APPLIED MATHEMATICS AND STATISTICS LABORATORIES
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OPTIMAL POLICY FOR DYNAMIC INVENTORY PROCESS
WITH NON-STATIONARY STOCHASTIC DEMANDS

By
D. IGLEHART and S. KARLIN

TECHNICAL REPORT NO. 44
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PREPARED FOR ARMY, NAVY AND AIR FORCE UNDER
CONTRACT N6onr-25126 (NR 042 002)
WITH THE OFFICE OF NAVAL RESEARCH
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0. Non-Mathematical Summary.

We shall treat an inventory model which considers a single commodity. A sequence of ordering decisions is made at the beginning of each of a number of discrete time intervals of equal length; e.g., every two months. These decisions may result in a replenishment of the stock of the commodity. Consumption during these time intervals may cause a depletion of the stock. The cumulative demand in each period is a random quantity. From period to period the type of randomness of the demand is allowed to vary.

The demands we assume include a number of important practical situations. In particular this model can deal with the following special cases:

a) The type of demand randomness is the same in all periods.

b) The type of demand randomness changes from period to period in a cyclical fashion. This type of demand structure would be appropriate if the demand for our commodity had a seasonal variation.

c) The type of demand randomness changes from period to period in a continuous fashion. This type of demand structure would be useful if the fluctuations in demand from period to period were sufficiently smooth.

Thus, in general the demand structure allows for uncertainties in demand within a period but permits fluctuations in the type of uncertainty from period to period.
Next we superimpose a cost structure on this inventory model. Three main costs are incurred during successive periods which influence the ordering decisions. There is an ordering cost, a holding cost associated with the excess of supply over demand, and a penalty or shortage cost associated with the excess of demand over supply. We also include a revenue term, which is regarded as a negative cost. Our objective is to determine the ordering policy which minimizes the cumulative average costs, where costs in future periods are discounted back to the present.

Under suitable conditions on the cost functions we are able to show that the optimal ordering policy is characterized by a set of critical numbers. The optimal policy is to order up to a critical number associated with the type of demand randomness present in the first period if the initial stock level is less than this critical number, and otherwise not to order.

The principal concern of this paper is to develop a computational algorithm for calculating these critical numbers. This algorithm involves solving for the unique roots of a set of transcendental equations.

As an extremely simple example of this algorithm, consider the case where our single commodity is summer uniforms. Assume furthermore that there are two types of demand (e.g., summer and winter) for this commodity and that they alternate in a cyclical manner. The ordering, holding, and penalty costs are all taken to be linear and
the revenue term to be zero. The demand densities we consider are negative exponentials with parameters \( \lambda_1 \) and \( \lambda_2 \). We let the unit ordering cost \( c \) be .1, the unit holding cost \( h \) be .01, the unit penalty cost be 1, the discount factor \( \alpha \) be .75, \( \lambda_1 \) be 0.1 and \( \lambda_2 \) be 0.01. To compute the two critical numbers which characterize the optimal policy we proceed as follows.

1. Compute \( y_1^{(1)} = \frac{1}{\lambda_1} \log \left[ \frac{h + p - \alpha c}{h + c(1-\alpha)} \right] \), \( i = 1, 2 \).

If we denote smallest critical number by \( \bar{x}_1 \), then

\[ \bar{x}_1 = \min(y_1^{(1)}, y_2^{(1)}) \]

In our example \( y_1^{(1)} = 32.8 \) and \( y_2^{(1)} = 328 \).

2. Set \( A = \alpha (h - \alpha c) \) and \( B = (A + \alpha p) \). We obtain \( A = -0.049 \) and \( B = .701 \).

3. Solve the following transcendental equation for \( x \).

\[
c + h(1 - e^{-\lambda_2 x}) - pe^{-\lambda_2 x} + \alpha c(1 - e^{-\lambda_2 \bar{x}_1}) e^{-\lambda_2 x} + A[1 - e^{-\lambda_2 (x - \bar{x}_1)}] \\
+ \frac{B\lambda_2}{(\lambda_1 - \lambda_2)} e^{-\lambda_2 x} [e^{-(\lambda_1 - \lambda_2) \bar{x}_1} - e^{-(\lambda_1 - \lambda_2) \bar{x}_1} - (\lambda_1 - \lambda_2) \bar{x}_1] = 0
\]

Let \( \bar{x}_2 \) denote the largest critical number. We obtain \( \bar{x}_2 = 277 \).
4. Our optimal policy when faced with an initial demand which is a negative exponential with parameter $\lambda_1 = 0.1$ is to order up to the level 33 if our stock level is less than 33, and not to order if the stock level is greater than 33. A similar statement applies when the initial demand is a negative exponential with parameter $\lambda_2 = 0.01$ where the value 277 is substituted for 33.
OPTIMAL POLICY FOR DYNAMIC INVENTORY PROCESS
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1. Introduction.

In this paper we deal with an inventory model where the demand is a stochastic process (called the demand process) in which the distributions of demand in successive periods are not identically distributed and in general are correlated.

Our model will consider a single commodity. A sequence of ordering decisions is to be made at the beginning of each of a number of discrete time intervals of equal length. These decisions may result in a replenishment of the stock of the commodity. Consumption during these time intervals may cause a depletion of the stock.

The cumulative demand in each period is a random variable whose distribution may change from period to period. We now make precise the form of this variation. Consider a finite collection of demand states labeled \( i = 1, 2, \ldots, k \). Let \( Y_n \) denote the demand state in the \( n \)th period. We assume that \( Y_n \) is a state variable obeying a Markov transition law with matrix \( P = \|p_{ij}\|, \ i, \ j = 1, 2, \ldots, k \) where \( p_{ij} \geq 0 \) and \( \sum_{j=1}^{k} p_{ij} = 1 \) for each \( i \).

With each state we associate a density function \( \varphi_i(x) \) such that the demand state in a given period tells us which demand density
is effective in that period. For example, if the demand state is \( i \) then the demand that results can be regarded as an observation based on the density \( \varphi_i(x) \). In the following period the state variable changes to state \( j \) with probability \( p_{ij} \) and the density \( \varphi_j \) underlies the demand in that period.

The following three special cases are illustrative.

a) If there exists only one state, then we have a stationary situation with the same distribution function underlying the demand in each period.

b) Let the Markov chain \( P \) have the form

\[
P = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

with \( k \) states (\( k > 1 \)). Then the demand pattern is periodic with cycle length of \( k \) periods duration whose demand densities are \( \varphi_1, \varphi_2, \ldots, \varphi_k \) corresponding to states \( 1, \ldots, k \) respectively. Such a demand structure may correspond to an inventory model where periods are related with different seasons and the periodic nature of the demand flow merely reflects seasonal variations.
c) Let the Markov chain $P$ have the form
\[
\begin{pmatrix}
  r_1 & p_1 & 0 \\
  q_2 & r_2 & p_2 & 0 \\
  q_3 & r_3 & p_3 \\
  0 & \cdots & \cdots \\
  0 & q_k & r_k
\end{pmatrix}
\]
with associated demand densities
\[
\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_k.
\]
In this case the demand density in successive periods consist of neighboring $\varphi_i$. If neighboring densities in the above set differ little from each other then the demand flow engendered by the matrix $P$ depicts a situation where the demand density in a sense varies continuously from period to period. A continuous time analog of such a demand flow would correspond to a stochastic process resembling a diffusion process.

The relationship of demand in successive periods can be regarded as a generalized Markovian process. Suppose first that the demand densities $\varphi_i$ are degenerate, each possessing a single observable value (say $i$). In other words, when the demand state is $i$ then necessarily a demand for $i$ units of stock arises. In the succeeding period a demand for $j$ units will happen with probability $p_{ij}$. Thus, in this case the actual demands can be identified with the demand states and they constitute a bona fide Markov chain.
On the other hand, if the $q_i$ are general densities then the demands in successive periods are no longer simply related as the realizations of a Markov chain. Nevertheless, to a large extent the demand that arises in a given period yields some knowledge as to the effective demand density for the following period. This is to be understood in the sense that each observation of demand permits by statistical methods an inference concerning the true demand density in that period or which is equivalent an estimate of the demand state of that period. Then the demand state in the next period is merely the outcome of a transition according to the Markov matrix $P_{ij}$. Generally we can say that the expected demands $E_i$ over successive periods form a Markov chain with transition matrix $||P_{ij}||$, even though the actual demands comprise a more general type process. The present formulation is quite general and appears to embrace a large variety of practical situations.

The stationary properties of an $(s,S)$ policy where the demands flow over time is a stochastic process of the type described above has been investigated in [6].

In this paper we superimpose a cost structure on this inventory model and develop an algorithm for calculating the critical numbers characterizing the optimal policy. In this connection we point out two recent papers [7] and [8] where various properties of the optimal policy for inventory situations of non-stationary demands were discussed. In [7] we set forth a computing procedure for
determining the explicit optimal policy where the demands vary in a
 cyclical manner. This discussion is now subsumed by our present
 formulation. The second paper studied the qualitative nature of the
 changes in the optimal policy when the demands in successive periods
 are suitably stochastically ordered.

 We now make precise the assumptions underlying the evaluation
 of the various costs of the model. Three main costs are incurred during
 successive periods which influence the ordering decisions. There is an
 ordering cost $c(z)$, where an amount $z$ is purchased; a holding cost
 $h(\cdot)$ is charged for the stock of inventories on hand (the cumulative
 excess of supply over demand); and finally a shortage or penalty cost
 $p(\cdot)$ associated with the failure to meet demands. Holding and penalty
 costs are charged at the end of each period. We also include a linear
 revenue factor $r$, which is regarded as a negative cost. We denote the
 one period expected holding, shortage and revenue costs when $y$ units
 of stock are available and the demand density in that period is $\varphi_1$ by

\begin{equation}
L_1(y) = \int_0^y [h(y-\xi) - r\xi] \varphi_1(\xi) \, d\xi + \int_y^\infty [p(\xi-y) - ry] \varphi_1(\xi) \, d\xi
\end{equation}

We assume throughout what follows that all cost functions are
sufficiently smooth and do not grow too rapidly so that all integrals
exist and all operations involving interchange of differentiation and
integration etc., are permissible.
Let \( f_i(x) \) denote the discounted expected cost which will be incurred during an infinite sequence of time periods if \( x \) is the initial level of stock, \( \varphi_i \) is the demand density in the first period, and an optimal ordering rule is used at each purchasing opportunity. Under the assumption that excess demand cannot be backlogged and that there is no time lag in delivery we obtain the system of functional equations

\[
(2) \quad f_i(x) = \min_{y \geq x} \{ c(y-x) + L_i(y) \}
\]

\[
+ \alpha \sum_{j=1}^{k} p_{ij} \left[ f_j(0) \int_{y}^{\infty} \varphi_j(\xi) \, d\xi + \int_{0}^{y} f_j(y-\xi) \varphi_i(\xi) \, d\xi \right] \quad i = 1, 2, \ldots, k,
\]

where \( \alpha \) denotes the discount factor and \( 0 < \alpha < 1 \). The proof of (2) follows a standard procedure (for example see [1], [2] and [3]).

If we allow backlogging, then the relevant functional equations are

\[
(3) \quad f_i(x) = \min_{y \geq x} \{ c(y-x) + L_i(y) \}
\]

\[
+ \alpha \sum_{j=1}^{k} \int_{0}^{\infty} f_i(y-\xi) \varphi_i(\xi) \, d\xi \quad i = 1, 2, \ldots, k.
\]

Here a negative value of \( x \) is to be interpreted as an amount owed to consumption. When there is a one period lag in delivery of ordered items and backlogging of excess demand is permitted, the appropriate functional equations are
\( f_1(x) = \min_{z \geq 0} \left[ c(z) + L_1(x) + \alpha \sum_{j=1}^{k} \beta_{ij} \int_{0}^{\infty} f_i(x+z-\xi) \nu_i(\xi) \, d\xi \right] \)

\[ i = 1, 2, \ldots, k. \]

Similar relations exist for inventory models of \( k \) period lags in delivery.

If the holding and shortage costs are convex increasing, continuous and vanish at the origin and \( c(z) = c \cdot z \), then the optimal policy is characterized by a set of \( k \) critical numbers \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k \). More precisely, if the demand density in the initial period is \( \nu_1 \), then purchase to the level \( \bar{x}_1 \) whenever less than \( \bar{x}_1 \) is available and otherwise do not buy. This is the character of the optimal policy in each of the cases corresponding to the functional equations (2), (3), and (4). These assertions are proved by induction on the number of periods, after first suitably truncating the infinite future to a finite number of periods. The techniques required follow those of Chapter 9, Section 3, and Chapter 10, Section 2, of [1]. The details will be omitted.

In Section 2 we present an algorithm for computing the critical numbers. It involves solving recursively for the unique root of \( k! \) transcendental equations. In Section 3 we describe a similar algorithm for the case when backlogging is permitted. In Section 4 the algorithm for the case of time lags in delivery is presented.
Some numerical examples are given in Section 5 for the case $k = 3$ and special choices of the cost factors. A proof of the optimality of the ordering rule of Section 2 is presented in Section 6.

Finally in Section 7 we relax the convexity assumptions imposed on the cost functions $h(\cdot)$ and $p(\cdot)$ at the expense of limiting the class of demand densities to Pólya Frequency Functions. Again we obtain an algorithm for computing the $\{x_i\}$ and prove the optimality of this policy.

2. The General Algorithm in the Non-Backlog Case.

In this section we treat the dynamic inventory model in which the demand distribution is allowed to vary from period to period according to a stochastic process as described in the Introduction.

We impose the following further conditions on the model:

(i) no backlogging of excess demand (see section 3 for backlogging case); (ii) no time lags in delivery (see section 4 for a discussion of time lags); (iii) linear purchasing cost $c(z) = c \cdot z$; (iv) holding and shortage costs are convex increasing, possess continuous derivatives, and vanish at the origin; and (v) there is a revenue term proportional to the quantity sold with unit revenue factor, $r$. 
To avoid pathological and uninteresting cases, we shall assume:

**Assumption I:** \( L'_i(0) + c < 0, \quad i = 1, 2, \ldots, k. \)

**Assumption II:** \( h'(0) + p'(0) + r - \alpha c > 0. \)

Both assumptions are satisfied if, for example, the marginal revenue factor exceeds the unit purchasing cost. The first assumption also will be satisfied if the expected marginal shortage cost in each period exceeds the unit purchasing cost. These are weak restrictions and will generally be fulfilled in practical situations. Actually if Assumption I is not satisfied, the optimal policy is never to order, i.e., demands are filled by priority shipments as they arise, thereby suffering a corresponding penalty charge.

In the Introduction we noted that the optimal policy is characterized by \( k \) critical numbers \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k \). We now proceed to describe an algorithm for computing \( \{\bar{x}_j\} \). The proof of the optimality for the policy characterized by the critical numbers calculated according to the algorithm will be presented in section .

Consider the \( k \) transcendental equations

\[
(5) \quad H^{(1)}_i(x) = c + L'_i(x) + \alpha \sum_{j=1}^{k} p_{ij} \int_0^x T_j^{(o)}(x-\xi) \varphi_i(\xi) \, d\xi
\]

\[
i = 1, 2, \ldots, k,
\]

* The superscript \((r)\) enclosed in parenthesis where \( r \) is an integer will not refer to differentiation. We reserve primes for this purpose.
where
\[ T_j^{(0)}(x) \equiv -c \quad j = 1, 2, \ldots, k. \]

Using Assumption I and the convexity properties of \( h(\cdot) \) and \( p(\cdot) \) we may verify by direct differentiation that each \( H_i^{(1)}(x) \) is increasing. By Assumption I, \( H_i^{(1)}(0) < 0 \) for each \( i \). Moreover, we have
\[ \lim_{x \to \infty} H_i^{(1)}(x) \geq c(1 - \alpha) > 0 \quad i = 1, 2, \ldots, k. \]

It follows that each equation \( H_i^{(1)}(x) = 0 \) possesses a unique root or a single closed interval of zeros. For definiteness, henceforth, whenever we speak of the root of such an equation we shall mean the smallest root. In most cases the root is in fact unique.

**Step 1.** We determine the \text{root} of each \( H_i^{(1)}(x) = 0 \) and denote them by \( y_1^{(1)}, y_2^{(1)}, \ldots, y_k^{(1)} \). Next we define
\[ y_{[1]}^{(1)} = \min_{1 \leq j \leq k} y_j^{(1)}. \]

The symbol \( [1] \) simply stands for the index value equal to that \( j \) for which \( y_j^{(1)} \) is minimum. We will continue as if \( y_{[1]}^{(1)} \) is unique. In the contrary event, the required modifications are the obvious ones.
Assertion I: The smallest critical number is \( \bar{x}_{[1]} = y_{[1]}^{(1)} \).

In order to calculate the second smallest critical number we form the functions

\[
H_{i}^{(2)}(x) = c + L_{i}^{1}(x) + \alpha \sum_{j=1}^{k} \pi_{ij} \int_{0}^{x} T_{j}^{(1)}(x-\xi) \varphi_{i}(\xi) \, d\xi
\]

\( i = 1, 2, \ldots, k \),

\[
T_{j}^{(1)}(x) = -c \quad \text{for all} \quad j \neq [1],
\]

\[
T_{[1]}^{(1)}(x) = \begin{cases} 
-c & 0 \leq x \leq \bar{x}_{[1]} \\
 g_{[1]}^{(1)}(x) & \bar{x}_{[1]} < x,
\end{cases}
\]

where the function \( g_{[1]}^{(1)}(x) \) is determined as the solution of the integral equation

\[
g_{[1]}^{(1)}(x) = L_{[1]}^{1}(x) + \alpha \sum_{j=1}^{k} \pi_{[1]j} \int_{0}^{x} T_{j}^{(1)}(x-\xi) \varphi_{[1]}(\xi) \, d\xi
\]

\( x > \bar{x}_{[1]} \).

We prove below the existence of a unique solution to (7). The practical problem of the explicit calculation of \( g_{[1]}^{(1)}(x) \) will be discussed at the close of this section.
Observe that the construction of \( g^{(1)}_{[1]}(x) \) compels \( T^{(1)}_{[1]}(x) \) to be continuous at \( \tilde{x}_{[1]} \), since \( H^{(1)}_{[1]}(\tilde{x}_{[1]}) = 0 \). In dealing with (7) it is convenient to introduce the function \( h^{(1)}_{[1]}(x) \) which is the translation of \( g^{(1)}_{[1]} \) given by

\[
(8) \quad h^{(1)}_{[1]}(x) = g^{(1)}_{[1]}(x + \tilde{x}_{[1]}) \quad \text{for } x > 0.
\]

In terms of \( h^{(1)}_{[1]}(x) \), (7) reduces to the renewal equation

\[
(9) \quad h^{(1)}_{[1]}(x) = L^{1}_{[1]}(x + \tilde{x}_{[1]}) - \alpha c \int_{0}^{x + \tilde{x}_{[1]}} \varphi_{[1]}(\xi) \, d\xi \\
\quad \quad + \alpha p_{[1]}(1) c \int_{0}^{x} \varphi_{[1]}(\xi) \, d\xi \\
\quad \quad + \alpha p_{[1]}(1) \int_{0}^{x} h^{(1)}_{[1]}(x - \xi) \varphi_{[1]}(\xi) \, d\xi \quad x > 0
\]

Differentiating (8) and using the continuity of \( T^{(1)}_{[1]}(x) \) at \( x = \tilde{x}_{[1]} \) we secure the further renewal equation

\[
(10) \quad h^{(1)}_{[1]}'(x) = L''_{[1]}(x + \tilde{x}_{[1]}) - \alpha c \varphi_{[1]}(x + \tilde{x}_{[1]}) \\
\quad \quad + \alpha p_{[1]}(1) \int_{0}^{x} h^{(1)}_{[1]}'(x - \xi) \varphi_{[1]}(\xi) \, d\xi \quad x > 0
\]

At this point, we record a familiar result pertaining to renewal equations, which we shall need.
Theorem A. [5] Suppose \( f(t) \) is a density function and that \( a(x) \) is continuous. The integral equation

\[
 u(x) = a(x) + \int_0^x u(x-t) f(t) \, dt \quad x \geq 0
\]

possesses a unique solution and if \( a(x) \geq 0 \) for all \( x \geq 0 \), then \( u(x) \geq 0 \).

Applying Theorem A to (1) we deduce that \( h_{[1]}^{(1)} \) exists and that \( h_{[1]}^{(1)}(x) \geq 0 \), since \( I_{[1]}^n(x + \tilde{x}_{[1]}) - \alpha c \varphi_{[1]}(x + \tilde{x}_{[1]}) \geq 0 \) by Assumption II. It follows that \( g_{[1]}^{(1)}(x) \) is non-decreasing and in particular

\[
 T_{[1]}^{(1)}(x) \geq -c \quad \text{for} \quad x \geq 0
\]

Furthermore \( T_{[1]}^{(1)}(x) \geq 0 \) for all \( x \geq 0 \) with the possible exception of the point \( x = \tilde{x}_{[1]} \) where only left and right hand bounded derivatives may exist. These facts insure that

\[
 \lim_{x \to \infty} H_{[1]}^{(2)}(x) \geq c(1 - \alpha) > 0 \quad 1 = 1, 2, \ldots, k,
\]

and that each \( H_{[1]}^{(2)}(x) \) is non-decreasing. On the other hand we know by Assumption I that \( H_{[1]}^{(2)}(0) < 0 \) for each \( i \). Hence each equation \( H_{[1]}^{(2)}(x) = 0 \) possesses a unique root or a single closed interval of zeros.
Step 2. We determine the roots of each $H_i^{(2)}(x) = 0$

and denote them by $y_1^{(2)}, y_2^{(2)}, \ldots, y_k^{(2)}$. Observe that $y_1^{(2)} \leq y_1^{(1)}$, since $H_i^{(2)}(x) \geq H_i^{(1)}(x)$ for all $x$. Moreover the minimum of the $y_i^{(2)}$ is exactly $y_{[1]}^{(1)} = \bar{x}_{[1]}$, since $H_i^{(1)}(x) = H_i^{(2)}(x)$ for $x \leq \bar{x}_{[1]}$. Now we define $y_{[2]}^{(2)}$ to be

$$y_{[2]}^{(2)} = \min_{1 \leq j \leq k, j \neq [1]} (y_j^{(2)}) .$$

Assertion II. The second smallest critical number is $\bar{x}_{[2]} = y_{[2]}^{(2)}$.

We proceed to find the third smallest critical number.

For this purpose we construct the functions

$$(11) \quad H_i^{(3)}(x) = c + L_i^{(1)}(x) + \alpha \sum_{j=1}^{k} p_{ij} \int_{0}^{x} T_j^{(2)}(x-\xi) \phi_i(\xi) \, d\xi$$

$$i = 1, 2, \ldots, k,$$

$$T_j^{(2)}(x) = \begin{cases} -c & \text{if } j \neq [1], [2] \\ g_{[1]}^{(1)} & 0 \leq x \leq \bar{x}_{[1]} \\ \bar{x}_{[1]} < x \leq \bar{x}_{[2]} \\ g_{[1]}^{(2)} & \bar{x}_{[2]} < x \end{cases}$$
\[
T_{[2]}^{(2)} = \begin{cases} 
-c & 0 \leq x \leq \bar{x}[2] \\
g_{[2]}^{(1)} & \bar{x}[2] < x 
\end{cases}
\]

where \(g_{[1]}^{(2)}\) and \(g_{[2]}^{(1)}\) represent the unique set of solutions of the pair of integral equations

\[
ge_{[1]}^{(2)}(x) = L_{[1]}^{(2)}(x) + \alpha \sum_{j=1}^{k} P_{[1]}^{(2)} \int_{0}^{x} T_{j}^{(2)}(x-\xi) \varphi_{[1]}^{(2)}(\xi) \, d\xi
\]

(12)

\[
ge_{[2]}^{(1)}(x) = L_{[2]}^{(2)}(x) + \alpha \sum_{j=1}^{k} P_{[2]}^{(2)} \int_{0}^{x} T_{j}^{(2)}(x-\xi) \varphi_{[2]}^{(2)}(\xi) \, d\xi
\]

\(x > \bar{x}[2]\)

Again reflecting on the meaning of \(y_{i}^{(2)}\) we easily see that \(T_{j}^{(2)}(x)\) are continuous throughout. We define

\[
h_{[1]}^{(2)}(x) = g_{[1]}^{(2)}(x + \bar{x}[2]) & x > 0
\]

(13)

\[
h_{[2]}^{(1)}(x) = g_{[2]}^{(1)}(x + \bar{x}[2]) & x > 0
\]

Expressing (12) in terms of (13) there results the pair of renewal equations
(14) \( h^{(2)}_{[1]}(x) \)

\[
= L_{[1]}^{(x+\bar{x}_{[2]}) - \alpha c} \int_{0}^{x+\bar{x}_{[2]}} \varphi_{[1]}(\xi) d\xi + \alpha c \psi_{[1]}^{(1)} \int_{0}^{x+\bar{x}_{[2]} - \bar{x}_{[1]}} \varphi_{[1]}(\xi) d\xi \\
+ \alpha c \psi_{[1]}^{(1)} \int_{x}^{x+\bar{x}_{[2]} - \bar{x}_{[1]}} \varphi_{[1]}(\xi) d\xi + \alpha c \psi_{[1]}^{(1)} \int_{0}^{x} \varphi_{[1]}(\xi) d\xi \\
+ \alpha c \psi_{[1]}^{(1)} \int_{0}^{x} \varphi_{[1]}^{(2)}(x-\xi) \varphi_{[1]}(\xi) d\xi + \alpha c \psi_{[1]}^{(1)} \int_{0}^{x} \varphi_{[1]}^{(2)}(x-\xi) \varphi_{[1]}(\xi) d\xi
\]

\( x > 0 \)

and

(15) \( h^{(1)}_{[2]}(x) \)

\[
= L_{[2]}^{(x+\bar{x}_{[2]}) - \alpha c} \int_{0}^{x+\bar{x}_{[2]}} \varphi_{[2]}(\xi) d\xi + \alpha c \psi_{[2]}^{(1)} \int_{0}^{x+\bar{x}_{[2]} - \bar{x}_{[1]}} \varphi_{[2]}(\xi) d\xi \\
+ \alpha c \psi_{[2]}^{(1)} \int_{x}^{x+\bar{x}_{[2]} - \bar{x}_{[1]}} \varphi_{[2]}(\xi) d\xi \\
+ \alpha c \psi_{[2]}^{(1)} \int_{0}^{x} \varphi_{[2]}^{(2)}(x-\xi) \varphi_{[2]}(\xi) d\xi + \alpha c \psi_{[2]}^{(1)} \int_{0}^{x} \varphi_{[2]}^{(2)}(x-\xi) \varphi_{[2]}(\xi) d\xi
\]

Differentiating (14) and (15) and taking account of the fact that \( t_{j}^{(2)}(x) \) are continuous, we derive the pair of renewal relations
(16) \[ h^{(2)}_1(x) = I''_1(x + \bar{x}_2) - \alpha_c \phi_1(x + \bar{x}_2) \]
+ \( \alpha_p [1][1] \int_x^{x+\bar{x}_2} g^{(1)}_1(x+\bar{x}_2 - \xi) \phi_1(\xi) d\xi \)
+ \( \alpha_p [1][1] \int_0^x h^{(2)}_1(x-\xi) \phi_1(\xi) d\xi \)
+ \( \alpha_p [1][2] \int_0^x h^{(1)}_2(x-\xi) \phi_1(\xi) d\xi \), \quad x > 0

and

(17) \[ h^{(1)}_2(x) = I''_2(x + \bar{x}_2) - \alpha_c \phi_2(x + \bar{x}_2) \]
+ \( \alpha_p [2][1] \int_x^{x+\bar{x}_2} g^{(1)}_1(x+\bar{x}_2 - \xi) \phi_2(\xi) d\xi \)
+ \( \alpha_p [2][1] \int_0^x h^{(2)}_1(x-\xi) \phi_2(\xi) d\xi \)
+ \( \alpha_p [2][2] \int_0^x h^{(1)}_2(x-\xi) \phi_2(\xi) d\xi \), \quad x > 0

We now cite a known result pertaining to solutions of systems of renewal equations.
Theorem B. Let \( f_1 \) denote density functions and suppose \( a_1(x) \) are continuous on the positive real axis. Then the system of renewal equations

\[
\begin{align*}
  u_i(x) &= a_i(x) + \sum_{j=1}^{k} \int_0^x u_j(x-t) f_j(t) \, dt \\
  x &\geq 0, \quad i = 1, 2, \ldots, k
\end{align*}
\]

possess a unique set of solutions. Moreover, if \( g_1(x) \geq 0 \) for \( x \geq 0, \quad i = 1, 2, \ldots, k, \) then \( u_i(x) \geq 0, \quad i = 1, 2, \ldots, k. \)

Applying theorem B to (17) and (18) we deduce that \( h_{[1]}^{(2)}(x) \geq 0 \) and \( h_{[2]}^{(1)}(x) \geq 0. \) By a similar argument to that followed in step 2 we infer that each equation \( H_i^{(3)}(x) = 0 \) possesses a unique root or a single closed interval of zeros.

**Step 3.** We determine the roots of each \( H_i^{(3)}(x) = 0 \) and denote them by \( y_1^{(3)}, y_2^{(3)}, \ldots, y_k^{(3)} \). The smallest and second smallest of these roots equal respectively \( \bar{x}_{[1]} \) and \( \bar{x}_{[2]} \).

Define \( y_{[3]}^{(3)} \) to be

\[
y_{[3]}^{(3)} = \min_{1 \leq j \leq k, j \neq [1],[2]} (y_j^{(3)}).
\]

**Assertion III:** The third smallest critical number is \( \bar{x}_{[3]} = y_{[3]}^{(3)}. \)
The general procedure should now be clear. Suppose the
r smallest critical numbers are determined,
\[ \bar{x}_1 = y_1 < \bar{x}_2 = y_2 < \ldots < \bar{x}_r = y_r. \]

We now show how to calculate the smallest critical number amongst the
set of remaining undetermined critical numbers. For this purpose,
we construct the functions
\begin{equation}
T_i^{(r)}(x) = c + L_i'(x) + \alpha \sum_{j=1}^{k} P_{ij} \int_{0}^{x} T_j^{(r)}(x) \varphi_j(\xi) \, d\xi
\end{equation}
\[ \quad i = 1, 2, \ldots, k, \]
where
\[ T_j^{(r)}(x) = -c \quad \text{for} \quad j \neq [1], [2], \ldots, [r] \]
\begin{equation}
T_{[j]}^{(r)}(x) = \begin{cases} 
-c & 0 \leq x \leq \bar{x}_{[j]} \\
\varepsilon_{[1]}^{(2)}(x) & \bar{x}_{[j]} < x \leq \bar{x}_{[j+1]} \\
\varepsilon_{[j]}^{(2)}(x) & \bar{x}_{[j+1]} < x \leq \bar{x}_{[j+2]}, \quad j = 1, 2, \ldots, r. \\
\varepsilon_{[r]}^{(x+1-j)}(x) & \bar{x}_{[r]} < x
\end{cases}
\end{equation}

Here, \( \varepsilon_{[j]}^{(i)}(x) \) for \( i = 1, 2, \ldots, r-1 \) and \( [j] = 1, 2, \ldots, r-1 \)
were determined during the analysis of the previous steps of the
algorithm. At this stage, we will solve for \( g^{(i)}_{[r]}, i = 1, 2, \ldots, r-1 \) as the solutions of the system of equations

\[
(19) \quad g^{(r+1-j)}_{[j]}(x) = L^{(r)}_{[j]}(x) + \alpha \sum_{\ell=1}^{k} p_{[j]} \ell \int_{0}^{x} T(\ell)(x-\xi) \varphi_{[\ell]}(\xi) \, d\xi
\]

\[x > \bar{x}_{[r]}, \quad j = 1, 2, \ldots, r.\]

In the process of evaluating the first \( r \) critical numbers we also deduced that \( g^{(r+1-i)}_{[j]}, j = 1, 2, \ldots, r, i = j, j+1, \ldots, r \) are non-decreasing functions. Furthermore, \( T^{(r-1)}_{[j]}(x), j = 1, 2, \ldots, k \) were shown to be continuous and non-decreasing. Forming the translates of \( g^{(j)}_{[r]} \) in the manner of (13) we obtain

\[
(20) \quad h^{(r+j-1)}_{[j]}(x) = L^{(r)}_{[j]}(x + \bar{x}_{[r]}) + \alpha \sum_{\ell=1}^{k} p_{[j]} \ell \int_{0}^{x+\bar{x}_{[r]}} T(\ell)(x + \bar{x}_{[r]} - \xi) \varphi_{[\ell]}(\xi) \, d\xi
\]

\[j = 1, 2, \ldots, r, \quad x > 0.\]

Differentiating (20) yields

\[
h^{(r+j-1)}_{[j]}'(x) = L^{(r)}_{[j]}(x + \bar{x}_{[r]}) - \alpha \varphi_{[j]}(x + \bar{x}_{[r]})
\]

\[+ \alpha \sum_{\ell=1}^{k} p_{[j]} \ell \int_{0}^{x+\bar{x}_{[r]}} T(\ell)'(x + \bar{x}_{[r]} - \xi) \varphi_{[\ell]}(\xi) \, d\xi
\]

\[j = 1, 2, \ldots, r, \quad x > 0.\]
By our construction we have that

\[(21) \quad T^{(r)}_\ell(x) = T^{(r-1)}_\ell(x) \geq 0 \quad \text{for} \quad x \leq \bar{X}_r, \ell = 1, 2, \ldots , k.\]

Referring to (21) plus Assumption I, we conclude that

\[h^{(r+1-j)}_j(x) = a_j(x) + \alpha \sum_{\ell=1}^{k} P[j][\ell] \int_{0}^{x} h^{(r+1-\ell)}_\ell(x-\xi) \varphi_{[j]}(\xi) \, d\xi\]

\[j = 1, 2, \ldots , r, \quad x > 0\]

and

\[a_j(x) \geq 0 \quad \text{for all} \quad x \geq 0.\]

Appealing once more to Theorem B on renewal equations we state

**General Assertion:** The \(r+1\)th smallest critical number is \(\bar{X}_{r+1}\) = \(y_{[r+1]}\), where \(y_{[r+1]}\) is the \(r+1\)th smallest root of the equations \(H_{[i]}^{(r+1)}(x) = 0, \quad i = 1, 2, \ldots , k.\)

In order to calculate all \(k\) critical numbers we must solve at most \(k!\) transcendental equations and \(k-1\) systems of renewal equations. A few comments are perhaps in order on the ease with which this algorithm can be implemented. To solve systems of renewal equations explicitly is in general a difficult task. Equations of this sort can be readily solved, however, when the demand distributions belong to the \(\Gamma\)-family of distributions, i.e.,

\[\varphi(x) = \begin{cases} 
\frac{k^{ \frac{k-1}{2} } x^{\frac{k-1}{2} - 1} e^{-\lambda x}}{\Gamma(k)} & \text{if } x > 0 \\
0 & \text{if } x \leq 0 
\end{cases}\]
or more generally when \( \varphi(x) \) is the density function of a sum of independent random variables belonging to exponential distributions with possibly different parameters. In this case the renewal equations can be converted into ordinary systems of linear differential equations with constant coefficients. When the densities are not members of the \( \Gamma \)-family, approximate solutions to the renewal equations can be obtained by an obvious iteration process. Thus, the algorithm could in the general case be suitably handled by a high-speed computer.

3. The Critical Numbers When Backlogging of Excess Demand is Permitted.

In this section we alter condition (i) of Section 2 to allow backlogging of excess demand. Conditions (ii) - (v) are unchanged from their statement as given in Section 2. In this case we can drop Assumption II. When backlogging is permitted, negative values \( x \) for the stock level may occur. A negative stock level should be interpreted as an amount of stock owed to consumption. Again the optimal policy is characterized by \( k \) critical numbers \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k \) corresponding to demands in the first period of \( \varphi_1, \varphi_2, \ldots, \varphi_k \).

The algorithm for evaluating the critical numbers \( \{\bar{x}_j\} \) is similar to that of Section 2, with appropriate adjustments. We briefly describe the necessary modifications. Consider the \( k \) equations

\[
(22) \quad K_1^{(1)}(x) = c + L_1(x) + \alpha \sum_{j=1}^{k} P_{ij} \int_{0}^{\infty} T_j^{(0)}(x-\xi) \varphi_i(\xi) \, d\xi
\]

\( i = 1, 2, \ldots, k \)
where
\[ T_j^{(0)}(x) = - c \]
and
\[
(23) \quad L_1(x) = \left\{ \begin{array}{ll}
\int_0^x h'(x-\xi) \varphi'(\xi) \, d\xi - \int_x^\infty [p'(x-\xi)+r] \varphi_1(\xi) \, d\xi & \quad x > 0 \\
- \int_0^\infty [p'(x-\xi)+r] \varphi_1(\xi) \, d\xi & \quad x \leq 0
\end{array} \right.
\]

Denote the roots of \( K_1^{(1)}(x) = 0, i = 1, 2, \ldots, k \) by \( y_1^{(1)}, y_2^{(1)}, \ldots, y_k^{(1)} \).
Then \( \bar{x}_{[1]} = y_{[1]}^{(1)} = \min_{1 \leq j \leq k} (y_j^{(1)}) \).

Next we form the equations
\[
K_1^{(2)}(x) = c + L_1(x) + \alpha \sum_{j=1}^k p_{1j} \int_0^\infty T_j^{(1)}(x-\xi) \varphi_1(\xi) \, d\xi
\]
\[
T_j^{(1)}(x) = - c \quad \text{for} \quad j \neq [1]
\]
\[
T_{[1]}^{(1)}(x) = \left\{ \begin{array}{ll}
- c & \quad x \leq \bar{x}_{[1]} \\
g_{[1]}^{(1)}(x) & \quad \bar{x}_{[1]} < x
\end{array} \right.
\]
where \( g_{[1]}^{(1)}(x) \) is determined as the solution of the renewal equation
\[
g_{[1]}^{(1)}(x) = L_{[1]}^{(1)}(x) + \alpha \sum_{j=1}^k p_{[1]j} \int_0^\infty T_j^{(1)}(x-\xi) \varphi_{[1]}(\xi) \, d\xi
\]
\[
x > \bar{x}_{[1]}
\]
If we denote the roots of $K_{i}^{(2)}(x) = 0$, $i = 1, 2, \ldots, k$ by $y_{1}^{(2)}, y_{2}^{(2)}, \ldots, y_{k}^{(2)}$, then the second smallest critical number $\bar{x}_{[2]}$ is $y_{[2]}^{(2)}$ where

$$\bar{x}_{[2]} = y_{[2]}^{(2)} = \min_{1 \leq j \leq k, j \neq [1]} (y_{j}^{(2)}).$$

The continuation of the algorithm is done in a similar manner to that of Section 2 with the functions $K_{i}^{(r)}(x)$ playing the role of the $H_{i}^{(r)}(x)$.

4. The Critical Numbers With Time Lag in Delivery.

We will consider only the case where the time lag is one period, however the method is applicable to the general case where the time lag is more than one period.

We alter correspondingly conditions (i) and (ii) of Section 2 to permit backlogging of excess demand and to take account of a time lag in delivery of one period duration. Purchasing decisions are done at the beginning of each period, but now delivery takes place at the start of the following period. In this context we replace Assumption II by the condition

$$\lim_{x \to \infty} p'(x) + r > \frac{1 - \alpha}{\alpha} c.$$

In practice this is an exceedingly mild restriction.
Subject to these conditions it is easily shown that the optimal policy is characterized by \( k \) critical numbers \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k \) corresponding to the initial demand densities \( \varphi_1, \varphi_2, \ldots, \varphi_k \). We proceed now to the algorithm for computing the \( \{\bar{x}_j\} \). For ease in exposition we shall consider the case \( k = 3 \) which sufficiently illustrates the arguments for the general case. Consider the expressions

\[
M_i^{(1)}(x) = c + \alpha \sum_{j=1}^{3} p_{ij} \int_0^\infty T_j^{(0)}(x-\xi) \varphi_i(\xi) \, d\xi \quad i = 1, 2, 3
\]

where

\[
T_j^{(0)}(x) = -c + L_j'(x)
\]

and \( L_j'(x) \) is defined by (23). Similar to the preceding analysis we find that each equation of \( M_1^{(1)}(x) = 0, i = 1, 2, 3 \) admits a unique root or a single closed interval of zeros. If we denote the roots of \( M_1^{(1)}(x) = 0 \) by \( y_1^{(1)}, y_2^{(1)}, y_3^{(1)} \), then the first critical number \( \bar{x}_{[1]} \) is given by

\[
\bar{x}_{[1]} = y_{[1]}^{(1)} = \min(y_1^{(1)}, y_2^{(1)}, y_3^{(1)}).
\]

Next we form the functions

\[
M_i^{(2)}(x) = c + \alpha \sum_{j=1}^{3} p_{ij} \int_0^\infty T_j^{(1)}(x-\xi) \varphi_i(\xi) \, d\xi \quad i = 1, 2, 3
\]
where

\[ T_{(1)}^j(x) = \begin{cases} -c + L_{(1)}^j(x) & \text{if } x \leq \bar{x}_{[1]} \\ g_{(1)}^{(1)}(x) & \bar{x}_{[1]} < x \end{cases} \]

and \( g_{(1)}^{(1)}(x) \) is determined as the unique solution of the renewal equation

\[ g_{(1)}^{(1)}(x) = L_{(1)}^j(x) + \alpha \sum_{j=1}^{3} p_{(1)j} \int_0^\infty T_{j}^{(1)}(x-\xi) \varphi_{(1)}(\xi) \, d\xi \]

\[ x > \bar{x}_{[1]} \]

Paraphrasing the earlier arguments we deduce that the equations \( M_i^{(2)}(x) = 0 \) have unique roots or single-closed intervals of zeros. Denote the roots of \( M_i^{(2)}(x) = 0 \) by \( y_1^{(2)}, y_2^{(2)}, y_3^{(2)} \). Then the second smallest critical number, \( \bar{x}_{[2]} \), is

\[ \bar{x}_{[2]} = y_{[2]}^{(2)} = \min_{1 \leq j \leq 3, j \neq [1]} (y_j^{(2)}) \]

To calculate the third critical number we consider the functions

\[ M_i^{(3)}(x) = c + \alpha \sum_{j=1}^{3} p_{ij} \int_0^\infty T_{j}^{(2)}(x-\xi) \varphi_i(\xi) \, d\xi \quad i = 1, 2, 3 \]
where

\[ T^{(2)}_j(x) = \begin{cases} 
- c + L^{(1)}_j(x) & x \leq \bar{x}_{[1]} \\
T^{(1)}_j(x) & \bar{x}_{[1]} < x \leq \bar{x}_{[2]} \\
T^{(2)}_j(x) & \bar{x}_{[2]} < x
\end{cases} \]

and

\[ T^{(2)}_{[1]}(x) = \begin{cases} 
- c + L^{(1)}_{[1]}(x) & x \leq \bar{x}_{[2]} \\
T^{(1)}_{[1]}(x) & \bar{x}_{[2]} < x
\end{cases} \]

The new functions \( g_{[1]}^{(2)}(x) \) and \( g_{[2]}^{(1)}(x) \) are determined by resolving the system of renewal equations

\[
g_{[1]}^{(2)}(x) = L^{(1)}_{[1]}(x) + \alpha \sum_{j=1}^{3} p_{[1],j} \int_{\bar{x}}^{\infty} T^{(2)}_j(x-\xi) \varphi_{[1]}(\xi) \, d\xi \]

(25)

\[
g_{[2]}^{(1)}(x) = L^{(1)}_{[2]}(x) + \alpha \sum_{j=1}^{3} p_{[2],j} \int_{\bar{x}}^{\infty} T^{(2)}_j(x-\xi) \varphi_{[2]}(\xi) \, d\xi \]

We deduce as before that the equations \( M^{(3)}_1(x) = 0, \ 1 = 1, 2, 3 \)

have unique roots or single closed intervals of zeros. Denote the roots of \( M^{(3)}_1(x) = 0, \ 1 = 1, 2, 3 \) by \( y^{(3)}_1, y^{(3)}_2, \) and \( y^{(3)}_3 \). The last critical number, \( \bar{x}_{[3]} \), will be given by \( y^{(3)}_j, \ j \neq [1], [2] \).
5. Some Numerical Examples.

In this section we offer some examples of the use of the algorithm of Section 2 for the case $k = 3$. The density functions considered are $\varphi_i(x) = \lambda_i e^{-\lambda_i x}$, $(x > 0)$, $i = 1, 2, 3$ where $\lambda_1 = 1$, $\lambda_2 = .5$, and $\lambda_3 = .1$. All cost functions are assumed to be linear and the revenue term is set equal to zero. The transition probability matrix, $P$, is specified to be

$$P = (p_{ij}) = \begin{pmatrix}
\frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{2} & \frac{1}{6}
\end{pmatrix}$$

The assumptions underlying the model are the same as those prescribed in Section 2. Table I lists the roots of $H_i^{(1)}(x) = 0$ for the various choices of the cost parameters as indicated.

<table>
<thead>
<tr>
<th>TABLE I. VALUES OF $y_1^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
</tr>
<tr>
<td>$c = 1$, $h = .1$</td>
</tr>
<tr>
<td>$p = 10$, $\alpha = .75$</td>
</tr>
<tr>
<td>$\varphi_1$</td>
</tr>
<tr>
<td>$\varphi_2$</td>
</tr>
<tr>
<td>$\varphi_3$</td>
</tr>
</tbody>
</table>
Following step I of the algorithm we deduce that the smallest critical numbers for these three choices of parameters correspond to densities \( \varphi_1 \) in the initial periods.

Continuing to step II we compute the roots of \( H_1^{(2)}(x) = 0 \), \( i = 2, 3 \). Observe that the root of \( H_1^{(2)}(x) = 0 \) will be the same as that of \( H_1^{(1)}(x) = 0 \) and hence need not be recalculated. Table II lists the roots required to implement step II.

<table>
<thead>
<tr>
<th>( y_1^{(2)} )</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_1 )</td>
<td>3.28</td>
<td>3.77</td>
<td>2.85</td>
</tr>
<tr>
<td>( \varphi_2 )</td>
<td>5.40</td>
<td>6.53</td>
<td>4.96</td>
</tr>
<tr>
<td>( \varphi_3 )</td>
<td>24.20</td>
<td>31.60</td>
<td>22.50</td>
</tr>
</tbody>
</table>

Hence according to the specifications of step II we see that the second smallest critical numbers correspond in each case to the density \( \varphi_2 \) in the initial period.

For step III we need only compute the root of \( H_3^{(3)}(x) = 0 \). Table III lists the roots required in step III. The values in Table III are also the critical numbers, \( \tilde{x}_1 \).
TABLE III. CRITICAL VALUES $x_i$

<table>
<thead>
<tr>
<th></th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c = 1$, $h = .1$</td>
<td>$c = 1$, $h = .2$</td>
<td>$c = 1$, $h = .5$</td>
</tr>
<tr>
<td></td>
<td>$p = 10$, $\alpha = .75$</td>
<td>$p = 20$, $\alpha = .75$</td>
<td>$p = 10$, $\alpha = .95$</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>3.28</td>
<td>3.77</td>
<td>2.85</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>5.40</td>
<td>6.53</td>
<td>4.96</td>
</tr>
<tr>
<td>$\phi_3$</td>
<td>22.10</td>
<td>30.60</td>
<td>20.60</td>
</tr>
</tbody>
</table>

In this example each case involved solving for the roots of 6 transcendental equations, obtaining the solution of one renewal equation, and resolving a system of two simultaneous renewal equations. Since the densities are exponential, the renewal equation in each case was dealt with by solving an equivalent linear differential equation.


In this section we prove that the policy obtained by the algorithm of Section 2 is indeed optimal. The optimality proof for the algorithms of Sections 3 and 4 can be carried out by appropriately modifying these arguments.
The conditions of Section 2 are assumed to prevail. As stated in Section 1, the optimal policy at each state is determined by a single critical number which is a function of the demand density of that period. Let $f_i(x)$ denote the minimum discounted expected cost if the initial stock level is $x$ and the demand density in the first period is $\varphi_1$. If $\bar{x}_i$ is the optimal critical number at the first period, then it is easily shown that $f_i(x)$ is convex, has a continuous derivative and

$$f_i'(x) = -c \quad \text{for} \quad x \leq \bar{x}_i$$

(see Chap. 9, Section 2 of [1], or [2]).

The optimal critical number $\bar{x}_i$, when confronted in the first period with the demand density $\varphi_1$, occurs at the absolute minimum of $G_i(x)$; where

$$G_i(x) = cx + L_i(x) + \alpha \sum_{j=1}^{k} p_{ij} \int_{0}^{x} f_j(x-\xi) \varphi_1(\xi) \, d\xi$$

$$+ \alpha \sum_{j=1}^{k} p_{ij} f_j(0) \int_{x}^{\infty} \varphi_i(\xi) \, d\xi$$

$$i = 1, 2, \ldots, k$$

The function (27) is related to the minimum expected cost function as follows

$$f_i(x) = \min_{y \geq x} \{-cx + G_i(y)\}.$$
Since \( G_i'(0) < 0 \) by Assumption I, we infer that \( \tilde{x}_i \) is a root of the equation

\[
(28) \quad G_i'(x) = c + L_i'(x) + \alpha \sum_{j=1}^{k} p_{ij} \int_{0}^{x} f_j'(x-\xi) \varphi_i(\xi) \, d\xi = 0
\]

\( i = 1, 2, \ldots, k. \)

Appealing to the convexity and monotonicity properties satisfied by the cost functions and the convexity property inferred for \( f_i(x) \), we deduce that \( G_i'(x) \) are monotone increasing. This fact combined with Assumption I implies that the equations \( G_i'(x) = 0 \) possess unique roots or single closed intervals of zeros. Define \( \tilde{x}_i \) as the root of (28). (As before, we mean the smallest root of an equation when we speak of the root.) Since \( G_i(x) \) is convex it follows that the optimal policy calls for replenishment of stock to the level \( \tilde{x}_i \) when \( x \leq \tilde{x}_i \), and no purchasing when \( x > \tilde{x}_i \). We show now that the \( \{\tilde{x}_i\} \) can be evaluated by the algorithm of Section 2.

For ease of exposition we assume that the \( \tilde{x}_i, i = 1, 2, \ldots, k \) are distinct. We employ the notation of Section 2. Since \( f_i'(x) = -c, \)

\( i = 1, 2, \ldots, k, \) for all \( x \leq \min_{1 \leq i \leq k} \{\tilde{x}_i\} \) (see (26)), we conclude that \( G_i'(x) \equiv H_i^{(1)}(x), i = 1, 2, \ldots, k, \) for \( x \leq \min_{1 \leq i \leq k} \{\tilde{x}_i\} \). Hence the smallest root of the equations \( H_i^{(1)}(x) = 0 \) is \( \tilde{x}_{[1]} = \min_{1 \leq i \leq k} \{\tilde{x}_i\} \).

This justifies Step 1 and Assertion I of Section 2.
Now when \( x \) ranges on the interval from \( \bar{x}_{[1]} \) to \( \bar{x}_{[2]} \),
the second smallest critical number, we have

\[
f'_i(x) = -c \quad x \leq \bar{x}_{[2]}, \; i \neq [1]
\]

and

\[
f'_1(x) = L'_1(x) + \alpha \sum_{j=1}^{k} p_{[1],j} \int_{0}^{x} f'_j(x-\xi) \varphi_{[1]}(\xi) \, d\xi
\]

for \( x > \bar{x}_{[1]} \).

Observe that \( s'_{[1]}(x) \equiv f'_1(x) \) on the range \( \bar{x}_{[1]} < x \leq \bar{x}_{[2]} \).

Hence \( G'_i(x) = H'_1(x) \) for \( x \leq \bar{x}_{[2]} \). Thus the second smallest
root of \( H'_1(x) = 0 \) is \( \bar{x}_{[2]} \), justifying Step 2 and Assertion 2.

Proceeding to Step 3 we note that

\[
f'_i(x) = -c \quad x \leq \bar{x}_{[3]}, \; i \neq [1],[2],
\]

\[
f'_2(x) = L'_2(x) + \alpha \sum_{j=1}^{k} p_{[2],j} \int_{0}^{x} f'_j(x-\xi) \varphi_{[2]}(\xi) \, d\xi
\]

for \( x > \bar{x}_{[2]} \),

and \( f'_1(x) \) for \( x > \bar{x}_{[1]} \) is given by (29). Examination of (29)
and (30) reveals that
\[-34.25\]

\[f_{[1]}(x) = \begin{cases} -c & x \leq \tilde{x}_{[1]} \\ g_{[1]}^{(1)}(x) & \tilde{x}_{[1]} < x \leq \tilde{x}_{[2]} \\ g_{[1]}^{(2)}(x) & \tilde{x}_{[2]} < x \leq \tilde{x}_{[3]} \end{cases}\]

and

\[f_{[2]}(x) = \begin{cases} -c & x \leq \tilde{x}_{[2]} \\ g_{[2]}^{(1)}(x) & \tilde{x}_{[2]} < x \leq \tilde{x}_{[3]} \end{cases}\]

Hence \(G_i^{(1)}(x) = H_i^{(3)}(x)\) for \(x \leq \tilde{x}_{[3]}\) and the third smallest root of \(H_i^{(3)}(x) = 0\) is \(\tilde{x}_{[3]}\).

The remainder of the optimality proof consists of a routine repetition of the above argument. This concludes the proof of optimality.

7. **Extensions to Non-convex Costs.**

Throughout the preceding sections the validity of the analysis and computation of the optimal policies depended crucially on the convexity of the cost functions. At the same time no restrictions were placed on the demand densities \(\varphi_1, \varphi_2, \ldots, \varphi_k\). In this section we shall permit more general assumptions concerning the cost functions at the price of restricting the form of the demand densities.
We make the following assumptions on the model:

(i) no backlogging of excess demand; (ii) no time lags in delivery;
(iii) linear ordering cost \( c(z) = c \cdot z \); (iv) holding cost is increasing,
shortage cost is concave increasing, and both vanish at the origin and
possess continuous derivatives for \( x > 0 \); (v) there is a revenue
term proportional to the amount sold with unit revenue factor, \( r \); and
(vi) the demand densities \( \varphi_1, \varphi_2, \ldots, \varphi_k \) are Pólya Frequency Functions
(abbreviated P.F.F.).

We also retain Assumption I of Section 2, but drop Assumption
II. Before proceeding to the analysis of this model, we review the
previous work in this area.

For the case of a one period inventory model with the cost
functions of (iv) and arbitrary demand density the optimal policy is
not necessarily determined by a single critical number (see [3] and
[4]). In contrast if the demand density is taken to be a P.F.F. it
was proved in Chapter 8 of [1] that the optimal policy again has
the desired simple form involving one critical number. The same result
was also achieved for the dynamic inventory model provided the demand
density is stationary over time (see Chapter 9 of [1]). Similar
results were obtained for more general cost functions under the re-
striction that the demand density is a P.F.F. The theory developed
below could equally well apply in the case of these more general
costs.
For other background material concerning Pólya Frequency Functions and their uses in the analysis of inventory problems we refer the reader to Chapters 8 and 9 of [ 1 ]. The key feature about Pólya Frequency Functions important to us is its smoothing properties when we form an integral convolution transform with \( \varphi \) as its kernel. More precisely, if \( \varphi \) is a P.F.F. and if \( m(u) \) is a function which changes sign \( r \) times as \( u \) traverses the real line in the direction from \( -\infty \) to \( +\infty \) then

\[
M(u) = \int_{-\infty}^{\infty} m(\eta) \varphi(u - \eta) \, d\eta
\]

changes sign at most \( r \) times. Moreover, if the number of sign changes of \( M(u) \) is the same as that of \( m(u) \), then the two functions have the same arrangement of signs, i.e., \( M(u) \) and \( m(u) \) have the same sign for \( u \) sufficiently large. This property is known as the variation diminishing property (see also page 114 of [ 1 ]). We will use this result for \( r = 2 \). In that case we can require less of \( \varphi \), namely, that \( \varphi \) be a P.F.F. of order 3. (A density is a P.F.F. of order 3 if the property stated for the integral transformation applies as long as \( r \leq 2 \).) This includes any Gamma density of parameter \( \alpha \geq 1 \), i.e.,

\[
\varphi(\xi) = \lambda^{\alpha+1} \frac{\alpha^\alpha}{\Gamma(\alpha+1)} \xi^\alpha e^{-\lambda \xi / \Gamma(\alpha+1)},
\]

the exponential density, the truncated normal density and many others.
We now prove that for the inventory model under the assumptions stated in the second paragraph of this section and the demand variation as stated in the introduction, the optimal policy at each stage is characterized by a single critical number. Moreover, the optimal critical numbers $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ associated with the initial demand densities $\Phi_1, \Phi_2, \ldots, \Phi_k$, respectively, may be calculated by means of the algorithm of Section 2.

The difficulties in the proof stem from the fact that the $f_i(x)$ are not necessarily convex and we cannot deduce simply the form of the optimal policy. A procedure of truncation and induction in terms of the time horizon appears very formidable again due to the lack of convexity. We shall analyze the functional equation system (2) directly. To elucidate the exposition we carry out the proof for the case $k = 3$, which already includes all the essential features of the reasoning.

Consider the system of equations

$$f_i(x) = \min_{y \geq x} \left\{ c(y-x) + L_i(y) + \alpha \sum_{j=1}^{3} p_{ij} \int_{0}^{y} f_j(y-\xi) \Phi_i(\xi) d\xi \right\}$$

$$+ \alpha \sum_{j=1}^{3} p_{ij} f_j(0) \int_{y}^{\infty} \Phi_i(\xi) d\xi \right\}$$

$$= \min_{y \geq x} \left\{ -cx + C_i(y) \right\}$$

$i = 1, 2, 3.$
In virtue of Assumption I, $G_1'(0) < 0$ while $G_1(y) \to \infty$ as $y \to \infty$. Hence, each $G_i(y)$, $i = 1, 2, 3$ attains its absolute minimum at, say, $\bar{y}_1, \bar{y}_2, \bar{y}_3$ respectively ($0 < \bar{y}_i < \infty$), where $G_i'(\bar{y}_i) = 0$. For definiteness we suppose that $\bar{y}_1 < \bar{y}_2, \bar{y}_3$. Clearly $f_1'(x) = -c$ provided $x < \bar{y}_1$ respectively.

Observe that $H_{1}^{(1)}(x)$ as defined in (5) can be written in the form

$$H_{1}^{(1)}(x) = \int_{-\infty}^{\infty} m(\xi) \varphi_1(x-\xi) \, d\xi$$

$x > 0$, $i = 1, 2, 3$

where

$$m(\xi) = \begin{cases} h'(\xi) + c(1 - \alpha) & \xi \geq 0 \\ c - r - p'(-\xi) & \xi < 0 \end{cases}$$

The assumptions on the cost functions trivially imply that $m(\xi)$ changes sign at most twice in the order $+ - +$ as $\xi$ varies from $-\infty$ to $+\infty$. Invoking the variation diminishing properties of $\varphi_1$, we conclude that each $H_{1}^{(1)}(x)$ changes sign at most twice. Moreover, since $H_{1}^{(1)}(0) < 0$ by Assumption I and $\lim_{x \to 0} H_{1}^{(1)}(x) \geq c(1 - \alpha) > 0$, it follows that each $H_{1}^{(1)}(x)$ possesses a unique zero in the interval $[0, \infty)$. But $G_1'(x) = H_{1}^{(1)}(x)$ for $x \leq \bar{y}_1$ which means that $\bar{y}_1$ is the only root of $H_{1}^{(1)}(x) = 0$. 
Since generally $f_i'(x) \geq -c$ for all $x$ (see page 139 [1]), it follows that

$$G_1'(x) \geq H_1'(x) > 0 \quad \text{for} \quad x > \bar{y}_1$$

Thus $G_1(x)$ is monotone increasing for $x > \bar{y}_1$ and this means that no purchasing should be done for stock levels $x > \bar{y}_1$, when $\varphi_1$ is the density of the first period. In summary, we have proved that the single critical number $\bar{x}_1 = \bar{y}_1$ characterizes the optimal policy for any period of demand $\varphi_i$. We must verify now that $\bar{x}_1$ is smaller than the zeros of $H_1^{(1)}(x)$ for $i = 2, 3$.

To this end, we first secure the existence of unique zeros $z_2$ and $z_3$ for $H_2^{(1)}(x)$ and $H_3^{(1)}(x)$ respectively following arguments like that conducted on $H_1(x)$ above. These numbers necessarily exceed $\bar{x}_1 = \bar{y}_1$. Indeed, in the contrary event, say $z_2 < \bar{y}_1$ happens, we can paraphrase the analysis applied for $H_1^{(1)}(x)$ above to the function $H_2^{(1)}(x)$ and we deduce that $z_2 = \bar{y}_2 < \bar{y}_1$, a contradiction. Thus the calculation of the smallest critical number can be obtained by the procedure of Step 1, Section 2.

Next, we prove that the second step of the algorithm of Section 2 determines the second smallest of the critical numbers. Consider the functions $H_1^{(2)}(x)$ defined in (6). These expressions can be written in the form

$$H_1^{(2)}(x) = \int_{-\infty}^{\infty} n_i(\xi) \psi_i(x-\xi) \, d\xi \quad \text{for} \quad x > 0, \ i = 1, 2, 3,$$
where
\[
n_1(\xi) = \begin{cases} 
  c + h'(\xi) - \alpha c p_{12} - \alpha c p_{13} + \alpha p_{11} T_1^{(1)}(\xi) & \xi \geq 0 \\
  c - r - p'(-\xi) & \xi < 0, \ i = 1, 2, 3
\end{cases}
\]

If we can show that \( T_1^{(1)}(\xi) \geq -c \) for \( \xi \geq 0 \), we will be able to deduce that \( n_1(\xi) \) has at most two sign changes in the order + - + as \( \xi \) traverses the real line from \(-\infty\) to \(+\infty\). From the definition of \( T_1^{(1)}(\xi) \) it is sufficient to verify that \( g_1^{(1)}(\xi) \geq -c \) for \( \xi > \tilde{x}_1 \).

By performing operations similar to those of Section 2 we obtain
\[
(h_1^{(1)}(x) + c) = c + L_1'(x + \tilde{x}_1) - \alpha c \int_0^{x + \tilde{x}_1} \varphi_1(\xi) \, d\xi \\
+ \alpha p_{11} \int_0^x [h_1^{(1)}(x - \xi) + c] \varphi_1(\xi) \, d\xi
\]

where \( h_1^{(1)}(x) = g_1^{(1)}(x + \tilde{x}_1), \ x > 0. \)

Because of the meaning of \( \tilde{x}_1 \), we know that
\[
H_1^{(1)}(x + \tilde{x}_1) = c + L_1'(x + \tilde{x}_1) - \alpha c \int_0^{x + \tilde{x}_1} \varphi_1(\xi) \, d\xi > 0
\]
for \( x > 0. \)

Now applying Theorem A on (32), we conclude that \( g_1^{(1)}(x) \geq -c \) for \( x > \tilde{x}_1 \) as desired.

Since each \( n_1(\xi) \) has at most two sign changes, we deduce by the same analysis used in determining \( \tilde{x}_1 \), that each \( H_1^{(2)}(x) \) has one positive zero, call them \( y_1^{(2)}, y_2^{(2)}, y_3^{(2)} \). Furthermore \( y_1^{(2)} = \tilde{x}_1 \).
since \( H_{1}^{(1)}(x) = H_{1}^{(2)}(x) \) for \( x \leq \bar{x}_{1} \). We claim that \( G_{2}'(x) \geq H_{2}^{(2)}(x) \) and \( G_{3}'(x) \geq H_{3}^{(2)}(x) \) for all \( x \). In order to prove this it suffices to verify the inequality \( f_{1}'(x) \geq T_{1}^{(1)}(x) \) for all \( x \). For the range \( x \leq \bar{x}_{1} \), the inequality trivially holds since both functions are equal to \(-c\). On the range \( x > \bar{x}_{1} \) we form the equation

\[
f_{1}'(x) - T_{1}^{(1)}(x) = \alpha p_{12} \int_{0}^{x} [f_{2}'(x-\xi) + c] \varphi_{1}(\xi) d\xi + \alpha p_{13} \int_{0}^{x} [f_{3}'(x-\xi) + c] \varphi_{1}(\xi) d\xi
\]

\[
+ \alpha p_{11} \int_{0}^{x} [f_{1}'(x-\xi) - T_{1}^{(1)}(x-\xi)] \varphi_{1}(\xi) d\xi, \quad x > \bar{x}_{1}
\]

Translating the function \( f_{1}'(x) - T_{1}^{(1)}(x) \) by \( \bar{x}_{1} \), using the general fact that \( f_{1}'(x) \geq -c \) for all \( x \), and appealing to Theorem A we deduce that \( f_{1}'(x) - T_{1}^{(1)}(x) \geq 0 \) for all \( x \). It now follows that \( G_{2}'(x) \geq H_{2}^{(2)}(x) \) and \( G_{3}'(x) \geq H_{3}^{(2)}(x) \) for all \( x \).

From here on we distinguish two cases.

**Case A:** \( \bar{y}_{2} < \bar{y}_{3} \). Clearly \( G_{1}'(x) \equiv H_{2}^{(2)}(x) \) and \( G_{2}'(x) \equiv H_{2}^{(2)}(x) \) for \( 0 \leq x \leq \bar{y}_{2} \). Since \( G_{2}'(\bar{y}_{2}) = 0, \bar{y}_{2} = y_{2}^{(2)} \) must be the unique root of \( H_{2}^{(2)}(x) = 0 \). It was shown above that \( G_{2}'(x) \geq H_{2}^{(2)}(x) > 0 \) for \( x > \bar{y}_{2} \), hence \( y_{2}^{(2)} = \bar{x}_{2} \) is the second critical number. To show that \( \bar{x}_{2} \) also results from the second step of the algorithm of Section 2, we must establish the set of inequalities \( y_{1}^{(2)} < y_{2}^{(2)} < y_{3}^{(2)} \). That \( y_{1}^{(2)} < y_{2}^{(2)} \) is correct follows immediately from the fact that \( H_{1}^{(2)}(x) \geq H_{1}^{(1)}(x) \), \( i = 1, 2, 3 \) for all \( x \), while \( H_{1}^{(1)}(x) = H_{1}^{(2)}(x) \) for \( x \leq \bar{x}_{1} \), and the knowledge that the root of \( H_{1}^{(1)}(x) = 0 \) is smaller than the root of
\( H_2^{(1)}(x) = 0 \). To show the second inequality \( y_2^{(2)} < y_3^{(2)} \), assume the contrary is true, namely \( y_2^{(2)} \geq y_3^{(2)} \). Then we have \( G_3'(x) \geq H_3^{(2)}(x) > 0 \) for \( x > y_2^{(2)} \). But \( G_3' \left( \bar{y}_3 \right) = 0 \), so we have a contradiction, and consequently \( y_2^{(2)} < y_3^{(2)} \). The discussion just completed proves that the calculation of the second smallest critical number can be obtained by the procedure of Step 2, Section 2.

Case B. \( \bar{y}_3 < \bar{y}_2 \).

The argument is identical to that used in Case A and will not be repeated here.

It remains to show that Step III of Section 2 determines the third critical number. We consider again Case A. In the usual manner we examine the functions \( H_i^{(3)}(x) \) defined by (11). These expressions may be written in the form

\[
H_i^{(3)}(x) = \int_{-\infty}^{\infty} \ell_i(\xi) \varphi_1(x-\xi) \, d\xi \quad x > 0, \ i = 1, 2, 3
\]

where

\[
\ell_i(\xi) = \begin{cases} 
\left[ c + h'(\xi) - \alpha p_{13} + \alpha p_{11} T_1^{(2)}(\xi) + \alpha p_{12} T_2^{(2)}(\xi) \right] & \xi \geq 0 \\
\left( c - r - p'(-\xi) \right) & \xi < 0 
\end{cases} \quad i = 1, 2, 3
\]

Again we must show that \( T_1^{(2)}(\xi) \geq -c \) and \( T_2^{(2)}(\xi) \geq -c \) for all \( \xi \geq 0 \) in order to be able to deduce that the \( \ell_i(\xi) \) have at most two sign changes. It clearly suffices to show that \( g_1^{(2)}(x) \geq -c \) and
\[ g_2^{(1)}(x) \geq -c \text{ for } x > \bar{x}_2. \] Rather than verifying these inequalities directly it appears easier to prove that \[ g_1^{(2)}(x) \geq g_1^{(1)}(x) \] and \[ g_2^{(1)}(x) \geq -c \text{ for } x > \bar{x}_2. \] The reader may recall that we already know by Step II that \[ g_1^{(1)}(x) \geq -c \text{ for } x > \bar{x}_1. \] We define the following translated function exactly as was done in Section 2.

\[
\begin{align*}
\begin{cases}
  h_1^{(2)}(x) &= g_1^{(2)}(x + \bar{x}_2) \\
  h_1^{(1)}(x) &= g_2^{(1)}(x + \bar{x}_2) \\
  h_1^{(1)}(x) &= g_1^{(1)}(x + \bar{x}_1)
\end{cases}
\] \quad x > 0
\]

From the defining integral equations for these functions as given in Section 2 we form the simultaneous renewal equations

\[ (33) \quad [h_1^{(2)}(x) - h_1^{(1)}(x + \bar{x}_2 - \bar{x}_1)] \]

\[
= L_1(x + \bar{x}_2 - \bar{x}_1) - \alpha c \int_0^{x+\bar{x}_2} \varphi_1(\xi) d\xi + \alpha c \int_0^{x+\bar{x}_2 - \bar{x}_1} \varphi_1(\xi) d\xi
\]

\[
+ \alpha c \int_0^{x+\bar{x}_2 - \bar{x}_1} h_1^{(1)}(x+\bar{x}_2 - \bar{x}_1 - \xi) \varphi_1(\xi) d\xi - h_1^{(1)}(x + \bar{x}_2 - \bar{x}_1)
\]

\[
+ \alpha c \int_0^x \left[ h_1^{(2)}(x-\xi) - h_1^{(1)}(x+\bar{x}_2 - \bar{x}_1 - \xi) \right] \varphi_1(\xi) d\xi
\]

\[
+ \alpha c \int_0^x \left[ h_2^{(1)}(x-\xi) + c \right] \varphi_1(\xi) d\xi \quad x > 0
\]

and
\[(34) \quad [h_2^{(1)}(x) + c] = c + L_2^{(1)}(x + \bar{x}_2) - \alpha c \int_0^{x + \bar{x}_2} \varphi_2(\xi) d\xi + \alpha p_{21} \int_0^{x + \bar{x}_2 - \bar{x}_1} [g_1^{(1)}(x + \bar{x}_2 - \xi) + c] \varphi_2(\xi) d\xi \\
+ \alpha p_{21} \int_0^x [h_1^{(2)}(x - \xi) - h_1^{(1)}(x + \bar{x}_2 - \bar{x}_1)] \varphi_2(\xi) d\xi \\
+ \alpha p_{22} \int_0^x [h_2^{(1)}(x - \xi) + c] \varphi_2(\xi) d\xi \quad x > 0\]

Observe that from (14)

\[(35) \quad h_1^{(1)}(x + \bar{x}_2 - \bar{x}_1) = L_1^{(1)}(x + \bar{x}_2) - \alpha c \int_0^{x + \bar{x}_2} \varphi_1(\xi) d\xi + \alpha c p_{11} \int_0^{x + \bar{x}_2 - \bar{x}_1} \varphi_1(\xi) d\xi \\
+ \alpha p_{11} \int_0^{x + \bar{x}_2 - \bar{x}_1} h_1^{(1)}(x + \bar{x}_2 - \bar{x}_1 - \xi) \varphi_1(\xi) d\xi \]

and from (6) that

\[(36) \quad H_2^{(2)}(x + \bar{x}_2) = c + L_2^{(1)}(x + \bar{x}_2) - \alpha c \int_0^{x + \bar{x}_2} \varphi_2(\xi) d\xi + \alpha p_{21} \int_0^{x + \bar{x}_2 - \bar{x}_1} [g_1^{(1)}(x + \bar{x}_2 - \xi) + c] \varphi_2(\xi) d\xi \]

But we have already shown that the unique root of $H_2^{(2)}(x) = 0$ is $\bar{x}_2$

and that $H_2^{(2)}(x) > 0$ for $x > \bar{x}_2$. Applying Theorem B to (33) and (34)

we deduce that
\[ h_1^{(2)}(x) - h_1^{(1)}(x+\bar{x}_2-\bar{x}_1) = g_1^{(2)}(x+\bar{x}_2) - g_1^{(1)}(x+\bar{x}_2) \geq 0 \text{ for } x > 0 \]

and

\[ h_2^{(1)}(x) + c = g_2^{(1)}(x+\bar{x}_2) + c \geq 0 \text{ for } x > 0. \]

With these facts we conclude in the standard way that \( f_1(x) \) has at most two sign changes of the character \(+ - +\). The variation diminishing properties of the \( \varphi_1 \) guarantees that each \( H_1^{(3)}(x) \) changes sign at most twice. By the same argument used before we infer that each \( H_1^{(3)}(x) \) possesses a unique positive zero. Call them \( y_1^{(3)}, y_2^{(3)}, \) and \( y_3^{(3)} \). Clearly \( G_3^{(1)}(x) \equiv H_3^{(3)}(x) \) for \( x < \bar{y}_3 \).

Since \( G_3^{(1)}(\bar{y}_3) = 0, \bar{y}_3 = y_3^{(3)} \) must be the unique root of \( H_3^{(3)}(x) = 0. \)

To finish the proof we must prove that \( G_3^{(1)}(x) \geq H_3^{(3)}(x) \) for all \( x \). By examining the formulas for \( G_3^{(1)}(x) \) and \( H_3^{(3)}(x) \) it is readily seen that it suffices to show that \( f_1^{(1)}(x) \geq t_1^{(2)}(x) \) and \( f_2^{(1)}(x) \geq t_2^{(2)}(x) \) for \( x > \bar{x}_2 \). We form the simultaneous renewal equations

\[
f_1'(x) - t_1^{(2)}(x) = \alpha p_{11} \int_0^x \left[ f_1'(x-\xi) - t_1^{(2)}(x-\xi) \right] \varphi_1(\xi) d\xi
\]

\[+ \alpha p_{12} \int_0^x \left[ f_2'(x-\xi) - t_2^{(2)}(x-\xi) \right] \varphi_1(\xi) d\xi
\]

\[+ \alpha p_{13} \int_0^x \left[ f_3'(x-\xi) + c \right] \varphi_1(\xi) d\xi \text{ for } x > \bar{x}_2.
\]
\[ f_2'(x) - T_2^{(2)}(x) = \alpha_{21} \int_0^x [f_1'(x - \xi) - T_1^{(2)}(x - \xi)] \varphi_2(\xi) d\xi \]
\[ + \alpha_{22} \int_0^x [f_2'(x - \xi) - T_2^{(2)}(x - \xi)] \varphi_2(\xi) d\xi \]
\[ + \alpha_{23} \int_0^x [f_3'(x - \xi) + c] \varphi_2(\xi) d\xi \quad x > \bar{x}_2 \]

Since \( f_1'(x) \geq -c \) for all \( x \), by translating the functions in the usual manner and appealing to Theorem B we deduce that \( f_1'(x) - T_1^{(2)}(x) \geq 0 \) and \( f_2'(x) - T_2^{(2)}(x) \geq 0 \) for all \( x \). Hence \( G_3'(x) \geq H_3^{(3)}(x) > 0 \) for \( x > \bar{y}_3 \), and \( \bar{x}_3 = \bar{y}_3 \) is the third critical number. The fact that \( \bar{x}_3 \) can be calculated by Step III of Section 2 follows by a routine repetition of the same arguments. The proof of our main assertion is now complete.
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