpha + n_2 d_2\]

\[\delta_1 = -\frac{c}{(1-q)(1-\beta)} + \frac{[qb_1 + (1-q)b_2 - e][h_2(1-\beta) - h_1\beta]}{(1-\beta)(1-q-s)} - n_2 d_2\]

\[\epsilon_1 = (b_1 - b_2)(p-q)\]

\[\xi_1 = 0\]

\[\alpha_2 = \frac{1}{(1-q)(1-\beta)}\]

\[\beta_2 = -\frac{[q + \beta(1-q)]}{(1-q)(1-\beta)}\]

\[\gamma_2 = \frac{(1-p)[h_1(1-\alpha) - h_2\alpha]}{s - 1 + p} - \frac{[h_2(1-\beta) - h_1\beta]}{(1-\beta)(1-q-s)} + n_1(1-p) - n_2\]

\[\delta_2 = \frac{[h_2(1-\beta) - h_1\beta]}{(1-\beta)(1-q-s)} + n_2\]

\[\epsilon_2 = 0\]
where \( h_1 = \frac{\log \frac{1-\alpha}{\beta}}{\log \frac{p(1-q)}{q(1-p)}} \)

\( h_2 = \frac{\log \frac{1-\beta}{\alpha}}{\log \frac{p(1-q)}{q(1-p)}} \)

\( s = \frac{\log \frac{p}{q}}{\log \frac{p(1-q)}{q(1-p)}} \)

Approximation to the optimal \( F \) for small \( g \).

\[
F = \sqrt{\frac{e(1-p)[h_1(1-\alpha) - h_2\alpha]}{c + \frac{s - 1 + p}{n_1(1-p)\alpha[d_1 + pb_1 + (1-p)b_2]}}} + \frac{1}{g(p-q)(b_1 - b_2) \left[ \frac{1}{(1-q)(1-\beta)} - \frac{1}{2} \right]}
\]

Approximate optimal income.

\( V^* = pb_1 + (1-p)b_2 \)

\[
-2\sqrt{g(p-q)(b_1 - b_2) \left[ \frac{1}{(1-q)(1-\beta)} - \frac{1}{2} \right] \left[ c + \frac{e(1-p)[h_1(1-\alpha) - h_2\alpha]}{s - 1 + p} + n_1(1-p)\alpha[d_1 + pb_1 + (1-p)b_2] \right]}
\]

Proof.

When the plan is in stage 2, the ideal situation would be for the plan to indicate passing to stage 3 if the process is in the inferior state, and returning immediately to stage 1 if the process is in the superior state.

The \( \alpha \) and \( \beta \) defined in this section are the errors of types I and II for the sequential sampling plan, and they should, therefore, be of the order of 0.05.
With these quantities given, the sequential sampling plan is completely defined. If $d_m$ is the number of defectives found after $m$ items have been inspected, the plan is:-

Return to stage 1 if $d_m \leq -h_1 + sm$

Continue sequential sampling if $-h_1 + sm < d_m < h_2 + sm$

Proceed to stage 3 if $d_m \geq h_2 + sm$

$h_1, h_2, s$ are as defined in the preceding paragraph [6, p. 755].

Then $\pi_1 = 1 - \alpha$, $\pi_2 = \beta$.

To find $E_1(\ell)$, $E_2(\ell)$, the fundamental identity of sequential sampling may be used [1, p. 274]. The sequential sampling plan may be considered as a random walk in one dimension, where motion begins at the origin, the barriers are at $-\frac{h_2}{s}$ and $\frac{h_1}{s}$, and the elements of the walk $z$ are random variables such that

$z = +1$ if the item is non-defective
$z = -\frac{1-s}{s}$ if the item is defective.

Let $Z_n = \sum_{i=1}^{n} z_i$

The fundamental identity states

$$E \left\{ e^{tZ_n} \left[ \varphi(t) \right]^{-n} \right\} = 1$$

Equating the coefficient of $t$ to zero

$$E(Z_n) = E(n) E(z)$$
When the process is in state I

\[ \frac{-h_2 \alpha}{s} + \frac{h_1 (1-\alpha)}{s} = E_1(\ell) \left[ p - \frac{(1-s)(1-p)}{s} \right] \]

\[ E_1(\ell) = \frac{h_1 (1-\alpha) - h_2 \alpha}{s - 1 + p} \]

Similarly

\[ E_2(\ell) = \frac{h_2 (1-\beta) - h_1 \beta}{1 - q - s} \]

Also

\[ E_1(i) = [pb_1 + (1-p)b_2 - e]E_1(\ell) \]

\[ = \frac{[pb_1 + (1-p)b_2 - e][h_1 (1-\alpha) - h_2 \alpha]}{s - 1 + p} \]

and

\[ E_2(i) = \frac{[qb_1 + (1-q)b_2 - e][h_2 (1-\beta) - h_1 \beta]}{1 - q - s} \]

Substitution of these quantities in the appropriate formulae gives the required results.

8. The Savage Three-Decision Plan (Attributes-Variables).

Procedure. The procedure for this plan is identical to that of the pure attributes case of section 7, except that here the sequential sampling is done by variables.

Remarks. It will be assumed for the purpose of the variables sampling, that the observations have a variables measurement \( x \) subject to an upper specification \( \mu \) [i.e. if \( x \leq \mu \), the item is non-defective, if
\[ x > U \text{ the item is defective}. \] It will also be assumed that \( x \) is normally distributed with standard deviation \( \sigma \), which need not be known. As before the sequential sampling plan will be determined to satisfy prescribed probabilities \( 1 - \alpha \) and \( \beta \) of returning to the stage 1 in the two states of the process.

Results.

Income coefficients (equation 3.1.1 page 11)

\[
\alpha_1 = \frac{qb_1 + (1-q)b_2}{(1-q)(1-\beta)}
\]

\[
\beta_1 = -\frac{(q + \beta(1-q))(qb_1 + (1-q)b_2)}{(1-q)(1-\beta)}
\]

\[
\gamma_1 = \frac{c[q + \beta(1-q)]}{(1-q)(1-\beta)} + \frac{(1-p)[pb_1 + (1-p)b_2 - e][h_1(1-\alpha) - h_2\alpha]}{s - \mu_1}
- \frac{[qb_1 + (1-q)b_2 - e][h_2(1-\beta) - h_1\alpha]}{(1-\beta)(\mu_2 - s)} - n_1 \frac{d_1(1-p)}{\alpha} + n_2 \frac{d_2}{\alpha}
\]

\[
\delta_1 = -\frac{c}{(1-q)(1-\beta)} + \frac{[qb_1 + (1-q)b_2 - e][h_2(1-\beta) - h_1\alpha]}{(1-\beta)(\mu_2 - s)} - n_2 \frac{d_2}{\alpha}
\]

\[
\epsilon_1 = (b_1 - b_2)(p-q)
\]

\[
\xi_1 = 0
\]

\[
\alpha_2 = \frac{1}{(1-q)(1-\beta)}
\]

\[
\beta_2 = -\frac{[q + \beta(1-q)]}{(1-q)(1-\beta)}
\]
\[ \gamma_2 = \frac{(1-p)[h_1(1-\alpha) - h_2\alpha]}{s - \mu_1} - \frac{[h_2(1-\beta) - h_1\alpha]}{(1-\beta)(\mu_2 - s)} + n_1(1-p)\alpha - n_2 \]

\[ \delta_2 = \frac{[h_2(1-\beta) - h_1\alpha]}{(1-\beta)(\mu_2 - s)} + n_2 \]

\[ \varepsilon_2 = 0 \]

where \( \mu_1, \mu_2 \) are defined by

\[ p = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu_1)^2} dx \]

and \( q = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu_2)^2} dx \)

and \( s = \frac{\mu_1 + \mu_2}{2} \)

\[ h_1 = \frac{-\sigma^2 \log \frac{\beta}{1-\alpha}}{\mu_2 - \mu_1} \]

\[ h_2 = \frac{\sigma^2 \log \frac{1-\beta}{\alpha}}{\mu_2 - \mu_1} \]

Approximate optimal value of \( F \) for small \( g \).

\[ F = \sqrt{\frac{e(1-p) [h_1(1-\alpha) - h_2\alpha]}{c + \frac{[h_2(1-\beta) - h_1\alpha]}{s - \mu_1} + n_1(1-p)\alpha [\tilde{d}_1 + pb_1 + (1-p)b_2]} + \frac{1}{g(p-q)(b_1-b_2) \left[ \frac{1}{(1-q)(1-\beta)} - \frac{1}{2} \right]}} \]
Approximate optimal income

\[ V^* = pb_1 + (1-p)b_2 - 2\sqrt{g} \left\{ \sqrt{(p-q)(b_1-b_2)} \left[ \frac{1}{(1-q)(1-\beta)} - \frac{1}{2} \right] \right. \]

\[ \left. \left[ \frac{c + \frac{e(1-p)[h_1(1-\alpha)-h_2\alpha]}{s - \mu_1} + n_1(1-p)\alpha[d_1+pb_1+(1-p)b_2]}{s - \mu_1} \right] \right\} \]

Proof. For given \( \alpha \) and \( \beta \) the sequential sampling is as follows.

Let \( x_1 \ldots x_n \) be the first \( m \) observations of \( x \). Then if

\[ \sum_{i=1}^{m} x_i \leq h_1 + sm, \text{ return to stage 1} \]

\[ -h_1 + sm < \sum_{i=1}^{m} x_i < h_2 + sm, \text{ continue sampling sequentially} \]

\[ \sum_{i=1}^{m} x_i \geq h_2 + sm, \text{ proceed to stage 3.} \]

\( h_1, h_2 \) and \( s \) are defined as in the above paragraph [6, p. 751].

A further application of Wald's fundamental identity for sequential sampling yields the results

\[ E_1(\ell) = \frac{h_1(1-\alpha) - h_2\alpha}{s - \mu_1} \]

\[ E_2(\ell) = \frac{h_2(1-\beta) - h_2\beta}{\mu_2 - s} \]

and

\[ E_1(i) = \frac{[pb_1 + (1-p)b_2 - e][h_1(1-\alpha) - h_2\alpha]}{s - \mu_1} \]

\[ E_2(i) = \frac{[qb_1 + (1-q)b_2 - e][h_2(1-\beta) - h_2\beta]}{\mu_2 - s} \]
Thus the results given may be derived using formulae (3.1.2), (3.1.4) and (3.1.5).

Comments. This plan is the most efficient of all regular plans in the sense that when a defective is found in stage 1, the amount of production time in stage 2 is minimized. The purpose of stage 2 is in effect to decide whether the process is in state I or state II, and hence this plan comes to the decision with the prescribed probabilities of errors with the minimum of time elapsed.

There are two possible disadvantages to using this plan. One is that the observations may not be suitable to a variables measurement. The other is that the sudden introduction of sequential sampling may cause considerable administrative problems.


9.1 Systematic Sampling.

Procedure. Inspect every $F$th item. When a defective item is found, if a further defective item is found within $k$ inspected items of the defective item found, stop the process and examine the machine. Otherwise continue inspecting every $F$th item.

This plan has the advantage that the rate of inspection remains constant throughout.

Stage 2 of this plan consists of the production after a defective has been found until either a further defective is found or $k$ non-defectives are
inspected. Since this involves sampling every \( F \)th item, the plan is not regular and the theorem is not applicable. It is shown, however, that under certain conditions, the same approximation may be employed.

Results.

The long-run average income is the ratio of equations (9.1.1) to equation (9.1.2) of the proof.

Approximation to the optimal \( F \) for small \( g \).

\[
F = \sqrt{\frac{c(2-p^k) + n_1(1-p)(1-p^k)(pb_1 + (1-p)b_2 + d_1)}{g(p-q)(b_1-b_2) \left[ \frac{q}{p-q} \left( 1 - \frac{(1-p)(1-p^k)}{(1-q)(1-q^k)} \right) + \frac{1}{(1-q)(1-q^k)} + (1-p^k) \left( \frac{1}{2} + \frac{q}{1-q} \right) \right]}}
\]

Approximate optimal income

\[
V^* = pb_1 + (1-p)b_2 - 2\sqrt{g} \left\{ \sqrt{\frac{n_1(1-p)(1-p^k)}{c + \frac{2-p^k}{pb_1+(1-p)b_2+d_1}} \left( \frac{p-q(b_1-b_2)}{2-p^k} \right) + \sqrt{\frac{q}{p-q} \left( 1 - \frac{(1-p)(1-p^k)}{(1-q)(1-q^k)} \right) + \frac{1}{(1-q)(1-q^k)} + (1-p^k) \left( \frac{1}{2} + \frac{q}{1-q} \right) \right]} \right\}
\]

Proof

(a). Computation of \( V \). (Method A).

Consider the production as blocks of size \( F \), the last item of which is inspected. Let \( I_s \) be the expected income remaining in the cycle (See section 3.1) given that the process is in state \( s \) at the beginning of a
sample block, conditional on the event that the last defective was found at least \(k+1\) inspected items ago.

There are eight terms in the expression for \(I_1\), each conditional on the outcome of the cycle. Write these conditional expected incomes multiplied by their probabilities as \(i_1, i_2, \ldots, i_8\).

1. The process remains in state I throughout the block and the inspected item is non-defective.

\[
i_1 = (1-g)^F p[(F-1)(pb_1 + (1-p)b_2) + b_1 - c + I_1]
\]

2. The process remains in state I throughout the cycle, a defective is found in the first block and a further defective is found in the next \(k\) inspected items.

\[
i_2 = (1-g)^F (1-p) \sum_{r=1}^{k} (1-g)^{rF} p^{r-1}(1-p)[(r+1)(F-1)(pb_1 + (1-p)b_2) + (r-1)b_1 + 2b_2 - (r+1)c - n_1 d_1]
\]

3. The process remains in state I throughout the cycle, a defective is found in the first block but no further defectives are found in the next \(k\) inspected items.

\[
i_3 = (1-g)^F (1-p) (1-g)^{kF} p^k[(k+1)(F-1)(pb_1 + (1-p)b_2) + kb_1 + b_2 - (k+1)c + I_1]
\]

4. The process remains in state I throughout the block, in which the inspected item is defective, a defective item is found in the next \(k\) inspected items, but before this the process changes state.

\[
i_4 = (1-g)^F (1-p) \sum_{r=1}^{k} \sum_{j=0}^{F-1} \sum_{i=0}^{(j+1)(r-1) + 1} (1-g)^{jF+1} q^j (1-q)^{r-j-1} p^{r-1} (1-q)[((j+1)(r-1) + 1)(pb_1 + (1-p)b_2) + ((r-j-1)(F-1) + F-1-1)(qb_1 + (1-q)b_2) + (r-1)b_1 + 2b_2 - (r+1)c - n_2 d_2]
\]
5. The process remains in state I throughout the block in which the inspected item is defective. The next k inspected items are non-defective, but the machine changes state in this period

\[ i_5 = (1-g)^F (1-p) \sum_{j=0}^{k-1} \frac{F-1}{j!} g^j \left( (j+1)(F-1) + i \right)(pb_1 + (1-p)b_2) \]

\[ + ((k-j-1)(F-1) + F-1-1)(qb_1 + (1-q)b_2) + kb_1 + b_2 - (k+1)c + I_2 \]

6. The process changes state during the block but the inspected item is non-defective.

\[ i_6 = \sum_{i=0}^{F-1} (1-g)^i g[ib_1 + (1-p)b_2] + (F-1-i)(qb_1 + (1-q)b_2) + b_1 - c + I_2 \]

7. The process changes state during the block, the inspected item is defective and a further defective is found within the next k inspected items

\[ i_7 = \sum_{i=0}^{F-1} (1-g)^i g(1-q) \sum_{r=1}^{k} q^{r-1}(1-q)[i(pb_1 + (1-p)b_2) + (r(F-1) + F-1-1)(qb_1 \]

\[ + (1-q)b_2) + (r-1)b_1 + 2b_2 - (r+1)c - n_2 d_2] \]

8. The process changes state during the block, the inspected item is defective but the next k inspected items are non-defective.

\[ i_8 = \sum_{i=0}^{F-1} (1-g)^i g(1-q)^k [i(pb_1 + (1-p)b_2) + (k(F-1) + F-1-1)(qb_1 + (1-q)b_2) \]

\[ + kb_1 + b_2 - (k+1)c + I_2] \]

\[ I_1 = i_1 + i_2 + i_3 + i_4 + i_5 + i_6 + i_7 + i_8. \]
Similarly \( I_2 \) may be expressed as the sum of three terms [or put \( g = 0 \), \( p = q \), \( n_1 = n_2 \), \( d_1 = d_2 \), in the expression for \( I_1 \)]

\[
I_2 = \frac{(2-q^k) \left[(qb_1 + (1-q)b_2)F-c\right]}{(1-q)(1-q^k)} - n_2 d_2
\]

and hence

\[
[1-\lambda p-\lambda(1-p)\lambda^k p^k]I_1
\]

\[
= \left[F(qb_1 + (1-q)b_2) - c\right] \left\{1 + \frac{\lambda(q(1-\lambda)(q-\lambda))}{(1-q)(1-\lambda p)} + \frac{\lambda(1-p)(1-\lambda q)}{(1-q)(1-\lambda p)} (1-\lambda^k p^k) \right. \\
+ \left. \frac{(1-\lambda)}{(1-q)(1-q^k)} + \frac{\lambda(1-\lambda)(1-p)(1-\lambda^k p^k)}{(1-q)(1-\lambda p)(1-q^k)} \right\} \\
+ (p-q)(b_1 - b_2) \frac{(1-\lambda)(1-g)}{g} \left[\frac{\lambda(1-p)(1-\lambda^k p^k)}{1-\lambda p} + 1\right] \\
- \frac{n_1 d_1 \lambda^2 (1-p)^2 (1-\lambda^k p^k)}{1-\lambda p} - n_2 d_2 \left[\frac{\lambda(1-p)(1-\lambda)(1-\lambda^k p^k)}{1-\lambda p} + (1-\lambda)\right] \\
\ldots (9.1.1)
\]

where \( \lambda = (1-g)^F \)
Put \( b_1 = b_2 = 1, c = 0, d_1 = d_2 = -1 \)

Then

\[
(1 - \lambda p - \lambda (1-p) \lambda p^k) L_1
\]

\[
= F \left\{ 1 + \frac{g(1-\lambda)(q-\lambda)}{(1-q)(q-\lambda p)} + \frac{\lambda(1-p)(1-\lambda q)}{(1-q)(1-\lambda p)} \left( 1 - \lambda p^k \right) + \frac{(1-\lambda)}{(1-q)(1-q^k)} + \frac{\lambda q(1-\lambda)(1-p)(1-\lambda p^k)}{(1-q)(q-\lambda p)(1-q^k)} \right\} + \frac{n_1 \lambda^2 (1-p)^2 (1-\lambda p^k)}{1-\lambda p}
\]

\[
+ n_2 \left[ \frac{\lambda(1-p)(1-\lambda)(1-\lambda p^k)}{1-\lambda p} + (1-\lambda) \right]
\]

\[
\ldots(9.1.2)
\]

\[
V = \frac{1}{L_1}
\]

Approximation to the optimal \( F \).

As a first approximation, it will be assumed that \( F = \frac{c}{\sqrt{E}} \)

The motivation behind this assumption is that if \( p \) is very close to 1, if \( k \) is a relatively small positive integer and if \( g \) is very small, then the expected time in stage 2 is small compared with that in stage 1.

The plan is then in effect approximated by the Basic Plan of section 4.

Let \( L_2 \) be the expected length in stage 2. Then \( L_2 \) is less than the expected length when the process is always in state 1.

\[
i.e. \quad L_2 < \frac{F(1-p)^k}{1-p}
\]

The expected length in stage 1, \( L_1 = \frac{F(1-\lambda q)}{(1-\lambda p)(1-q)} \)

Using a similar argument to that of theorem 3.2 it may be shown that as \( g \to 0 \), the optimal \( F \) must be such that \( \lambda \to 1 \).
Hence \( L_1 \rightarrow \frac{F}{1-p} \) which is considerably greater than \( \frac{F(1-p^k)}{1-p} \) if \( p \) is close to unity and \( k \) is a small positive integer.

The assumptions on \( p, g \) and \( k \) are plausible and the substitution \( F = \theta/\sqrt{g} \) will be made.

\[
\lambda = 1 - \theta \sqrt{g} + o(\sqrt{g}), \quad \lambda^k = 1 - k \theta \sqrt{g} + o(\sqrt{g})
\]

\[
\frac{(1-g)(1-\lambda)}{g^2} = 1 - \frac{\theta^2}{g} + o(\sqrt{g})
\]

\[
\frac{\lambda}{F} = \frac{\sqrt{g}}{\theta} + o(\sqrt{g})
\]

Then

\[
V = pb_1 + (1-p)b_2 - \sqrt{g} \left\{ \frac{1}{\theta} \left[ c + \frac{n_1(1-p)(1-p^k)}{2-p^k} (pb_1 + (1-p)b_2 + d_1) \right] + \theta \frac{(p-q)(1-q)\bar{b}_1 \bar{b}_2)}{(2-p^k)} \left[ \frac{q}{p-q} \left( 1 - \frac{(1-p)(1-p^k)}{(1-q)(1-q^k)} \right) + \frac{1}{(1-q)(1-q^k)} + (1-p^k) \left( \frac{1}{2} + \frac{q}{1-q} \right) \right] \right\} + o(\sqrt{g})
\]

\[
= pb_1 + (1-p)b_2 - \sqrt{g} \left\{ \frac{\Delta'}{\theta} + \theta \Gamma' \right\} + o(\sqrt{g}), \quad \text{say, where}
\]

\( \Delta' > 0, \quad \Gamma' > 0 \)

\[
V = pb_1 + (1-p)b_2 - \sqrt{g} \left[ \left( \frac{\Delta'}{\theta} - \sqrt{\theta \Gamma'} \right)^2 + 2\sqrt{\theta \Gamma'} \Delta' \right] + o(\sqrt{g})
\]

Maximizing \( \theta = \sqrt{\frac{\Delta'}{\Gamma'}} \)

Hence an approximation to the optimal \( F \) is given by
\[ F = \sqrt{\frac{\sum_i}{F'g}} \]

\[
\sqrt{c + \frac{n(1-p)(1-p^k)}{(2-p^k)}} \left( pb_1 + (1-p)b_2 + d_1 \right)
\]

\[ \frac{1}{\sqrt{g \left\{ \frac{(p-q)(b_1-b_2)}{(2-p^k)} \right\}} \left[ \frac{q}{p-q} \left( 1 - \frac{(1-p)(1-p^k)}{(1-q)(1-q^k)} \right) + \frac{1}{(1-q)(1-q^k)} + (1-p^k) \left( \frac{1}{2} + \frac{q}{1-q} \right) \right] } \]

and

\[ V^* = pb_1 + (1-p)b_2 \]

\[-2\sqrt{g} \left\{ \sqrt{c + \frac{n(1-p)(1-p^k)}{(2-p^k)}} \left( pb_1 + (1-p)b_2 + d_1 \right) \right\} \frac{(p-q)(b_1-b_2)}{(2-p^k)} \]

\[ \left[ \frac{q}{p-q} \left( 1 - \frac{(1-p)(1-p^k)}{(1-q)(1-q^k)} \right) + \frac{1}{(1-q)(1-q^k)} + (1-p^k) \left( \frac{1}{2} + \frac{q}{1-q} \right) \right] \left[ \sqrt{ \frac{q}{p-q} \left( 1 - \frac{(1-p)(1-p^k)}{(1-q)(1-q^k)} \right) + \frac{1}{(1-q)(1-q^k)} + (1-p^k) \left( \frac{1}{2} + \frac{q}{1-q} \right) } \right] \]

This completes the discussion for the case when systematic sampling is used.

The computation for stratified sampling is similar.

9.2 Random Sampling.

Procedure. Inspect each item with probability \( f \). If a further defective is found within \( k \) inspected items of a defective item found, stop the machine and examine the process. Otherwise continue inspecting with probability \( f \).
This plan does not satisfy the conditions of theorem 3.2 for the purposes of approximating $f$.

**Results.**

**Income.** The long run average income is the ratio of equation (9.2.17) to equation (9.2.16) of the proof.

**Approximation to the optimal $f$ for small $g$.**

$$f = \sqrt{g(b_1 - b_2) \left( \frac{q}{p} - \frac{(1-p)(1-p^k)}{1-q} \right) + \frac{(p-q)(1-p^k)}{1-q} + \frac{p-q}{(1-q)(1-q^k)} \cdot \frac{c(2-p^k) + n_1(1-p)(1-p^k)(d_1 + pb_1 + (1-p)b_2)}{c(2-p^k) + n_1(1-p)(1-p^k)(d_1 + pb_1 + (1-p)b_2)} }$$

**Approximate optimal income.**

$$V^* = pb_1 + (1-p)b_2 - 2\sqrt{g} \left\{ \sqrt{\left( c + \frac{n_1(1-p)(1-p^k)(d_1 + pb_1 + (1-p)b_2)}{2-p^k} \right) \left( \frac{b_1 - b_2}{2-p^k} \right)} \right\} \sqrt{\left( \frac{q}{p} - \frac{(1-p)(1-p^k)}{1-q} \right) + \frac{(p-q)(1-p^k)}{1-q} + \frac{p-q}{(1-q)(1-q^k)} }$$

**Proof.** Method B of computing the average income will be used.

Consider the triple $(s, a, n)$ where

- $s = 1$ if the machine is in state I.
- $s = 2$ if the machine is in state II.
- $a = 0$ if an item is produced and passed without inspection.
- $a = 1$ if an item is produced and inspected.
- $a = 2$ if the machine is shut down.
n = number of time units remaining before the machine goes back into
production if \( a = 2 \).

\( = 1, \ldots k \) when a defective has been found and since then \( k-n \) units have
been inspected \( (a = 0, l) \)

\( = 0 \) otherwise.

The triple \( (s, a, n) \) is a Markov chain with \( 1 + 4k + n_1 + n_2 \) states.

There are no transient states and there is only one ergodic class.

Computation of the elements of the stochastic matrix is a routine
matter. Most of them are zero, and typical non-zero elements are:

<table>
<thead>
<tr>
<th>Transition</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 0, 0) \to (1, 0, 0))</td>
<td>((1-g)(1-f))</td>
</tr>
<tr>
<td>((1, 1, 0) \to (2, 0, 0))</td>
<td>(pg(1-f))</td>
</tr>
<tr>
<td>((1, 0, r) \to (2, 1, r))</td>
<td>(gf)</td>
</tr>
<tr>
<td>((1, 1, r) \to (2, 1, r-1))</td>
<td>(pgf)</td>
</tr>
<tr>
<td>((2, 1, r) \to (2, 2, \ a_2)) for ( r = 1, \ldots k )</td>
<td>(1-q)</td>
</tr>
<tr>
<td>((2, 2, 1) \to (1, 1, 0))</td>
<td>((1-g)f)</td>
</tr>
</tbody>
</table>

Let \( A = P(1, 0, 0) + pP(1, 1, 0) + pP(1, 1, 1) + pP(1, 2, 1) + pP(2, 2, 1) \)

\( A \) is chosen so that \((1-g)(1-f)A\) is the scalar product of the row vector
of stationary probabilities and the column of the stochastic matrix

\( \) corresponding to the state \((1, 0, 0)\).

Then

\[ P(1, 0, 0) = (1-g)(1-f)A \]  \( \quad \text{...(9.2.1)} \)

\[ P(1, 1, 0) = (1-g)fA \]  \( \quad \text{...(9.2.2)} \)
\[ P(1, 0, k) = (1-p)(1-g)(1-f)P(1, 1, 0) + (1-g)(1-f)P(1, 0, k) \]

\[ P(1, 0, k) = \frac{(1-p)(1-g)^2 f(1-f)A}{[1 - (1-g)(1-f)]} \quad \cdots (9.2.3) \]

\[ P(1, 1, k) = \frac{f}{1-f} P(1, 0, k) = \frac{(1-p)(1-g)^2 f^2 A}{[1 - (1-g)(1-f)]} \quad \cdots (9.2.4) \]

\[ P(1, 0, r) = (1-g)(1-f)P(1, 0, r) + p(1-g)(1-f)P(1, 1, r+1) \]

\[ \text{for } r = 1, \ldots k-1. \]

\[ P(1, 0, r) \left[ 1 - (1-g)(1-f) \right] = p(1-g)(1-f)P(1, 1, r+1) \]

\[ P(1, 1, r) = (1-g)fP(1, 0, r) + p(1-g)fP(1, 1, r+1) \]

\[ = \frac{p(1-g)f}{[1 - (1-g)(1-f)]} P(1, 1, r+1) \]

Hence

\[ P(1, 1, r) = \left[ \frac{p(1-g)f}{[1 - (1-g)(1-f)]} \right]^{k-r} P(1, 1, k) \]

Let \( \alpha = \frac{p(1-g)f}{1 - (1-g)(1-f)} \)

\[ P(1, 1, r) = \alpha^{k-r} P(1, 1, k), \quad r = 1, \ldots k. \]

\[ = \frac{\alpha^{k-r}(1-p)(1-g)^2 f^2 A}{[1 - (1-g)(1-f)]} \quad \cdots (9.2.5) \]

\[ P(1, 0, r) = \frac{1-f}{f} P(1, 1, r) \]

\[ = \frac{\alpha^{k-r}(1-p)(1-g)^2 f(1-f)A}{[1 - (1-g)(1-f)]} \quad \cdots (9.2.6) \]
\[ P(1, 2, k) = P(1, 2, n_1) \]
\[ = (1-p) \sum_{r=1}^{\infty} P(1, 1, r) \]
\[ = \frac{(1-p)^2 (1-g)^2 r^2 (1-\alpha^k)A}{g + f(1-g)(1-p)}, \quad k = 1, \ldots, n_1 \quad \ldots(9.2.7) \]

Let \( B = gP(1, 0, 0) + pgP(1, 1, 0) + pgP(1, 1, 1) + gP(1, 2, 1) \)
\[ + P(2, 0, 0) + qpP(2, 1, 0) + qP(2, 1, 1) + gP(2, 2, 1) \]
where \((1-f)B\) is the scalar product of the vector of the stationary probabilities and the column of the stochastic matrix corresponding to the state \((2, 0, 0)\).

Then \( P(2, 0, 0) = (1-f)B \quad \ldots(9.2.8) \)
\[ P(2, 1, 0) = fB \quad \ldots(9.2.9) \]
\[ fP(2, 0, k) = (1-q)(1-f)P(2, 1, 0) + g(1-p)(1-f)P(1, 1, 0) + g(1-f)P(1, 0, k) \]
\[ P(2, 0, k) = (1-q)(1-f)B + \frac{g(1-p)(1-g)(1-f)A}{[1-(1-g)(1-f)]} \quad \ldots(9.2.10) \]
\[ P(2, 1, k) = \frac{f}{1-f} P(2, 0, k) = (1-q)fB + \frac{g(1-p)(1-g)fa}{[1-(1-g)(1-f)]} \quad \ldots(9.2.11) \]
\[ fP(2, 0, r) = g(1-f)P(1, 0, r) + pg(1-f)P(1, 1, r+1) + q(1-f)P(2, 1, r+1) \]
\[ P(2, 1, r) = \frac{f}{1-f} P(2, 0, r) \]
\[ = gP(1, 0, r) + pgP(1, 1, r+1) + qP(2, 1, r+1) \]
\[ P(2, 1, r) = \frac{gF(1-p)(1-g)^2(1-f)A}{1-(1-g)(1-f)} \alpha^{k-r} + \frac{pgF(1-p)(1-g)^2A}{1-(1-g)(1-f)} \alpha^{k-r-1} + qP(2, 1, r+1) \]

using equations (9.2.5) and (9.2.6)
\[ = \frac{g(1-p)(1-g)fA}{1-(1-g)(1-f)} \alpha^{k-r} + qP(2, 1, r+1) \]
This difference equation has the solution, using equation (9.2.11)

\[ P(2, l, r) = \frac{g(1-p)(1-g)fA}{[(p-q)f(1-g)-gq]} \alpha^{k-r+1} + q^{k-r}(1-q)FB \]

\[ - \frac{g(1-p)(1-g)fAq^{k-r+1}}{[(p-q)f(1-g)-gq]}, \ r=1, \ldots, k. \] ....(9.2.12)

\[ P(2, 0, r) = \frac{(1-f)}{f} P(2, 1, r) \]

\[ = \frac{g(1-p)(1-g)(1-f)A}{[(p-q)f(1-g)-gq]} \alpha^{k-r+1} + q^{k-r}(1-q)(1-f)B \]

\[ - \frac{g(1-p)(1-g)(1-f)Aq^{k-r+1}}{[(p-q)f(1-g)-gq]} \] ....(9.2.13)

\[ P(2, 2, \ell) = P(2, 2, n_2) \]

\[ = (1-q) \sum_{r=1}^{k} P(2, l, r) \]

\[ = \frac{g(1-p)(1-g)^2f^2p(1-q)A(1-\alpha^k)}{[(p-q)f(1-g)-gq][g+f(1-g)(1-p)]} \]

\[ + B(1-q)f(1-q^k) - \frac{g(1-p)(1-g)fA(1-q^k)}{[(p-q)f(1-g)-gq]}, \ \ell = 1, \ldots, n_2 \] ....(9.2.14)

By definition and using the above equations,

\[ A = (1-g)(1-f)A + p(1-g)fA + \frac{g(1-p)(1-g)^2f^2A\alpha^{k-1}}{[1-(1-g)(1-f)]} \]

\[ + \frac{(1-p)^2(1-g)^2f^2A(1-\alpha^k)}{g+f(1-g)(1-p)} + \frac{g(1-p)(1-g)^2f^2Ap(1-q)(1-\alpha^k)}{[(p-q)f(1-g)-gq][g+f(1-g)(1-p)]} \]

\[ + (1-q)f(1-q^k)b - \frac{g(1-p)(1-g)fA(1-q^k)}{[(p-q)f(1-g)-gq]} \]

\[ gA = \frac{gA(1-p)(1-g)fA(1-\alpha^k)}{(p-q)f(1-g)-gq} - \frac{g(1-p)(1-g)fA(1-q^k)}{(p-q)f(1-g)-gq} + (1-q)f(1-q^k)b \] ....(9.2.15)
Sum of the probabilities of the states = 1.

\[
1 = (1-g)(1-f)A + (1-g)fA + \frac{(1-p)(1-g)^2f(1-f)A(1-\alpha^k)}{g + f(1-g)(1-p)}(1-f)
+ \frac{n_1(1-p)^2(1-g)^2f^2A(1-\alpha^k)}{g + f(1-g)(1-p)} + (1-f)B + fB + \frac{g(1-p)(1-g)^2fPA(1-\alpha^k)}{g + f(1-g)(1-p)[(p-q)(f(1-g)-gq)]}
+ \frac{(1-q)B}{(p-q)(f(1-g)-gq)} - \frac{g(1-p)(1-g)AgA(1-\alpha^k)}{(p-q)(f(1-g)-gq)[(p-q)(f(1-g)-gq)]} + \frac{n_2g(1-p)(1-g)^2f^2p(1-q)A(1-\alpha^k)}{(p-q)(f(1-g)-gq)[(p-q)(f(1-g)-gq)]}
+ \frac{n_2B(1-q)f(1-q)}{(p-q)(f(1-g)-gq)} - \frac{n_2g(1-p)(1-g)fQA(1-\alpha^k)}{(p-q)(f(1-g)-gq)}
\]

From this and (9.2.15)

\[
\frac{1}{A} = n_2g + \frac{n_2f(g(1-p)(1-g)(1-\alpha^k))}{g + f(1-g)(1-p)} + \frac{n_1(1-p)^2(1-g)^2f^2(1-\alpha^k)}{g + f(1-g)(1-p)}
- \frac{g(1-p)(1-g)(1-\alpha^k)}{(p-q)(f(1-g)-gq)[(p-q)(f(1-g)-gq)]} + \frac{g}{f(1-q)(1-q)}
+ \frac{(1-p)(1-g)[g + f(1-g)(1-q)](1-\alpha^k)}{(1-q)[g + f(1-g)(1-p)]} + 1-g + \frac{g[(1-q)fP(1-g) - gq]}{f(1-q)[(p-q)f(1-g) - gq]}
\]

\[\ldots(9.2.16)\]

Expected Incomes.

\[
\begin{align*}
I(1, 0, r) &= pb_1 + (1-p)b_2 & r &= 0, 1, \ldots k \\
I(1, 1, r) &= pb_1 + (1-p)b_2 - c & r &= 0, 1, \ldots k \\
I(1, 2, h) &= -d_1 & h &= 1, \ldots n_1 \\
I(2, 0, r) &= qb_1 + (1-q)b_2 & r &= 0, 1, \ldots k \\
I(2, 1, r) &= qb_1 + (1-q)b_2 - c & r &= 0, 1, \ldots k \\
I(2, 2, h) &= -d_2 & h &= 1, \ldots n_2
\end{align*}
\]
Average income \( V = \Sigma I(s, a, n)P(s, a, n) \) and using equations (9.2.1) through (9.2.14) we have

\[
V = (pb_1 + (1-p)b_2 - fc) \left[ (l-g)A + \frac{(1-p)(1-g)^2fA(1-\alpha^k)}{g + f(1-g)(1-p)} \right] \\
+ (qb_1 + (1-q)b_2 - fc) \left[ B + \frac{g(1-p)(1-g)^2fpA(1-\alpha^k)}{[g+f(1-g)(1-p)][(p-q)f(1-g)-gq]} + (1-q^k)B \right. \\
- \left. \frac{g(1-p)(1-g)qA(1-q^k)}{[(p-q)f(1-g)-gq](1-q)} \right] - \frac{n_2a_1(1-p)^2(1-g)2f^2(1-\alpha^k)A}{g + f(1-g)(1-p)} \\
- n_2d_2 \left[ g + \frac{fg(1-p)(1-g)(1-\alpha^k)}{g + f(1-g)(1-p)} \right] A,
\]

Substitute for \( B \) from equation (9.2.15)

\[
V = A \left( (pb_1 + (1-p)b_2 - fc) \left( 1 - g + \frac{(1-p)(1-g)^2f(1-\alpha^k)}{g + f(1-g)(1-p)} \right) \right) \\
+ (qb_1 + (1-q)b_2 - fc) \left( \frac{g}{f(1-q)(1-q^k)} - \frac{g(1-p)(1-g)(1-\alpha^k)}{[(p-q)f(1-g)-gq](1-q)(1-q^k)} \right) \\
+ \frac{g(l-p)(1-g)(1-\alpha^k)}{(1-q)[g+f(l-g)(1-p)]} + \frac{g[(1-g)(1-q)-gq]}{f(1-q)[(p-q)f(1-g)-gq]} \right) - \frac{n_1d_1(1-p)^2(1-g)2f^2(1-\alpha^k)}{g + f(1-g)(1-p)} \\
- n_2d_2 \left( g + \frac{fg(1-p)(1-g)(1-\alpha^k)}{g + f(1-g)(1-p)} \right) \right) \].
\]

The average income \( V \) is found from the above equation and equation (9.2.16).

The optimum plan is given by the values of \( f \) and \( k \) which maximize \( V \).
Approximation to the Optimal Plan.

As in section 9.1 a first approximation to the optimal $f$ will be $f = \theta g$. This may be justified by a similar argument to that used in the afore-mentioned section.

$$\lambda = 1 - \theta g + o(g)$$

and

$$\alpha = p(1 - \frac{g}{\theta}) + o(g)$$

The income function then reduces to

$$V = pb_1 + (1-p)b_2 - \frac{g}{2-p} \left\{ \delta[c(2-p^k) + n_1(l-p)(1-p^k)(d_1 + pb_1 + (1-p)b_2)]
+ \frac{(b_1-b_2)}{\theta} \left[ q\left( \frac{p}{q} - \frac{1-p}{1-q} \right) \left( \frac{1-p^k}{1-q} \right) + \frac{(p-q)(1-p^k)}{(1-q)} + \frac{p-q}{(1-q)(1-q^k)} \right] + o(g) \right\}$$

i.e.

$$V = pb_1 + (1-p)b_2 - \frac{g}{2-p} \left\{ \Delta' + \delta r' \right\} + o(g)$$

where $\Delta' > 0$, $r' > 0$.

The optimal $\theta$ is then equal to $\sqrt{\frac{\Delta'}{r'}}$. Hence an approximation to the optimal $f$ is $\sqrt{\frac{g\Delta'}{r'}}$ and the corresponding $V$ is $V^* = pb_1 + (1-p)b_2 - 2g \sqrt{r'\Delta'}$.

10. The Derman-Littauer-Solomon Multi-Level Plan MLP-T.

Random Sampling

Procedure. The plan consists of $k+1$ levels of inspection.

Level 0. Inspect every item. If 1 consecutive non-defectives are found, proceed to level 1. If a defective is found before this happens, stop the
process and examine the machine.

Level $j$, $1 \leq j \leq k-1$. Inspect each item with probability $f^j$. If the next $i$ inspected items are non-defective, proceed to level $j+1$. If a defective occurs before this happens, return to level 0.

Level $k$. Inspect each item with probability $f^k$. If a defective occurs, return to level 0. When the machine is started up after a period of inspection, the plan begins again at level 0.

This is not a regular plan.

The assumption is made that the machine only changes state at level $k$. This is explained by the fact that under ideal conditions the plan will be at level $k$ and any other level implies closer inspection and hence a closer watch on the process.

A further refinement, not hitherto mentioned, will be the introduction of a probability $\theta$ that, after the machine has been shut down in state II, the machine is not repaired (i.e. it remains in state II).

Method B of evaluating the average income is used.

Results. An explicit form for the average income $V$ is not presented since the formulae for the probabilities of the states of the Markov chain are somewhat cumbersome. Nor has any approximation to the maximizing parameters of the plan been attempted. However, the components which make up the equation for $V$ are given and a trial and erro method might be used to find good, if not optimum, values of $f$, $k$ and $i$. 
Consider the quadruple \((s, a, \ell, w)\) where

- \(s = 1\) if the machine is in state I.
- \(s = 2\) if the machine is in state II.

- \(a = 0\) if an item is produced and passed without inspection.
- \(a = 1\) if an item is produced and inspected.
- \(a = 2\) if the machine is shut down.

- \(\ell = -1, 0, 1, \ldots, k\), the level of inspection, where the level \(-1\) is used for the time when the machine is shut down.

- \(w = 0, 1, \ldots, i-1\) if \(\ell = 0, 1, \ldots, k-1\), the number of items already inspected at this level.
- \(w = 0\) if \(\ell = k\)
- \(w\) is the number of time units remaining before the machine is started up again if \(\ell = -1(a=2)\).

The process is a Markov chain with \(4k + 4 + n_1 + n_2\) states. There are no transient states and there is only one ergodic class.

Computation of the elements of the stochastic matrix is again a routine matter. A large majority of the elements are zero, and typical non-zero elements are:

\[
\begin{align*}
(1, 0, j, r) &\rightarrow (1, 1, j, r), j=1, \ldots, k-1; \ r=0, \ldots, i-1 & f^j \\
(2, 1, j, r) &\rightarrow (2, 0, j, r+1), j=1, \ldots, k-1; \ r=0, \ldots, i-2 & q(1-f^j) \\
(1, 1, j, i-1) &\rightarrow (1, 1, 0, 0), j=1, \ldots, k-1 & 1-p \\
(2, 1, k, 0) &\rightarrow (2, 1, k, 0) & qf^k \\
(1, 1, 0, r) &\rightarrow (1, 2, -1, n), \ r=0, \ldots, i-1 & 1-p \\
(2, 2, -1, 1) &\rightarrow (2, 1, 0, 0) & \theta
\end{align*}
\]
Then \( P(1,1,j,r) = p^{j+r} P(1,1,0,0) \)
and \( P(2,1,j,r) = q^{j+r} P(2,1,0,0), \ j=0, \ldots \ k-1; \ r=0, \ldots \ i-1. \)

\[
P(1,0,j,r) = \frac{1-f^j}{f^j} P(1,1,j,r) = \frac{(1-f^j)p^{j+r}}{f^j} P(1,1,0,0)
\]

\[
P(2,0,j,r) = \frac{1-f^j}{f^j} P(2,1,j,r) = \frac{(1-f^j)q^{j+r}}{f^j} P(2,1,0,0), \ j=0, \ldots \ k-1; \ r=0, \ldots \ i-1.
\]

\[
P(1,2,-1,h) = P(1,2,-1,n) = (1-p) \sum P(1,1,0,r)
\]

\[
= (1-p^i)P(1,1,0,0)
\]

\( h = 1, \ldots \ n. \)

\[
P(2,2,-1,h) = (1-q^i)P(2,1,0,0)
\]

\( h = 1, \ldots \ n. \)

\[
P(1,1,k,0) = (1-g)p^{k} P(1,1,k-1,1-1) + (1-g)f^k P(1,0,k,0) + p(1-g)f^k P(1,1,k,0)
\]

Hence \( P(1,1,k,0) = \frac{(1-g)f^k P(1,1,0,0)}{g + f^k(1-p)(1-g)} \)

and \( P(1,0,k,0) = \frac{(1-g)(1-f^k)p^{k} P(1,1,0,0)}{g + f^k(1-p)(1-g)} \)

Also \( P(2,1,k,0) = gpf^k P(1,1,k-1,1-1) + qf^k P(2,1,k-1,1-1) + pf^k P(1,0,k,0) + pgf^k P(1,1,k,0) + f^k P(2,0,k,0) + qf^k P(2,1,k,0) \)

Hence \( (1-q)P(2,1,k,0) = \frac{gpf^k P(1,1,0,0)}{g + f^k(1-p)(1-g)} + qf^k P(2,1,0,0) \)

and \( f^k(1-q)P(2,0,k,0) = \frac{g(1-f^k)p^{k} P(1,1,0,0)}{g + f^k(1-p)(1-g)} + qf^k(1-f^k)P(2,1,0,0) \)

All the stationary probabilities have been expressed in terms of \( P(1,1,0,0) \) and \( P(2,1,0,0). \) The relationship between these two probabilities is
\[
\begin{align*}
&\text{P}(1,1,0,0) = (1-\varphi) \sum_{r=0}^{\infty} \sum_{j=1}^{i-1} \text{P}(1,1,j,r) + (1-\varphi)\text{P}(1,1,k,0) + \text{P}(1,1,-1,1) \\
&\quad + (1-\varphi)\text{P}(2,2,-1,1) \\
&\frac{\sum_{k} \text{P}(1,1,0,0)}{g + \varphi(1-\varphi)\text{P}(2,1,0,0)} = (1-\varphi)(1-\varphi)\text{P}(2,1,0,0)
\end{align*}
\]

Using the relationship \( \sum_{s,a,e,w} \text{P}(s,a,e,w) = 1 \) the actual values of the stationary probabilities can be found.

**Expected incomes.**

\[
\begin{align*}
&\text{I}(1,0,j,r) = p_{b_{1}} + (1-p)b_{2} \\
&\text{I}(1,1,j,r) = p_{b_{1}} + (1-p)b_{2} - c, \quad j=1, \ldots, k-1; \quad r=0, \ldots, i-1 \\
&\text{I}(1,0,k,0) = p_{b_{1}} + (1-p)b_{2} \\
&\text{I}(1,1,k,0) = p_{b_{1}} + (1-p)b_{2} - c, \quad j=k. \\
&\text{I}(1,1,0,r) = p_{b_{1}} + (1-p)b_{2} - c, \quad j=0; \quad r=0, \ldots, i-1. \\
&\text{I}(1,2,-1,h) = -d_{1}, \quad j=-1; \quad h=1, \ldots, n_{1}
\end{align*}
\]

The incomes from state II are found by substituting \( q \) for \( p, \varphi \) for \( d_{1} \) in the above. Hence the long-run average income

\[
V = \sum_{s,a,e,w} \text{I}(s,a,e,w)\text{P}(s,a,e,w)
\]

is determined. No approximation to the optimal \( f \) has been discovered, and it is suggested that in order to find a "good" value of \( f \), the method of trial and error be used.

This is one of the simplest forms of multi-level sampling. Other multi-level plans can be analysed in the same way, although the computations may present some difficulty.
11. The Continuous Case - General Procedure and Remarks.

Suppose now that the production consists of a continuous flow of material. The analogous concept to the inspection of a single item in the discrete case is the examination of a small length $h$ of production. This will be known as an inspection period.

Definition. Sampling at the rate $F$ can be interpreted in three ways:

1. **Systematic Sampling.** Begin an inspection period after every length $F$ of production.

2. **Stratified Sampling.** Divide the production into lengths $F$ and choose a point at random from each length. Begin an inspection period at this point.

3. **Random Sampling.** Sample so that the length between the points at which an inspection period begins has an exponential distribution with mean $F$.

The assumption is made that $h$ is small compared with $F$, so that the probability that inspection periods overlap may be neglected.

The analogous concept to one hundred percent inspection is the inspection of a length $j$ units of production. Also when a defect is found, it is not removed.

With the above definition of sampling there is little difference in the procedures used in the continuous and the discrete cases. The plans already described extend naturally and the method of computing the long-run average income needs little modification. The theorem of Girshich and Rubin [5, p. 118], theorem 3.1, holds for the continuous case and hence method A has been used.
Method B could not be applied directly and in the work done so far on the continuous case, method A has proved satisfactory.

The definition of a regular plan needs no modification and the theorem 3.2 on the approximation to the optimal sampling rate holds with \( \beta \), the parameter of the exponential distribution of the time before a transition from state I to state II takes place, replacing the transition probability \( g \).

The proof of this theorem would differ essentially only in the constants of the income function \( V \) and will not be given. Similarly the evaluation of the income coefficients, the approximate optimal sampling rate and the approximate optimal income for any particular plan differs only in detail and not in principle.

12. A \textbf{Minimax Method of Choosing the Plan}.

In the theory considered in the foregoing sections it has been assumed that values can be given to all the parameters, in particular to \( p \) and \( q \) the probabilities of non-defective items in the two states. The parameter \( p \) represents the proportion of non-defective items when the machine is in good running order and it is felt that little difficulty will be met in obtaining an accurate estimate of \( p \). However, \( q \) is an indication of the degree to which the process has gone out of control and may consequently be rather vague. Suppose then that \( p \) is fixed and it is desired to choose a plan which will be reasonable for any possible value of \( q \).

The situation may be considered as a statistical game where one player
(nature) choose the parameter $q$ and the other player (the statistician) chooses the parameters of the plan. The payoff to the statistician is the expected income. One plausible strategy for the statistician is to use a "minimax" approach, which in this case amounts to choosing the parameters which maximize the minimum (over $q$) income. It will be shown how this can be done for the basic sampling plan.

Consider the basic sampling plan where systematic sampling is used. From section 4.1, the average income

$$V = \frac{(b_1 F - c)(1 - \lambda q)}{(1 - q)^2} \left[ F(1 - \lambda q) - \frac{(1 - q)(1 - \lambda)(p - q)}{g} \right] - n_1 d_1 \lambda (1 - p) - n_2 d_2 (1 - \lambda)$$

$$\frac{F(1 - \lambda q)}{(1 - q)^2} + n_1 \lambda (1 - p) + n_2 (1 - \lambda)$$

$V$ is a function of $F$ and $q$. As a function of $q$, it is the ratio of a quadratic function to a linear function. The relationship is then a hyperbola.

The asymptotes are parallel to

$$q \left[ q(b_1 - b_2) \left( \frac{(1 - \lambda)(1 - g)}{g} - \lambda F \right) + V(\lambda F + n_2 (1 - \lambda) + n_1 \lambda (1 - p)) \right] = 0$$

and since $0 \leq \sum_{r=0}^{F-1} r(1 - g)^r = \frac{1}{g^2} [(1 - \lambda)(1 - g) - \lambda g F]$, then $\frac{(1 - \lambda)(1 - g)}{g} \geq \lambda F$.

One asymptote has a negative gradient, whilst the other is parallel to the $V$ axis. The equation of the latter is actually

$$q = 1 + \frac{F(1 - \lambda)}{\lambda F + n_1 \lambda (1 - p) + n_2 (1 - \lambda)}$$
Hence the hyperbola is of the form

\[ V \]

\[ \rightarrow q \]

Figure 12.1.1

or

\[ V \]

\[ \rightarrow q \]

Figure 12.1.2
In both cases the loss function (when the statistician is considered as player I and Nature as player II) is for each $F$ a continuous convex function of $q$ over player II's range of strategies.

Player II chooses a value $q$ from the closed convex set $[0, 1]$ and player I chooses a value of $F$ from the integers $1, 2, 3,$ etc. Then, using a theorem of the theory of games, [1, p. 51] the game has a value, player II has a good pure strategy and player I has an $\epsilon$-good strategy which is the mixture of at most two pure strategies. It is the $\epsilon$-good strategy of player I which is to be found. Since in practice it would be undesirable to randomize between two sampling plans, it is suggested that the procedure adopted be to find for each $F$ the $q$ which minimizes the income and select that $F$ which gives the largest minimum income. The theorem quoted above establishes that this can always be done.


When a sampling plan calls for "sampling one in every $F$ items", the phrase may be interpreted in several ways. The purpose of this section is to consider three possible interpretations and, from the point of view of maximization of the average income, to determine the criteria which would indicate the appropriate method. Other factors impossible to evaluate mathematically may have considerable influence on the choice, and the intention here is not to give a rigid set of rules, but rather to present a guide to be used if these factors are not completely conclusive.

The interpretations considered are those already used in the preceding
sections, namely:

1. Systematic sampling. Inspect every \( F \)th item.

2. Stratified sampling. Divide the production into blocks of size \( F \) and select an item at random from each block.

3. Random sampling. Inspect every item with probability \( 1/F \).

The comparison that is made in this section is of the approximate optimum incomes, \( V^* \).

**Theorem 13.1.** For any regular plan let \( V^*_{sy}, V^*_{st} \) and \( V^*_{tr} \) be the approximate optimal incomes to the order \( o(\sqrt{E}) \) for systematic, stratified and random sampling respectively in stage 1.

Then \( V^*_{sy} > V^*_{st} > V^*_{tr} \) and \( V^*_{st} > V^*_{tr} \) if \( p < 1 \).

**Proof.** (a) \( V^*_{sy} > V^*_{st} \).

For either plan \( V^* = pb_1 + (1-p)b_2 - 2\sqrt{E} \sqrt{\Delta} \).

Thus \( V^*_{sy} > V^*_{st} \) if \( \gamma_{sy} \Delta_{sy} < \gamma_{st} \Delta_{st} \).

But \( \Delta_{sy} = \Delta_{st} = c + n_1(1-p)(1-\pi_1)(pb_1 + (1-p)b_2 + d_1) + (1-p)[(pb_1 + (1-p)b_2)E_1(\eta) - E_1(\eta)] \)

Hence \( V^*_{sy} > V^*_{st} \) if \( \gamma_{sy} < \gamma_{st} \)

\[
\gamma_{sy} = \frac{(b_1-b_2)(p-q)}{(1-q)} \left[ \frac{1}{1-\pi_2} - \frac{1-q}{2} \right]
\]

\[
\gamma_{st} = \frac{(b_1-b_2)(p-q)}{(1-q)} \left[ \frac{1}{1-\pi_2} - \frac{1-p}{2} \right]
\]

Thus \( \gamma_{sy} < \gamma_{st} \) if
\[ \frac{1}{1-\pi_2} \left( \frac{1-q}{2} \right) < \frac{1}{1-\pi_2} - \frac{1-p}{2}, \text{ since } b_1 > b_2, p > q, l > q. \text{ i.e. if } q < p, \]

which is true.

(b) \ V^*_{st} > V^*_r

By equation (3.1.9)
\[ V^*_r = pb_1 + (1-p)b_2 - 2\sqrt{g \frac{(p-q)(b_1-b_2)}{(1-q)(1-\pi_2)}} \Delta \]

Thus \ V^*_{st} > V^*_r

if \[ \frac{(b_1-b_2)(p-q)}{(1-q)} \left( \frac{1}{1-\pi_2} - \frac{1-p}{2} \right) \leq \frac{(b_1-b_2)(p-q)}{(1-q)(1-\pi_2)} \]

i.e. if \ -(1-p) \leq 0, \text{ since } b_1 > b_2, p > q, l > q.

\[ p \leq l, \text{ which is true.} \]

Also \ V^*_{st} > V^*_r \text{ if } p < l, \text{ and the theorem is proved.}

Thus it appears that as a general rule systematic sampling is superior to stratified sampling, which in turn is superior to random sampling.

However, systematic sampling can not be used where there is a possibility of periodicity in the production, or, in fact, any chance that the machine operator may be able to predict which items of his production will be inspected.

The parameters given below describe a typical situation on which a continuous sampling plan might be used.

\[
\begin{align*}
b_1 &= 10 & n_1 &= 5 \\
b_2 &= 0 & n_2 &= 10 \\
c &= 1 & g &= 0.0005 \\
d_1 &= 10 & p &= 0.99 \\
d_2 &= 10 & q &= 0.90.
\end{align*}
\]

Then, if the Basic Sampling Plan of Section 4 were to be applied, the following results would be obtained.

(a) Systematic Sampling.

\[
\begin{align*}
\text{Approximate optimal } F &= 22 \\
\text{Exact optimal } F &= 25 \\
\text{Approximate optimal income} &= 9.7152 \\
\text{Exact optimal income} &= 9.6417
\end{align*}
\]

(b) Stratified Sampling.

\[
\begin{align*}
\text{Approximate optimal } F &= 21 \\
\text{Exact optimal } F &= 25 \\
\text{Approximate optimal income} &= 9.7110 \\
\text{Exact optimal income} &= 9.6385
\end{align*}
\]
(c) Random Sampling.

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<td>Exact optimal f</td>
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<tr>
<td>Exact optimal income</td>
<td>9.6377</td>
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The exact optimal parameters are found by trial-and-error methods. It will be noticed that there is a close similarity between the corresponding figures for the different methods of sampling.

Other plans can be approached in the same manner. However, the remaining plans involve parameters other than the sampling rate. If these parameters are fixed in advance, then the problem is no greater than that dealt with in the case of the basic plan above. Maximization of the income over any set of these parameters would multiply appreciably the computations involved, although the problem is well within the scope of an electronic calculating machine. It has, in fact, also been done on an automatic desk calculator.
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70