QUALITY CONTROL FOR EXPENSIVE ITEMS

By

G. ELFVING

TECHNICAL REPORT NO. 57
FEBRUARY 28, 1962

PREPARED FOR ARMY, NAVY, AND AIR FORCE UNDER
CONTRACT Nonr-225(53) (NR-042-002)
WITH THE OFFICE OF NAVAL RESEARCH
QUALITY CONTROL FOR EXPENSIVE ITEMS

by

G. Elfving

TECHNICAL REPORT NO. 57
February 28, 1962

Prepared for Army, Navy and Air Force under
Contract Nonr-225(53) (NR 042-002)
with the Office of Naval Research

Gerald J. Lieberman, Project Director

Reproduction in Whole or in Part is Permitted for
any Purpose of the United States Government

APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
QUALITY CONTROL FOR EXPENSIVE ITEMS

by

G. Elfving

Introduction

This paper will be concerned with the continuous control of a manufacturing process of the following type: Expensive items (say, airplane motors) are produced one by one and, in general, turned out without inspection. ("Inspection" may imply, e.g., a thorough running test which is expensive, perhaps even destructive.) Their quality varies randomly, with a strong correlation, however, between adjacent items. The turning out of an inferior item is assumed to inflict a certain loss, depending on the degree of inferiority. In order to keep the production process from deteriorating too much, now and then an item is sampled and inspected. The purpose of sampling is not to replace poor items by good ones, but to check the manufacturing process. After each inspection, on the basis of its outcome (and perhaps also of previous information) one of the following actions is taken: (i) continue production, and take another sample after a certain time, depending on the quality observed; (ii) stop and revise the manufacturing process. Inspection as well as stopping production brings about costs, the latter also a delay in time until the process can start afresh. The objective of the theory is to find a rule for the decision to be made after each inspection. This rule has to balance the advantage of revising the process against the costs of inspecting, and perhaps stopping it.

Our first aim must be to establish a model for the production process. Since estimating parameters is likely to be difficult, due to scarce material, the scheme must be a relatively simple one. We shall adopt a Markov model with states $i=1,2,\ldots,k$; 0 describing the true quality level of the
production process. In practice, this level will often be characterized by one or more continuous variables; the states then must be defined by a convenient partition of the space of these variables. State 0 is introduced in Sections 1 – 5 as one that calls for immediate revision of the process; a more elaborate theory should perhaps admit several, qualitatively different such states. In Sections 6 – 9, state 0 is omitted.

We shall use the following notions and notations:

\[ p_{i,j} \text{ and } p_{i,j}^{(t)} = \text{probability of transition from state } i \text{ to state } j, \]

\[ (i, j = 1, \ldots, k; 0), \text{ in } l \text{ or } t \text{ steps, respectively}; \]

\[ h_i = \text{loss incurred by turning out an item of quality } i (i = 1, \ldots, k; 0); \]

\[ c^* = \text{inspection cost}; \]

\[ c_o = \text{costs of revising manufacturing process}; \]

\[ d_o = \text{time delay caused by such revision}. \]

Here, \( c_o \) is thought of as covering also the loss incurred by not producing any acceptable items during the revision of the process; and similarly for \( c^* \) if the inspection is destructive. If one prefers to think in terms of gain instead of loss, one has to subtract the above losses and costs from a constant \( g \), representing the value of one good item.

The time between two subsequently completed revisions will be called a cycle. After revision, the process is assumed to be in state 1.

The theory depends essentially on whether the true state of the process is observable on inspection, or not.
I. EXHAUSTIVE SAMPLING INFORMATION

1. The Inspection Procedure and its Efficiency

In this and the four following sections we shall assume that the true state is observable, i.e., that the quality of an inspected item not only is indicative of, but actually indicates the state of the process. As a consequence, the inspection rule will be of the following form:

When in state 0, stop and revise the process. When in state \( i = 1, \ldots, k \), take next sample after \( d_i \) time units (= produced items).

The inspection rule is quantitatively described by the vector \( d = (d_1, \ldots, d_k) \).

The efficiency of an arbitrary inspection procedure may be defined in different ways. One reasonable criterion is presented below; another one will be indicated in Section 4.

Denote

\[ L_i = \text{expected loss and costs, up to end of cycle, when the process has been found to be in state } i. \]

\[ N_i = \text{expected time, up to end of cycle, when the process has been found to be in state } i. \]

It can easily be shown that, for any ergodic, non-cyclic chain \( (p_{ij}) \), and for any finite procedure vector \( (d_1, \ldots, d_k) \), the length of a cycle, and also the corresponding loss, is almost surely finite, with finite moments of all orders.

Let \( \ell_\nu \) be the total loss incurred during the \( \nu^{th} \) cycle of a process, and \( n_\nu \) the length of that cycle. Then \( E(\ell_\nu) = L_1, \ E(n_\nu) = N_1 \). The average loss per time unit during \( m \) cycles,

\[
\frac{\ell_1 + \cdots + \ell_m}{n_1 + \cdots + n_m} = \frac{1}{m} \sum_{\nu=1}^{m} \frac{\ell_\nu}{n_\nu}
\]
will, by the law of large numbers, almost surely converge to

\[ Q(\delta) = \frac{E(\ell)}{E(n)} = \frac{L_1}{N_1} \]

where the argument \( \delta \) indicates the dependence on the procedure vector.

We shall take \( Q(\delta) \) for optimality criterion, trying to minimize this function by a proper choice of \( \delta \).

2. The Two States Case: \( k = 1 \)

Since this model — involving only two transition probabilities to be estimated — is perhaps most likely to be used in practice, and since finding the optimal sampling rule requires only quite elementary computations, we shall here treat it separately. We may assume \( h_\perp = 0 \) and denote \( d_\perp = \delta \).

We recall that \( p_{i,j}^{(t)} \) denotes the probability that the process, starting from state \( i \), is in state \( j \) after \( t \) steps. Dividing the expected loss \( L_1 \) into the expected loss incurred before the next inspection, the inspection costs, and the expected losses connected with the two possible decisions following the inspection, and noting that \( p_{10}^{(0)} = 0 \), we find

\[ L_1 = \sum_{t=0}^{d-1} p_{10}^{(t)} h_0 + c^* + p_{11}^{(d)} L_1 + p_{10}^{(d)} c_0 \]

and similarly, for the expected length of a cycle,

\[ N_1 = \delta + p_{11}^{(d)} N_1 + p_{10}^{(d)} d_0 \]
The iterated transition probabilities are elementarily obtained from the obvious recurrence relation

\[(2.3) \quad p_{10}^{(t)} = p_{10}^{(t-1)}(1-p_{01}) + (1-p_{10}^{(t-1)}) p_{10} = p_{10} + \lambda p_{10}^{(t-1)} \cdot\]

where

\[(2.4) \quad \lambda = 1 - p_{10} - p_{01}\]

is the non-one eigenvalue of the transition matrix \( \{ p_{ij} \} \). For the conservative type of process that we are considering, \( p_{10} \) and \( p_{01} \) will be small, and it is, from the point of view of the applications, no restriction to assume \( \lambda > 0 \). The equation (2.3) is fulfilled by

\[ p_{10}^{(t)} = p_{10}^{(t-1)} = \pi_0, \quad \text{where} \]

\[(2.5) \quad \pi_0 = \frac{p_{10}}{p_{10} + p_{01}}\]

is the stationary probability of state 0. Inserting (2.5) in (2.3), and subtracting the equation thus obtained from the original one, we find

\[ p_{10}^{(t)} - \pi_0 = \lambda(p_{10}^{(t-1)} - \pi_0) = \cdots = \lambda^t(-\pi_0), \]

\[(2.6) \quad p_{10}^{(t)} = \pi_0(1 - \lambda^t), \]

\[(2.7) \quad \sum_{t=0}^{d-1} p_{10}^{(t)} = \pi_0 \left( d - \frac{1 - \lambda^d}{1 - \lambda} \right). \]
Substituting from (2.6-7) in (2.1-2), solving for \( L_1, N_1 \), and dividing, we finally obtain

\[
Q(d) = \frac{L_1}{N_1} = \frac{c^* + \pi_0 h_0 d + \pi_0 [c_o - h_o/(1-\lambda)](1-\lambda^d)}{d + \pi_0 d_0 (1-\lambda^d)}
\]

Our objective is to find that \( d \) for which \( Q(d) \) is a minimum. For this purpose we note, to begin with, that \( Q(d) \to \pi_0 h_0 \) as \( d \to \infty \); obviously, \( \pi_0 h_0 \) is the average loss incurred when no inspection is performed. To simplify the minimization, consider the transform of (2.8),

\[
\frac{Q(d) - \pi_0 h_0}{c^*} = \frac{1-\alpha (1-\lambda^d)}{d + \beta (1-\lambda^d)}
\]

where we have denoted

\[
\alpha = \left( \frac{\pi_0 h_0}{c^*} \right) \left[ h_o/(1-\lambda) + \pi_0 h_o d_0 - c_o \right],
\]

\[
\beta = \pi_0 d_o .
\]

From (2.9) is seen that \( Q(d) \) can become less than \( Q(\infty) = \pi_0 h_0 \) for positive \( d \) if, and only if, \( \alpha > 1 \), i.e., if

\[
h_o > \left( \frac{c^*}{\pi_0} + c_o \right) / \left( \frac{1}{1-\lambda} + \pi_0 d_o \right)
\]

The intuitive interpretation of (2.12) is evident: The costs \( c^* \) and \( c_o \) are the arguments speaking against inspection; their effect is
mitigated by the time delay $d_0$. All these data, together with the probability structure of the process, determine a certain level which the per item loss $h_0$ must exceed in order for inspection to be at all profitable.

If (2.12) is fulfilled, a relation for the minimizing $d$ is derived from (2.9) by equating to 0 the logarithmic derivative of the right-hand side. Denoting

$$
(2.13) \quad \theta = \log \left( \frac{1}{\lambda} \right),
$$

$$
(2.14) \quad \gamma = \frac{\alpha}{(\alpha - 1)},
$$

$$
(2.15) \quad \delta = \frac{(\alpha + \beta \theta)}{(\alpha - 1)}
$$

$$
(2.16) \quad x = \theta d,
$$

one obtains after some computation the equation

$$
(2.17) \quad e^x = \gamma x + \delta.
$$

If (2.12) is fulfilled, and hence $\alpha > 1$, it is seen that $\gamma > 1$, $\delta > 1$. Equation (2.17) then obviously has a single positive root which can be obtained graphically. The corresponding $d = x/\theta$ has, in practice, to be replaced by the nearest integer.
3. The k + 1 States Case

In this section, we shall find the average loss $Q(d)$ for an arbitrary number $k + 1$ of states, $d$ denoting the procedure vector $(d_1, \ldots, d_k)$.

Using the notations introduced in Section 1, we may derive recurrence relations for the expected losses $L_i$ and durations $N_i$, similar to equations (2.1-2). Dividing these expectations into components corresponding to their different sources, we find

\[ L_i = c^* + \sum_{t=1}^{d_i-1} \sum_{j=0}^{k} p_{ij}^t h_j + p_{io}^t c_o + \sum_{j=1}^{k} p_{ij} d_j, \]

\[ N_i = d_i + p_{io} d_o + \sum_{j=1}^{k} p_{ij} N_j, \]

for $i = 1, \ldots, k$. Let

\[ P_d = \begin{bmatrix}
(d_1) & (d_1) \\
\vdots & \vdots \\
(d_k) & (d_k)
\end{bmatrix}, \]

\[ p_{il} \ldots p_{ik} \]

(note that this is not a stochastic matrix, since the state 0 is missing), and

\[ L_i = c^* + \sum_{t=1}^{d_i-1} \sum_{j=0}^{k} p_{ij}^t h_j + p_{io}^t c_o, \]

\[ N_i = d_i + p_{io} d_o. \]
Let \( L, N, \lambda, \nu \) be the column vectors having for components \( L_1, N_1, \lambda_1, \nu_1 \). The equations (3.1-2) may then be written

\[
(3.6) \quad (I - P_d) L = \lambda_d ,
\]

\[
(3.7) \quad (I - P_d) N = \nu_d .
\]

From these systems, \( L_1 \) and \( N_1 \), and hence \( Q(d) = \frac{L_1}{N_1} \), may in principle be found for any inspection plan \( d \).

The function \( Q(d) \) is rather complex, and its minimization by analytic methods seems to offer considerable difficulties. We may, however, gain some insight into the problem by studying the behavior of \( Q(d) \) for large \( d \).

In most applications, we may without essential restriction assume the Markov chain \( \{p_{ij}\} \) to be ergodic and non-cyclic. Then there is a unique set of stationary probabilities \( \pi_j \ (j=0,1, \ldots, k) \); moreover, it is well known that for \( t \to \infty \)

\[
(3.8) \quad P_{ij}^{(t)} = \pi_j + O(\theta^t)
\]

where \( \theta \) is the largest of the non-one eigenvalues of \( \{p_{ij}\} \). Consequently, as \( d_1, \ldots, d_k \to \infty \), the matrix on the left hand side of (3.6-7) approaches \( I - P_\infty \), where \( P_\infty \) has all rows equal to \( (\pi_1, \ldots, \pi_k) \).

Noting that \( \pi_0 + \pi_1 + \cdots + \pi_k = 1 \), it is easily verified that \( I - P_\infty \) has for inverse the matrix \( I + \pi_0^{-1} P_\infty \). For the calculation of \( L_1 \) and \( N_1 \), we need only the first row of the inverse of \( I - P_d \), and this row
obviously is of form

\[(3.9) \quad (I - P_d)^{l*} = \pi_o^{-1}(\pi_o + \pi_1, \pi_2, \ldots, \pi_k) + \varepsilon_d \]

where \( \varepsilon_d \) will be used as generic notation for a quantity that vanishes as \( d \to \infty \).

Turning, then, to the right-hand sides of (3.4-5) and (3.6-7) we first note that \( p_{i0}^{(d_i)} \to \pi_0 \) as \( d_i \to \infty \). Furthermore, writing \( p_{ij}^{(t)} = \pi_j + (p_{ij}^{(t)} - \pi_j) \), and noting that \( \Sigma_t (p_{ij}^{(t)} - \pi_j) \) converges like a geometric series, we find that, for \( i = 1, \ldots, k \),

\[(3.4)' \quad \lambda_{di} = c^* + \pi_o c_o + \kappa(d_i - l) + \sum_{j=0}^{\infty} \sum_{t=1}^{\infty} (p_{ij}^{(t)} - \pi_j) h_j + \varepsilon_d \]

\[(3.5)' \quad \nu_{di} = \pi_o d_o + d_i + \varepsilon_d \]

where

\[(3.10) \quad \kappa = \sum_{j=0}^{k} \pi_j h_j \cdot \]

We now perform the vector multiplication of (3.9) with the column vectors given by (3.4-5)'. To simplify the result, denote

\[(3.11) \quad \tilde{d} = (\pi_o + \pi_1)d_1 + \pi_2 d_2 + \ldots + \pi_k d_k \]

and write the product of the term \( \pi_o \) in the first component of (2.9), and the inner sum in (3.4)' that corresponds to \( i = l \), as
\[(3.12) \quad \pi_o \sum_{t=1}^{\infty} (p_{1j}^{(t)} - \pi_j) h_j = \pi_o \sum_{t=1}^{\infty} (p_{oj}^{(t)} - \pi_j) h_j - \pi_o \mu_j h_j \]

where

\[(3.13) \quad \mu_j = \sum_{t=1}^{\infty} (p_{oj}^{(t)} - p_{1j}^{(t)}) . \]

Noting that \( \sum_{i=0}^{k} \pi_i (p_{ij}^{(t)} - \pi_j) = \pi_j - \pi_j = 0 \), and denoting, finally,

\[(3.14) \quad \mu = \sum_{j=0}^{k} \mu_j h_j , \]

we find

\[\pi_o L_{\perp} = \kappa d + c^* - \kappa + \pi_o c_o - \pi_o \mu + \epsilon_d , \]

\[(3.15) \quad \pi_o N_{\perp} = d_o + \pi_o d_o + \epsilon_d . \]

From (3.15) it is obvious that \( Q(d) = L_{\perp} / N_{\perp} \to \kappa \) as \( d \to \infty \).

Moreover,

\[(3.16) \quad Q(d) - \kappa = \frac{1}{\pi_o N_{\perp}} [c^* - \kappa + \pi_o (c_o - \mu - \kappa d_o)] + \epsilon_d . \]

A sufficient condition for the existence of finite sampling plans with \( Q(d) < \kappa \) is, then, that
(3.17) \[ \kappa > \left( \frac{c^*}{\pi_o} + c_o - \mu \right) \left( d_o + \frac{1}{\pi_o} \right). \]

The interpretation of (3.17) is analogous to that of the corresponding condition (2.12) in the case \( k=1 \) (note that \( \kappa = \pi_o h_o \) when \( h_1 = 0 \)).

The left hand side is the average loss per unit when the process is left alone. In order for inspection and revision to be profitable, this loss must exceed a certain level that increases with increasing inspection and revision costs, and with decreasing revision time. The quantity \( \mu \), defined by (3.13-14), measures, roughly speaking, the gain due to the fact that the process is more likely to get into the "good" states 1, 2, ... from state 1 than from state 0; thus, it measures the positive effect of interfering with the process.

4. **Discounted Loss Criterion.**

The developments of the last three sections have been based on the criterion \( Q(d) = L_1/N_1 \), this being the limit of the average loss (1.1). Maintaining all assumptions concerning the probabilistic structure of the process, and the form of the inspection and revision procedure, we shall now briefly present the consequences of another goodness criterion, viz., the following. Every loss shall be discounted to a fixed time-point — say, the start of the manufacturing process — at a given rate \( p < 1 \) per time unit. The process will be considered as infinite. The expectation \( Q^*(d) \) of the total discounted loss will be used as criterion for the efficiency of \( d \).
In order to compute $Q^*(d)$ in terms of the process data, let (as in Section 1) $n_1, n_2, \ldots$ be the lengths of the subsequent cycles, and let $l_1^*, l_2^*, \ldots$ be the corresponding losses, duly discounted to the beginning of each cycle.

The total discounted loss at the beginning of the whole process is then seen to be

\begin{equation}
(4.1) \quad l_1^* + \rho l_2^* + \rho^2 l_3^* + \ldots .
\end{equation}

The pairs $(l_i^*, n_i)$ being independent and equally distributed random vectors, the expectation of $(4.1)$ is

\begin{equation}
(4.2) \quad Q^*(d) = E(l^*) \left[ l + E(\rho^n) + E(\rho^n)^2 + \ldots \right] = \frac{E(l^*)}{1 - E(\rho^n)} .
\end{equation}

**Calculation of $E(\rho^n)$.** Generically, let $t_i (i=1, \ldots, k)$ denote the remaining time, up to the end of the cycle, when the process has been inspected and found to be in state $i$. Let $t'_i = t_i - d_i$ be the length of that part of the cycle that is still ahead at the following inspection. Here, $t_i$ is a random variable with distribution depending only on $i$; $t_1$ has the same distribution as the variable $n$ in the previous paragraph. The variable $t_i'$ has for distribution a weighted average of the distributions of $t_1, \ldots, t_k$, with weights $\left(\frac{\rho}{d_i}\right)$, $\left(\frac{d_i}{\rho}\right)$, $p_{i1}, \ldots, p_{ik}$, and the non-random number $d_0$ (the revision delay) with weight $p_{i0}$. Since $t_i = d_i + t_i'$, hence $\rho^{t_i} = \rho^{d_i} \rho^{t_i'}$, we find

\begin{equation}
(4.3) \quad E(\rho^{t_i}) = \rho^{d_i} \sum_{j=1}^{k} p_{ij} \left(\frac{d_i}{\rho}\right) E(\rho^{t_j'}) + \rho^{d_i} \frac{d_i}{p_{i0}} \rho^n .
\end{equation}
Denoting

\[ E(\rho^t) = T_t, \]  
\[ (4.4) \]

\[ \rho \cdot \frac{d_i}{p_{ij}} = \frac{-d_i}{p_{ij}}, \]  
\[ (4.5) \]

\[ \tau_{di} = \frac{-d_i \cdot d_o}{p_{ii}} = \rho^{d_i+d_o} \cdot p_{ii}, \]  
\[ (4.6) \]

we may write (4.3), in obvious matrix notation, as

\[ (4.7) \]

\[ (I - \bar{P}_d) T = \tau_d. \]

This system determines the denominator \( 1 - T_1 \) on the right hand side of (4.2).

**Calculation of \( E(\ell^*) \).** This is entirely analogous to the calculation of \( L_1 \) in Section 3. Let \( \bar{L}_1 \) denote the expected discounted loss, up to the end of the cycle, at a moment when the process has been inspected and found to be in state \( i \). We then have the recursive relations

\[ (4.8) \]

\[ \bar{L}_1 = \sum_{j=1}^{k} \frac{-d_i}{\bar{p}_{ij}} \bar{L}_j + c^* \rho^{d_i} + \frac{-d_i}{p_{ii}} c_o \]

\[ + \sum_{t=1}^{d_i-1} \sum_{j=0}^{k} \frac{(t)}{\bar{p}_{ij}} h_j, \quad (i=1, \ldots, k), \]

i.e., the same relations as in (3.1), with \( c^* \) replaced by \( c^* \bar{d}_i \), and with \( \bar{p} \) instead of \( p_i \), indicating the presence of a discount factor.
Denoting

\[(4.9) \quad \bar{\lambda}_{d_i} = c^* \rho_i^{d_i} + \sum_{t=0}^{d_i-1} c_0 \sum_{j=0}^{k} p_{ij}^t h_j ,\]

we get the system

\[(4.10) \quad (I - \overline{P}_d) \overline{L} = \bar{\lambda}_d\]

for the calculation of \( \overline{L}_1 = E(\delta^*) \).

Asymptotic behavior of \( Q^*(d) \). The limiting value of \( Q^*(d) \) when all \( d_i \to \infty \) is easily obtained. In fact, \( p_{ij}^t = 0(\rho_j^t) \to 0 \) as \( t \to \infty \). Hence \( I - \overline{P}_d \to I \) and \( (I - \overline{P}_d)^{-1} \to I \). As a consequence, the limiting value of \( T_1 \) equals \( \lim \tau_{d_1} = 0 \), the limiting value of \( \overline{L}_1 \) equals \( \lim \bar{\lambda}_{d_1} \) as obtained from (4.9), and hence

\[(4.11) \quad Q^*(\infty) = \sum_{j=0}^{k} h_j \sum_{t=1}^{\infty} \rho_j^t p_{ij}^t .\]

This is simply the expectation of the total discounted loss when the process is not inspected at all.

5. The Estimation Problem

In the applications, the data about loss, costs, and time delay, will presumably be readily available. The transition probabilities \( p_{ij} \), however, must be estimated from past records of the manufacturing process.
We shall only consider the case that the process has been sampled at regular time intervals of constant length $n$, say, without any interruption or revision. We then are confronted with a sample of the chain $P^n$. It is well known that the maximum likelihood estimates of the transition probabilities of a Markov chain are simply the relative frequencies $\hat{p}_{ij} = \frac{n_{ij}}{n_1}$ of the different transitions. In the present case, we are thus in possession of a maximum likelihood estimate $\hat{P}^n$ from which we have to derive the corresponding estimate $\hat{P}$. From our practical point of view, we may without any essential restriction assume the Markov chain to be ergodic and non-cyclic. The matrix $\{p_{ij}\}$ then has exactly one eigenvalue $\lambda_0 = 1$, the remaining eigenvalues $\lambda_1, \ldots, \lambda_k$ being all absolutely less than one. We shall assume that these eigenvalues are all simple (this is no serious restriction either, since the $p_{ij}$ have to be empirically determined, and thus more or less inaccurate anyway). Under these assumptions, it is well known that the iterated transition matrix $P^n$ may be written

$$(5.1) \quad P^n = V \Lambda^n V^{-1}$$

where $\Lambda$ is the diagonal eigenvalue matrix, and $V$ has for columns the corresponding eigenvectors. The first column, corresponding to $\lambda_0 = 1$, has all elements equal, and may be taken to be $(1, 1, \ldots, 1)'$; the other columns can be arbitrarily normalized. Both $\Lambda$ and $V$ may contain complex elements.

Using the representation (5.1), $\hat{P}$ can now be found as follows:

(i) Find the eigenmatrix $\hat{V}$ of $\hat{P}^n$; (ii) find the eigenvalues (say)
\( \hat{\theta}_1 \) of \( \hat{P}^n \), take \( \hat{\lambda}_1 = (\hat{\theta}_1)^{1/n} \), and form the diagonal matrix \( \hat{\Lambda} \) with elements \( \hat{\lambda}_1 \); (iii) form \( \hat{P} = \hat{\Lambda}^{-1} \). Theoretically, each root \( \hat{\lambda}_1 \) has \( n \) values, and hence there would be \( n^k \) determinations of \( \hat{P} \), partly with complex elements. In view of the supposedly conservative character of our process, it seems reasonable always to choose the value of \( (\hat{\theta}_1)^{1/n} \) that is closest to 1; the roots of two conjugate \( \hat{\theta}_1 \)'s of course have to be conjugate.

In the case \( k = 1 \), the taking of the \( n^{th} \) root can be performed quite elementarily. The equations (2.4-6), and the corresponding equations with 0 and 1 interchanged, give

\[
\lambda = 1 - p_{10} - p_{01},
\]

\[
\pi_0 = \frac{p_{10}}{p_{10} + p_{01}}, \quad \pi_1 = \frac{p_{01}}{p_{10} + p_{01}}.
\]

\[
p_{10}^{(n)} = \pi_0 (1 - \lambda^n), \quad p_{01}^{(n)} = \pi_1 (1 - \lambda^n).
\]

Given \( p_{10}^{(n)}, p_{01}^{(n)} \) (or rather their estimates) we obtain, conversely,

\[
\pi_0 = \frac{p_{10}^{(n)}}{p_{10}^{(n)} + p_{01}^{(n)}}, \quad \pi_1 = \frac{p_{01}^{(n)}}{p_{10}^{(n)} + p_{01}^{(n)}},
\]

\[
\lambda = \sqrt[n]{1 - p_{01}^{(n)} - p_{10}^{(n)}},
\]

and finally, applying (5.1) with \( n = 1 \),

\[
p_{10} = \pi_0 (1 - \lambda), \quad p_{01} = \pi_1 (1 - \lambda).
\]
In (5.6) the positive $n^{th}$ root has to be taken.

In practice, it will probably often be difficult to get an estimate of $p_{01}^{(n)}$ since it is desirable to stop and revise the process whenever state 0 is observed. Sometimes it may be justifiable to simplify the model by assuming $p_{01} = 0$ (no return from the bad to the good state); in this case the above solution still works, with $p_{01} = p_1 = p_{01}^{(n)} = 0$. Otherwise, if $p_{01}^{(n)}$ is not 0 but simply unknown, it is still possible to find estimates for $\pi_0$ and $\lambda$ (and hence for $p_{10}$, $p_{01}$) from the first equation (5.4) if $p_{10}^{(n)}$ is known (i.e., estimated) for two different $n$'s. For instance, if $p_{10}^{(n)}$ and $p_{10}^{(2n)}$ are known, $\lambda^n$ is obtained from the equation $\lambda^n + 1 = p_{10}^{(2n)} / p_{10}^{(n)}$. 
II. NONEXHAUSTIVE SAMPLING INFORMATION


All the previous considerations are based on the assumption that the true state of the production process is observable. What is really observed is, however, usually a single produced item, the quality of which need not give exhaustive information about the true state of the process itself. We shall now attempt a treatment of this more general situation. For the fundamental ideas of the theory we are indebted to a paper by Girshick and Rubin, "A Bayes approach to a quality control model," *Ann. Math. Stat.* 23 (1952), pp. 114-125.

We shall maintain our assumption concerning the Markovian character of the fundamental process, and also our notations for the transition probabilities; the state 0, however, will be omitted. The inspection procedure will still consist of a decision rule which, after each inspection or revision, prescribes how soon the next sample shall be taken, or whether the process has to be revised. The argument of the decision function, however, will no more be the true state (which is not observable), but the accumulated information contained in the last and the previous observations made during the current cycle.

We shall denote the true state of the process by $x$, or by $x(t)$ when we wish to stress the time argument. Similarly, $y$, or $y(t)$, will denote a single observation on the process. While $x$ assumes only the values $1, \ldots, k$, $y$ may be discrete or continuous, one- or multi-dimensional, as the case may be. Under the condition $x(t) = 1$, $y(t)$ will be assumed to have a known distribution, depending only on $1$ and
independent of any x- or y-values pertaining to the past. The density of y with respect to some convenient basic measure will be denoted by $f_i(y)$.

Consider a cycle starting at $t=0$, and let $t_1 < t_2 < \ldots < t_m$ be the m first inspection time-points. We shall write $x(t_{ij}) = x_{ij}$, $y(t_{ij}) = y_{ij}$.

All the information available at time $t_m$ is contained in the observation vector $Y_m = (y_{i1}, \ldots, y_{im})$ combined with the inspection-time vector $T_m = (t_{i1}, \ldots, t_{im})$. When the inspection rule up to the mth inspection is fixed, $T_m$ is completely determined by $Y_m$. The decision of the statistician at $t_m$ must be a function $d(Y_m)$ with values 0, 1, 2, \ldots indicating after how long time the next sample has to be taken. The value $d = 0$ will be used to indicate revision.

The vector $Y_m$ (of increasing dimensionality!) is a rather awkward argument for a decision function. Fortunately, the information contained in this vector can be condensed. Let $g_m(i, Y)$ be the joint probability function of $x_m$ and $Y_m$; i.e., for any measurable set $B$ in $Y$-space, we have

$$P(x = i, Y \in B) = \int_B g_m(i, Y) \, dY$$

where $dY$ represents the basic measure in $Y$-space. The random vector $q(m)$, with components

$$(6.1) \quad q_i(m) = g_m(i, Y_m)$$
will be referred to as the information vector of the process, as evaluated immediately after the \( m \)th inspection. Note that in (6.1), \( Y_m \) is a random vector while 1 appears as component subscript.

7. Sufficiency of \( q(m) \).

Since \( q(m) \) determines the conditional probabilities, given \( Y_m' \) of the values \( x_m = 1, \ldots, k \), and since the value of \( x_m \) determines the distribution of all stochastic variables related to the process \( \{x(t), y(t); t > t_m\} \), it is intuitively likely that \( q(m) \) be a sufficient summary of all past observations. We shall now study this question a little closer.

For each cycle, it is mathematically convenient to imagine the Markov process \( \{x(t)\} \) as going on ad infinitum, even though the actual production process is stopped when a revision is undertaken, and then started afresh with a new cycle. Similarly, we may imagine \( y(t) \) as defined for every \( t = 1, 2, \ldots \) even though in practice the selected sequence \( \{y(t)\} \) only is observed. The process

\[
(7.1) \quad \{x(t), y(t); t = 1, 2, \ldots\}
\]

is defined independently of any inspection scheme. We shall for brevity denote a segment such as \( \{x(s), \ldots, x(t)\} \) by \( X_s^t \), and similarly for \( y(t) \).

Any inspection procedure (decision rule) \( D \) of the type envisaged in the present section has the following form:
Take \( t_1 = d_1 \), a constant given by \( D \), and observe \( y_1 = y(t_1) \). Form \( Y_1 = (y_1) \).

Find \( d_2 = d_2(Y_1) \), the function \( d_2(\cdot) \) being given by \( D \).

Take \( t_2 = t_1 + d_2 \) and observe \( y_2 = y(t_2) \). Form \( Y_2 = (y_1, y_2) \).

Find \( d_3 = d_3(Y_2) \), the function \( d_3(\cdot) \) being given by \( D \). Etc.

When the procedure yields a value \( d_v = 0 \), revision has to be undertaken, whereupon the cycle is finished.

The segment of \( D \) that comprises the \( m \) first inspections will be denoted by \( D_m \).

After a completed inspection (the \( m \text{th} \), say), the remaining part \( d^* \) of the sampling procedure is of the same general form as \( D \), with \( d_1, d_2, \ldots \) replaced by \( d^*_1 = d_{m+1} \), etc. The rule \( D^* \), however, will in general depend on the information gained from the past \( m \) inspections, as given by the vector \( Y_m \). We may indicate this by writing \( D^* = D^*(Y_m) \); \( D^*(\cdot) \) will be called a conditional inspection rule. A complete inspection rule \( D \) may, for any \( m \), be decomposed into two parts \( D_m \) and \( D^*(\cdot) \), prescribing the first \( m \), and all the subsequent inspections, respectively. We wish to show that \( D^*(\cdot) \) need depend only on the information vector \( g(m) \) given by (7.1). The argument will be somewhat heuristic.

Consider, for a generic cycle, any "cumulative" random variable \( Z \) such as the duration time \( N \), the total loss \( L \), or the like. For any given \( m \), and for any inspection scheme \( D = (D_m, D^*(\cdot)) \), the variable \( Z \) may be written as \( Z = Z + Z \) where
\[ \zeta = \zeta(X_{\bar{m}}^t, Y_{\bar{m}}^t, D_m) , \]

\[ z = z(X_{\bar{m}+1}^\infty, Y_{\bar{m}+1}^\infty, D^*(Y_m)) . \]

Consider the conditional distribution of \( z \) for a given \( x_m = i \) and \( Y_m = Y \). Due to the Markovian character of the process \( \{x(t)\} \), the distribution of the sequences \( X_{\bar{m}+1}^\infty, Y_{\bar{m}+1}^\infty \) is completely determined by the value \( x(t_m) = i \), and hence the distribution of \( z \) (under the said conditions and for fixed \( D_m \)) depends on \( Y \) through \( D^*(Y) \) only.

Denoting

\[ (7.4) \quad E(z|x_m = i, Y_m = Y) = \frac{z_i}{z} [D^*(Y)] , \]

we may write

\[ (7.5) \quad E(Z) = E(\zeta) + \int \sum_i g_m(i,Y) \frac{z_i}{z} [D^*(Y)] \, dY . \]

Assume, first, that the purpose of the inspection procedure is to minimize \( E(Z) \), and suppose that \( D_m \) is already chosen whereas \( D^*(\cdot) \) has to be selected. The minimization of \( E(Z) \) can then be achieved by choosing, for each \( Y \) separately, \( D^* \) so as to make

\[ (7.6) \quad \sum_i g_m(i,Y) \frac{z_i}{z} (D^*) = \min. \]
The solution to this problem need not depend on $Y$ but through the coefficients $g_m(i,Y)$, i.e., through the information vector $q(m)$. In this case, then, $q(m)$ provides sufficient information for the decision to be made after the $m^{th}$ inspection. Moreover, once $q(m)$ is given, this decision does not depend on $m$ anymore.

If our criterion for the goodness of our inspection procedure is a function of several expected values of type $E(Z)$, the previous arguments require some modification. Consider the average loss criterion (1.2). Let $\lambda$ denote the loss up to and including the $m^{th}$ inspection, and $\ell$ the loss after this inspection; and similarly for the durations $\nu$ and $n$. We then may write

$$Q = \frac{\frac{L}{N}}{\frac{E(\nu)}{E(\nu)} + \frac{\sum_i g_m(i,Y) \bar{L}_i(D^*(Y))}{\sum_i g_m(i,Y) \bar{n}_i(D^*(Y))}} dY.$$  

(7.7)

Consider an infinitesimal element $dY$ and the corresponding elements $dL$ and $dN$ of the integrals, in (7.7). Their contribution to $Q$ is given by

$$dQ = \frac{NaL - LdN}{N^2} = \frac{\sum_i g_m(i,Y) [N \bar{L}_i(D^*) - L \bar{n}_i(D^*)]}{N^2} dY.$$ 

(7.8)

If $D^*$ minimizes $Q$, then $D^*(Y)$ has to minimize the numerator on the right hand side; and since $Y$ enters into this problem only through the information vector $q(m) = (g_m(i,Y))$, it is seen that $D^*$ may always be taken as a function of $q$. We note, in particular, that the length $d^*$ of the next inspection interval may be taken as a function $d(q)$ of the information vector.
8. Recurrence Formula for \( q(m) \).

Knowing that \( q(m) \) is (for our purposes) a sufficient summary of all inspections up to the \( m \)th one, we may expect, in particular, that \( q(m+1) \) be completely determined by \( q(m) \) and the additional observation \( y_{m+1} \). As a matter of fact, we find

\[
q_j(m+1) = g_{m+1}(j, y_{m+1})
\]

\[(8.1)\]

\[
= \sum_i g_m(i, y_m) p_{ij}^{(d)} r_j(y_{m+1})
\]

\[
= \sum_i q_i(m) p_{ij}^{(d)} r_j(y_{m+1})
\]

where the inspection interval

\[(8.2)\]

\[d = d(q(m))\]

is a function of the information vector, according to the previous section.

The third line in (8.1) constitutes a recurrence formula for \( q(m) \). On the right hand side, the past is summarized in the vector \( q(m) \), which also determines \( d \), while the additional information gained from the \((m+1)\)st observation is represented by the likelihood vector \( \{f_j(y_{m+1})\} \). In the course of an actual production process, we may calculate \( q(1), q(2), \ldots \) successively according to (8.1).

We note that (8.1), for fixed \( q(m) = q \), constitutes a mapping from the \( y \)-space to (say) \( q^* \)-space, a mapping to be denoted by
(8.3) \[ q^* = q^*(y|a) \]

Any distribution of \( y_{m+1} \) (as given by one of the conditions \( x_{m+1} = j, \)
\( j=1, \ldots, k \)) induces, by this mapping, a probability measure

(8.4) \[ \mu_j(q^*|a) \]

in \( q^*-\text{space}, \) viz., the conditional distribution of \( q^* = q(m+1) \) given
\( q(m) = q \) and \( x_{m+1} = j. \)


We are now in a position to establish relations corresponding to
equations (3.1-2) in the case of exhaustive sampling information. At
each inspection – the \( m^{th} \); say – there are now two quantities rele-
vant to the future of the process: the "latent" true state, \( x_m = i, \)
which determines the distribution of all subsequent \( x(t), y(t); \) and the
known information vector \( q(m) = q, \) on which the decision \( d(q) \) about
the next inspection interval has to be based. Accordingly, we shall
introduce the functions

\[ L_i(q) = \text{expected loss}, \] \[ N_i(q) = \text{expected time}, \]

up to the end of the cycle, when an inspection has just been performed,
when the process is in state \( i \), and when the information vector is \( q \). All these functions depend, of course, on the inspection rule \( d(\cdot) \) adopted. What is really relevant is the expected loss and duration as evaluated at the beginning of a cycle. Since, after revision, the process is known to be in state \( l \), we have \( x_0 = l \) and may take

\[
q(0) = e_1 = (1, 0, \ldots, 0)
\]

(it is easily checked that this is in accordance with the recurrence formula (8.1)). Our final aim, then, is to minimize the average loss

(9.1) \[
q(D) = \frac{L_1(e_1)}{N_1(e_1)}
\]

with respect to the decision function \( D \).

The equations for \( L_1(q) \) and \( N_1(q) \) are derived in the same way as (2.1-2). When the process is in state \((i, q)\), the expected future loss \( L_1(q) \) may be divided into (i) the expected loss during the next inspection period, (ii) the costs of the following inspection, and (iii) the expected loss after that inspection. The expectation (iii) can be expressed as a weighted mean of \( L_j(q^*) \) over \( j \) and \( q^* \); the integration over \( q^* \) may be performed either in \( q^*-\)space, using the induced measures (8.4), or in \( y\)-space, using the mapping (8.3). We thus obtain the following systems:
\[ L_1(q) = c_o \quad \text{when } d(q) = 0; \text{ otherwise} \]

\[
L_1(q) = \sum_{t=1}^{d-1} \sum_{j=1}^{k} p_{ij} h_j + c^* + \sum_{j=1}^{k} p_{ij} \int_{L_j(q^*)} \mu_j(dq^*|q) \\
= \sum_{t=1}^{d-1} \sum_{j=1}^{k} p_{ij} h_j + c^* + \sum_{j=1}^{k} p_{ij} \int_{L_j[q^*(y|q)]} f_j(y) \, dy,
\]

\[ d = d(q) . \]

Similarly, we obtain for the expected duration time

\[ N_1(q) = d_0 \quad \text{when } d(q) = 0; \text{ otherwise} \]

\[
N_1(q) = d + \sum_{j=1}^{k} p_{ij} \int_{N_j(q^*)} \mu_j(dq^*|q) \\
= d + \sum_{j=1}^{k} p_{ij} \int_{N_j[q^*(y|q)]} f_j(y) \, dy,
\]

\[ d = d(q) . \]

The actual solution of (9.2) and (9.3) — for some given decision rule \( d(\cdot) \) — can hardly be performed otherwise than by a discretizing approximation. For this purpose \( q \)-space has to be partitioned into a convenient number of cells, the integrals in (9.2-3) replaced by sums, and the integral equations solved as a linear system. It may be noted that what matters in \( q \) is the ratio of the components (i.e., the conditional distribution of \( x_m \) given \( y_m \)); hence the partition may
properly be performed in the barycentric simplex of the $q$-space.

Even if the solution of (9.2-3) can be accurately performed the minimization of $Q = L_1(e_1)/N_1(e_1)$ with respect to $d(\cdot)$ remains a forbiddingly complex problem. It seems that, in practice, hardly more than a comparison of a few tentative inspection rules can be attempted.
STANFORD UNIVERSITY
TECHNICAL REPORT DISTRIBUTION LIST
CONTRACT N00014-73-C-0401 (IN 042-002)

Chief, Bureau of Ordinance
Quality Control Divison (QCD)
Department of the Navy
Washington 25, D. C.
Att: Dr. W. R. Patek, Jr.

Chief, Navy Procurement Dist.
Navy Material Area
2865 Sky Harbor Blvd.
Phoenix, Arizona

Chief, San Diego Procurement Dist.
Navy Material Area
Att: Quality Control Divison
P.O. Box 1948
San Diego 10, California

Chief, Los Angeles Air Proc. Dist.
Navy Material Area
Att: Quality Control Divison
1206 S. Maple Street
Los Angeles, California

Chief, San Francisco Air Procurement Dist.
Sacramento Naval Material Area
Att: Quality Control Divison
1515 Clay Street
Oakland 12, California

Chief, Atlantic Air Procurement Dist.
Warner Robins Air Material Area
Att: Quality Control Divison
441 West Peachtree Street N.E.
Atlanta, Georgia

Chief, Boston Air Procurement Dist.
Middleton Air Material Area
Att: Quality Control Divison
1114 Market Street
St. Louis 1, Missouri

Chief, Indianapolis Air Procurement Dist.
Mobile Air Material Area
Att: Quality Control Divison
54 Monument Circle
Indianapolis 6, Indiana

Chief, Rochester Air Procurement Dist.
Middleton Air Material Area
Att: Quality Control Divison
20 Symington Place, P.O. Box 1669
Rochester 3, New York

Chief, Cleveland Air Procurement Dist.
Mobile Air Material Area
Att: Quality Control Divison
1279 W. Third Street
Cleveland 13, Ohio

Contract Natl 225153
May 1961
Chief
Dallas Air Procurement Dist.
San Antonio Air Materiel Area
Attn: Quality Control Division
Wilson Building - Room 338
Dallas, Texas 1

Commanding Officer
U.S. Navy Supply Agency
235 South 40th Street
Philadelphia 3, Pennsylvania
Attn: Chief, SIGSUS-43d 2

Commander
U.S. Naval Air Development Center
Johnsville, Pennsylvania 1

Commander
U.S. Naval Air Missile Test Center
Point Mugu, California
Attn: Chief, Scientific 1

Commander
Naval Construction Battalion Center
Davisville, Rhode Island 1

Commander
Naval Construction Battalion Center
Gulfport, Mississippi 1

Commander
Naval Construction Battalion Center
Port Hueneme, California 1

Commander
U.S. Naval Ammunition Depot (QEL)
Navy Yrd. 667 Fleet Post Office
San Francisco, California 1

Commander
U.S. Naval Ordnance Test Station
China Lake, California
Attn: R. J. Gardner - E. J. Cottrell 3

Commander
Naval Ordnance Test Station
China Lake, California
Attn: Dr. J. D. Villars 3

Commander
U.S. Naval Ordnance Test Station
Keppert, Washington
Attn: Technical Library

Commander
U.S. Naval Ordnance Test Station
Bremerton, Washington 1

Commander
U.S. Naval Ordnance Test Station
Philadelphia 3, Pennsylvania 1

Commander
U.S. Navy Ordnance Laboratory
San Diego 52, California 1

Commander
Material Laboratory
New York Naval Shipyard, Naval Base
Brooklyn, New York
Attn: A. W. Walzer 1

Commander
U.S. Navy Radiological Defense Lab.
San Francisco, California 1

Commander
U.S. Navy Mine Defense Lab.
Panama City, Florida 1

Commander
U.S. Naval Ordnance Laboratory
San Francisco, California 1

Commander
U.S. Naval Ordnance Laboratory
San Francisco, California
Attn: Contract No. 225(53) May 1961 1
<table>
<thead>
<tr>
<th>Commanding Officer &amp; Director</th>
<th>U. S. Navy Underwater Sound Lab. Fort Trumbull New London, Connecticut</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commander Mobile Air Materiel Area</td>
<td>Attn: Assistant for Quality Director of Maintenance Engineering Brookley Air Force Base Alabama</td>
</tr>
<tr>
<td>Commander Mobile Air Materiel Area</td>
<td>Attn: Material Quality Division Directorate of Supply &amp; Services Brookley Air Force Base Alabama</td>
</tr>
<tr>
<td>Commander Mobile Air Materiel Area</td>
<td>Attn: QC Division Directorate of Procurement &amp; Production Brookley Air Force Base Alabama</td>
</tr>
<tr>
<td>Commander Southern Air Materiel Area, Pacific</td>
<td>Attn: Quality Control Office Clark AFB, AP074 San Francisco, California</td>
</tr>
<tr>
<td>Commander Air Materiel Force, Pacific Area</td>
<td>Attn: Quality Control Office FLEAMCOM Air Base, AP0 323 San Francisco, California</td>
</tr>
<tr>
<td>Commander Northern Air Materiel Area, Pacific</td>
<td>Attn: Quality Control Office FLEAMCOM Air Base, AP0 323 San Francisco, California</td>
</tr>
<tr>
<td>Commander Sacramento Air Materiel Area</td>
<td>Attn: Assistant for Quality Director of Maintenance Engineering McClellan AFB, California</td>
</tr>
<tr>
<td>Commander Sacramento Air Materiel Area</td>
<td>Attn: Quality Control Division Directorate of Procurement and Production McClellan AFB, California</td>
</tr>
<tr>
<td>Commander Sacramento Air Materiel Area</td>
<td>Attn: Material Quality Division Directorate of Supply and Services McClellan AFB, California</td>
</tr>
<tr>
<td>Commander Sacramento Air Materiel Area</td>
<td>Attn: Quality Control Offices, SQ McClellan AFB, California</td>
</tr>
<tr>
<td>Commander San Bernardino Air Materiel Area</td>
<td>Attn: Assistant for Quality Directorate of Maintenance Engineering Norton AFB, California</td>
</tr>
<tr>
<td>Commander San Bernardino Air Materiel Area</td>
<td>Attn: Quality Control Division Directorate of Procurement and Production Norton AFB, California</td>
</tr>
<tr>
<td>Commander San Bernardino Air Materiel Area</td>
<td>Attn: Material Quality Division Directorate of Supply and Production Norton AFB, California</td>
</tr>
<tr>
<td>Commander Warner Robins Air Materiel Area</td>
<td>Attn: Assistant for Quality Director of Maintenance Engineering Robins AFB, Georgia</td>
</tr>
<tr>
<td>Commander Warner Robins Air Materiel Area</td>
<td>Attn: Quality Control Division Directorate of Procurement and Production Robins AFB, Georgia</td>
</tr>
<tr>
<td>Commander Warner Robins Air Materiel Area</td>
<td>Attn: Quality Control Offices, WRQ Robins AFB, Georgia</td>
</tr>
<tr>
<td>Commander Oklahoma City Air Materiel Area</td>
<td>Attn: Assistant for Quality Director of Maintenance Engineering Tinker Air Force Base Oklahoma City, Oklahoma</td>
</tr>
<tr>
<td>Commander Oklahoma City Air Materiel Area</td>
<td>Attn: Material Quality Division Directorate of Supply and Services Oklahoma City, Oklahoma</td>
</tr>
<tr>
<td>Commander Oklahoma City Air Materiel Area</td>
<td>Attn: Quality Control Division Directorate of Procurement and Production Tinker Air Force Base Oklahoma City, Oklahoma</td>
</tr>
<tr>
<td>Commander Oklahoma City Air Materiel Area</td>
<td>Attn: Quality Control Offices, OQQ Tinker AFB, Oklahoma</td>
</tr>
<tr>
<td>Commander Rome Air Force Depot</td>
<td>Attn: Assistant for Quality Director of Maintenance Engineering Griffiss AFB, New York</td>
</tr>
<tr>
<td>Commander Rome Air Force Depot</td>
<td>Attn: Directorate of Supply and Services Griffiss AFB, New York</td>
</tr>
<tr>
<td>Commander Air Technical Intelligence Center</td>
<td>Attn: (AFPN-4GI) Wright-Patterson AFB, Ohio</td>
</tr>
<tr>
<td>Commander Air Materiel Command</td>
<td>Attn: Quality Control Offices, MQ Wright-Patterson AFB, Ohio</td>
</tr>
<tr>
<td>Commander Middletown Air Materiel Area</td>
<td>Attn: Assistant for Quality Director of Maintenance Engineering Otis Air Force Base Middletown, Pennsylvania</td>
</tr>
<tr>
<td>Commander Middletown Air Materiel Area</td>
<td>Attn: Material Quality Division Directorate of Supply and Service Otis Air Force Base Middletown, Pennsylvania</td>
</tr>
<tr>
<td>Commander Middlesex Air Materiel Area</td>
<td>Attn: Quality Control Division Directorate of Procurement and Production Otis Air Force Base Middletown, Pennsylvania</td>
</tr>
<tr>
<td>Commander Middlesex Air Materiel Area</td>
<td>Attn: Quality Control Office, MQ Otis Air Force Base Middletown, Pennsylvania</td>
</tr>
<tr>
<td>Commander Memphis Air Force Depot</td>
<td>Attn: Directorate of Supply and Services 3300 Jackson Avenue Memphis, Tennessee</td>
</tr>
<tr>
<td>Commander Memphis Air Force Depot</td>
<td>Attn: Quality Control Division Directorate of Maintenance Engineering 3300 Jackson Avenue Memphis, Tennessee</td>
</tr>
<tr>
<td>Commander San Antonio Air Materiel Area</td>
<td>Attn: Quality Control Office, SQ Norton AFB, California</td>
</tr>
</tbody>
</table>

Contract No. 225053 May 1961
Mr. Bernard F. Goldsmith  
Associate Professor  
Northwestern University  
Huntington Avenue  
Boston 15, Massachusetts

Professor Lee A. Goodman  
Statistical Research Center  
University of Chicago  
Chicago 37, Illinois

Mr. Leon Gilford  
Operations Research Inc.  
8005 Cameron Street  
Silver Springs, Maryland

Dr. J. Greenwood  
Directorate of Intelligence  
HQ., U. S. Air Force  
Washington 25, D. C.

Professor Frank M. Grzyma, Jr.  
University College  
Rutgers University  
New Brunswick, New Jersey

Dr. Donald Guthrie  
Stanford Research Institute  
333 Ravenswood Avenue  
Menlo Park, California

Dr. Theodore E. Harris  
The RAND Corporation  
1700 Main Street  
Santa Monica, California

Dr. Leon H. Herback  
Department of Mathematics  
New York University  
New York 3, New York

Professor W. Hirsch  
Institute of Mathematical Sciences  
New York University  
New York 3, New York

Dr. Paul G. Hoe  
Department of Mathematics  
University of California  
Los Angeles 24, California

Professor Harold Hotelling  
Associate Director  
Institute of Statistics  
University of North Carolina  
Chapel Hill, North Carolina

Professor L. Humicz  
School of Business Administration  
University of Minnesota  
Minneapolis, Minnesota

Mr. Rudolph Hussar  
Visiting Research Mathematician  
Numerical Analysis Research  
University of California  
Los Angeles 24, California

Dr. James R. Jackson  
Management Sciences Research Project  
65 administrative building  
University of California  
Los Angeles 24, California

Dr. W. C. Jacob  
Astronomy Department  
University of Illinois  
Urbana, Illinois

Professor W. D. Jones  
Department of Statistics  
Michigan State University  
East Lansing, Michigan

Mr. J. P. Kearney  
Quality Control Division  
General Services Administration  
Room 6316, Region 3 Building  
Washington 25, D. C.

Professor Oscar Kempton  
Statistics Laboratory  
Iowa State College  
Ames, Iowa

Professor Solomon Klefisch  
Department of Statistics  
George Washington University  
Washington 7, D. C.

Dr. Carl F. Kossack  
Statistics and Operations Research Center - IBM  
P. O. Box 218, Lamb Estate  
Yorktown Heights, New York

Mr. Howard Latiu  
The RAND Corporation  
1700 Main Street  
Santa Monica, California

Dr. E. L. LeClerc, Chief  
Biometrical Services  
Agricultural Research Service, USDA  
Beltsville, Maryland

Professor Sebastian B. Littauer  
431 Engineering Building  
Columbia University  
New York 27, New York

Dr. Eugene Lukacs  
Department of Mathematics  
Catholic University  
Washington 17, D. C.

Dr. Robert Landegard  
Logistics and Mathematical Statistics Branch  
Office of Naval Research  
Washington 25, D. C.

Professor Frank Massey  
School of Public Health  
University of California  
Los Angeles 24, California

Professor G. W. McElreath  
Department of Industrial Engineering  
University of Minnesota  
Minneapolis 34, Minnesota

Dr. Paul Meyer  
Department of Mathematics  
Washington State University  
Pullman, Washington

Dean Paul E. Mohn  
School of Engineering  
University of Buffalo  
Buffalo, New York

Mr. R. B. Murphy  
Bell Telephone Laboratories, Inc.  
463 West Street  
New York 34, New York

D. E. Neufeld  
Chief, Industrial Engineering Division  
Comptroller  
Hilltop, San Bernardino Air Materiel Area  
Norton Air Force Base, California

Professor J. Neuman  
Department of Statistics  
University of California  
Berkeley 4, California

Mr. Monroe Norden  
Research Division  
College of Engineering  
New York University  
New York 53, New York

Mr. Fred Okano  
National Aeronautics & Space Administration  
Reliability & Systems Analysis Office  
Room A-131  
1520 H St., N. W.  
Washington 25, D. C.

Professor E. G. Olds  
Department of Mathematics  
Cornell Institute of Technology  
Pittsburgh 13, Pennsylvania

Professor E. R. Ott  
Department of Mathematics  
Rutgers University  
New Brunswick, New Jersey

Mr. Cyril Peckham  
Project Gnome  
University of Dayton  
Dayton, Ohio

Dr. Richard Post  
Department of Mathematics  
San Jose State College  
San Jose, California

Professor P. H. Randolph  
Purdue University  
Department of Industrial Engineering  
Lafayette, Indiana

Professor George J. Resnikoff  
Department of Industrial Engineering  
Illinois Institute of Technology  
Chicago 16, Illinois

Dr. Paul R. Rider  
Chief Statistician  
Aeronautical Research Lab., WADC  
Wright-Patterson AFB, Ohio

Professor Herbert Robbins  
Mathematical Statistics Department  
Columbia University  
New York 27, New York

Dr. Harry G. Romig  
351 Alma Real Drive  
Pacific Palisades, California

Dr. Harry Rosenblatt  
Statistical Research Division  
Bureau of Census  
Washington 25, D. C.

Professor Murray Rosenblatt  
Department of Mathematics  
Brown University  
Providence, Rhode Island

Contract No. 225(53)  
May 1961
Professor Herman Rubin  
Department of Statistics  
Michigan State University  
East Lansing, Michigan

Professor Mason Wescott  
Editor, Industrial Quality Control  
Rutgers University  
New Brunswick, New Jersey

Professor S. S. Wilks  
Department of Mathematics  
Princeton University  
Princeton, New Jersey

Professor J. Wolfowitz  
Department of Mathematics  
Cornell University  
Ithaca, New York

Dr. Max A. Wooldridge  
Department of Mathematics  
College of Engineering  
New York University  
New York, New York

Distribution via ONR London

Commanding Officer  
Branch Office  
Navy No. 100  
Fleet Post Office  
New York, New York

Dr. William R. Buckland  
22 Ryder Street  
London S. W. 1  
England

Professor Georges Darmois  
Director, Institute de Statistique  
University of Paris  
11 Rue Pierre Curie  
Paris 5, France

Professor R. Fortet  
Institut Henri Poincare  
Paris, France

Dr. Geoffrey Gregory  
4, Osborne Grove  
Galley, Chaddle  
Cheshire, England

A. Hald  
Fælledvej, 83  
Vinder, Denmark

Dr. H. C. Harroker  
Philips Research Laboratories  
Eindhoven, Netherlands

Mr. I. D. Hill  
Statistical Advisory Unit  
Ministry of Supply  
London W. 1, England

Professor M. G. Kendall  
London School of Economics  
London, England

Professor A. Walther  
Technische Hochschule  
Darmstadt, Germany

Other Foreign Addresses

Professor Maurice H. Beltz  
University of Melbourne  
Carlton N. 3  
Victoria, Australia

Professor Toyo Kitagawa  
Mathematical Institute  
Faculty of Science  
Kyoto University  
Fukuoka, Japan

Kensichi Koyama  
Managing Director  
Union of Japanese Scientists & Engineers  
2, 1-chome, Yokohashi  
Chiba-shi, Tokyo, Japan

Dr. Lai Verman  
Director  
Indian Standards Institute  
New Delhi, India

Professor P. C. Mahalanobis  
Indian Statistical Institute  
St. 28 Baranagore Trunk Road  
Calcutta 3, India

W. T. Mathew, SRC Unit  
Indian Statistical Institute  
8 King George Road  
New Delhi, India

Professor Sigeiti Moriguti  
Syoan Minamimachi 6  
Sugamadzu, Tokyo, Japan

R. G. Narasimhan  
SRC Unit, Calcutta  
9B Esplanade East  
Calcutta 3, India

H. D. Shourie  
Director  
National Productivity Council  
Golf Links  
New Delhi, India

P. V. Sivaramakrishnan  
SRC Unit, Indian Statistical Inst.  
Queens Road, Government House  
Bengaluru, India

Srinagabhusgana  
S2K Unit Bengaluru, Gadse SSKS Technologi Inst.  
Bengaluru 1, India

Professor Eduardo Valenzuela  
Republica 517-CIEF  
Santiago, Chile

Mr. Cesareo Villegas  
Inst. de Matemática y Estadística  
Av. J. Herrera y Reissig  
Montevideo, Uruguay

Additional copies for project leaders and assistants, office files, and reserve for future requirements 70