APPLIED MATHEMATICS AND STATISTICS LABORATORIES

STANFORD UNIVERSITY
CALIFORNIA

CONTINUOUS SAMPLING PLANS UNDER
DESTRUCTIVE TESTING

BY
FREDERICK S. HILLIER

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1. Introduction

Continuous sampling plans are often used to exercise some control over the quality of outgoing items, especially when the formation of inspection lots for lot-by-lot acceptance is impractical or artificial due to the continuous nature of the production process. The primary measure of effectiveness of practically all existing continuous sampling plans is the average outgoing quality limit or AOQL. To insure that the actual average outgoing quality does not exceed the AOQL, these plans stipulate that 100% inspection shall be used, with non-defective items replacing defective items, whenever incoming quality is found to be sufficiently poor. Unfortunately, there are a number of products or parts which require destructive testing in order to ascertain whether the item is defective or not. For such items, it is clearly not feasible to permit 100% inspection. Therefore, new continuous sampling plans designed especially for destructive testing seem to be needed.

The formulation presented here departs almost completely from the usual scheme for existing continuous sampling plans.\(^1\) It explicitly takes into

\(^1\) See [4] for a comprehensive discussion of existing continuous sampling plans. Special mention should be made of two of these plans which introduced simplified versions of some of the unusual features of this formulation. Girschick and Rubin [1] first proposed a model for a deteriorating process. Their proposed plan, which includes a provision for stopping the process to take corrective action, is also based explicitly on cost considerations. Savage's proposed plan [5] includes these features and, additionally, is designed for destructive testing in that 100% inspection is never used. Unfortunately, Savage only considered the restrictive class of plans where only a single sampling rate is ever used and the process is always stopped and investigated as soon as a single defective item is inspected.
account the possibility that the process quality may deteriorate to an undesirable level. It uses certain a priori information about the relative likelihood of the possible patterns of process quality over time. It then improves upon this a priori information by a limited amount of sampling inspection in order to more reliably ascertain when action needs to be taken to reestablish the process quality at a desirable level. The resulting continuous sampling scheme is designed to approximate the procedure which would minimize the total expected cost and imputed cost, including the expected cost of inspection, the expected cost and imputed cost of clearing defective items for use, and the expected cost and imputed cost of investigating and adjusting the process in order to improve the process quality.

Section 2 presents the formulation of the problem. Section 3 begins by determining when the process should be adjusted if no inspections are permitted to supplement the a priori information. It then proceeds to determine when the additional information on the process quality level to be obtained from inspection justifies the additional cost of inspection. Section 4 summarizes the continuous sampling plan arising out of the analysis of the preceding section. The resulting operating characteristics are presented in Section 5. Section 6 illustrates the use of this plan and Section 7 summarizes the conclusions.

2. **Formulation of the Problem**

   Consider any production process producing homogeneous items sequentially where the quality of each item can be meaningfully classified, after inspection, as either defective or non-defective. The
defective vs. non-defective status of the respective items will be treated as independent Bernoulli trials where \( p(m) \) is the unknown probability that the \( m^{th} \) item is defective. The traditional continuous sampling approach to this situation is to assume that \( p(m) = p \) for all \( m \), i.e., that the process is always in statistical control. However, while this is a convenient assumption for deriving operating characteristics of the plans, an inspection plan as elaborate as the continuous sampling plan is needed and used only because protection is also desired against the process going out of control.\(^1\) In this formulation, the more realistic assumption will be made that the process can go out of control, i.e., that the process quality level, \( p(m) \), can shift. Then, when it is detected that \( p(m) \) is no longer acceptable, the process is investigated and adjusted in an attempt to reestablish the quality level at an acceptable level.

Another usual practice with existing continuous sampling plans is to ignore any available a priori information as to the relative likelihood of the various values of \( p(m) \). These plans instead rely entirely upon the results of the sampling inspection to, in effect, estimate \( p(m) \) and draw the consequent conclusions about what further sampling inspection is desirable. Unfortunately, this approach leads to considerable 100% inspection, which is not feasible where destructive testing is required. It thus appears that, under destructive testing, one cannot afford the luxury of ignoring a priori information about the process quality level, even though, in reality, this information may only consist

\(^1\) This contention and the resulting consequences are fully developed in [2].
of a competent educated guess supported by limited historical quality data. Therefore, the following formulation assumes the existence of a relatively modest amount of such a priori information.

Assume that the state of the process quality level depends only on the number of items produced since the last adjustment of the production process.\(^1\) Hence, let the stochastic process \(\{X(t) : t \geq 1\}\) represent the state of the process quality level following each process adjustment, where, for integer values of \(t\), \(X(t)\) takes on the value of the probability that the \(t\)th item since the last process adjustment is defective. In other words, for integer values of \(t\), \(X(t)\) takes on the values \(p(m)\) for those values of \(m\) such that the \(m\)th item is the \(t\)th item since the last adjustment of the process. While \(X(t)\) has no obvious physical interpretation when \(t\) is not an integer, it will be analytically convenient to permit non-integers. Denote the probability density function, mean, and variance of \(X(t)\) by \(f_{X(t)}(x)\), \(\mu_t\), and \(\sigma_t^2\), respectively.

Assume that the a priori distribution of \(X(1)\) is a Beta distribution with a known mean \(\mu_1\) and a known variance \(\sigma_1^2\). Therefore,

\[
f_{X(1)}(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1) \Gamma(\beta+1)} x^\alpha (1-x)^\beta, \quad 0 \leq x \leq 1,
\]

where \(\alpha\) and \(\beta\) are specified by the relationships,

\(^1\) It does not appear that the presence of a continuous "learning curve" effect - so that the process quality level is, on the average, improving over time - would affect the subsequent results. However, the increased notational, conceptual, and analytical difficulties recommended against this more general formulation.
\[ \mu_1 = \frac{\alpha_1 + 1}{\alpha + \beta + 2}, \]

\[ \sigma_1^2 = \frac{\alpha + 2}{(\alpha + \beta + 1)^2} \mu_1^2 = \frac{(\alpha + 1)(\beta + 1)}{(\alpha + \beta + 2)^2(\alpha + \beta + 3)}. \]

Assume that the stochastic process \( \{X(t) : t \geq 1\} \) is a superposition process based upon a Poisson counting process \( \{M(t) : t \geq 1\} \), where \( M(t) \) is the number of changes in the value of \( X(t) \) that have occurred in the "time" interval from \( 1 \) to \( t \). Thus,

\[ f_{M(t+1)}(m) = \frac{\theta^m e^{-\theta t}}{m!}, \quad m = 0, 1, 2, \ldots. \]

An equivalent assumption is that the distribution of the increment in \( t \) between consecutive changes in \( X(t) \) is exponential, so that its cumulative distribution function, \( G(\Delta t) \), is

\[ G(\Delta t) = 1 - e^{-\theta \Delta t}, \quad \Delta t \geq 0, \]

where \( \frac{1}{\theta} \) is the mean. Assume that \( \theta \) is known.

Let the random variable \( Y(t) \) be the change in \( X(s) \) at \( s = t \), given that a change occurs then. In other words, letting \( X(t-) \) and \( M(t-) \) denote the limit of \( X(s) \) and \( M(s) \) as \( s \) approaches \( t \) from below,

\[ Y(t) = X(t) - X(t-), \quad \text{where} \quad 0 \leq X(s) \leq 1 \quad \text{for} \quad s \geq 1, \]
provided that

\[ M(t) - M(t-) = 1. \]

Thus, using Riemann-Stieltjes integration,

\[ X(t) = \int_{t}^{t} Y(\tau) \, dM(\tau) + X(1). \]

Assume that

\[ E[Y(t) \mid X(t-)] = \begin{cases} \mu_{\Delta \rho}, & \text{if } X(t-) < 1 - \mu_{\Delta \rho} \\ 1 - X(t-), & \text{otherwise} \end{cases} \]

and

\[ \text{Var} \{Y(t) \mid X(t-)\} = \begin{cases} \sigma_{\Delta \rho}^2, & \text{if } X(t-) < 1 - \mu_{\Delta \rho} \\ 0, & \text{otherwise} \end{cases} \]

where \( \mu_{\Delta \rho} > 0 \) and \( \sigma_{\Delta \rho}^2 \) are both known. Beyond this, the probability distribution of \( Y(t) \) is not specified.

In order to prevent a cumbersome analysis, several simplifications in this formulation will be introduced into the analysis. The first of these involves the truncation of \( Y(t) \). Production processes of practical interest will be adjusted before \( \mu_t \) becomes very large. Therefore, the probability that \( X(t) \) will reach \( 1 \) is small. Hence, \( Y(t) \) will be treated as if \( E[Y(t)] = \mu_{\Delta \rho} \) and \( \text{Var} \{Y(t)\} = \sigma_{\Delta \rho}^2 \). Furthermore, \( \{Y(t) : t \geq 1\} \) will be treated as a family of independently distributed random variables which is independent of \( \{X(t) = t \geq 1\} \) for purposes of finding \( f_{X(t)}(x) \) and \( f_{X(t)}(x \mid M(t)) \). This assumption should be
sufficiently valid for the small values of \( t \) of interest that it would introduce no significant error. Therefore,

\[
E(X(t) | M(t) = m) = \mu_L + m\mu_D \\
\text{Var}(X(t) | M(t) = m) = \sigma_L^2 + m\sigma_D^2.
\]

Finally, assume that the distribution of \( Y(t) \) is such that the distribution of \( X(t) \) can be taken, at least as a close approximation, to be a Beta distribution. In other words, while the a priori information does not prescribe the distribution of \( Y(t) \) directly, it does prescribe that the distribution of \( X(t) \) would be reasonably expected to continue to resemble the Beta distribution. Thus,

\[
f_{X(t)}(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha}(1-x)^{\beta}, \quad 0 \leq x \leq 1,
\]

where \( \alpha \) and \( \beta \) are specified by the relationships,

\[
\mu_t = \frac{\alpha+1}{\alpha+\beta+2},
\]

\[
\sigma_t^2 = \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)},
\]

where \( \mu_t \) and \( \sigma_t^2 \) will be subsequently derived on the basis of the other a priori information.

In order to design an economical continuous sampling plan, it is necessary to specify the relevant costs. Let \( C_I \) be the total expected
cost of the actual inspection of an item, including the cost of the loss of the item through destructive testing. Let $C_D$ be the total expected cost and imputed cost of clearing a defective item for use. Let $C_A$ be the total expected cost and imputed cost of taking corrective action to adjust an out of control process, including the cost of the investigation.

It is obvious that, for an actual application, it would usually be difficult to specify satisfactory values for the eight imput parameters defined above. However, this writer feels that, by carefully studying the meaning and effect of each of these parameters, this task becomes entirely feasible. The example presented in Section 6 illustrates the thought process which should lead to the specification of satisfactory parameter values.

Assuming the availability of the indicated information, the ideal result would be to develop a sampling procedure which, not only is sufficiently simple for practical use, but also minimizes total expected cost. Unfortunately, it does not appear feasible to fully achieve both objectives. Since an elegant optimal solution described in terms of numerous recursive relationships, etc., would have little practical value unless the solution can be used, the subsequent analysis places a higher priority on obtaining usable results than on maintaining optimality. Therefore, there will be a few cases where heuristic simplifications and approximations will be introduced into the analysis. In brief, the problem that will now be considered is the development of a continuous sampling plan under destructive testing which, on the basis of available information, appears to be suitable for practical use.
3. Analysis

The analysis will begin by investigating special cases. The preliminary results thus obtained will then be extended to the general case. Thus, the case where no inspections are allowed, so that only the a priori information on the process quality level over time is available, will first be considered. The form of the rule for stopping the process for this case will then be extended to the general case. Next, certain special classes of sampling procedures will be analyzed; the results will lead to a recommended sampling procedure for the general case where any sampling procedure is admissible.

Before proceeding further, it will be necessary to introduce additional notation. Let

\[ Z = \{z : z = (z_1, z_2, \ldots)\} \]

be the set of possible outcomes of the item-by-item production process, where \( z_t = 1 \) if the \( t^{th} \) item is defective and \( z_t = 0 \) otherwise. Thus, \( E[z_t] = X(t) \). Let

\[ D = \{d(z) : d(z) = (d_1(z), d_2(z), \ldots), z \in Z\} \]

be the set of possible sampling procedures, where \( d_t(z) = 1 \) if the \( t^{th} \) item is to be inspected and \( d_t(z) = 0 \) otherwise. Define \( d^{(t)}(z) = (d_1(z), d_2(z), \ldots, d_t(z), 0, 0, \ldots) \). Let

\[ R = \{r(d,z) : r(d,z) = (r_1(d,z), r_2(d,z), \ldots), d \in D, z \in Z\} \]

be the set of possible results of the sampling process, where \( r_t(d,z) = \)
\( d_t(z)[d_t(z) + z_t] \). Let \( r_t(d,z) = (r_1(d,z), r_2(d,z), \ldots, r_t(d,z)) \).

Let
\[
S = \{s(r(s)) : s(r(s)) \in \{0, 1, 2, \ldots, \text{reR}\}\}
\]
be the set of possible stopping rules, where the process is stopped after \( s(r(s)) \) items. For the sake of notational simplicity, the arguments will usually be omitted from the elements of \( D, R, \) and \( S \).

Let \( C(d,s,z) \) be the total cost of a cycle, given \( z \) and the rule \( (d,s) \). Thus,
\[
C(d,s,z) = C_A + \sum_{t=1}^{s} C_D z_t (1 - d_t) + \sum_{t=1}^{s} C_I d_t
\]
\[
= C_A + C_D \sum_{t=1}^{s} z_t + \sum_{t=1}^{s} d_t (C_I - C_D z_t).
\]

Let \( N(d,s,z) \) be the number of items not inspected during the cycle.

Thus,
\[
N(d,s,z) = s - \sum_{t=1}^{s} d_t
\]
so that \( N(d,s,z) \) is just the number of items not subjected to destructive testing and therefore available for use. Consider a sequence of \( m \) cycles. Let \( z^{(i)} \in Z \) be the outcome of the process for the \( i^{th} \) cycle, where \( i = 1, 2, \ldots, m \). Notice that
\[
\frac{\sum_{i=1}^{m} C(d,s,z^{(i)})}{\sum_{i=1}^{m} N(d,s,z^{(i)})}
\]
is just the average cost per outgoing item for the $m$ cycles. Therefore, an obvious and reasonable criterion to use in selecting $(d,s)$ is to minimize, or at least come arbitrarily close to minimizing,

$$
\lim_{m \to \infty} E \left\{ \frac{\sum_{i=1}^{m} C(d,s,z^{(i)})}{\sum_{i=1}^{m} N(d,s,z^{(i)})} \right\},
$$

the asymptotic expected cost per outgoing item. Johns and Miller [3] show that, under certain weak conditions satisfied here for a fixed rule $(d,s)$ such that $N(d,s,z^{(i)}) > 0$ almost surely,

$$
\lim_{m \to \infty} E \left\{ \frac{\sum_{i=1}^{m} C(d,s,z^{(i)})}{\sum_{i=1}^{m} N(d,s,z^{(i)})} \right\} = \frac{E_{Z}[C(d,s,z)]}{E_{Z}[N(d,s,z)]}.
$$

Furthermore, by the Law of Large Numbers, for any fixed rule $(d,s),$

$$
\frac{\sum_{i=1}^{m} C(d,s,z^{(i)})}{\sum_{i=1}^{m} N(d,s,z^{(i)})} \xrightarrow{a.s.} \frac{E_{Z}[C(d,s,z)]}{E_{Z}[N(d,s,z)]}.
$$

Let

$$
A(d,s) = \frac{E_{Z}[C(d,s,z)]}{E_{Z}[N(d,s,z)]}.
$$

Therefore, the natural objective becomes to minimize, or almost minimize
A(d, s). To eliminate the question as to whether an actual minimum occurs, s will be restricted from becoming arbitrarily large. Thus, let

$$ S_L = \{ s(r(s)) : s(r(s)) \in \{0, 1, 2, \ldots , L\}, reR \}. $$

Since the range of A(d, s) over (D, S_L) forms a finite set, a minimum must exist. Select L as an extremely large integer, so that this restriction has no significant practical effect. Hence, the objective will be, as far as possible, to find the rule \((d^*, s^*) \in (D, S_L)\) such that

$$ A(d^*, s^*) = \inf_{(d, s) \in (D, S_L)} \{ A(d, s) \}. $$

To begin, consider the case where no inspection are allowed. Thus, d'o(z) = (0, 0, 0, ...). Furthermore, r becomes a constant independent of z, so that s(r) becomes independent of z. Thus, the problem becomes one of finding that fixed integer s* which minimizes A(d'o, s),

$$ A(d'o, s) = \frac{c_A + c_D E(D)}{s}, $$

where \( E(D) = \sum_{t=1}^{s} E(x_t) \) is the expected number of defectives among the s items. Thus,

$$ E(D) = \sum_{t=1}^{s} E(x_t) $$

$$ = \sum_{t=0}^{s-1} \left( \mu_1 + t \theta \mu_\Delta \right) $$

$$ = s\mu_1 + \frac{s(s-1)}{2} \theta \mu_\Delta $$

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It will be convenient at times to approximate the actual item-by-item production process by a strictly continuous process where $s$ need not be an integer. While such a strictly continuous process has no meaningful physical interpretation, it will greatly simplify the analysis while providing results which, when rounded off to the nearest integer, will be almost exact for the item-by-item process. Assuming a strictly continuous process, the expression for $E(D)$ becomes

$$E(D) = \int_{-\frac{1}{2}}^{s - \frac{1}{2}} (\mu_1 + t \theta \mu_\Delta) dt,$$

so that, once again,

$$E(D) = s\mu_1 + \frac{s(s-1)}{2} \theta \mu_\Delta.$$

Therefore,

$$A(d^{(o)}, s) = C_D(\mu_1 + \frac{s-1}{2} \theta \mu_\Delta) + \frac{C_A}{s}.$$

To determine the minimizing value of $s$, set the derivative equal to zero,

$$\frac{d}{ds} A(d^{(o)}, s) = \frac{C_D \theta \mu_\Delta}{2} - \frac{C_A}{s^2} = 0.$$

Hence, since the second derivative is positive for positive $s$, the optimal value of $s$, denoted by $s^*$, is

$$s^* = \sqrt{\frac{2 C_A}{C_D \theta \mu_\Delta}}.$$
Thus, letting \( P \) be the expected value of \( X(t) \) when the value of \( t \) is such that the process should be adjusted,

\[
P = \mathbb{E}(X(s^*)) = \mu_1 + \left( \sqrt{\frac{2 \cdot C_A}{C_D \theta_\mu \Delta p}} - \frac{1}{2} \right) \theta_\mu \Delta p.
\]

It is now important to notice that

\[
A(d^{(o)}, s^*) = C_D \left( \mu_1 + \frac{\sqrt{\frac{2 \cdot C_A}{C_D \theta_\mu \Delta p}} - 1}{2} \theta_\mu \Delta p \right) + C_A \sqrt{\frac{C_D \theta_\mu \Delta p}{2 C_A}}
\]

\[
= C_D \mu_1 + \left( \sqrt{\frac{2 \cdot C_A}{C_D \theta_\mu \Delta p}} - 1 \right) \theta_\mu \Delta p + C_D \theta_\mu \Delta p
\]

\[
= C_D P.
\]

Thus, the optimal stopping rule says to stop the process as soon as the expected incremental cost of continuing, \( C_D \mathbb{E}(X(t)) \), would increase to the point that it would exceed the total expected cost per item for the cycle. It will now be shown that the same form applies for a more general case where inspections are allowed.

Lemma 1: Define \( s^* \in S_L \) by

\[
A(d, s^*) = \inf_{s \in S_L} \{ A(d, s) \}.
\]

Consider any \( t \in \{1, 2, \ldots, L\} \) and any \( d \in D \) such that \( d_n(z) = 0 \).
for \( n > \min \{t, s^* (r^{(s^*)}(d, z))\} \). Define

\[
Z_0 = \{z : z \notin Z, \ t < s^* (r^{(s^*)}(d, z))\}.
\]

Then,

\[
C_D E[X(t+1) | r(t)(d, z)] \leq A(d, s^*) \quad \text{for } z \in Z_0 \quad \text{almost surely;}
\]

\[
C_D E[X(t+1) | r(t)(d, z)] \geq A(d, s^*) \quad \text{for } z \in \overline{Z_0} \quad \text{almost surely.}
\]

**Proof:** Assume that there exists a set \( Z_1 \subseteq Z_0 \) such that \( P(Z_1) > 0 \) and for any \( z \in Z_1 \),

\[
C_D E[X(t+1) | r(t)(d, z)] > A(d, s^*).
\]

Let

\[
s_1(r^{(s_1)}(d, z)) = \begin{cases} 
  s^* (r^{(s^*)}(d, z)), & z \in Z - Z_1, \\
  t, & z \in Z_1. 
\end{cases}
\]

Since \( \mu_{D_p} > 0 \), \( E[X(t+n) | r(t)(d, z)] \) increases as \( n \) increases over the positive integers. Therefore, for \( z \in Z_1 \),

\[
C(d, s^*, z) - C(d, s_1, z) > (s^* - s_1) A(d, s^*);
\]

\[
N(d, s^*, z) - N(d, s_1, z) = s^* - s_1 \geq 1.
\]

Hence,

\[
E_z [C(d, s^*, z)] - E_z [C(d, s_1, z)]
\]

\[
> [E_z (N(d, s^*, z)) - E_z (N(d, s_1, z))] A(d, s^*).
\]
Therefore,

\[ A(d,s^*) = \frac{E_z[C(d,s^*_1,z)] + E_z[C(d,s^*,z)] - E_z[C(d,s^*_1,r)]}{E_z[N(d,s^*,z)l]} \]

\[ > \frac{E_z[C(d,s^*_1,z)] + [E_z[N(d,s^*,z)] - E_z[N(d,s^*_1,z)]] A(d,s^*)}{E_z[N(d,s^*,z)l]} \]

\[ = A(d,s^*_1) \frac{E_z[N(d,s^*_1,z)]}{E_z[N(d,s^*,z)l]} + A(d,s^*) \left( 1 - \frac{E_z[N(d,s^*_1,z)]}{E_z[N(d,s^*,z)l]} \right) . \]

Hence,

\[ A(d,s^*) > A(d,s^*_1) , \]

so that

\[ A(d,s^*_1) < \inf_{s \in S_L} \{ A(d,s) \} , \]

which is impossible. This contradiction verifies the first part of the lemma.

Assume that the second part of the lemma is false, which therefore assumes that there exists a set \( Z_\subseteq Z - Z_0 \) such that \( P(Z_\subseteq) > 0 \) and for any \( z \in Z_\subseteq \),

\[ C_D E[X(s^*_1) \mid r(s^*)(d,z)] < A(d,s^*) . \]

Let

\[ s^2_r(s^*)(d,z) = \begin{cases} s^*(r(s^*)(d,z)), & z \in Z - Z_2 \\ (s^*(r(s^*)(d,z)) + 1, & z \in Z_2 . \end{cases} \]
Therefore,

\[
A(d,s_2) < \frac{\mathbb{E}_z[C(d,s^*, z)] + P(Z \neq \emptyset) \cdot A(d,s^*)}{\mathbb{E}_z[N(d,s^*, z)] + P(Z \neq \emptyset)}
\]

\[= A(d,s^*)\]

so that

\[A(d,s_2) < \inf_{s \in S_L} (A(d,s))\]

which is impossible. This contradiction completes the proof of the lemma.

Lemma 1 indicates that the optimal stopping rule would stop the process if and only if the expected probability of defectiveness has risen so high that the expected incremental cost of continuing exceeds the minimum total expected cost per outgoing item, given a fixed sampling rule which prescribes no further inspection if the process were to continue. While this is an exact result, it is not quite general enough. Therefore, two simplifying approximations will now be made so that the general form of the optimal stopping rule can be prescribed.

First, it will be assumed that Lemma 1 applies to any sampling rule. In other words, given any fixed sampling rule, the question of when the process should be stopped is independent of whether more inspections might be conducted otherwise. This approximation appears to be a plausible one for the following reason. If the process has deteriorated badly so that an adjustment is needed, more inspections will accomplish nothing beyond providing further verification of this condition. Therefore, the more likely it is that the process should be
adjusted, the more likely it is that waiting in order to conduct more inspections will be a waste of time and money. Thus, it is more efficient to conduct the inspections earlier when they are more likely to give valuable evidence that the process is either better than expected, so that the eventual adjustment can be postponed, or worse than expected, so that an adjustment should be considered soon. Hence, if an adjustment should be made if no further inspections would be conducted otherwise, it appears reasonable not to postpone the adjustment in order to conduct more inspections in the hope of justifying a further postponement.

Notice that the optimal stopping rule is based on the value of \( \inf_{s \in S_L} \{A(d, s)\} \), which would usually be difficult to ascertain. Furthermore, this difficulty would soon be compounded when it becomes necessary to find the optimal sampling rule, given the use of the optimal stopping rule. Therefore, the second simplifying approximation involves the specification of an approximate value for \( \inf_{s \in S_L} \{A(d, s)\} \) which should be appropriate for sampling rules of interest. Thus, the value \( C_D^P \) will be used henceforth as the value of \( \inf_{s \in S_L} \{A(d, s)\} \). Recall that \( P \) was specified during the discussion of the no inspection case, and that it was shown that \( C_D^P \) equals \( A(d^{(o)}), \) which is the rationale behind the selection of this value. Thus, when no inspections are justified, \( C_D^P \) will be exact for the optimal sampling rule \( d^{(o)} \), whereas when a few inspections are justified, \( C_D^P \) will be slightly too large for the best sampling rules. It will be seen that \( C_D^P \) is almost exact for the recommended sampling rule that will
be developed.

In summary, the recommended stopping rule, hereafter denoted as $s^*$, may be defined as that rule such that, for $t \in \{1, 2, \ldots\}$,

$$E[X(t+1)|r(t)(d,z)] \leq P \text{ for } t < s^*(r^*(d,z)),$$

and

$$E[X(t+1)|r^*(d,z)] \geq P \text{ for } t \geq s^*(r^*(d,z)).$$

The next objective will be to determine the proper sampling rule. Before this is done, some preliminary results will be obtained.

**Lemma 2.** Assume that $d_t(z) = 1$. Let $\alpha$ and $\beta$ be such that the a priori distribution of $X(t)$ is

$$f_{X(t)}(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1) \Gamma(\beta+1)} x^{\alpha}(1-x)^{\beta}, \quad 0 \leq x \leq 1,$$

so that

$$E[X(t)] = \frac{\alpha+1}{\alpha+\beta+2}, \quad \text{Var}(X(t)) = \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2 (\alpha+\beta+3)}.$$

Then the a posteriori distribution of $X(t)$ is

$$f_{X(t)}(x|z_t=1) = \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2) \Gamma(\beta+1)} x^{\alpha+1}(1-x)^{\beta}, \quad 0 \leq x \leq 1,$$

or

$$f_{X(t)}(x|z_t=0) = \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+1) \Gamma(\beta+2)} x^{\alpha}(1-x)^{\beta+1}, \quad 0 \leq x \leq 1.$$
Furthermore,

\[ E[X(t) | z_t = 1] = E[X(t)] + \frac{\operatorname{Var}(X(t))}{E[X(t)]} ; \]

\[ E[X(t) | z_t = 0] = E[X(t)] - \frac{\operatorname{Var}(X(t))}{1-E[X(t)]} ; \]

\[ \operatorname{Var}(X(t) | z_t = i) = \frac{E[X(t) | z_t = i][1-E[X(t) | z_t = i]]}{E[X(t)][1-E[X(t)]] + \operatorname{Var}(X(t))} \operatorname{Var}(X(t)), \quad i = 0, 1. \]

**Proof:** By Bayes rule,

\[ f_{X(t)}(x | z_t = 1) = \frac{f_{X(t)}(x)}{E[X(t)]} = \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2) \Gamma(\beta+1)} x^{\alpha+1}(1-x)^\beta, \quad 0 \leq x \leq 1, \]

and

\[ f_{X(t)}(x | z_t = 0) = \frac{f_{X(t)}(x)(1-x)}{1-E[X(t)]} = \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+1) \Gamma(\beta+2)} x^{\alpha}(1-x)^{\beta+1}, \quad 0 \leq x \leq 1. \]

The conditional expected values and variances are then easily derived with some algebraic manipulation. First, a little work yields

\[ \alpha = -1 - E[X(t)] + \frac{E^2[X(t)]}{\operatorname{Var}(X(t))} \left[1-E[X(t)]\right] ; \]

\[ \beta = -2 + E[X(t)] + \frac{E[X(t)]}{\operatorname{Var}(X(t))} \left[1-E[X(t)]\right]^2. \]

Therefore,

\[ E[X(t) | z_t = 1] = \frac{\alpha+2}{\alpha+\beta+3}. \]

20.
\[1 + \frac{E^2(X(t))}{\text{Var}(X(t))} [1 - E(X(t))] - E(X(t))\]

\[= \frac{E(X(t))}{\text{Var}(X(t))} [1 - E(X(t))]\]

\[= E(X(t)) + \frac{\text{Var}(X(t))}{E(X(t))};\]

\[E(X(t)|z_t=0) = \frac{\alpha + 1}{\alpha + \beta + 3}\]

\[= \frac{E^2(X(t))}{\text{Var}(X(t))} [1 - E(X(t))] - E(X(t))\]

\[= \frac{E(X(t))}{\text{Var}(X(t))} [1 - E(X(t))];\]

\[\text{Var}(X(t)|z_t=1) = \begin{pmatrix} \alpha + 2 \\ \alpha + \beta + 3 \end{pmatrix} \begin{pmatrix} \beta + 1 \\ \alpha + \beta + 3 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha + \beta + 4 \end{pmatrix}\]

\[= \frac{E(X(t)|z_t=1) [1 - E(X(t)|z_t=1)]}{\text{Var}(X(t)) [1 - E(X(t))] + 1};\]

\[\text{Var}(X(t)|z_t=0) = \begin{pmatrix} \alpha + 1 \\ \alpha + \beta + 3 \end{pmatrix} \begin{pmatrix} \beta + 2 \\ \alpha + \beta + 3 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha + \beta + 4 \end{pmatrix}\]

\[= \frac{E(X(t)|z_t=0) [1 - E(X(t)|z_t=0)]}{\text{Var}(X(t)) [1 - E(X(t))] + 1};\]
so that

$$\text{Var}(X(t) \mid z_t = i) = \frac{E[X(t) \mid z_t = i][1 - E[X(t) \mid z_t = i]]}{E[X(t)] - [1 - E[X(t)]] + \text{Var}(X(t))} \text{Var}(X(t)), \ i = 0, 1.$$  

Thus, the lemma is verified.

**Lemma 3.** Assume that $d_n = 0 \text{ if } t \leq n \leq t + \Delta t$.

Then,

$$E[X(t + \Delta t)] = E[X(t)] + \theta \Delta t \mu_{\Delta P};$$

$$\text{Var}(X(t + \Delta t)) = \text{Var}(X(t)) + \theta \Delta t \left( \sigma^2_{\Delta P} + \mu_{\Delta P}^2 \right).$$

**Proof:** The first part of the lemma is immediate. The expression for $\text{Var}(X(t + \Delta t))$, which is not as evident, will now be derived. By definition,

$$\text{Var}(X(t + \Delta t)) = \int_0^1 [X(t + \Delta t) - E[X(t + \Delta t)]]^2 f_{X(t + \Delta t)}(x) \, dx.$$  

Hence,

$$\text{Var}(X(t + \Delta t)) = \int_0^1 \left[ X(t + \Delta t) - (E[X(t)] + \theta \Delta t \mu_{\Delta P}) \right]^2 f_{X(t + \Delta t)}(x) \, dx$$

$$= \int_0^1 \left[ X(t + \Delta t) - (E[X(t)] + \theta \Delta t \mu_{\Delta P}) \right]^2 \left[ \sum_{m=0}^{\infty} p(M=m) f_{X(t + \Delta t)}(x \mid M=m) \right] \, dx.$$  

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\[
\begin{align*}
= \sum_{m=0}^{\infty} P(M=m) \int_{0}^{1} \left[ X(t+\Delta t) - (E[X(t)] + \theta \Delta t \mu_{\Delta \rho}) \right]^2 f_{X(t+\Delta t)}(x | M=m) dx \\
= \sum_{m=0}^{\infty} P(M=m) \int_{0}^{1} \left[ X(t+\Delta t) - E[X(t)] - m \mu_{\Delta \rho} + (m-\theta \Delta t) \mu_{\Delta \rho} \right]^2 f_{X(t+\Delta t)}(x | M=m) dx \\
= \sum_{m=0}^{\infty} P(M=m) \left[ \text{Var}(X(t)) + m \mu_{\Delta \rho}^2 + (m - \theta \Delta t) \mu_{\Delta \rho} \right] \\
= \text{Var}(X(t)) + \sigma_{\Delta \rho}^2 \sum_{m=0}^{\infty} m P(M=m) + \mu_{\Delta \rho}^2 \sum_{m=0}^{\infty} m^2 - 2m \theta \Delta t + [\theta \Delta t]^2 \right] P(M=m) \\
= \text{Var}(X(t)) + \sigma_{\Delta \rho}^2 \theta \Delta t + \mu_{\Delta \rho}^2 \left[ \theta \Delta t + (\theta \Delta t)^2 \right] - 2(\theta \Delta t)^2 + (\theta \Delta t)^2 \\
\end{align*}
\]

Therefore,

\[
\text{Var}(X(t+\Delta t)) = \text{Var}(X(t)) + \theta \Delta t \left( \sigma_{\Delta \rho}^2 + \mu_{\Delta \rho}^2 \right)
\]

and the lemma is proved.

In order to compare the expected costs for alternative sampling rules, an artificial situation will be constructed and the associated expected cost used as a reference point. This artificial situation assumes that no inspections will be conducted, but that at time \( t < s^* \), the actual exact value of \( X(t) \) becomes known. With this additional information, the process will thus be stopped when \( E(X(t+\Delta t) | X(t)) = P \)
or immediately if \( X(t) \leq P \). Denote this stopping rule predicated on the knowledge of \( X(t) \) by \( s^*_{t} \).

In order to simplify the following analysis, the requirement that \( s^* \) must be an integer will be ignored.

**Lemma 4.** Assume that \( t \in \{1, 2, \ldots \} \) and \( t < \min(s^*(r(s^*)^{-1}(d(t), z)), s^*(r(s^*)^{-1}(d(t), z))) \). For notational simplicity, let it be understood that \( f_X(t)(x) \), \( E[X(t)] \), and \( \text{Var}[X(t)] \) denote the a posteriori density function, mean, and variance of \( X(t) \), given \( r(t)(d(t), z) \). Then,

\[
E_z\{[C(d(t), s^*, z) - C(d(o), s^*, z)] \mid r(t)(d(t), z)\}
\]

\[
= \frac{C_D}{2\theta u} \text{Var}[X(t)] + E_z\left\{ \sum_{n=1}^{t} d_n(z)[C_I - C_D z_n] \mid r(t)(d(t), z)\right\}
\]

\[
- \frac{C_D}{\theta u} \int_{P}^{1} \left[ \frac{(x-E[X(t)])^2}{2} - \frac{(P-E[X(t)])^2}{2} + (x-P)E[X(t)] \right] f_X(t)(x)dx.
\]

**Proof:** There are three cost differences between the \( (d(t), s^*) \) and \( (d(o), s^*) \) rules. One arises because the process would be stopped at different times, creating differences in the expected number of defective items. A second difference is that \( (d(t), s^*) \) would usually create inspection costs. Finally, these inspections may remove defectives and thereby decrease the cost of defective items. These cost differences are summarized in the following equation.
\[
E_z[[C(d(t), s^*, z) - C(d(0), s^*_t, z)] | r(t)(d(t), z)]
\]

\[
= \int_0^1 \int_0^{P-E[X(t)]} (x+y) C_D \frac{dy}{\theta \mu \Delta \rho} f_X(t)(x) dx + \int_0^P \int_0^{x-E[X(t)]} (P+y) C_D \frac{dy}{\theta \mu \Delta \rho} f_X(t)(x) dx
\]

\[
- \int_0^{E[X(t)]} \int_0^{(P+y) C_D \frac{dy}{\theta \mu \Delta \rho} f_X(t)(x) dx} E_X(t) \left\{ \sum_{n=1}^{t} d_n(z) [C_I - C_D z_n] | r(t)(d(t), z) \right\}
\]

\[
= \frac{C_D}{\theta \mu \Delta \rho} \int_0^1 \left[ x + \frac{P-E[X(t)]}{2} \right] [P-E[X(t)]] f_X(t)(x) dx
\]

\[
+ \frac{C_D}{\theta \mu \Delta \rho} \int_0^P \left[ P + \frac{x-E[X(t)]}{2} \right] [x-E[X(t)]] f_X(t)(x) dx
\]

\[
+ E_z \left\{ \sum_{n=1}^{t} d_n(z) [C_I - C_D z_n] | r(t)(d(t), z) \right\}
\]

\[
= \frac{C_D}{\theta \mu \Delta \rho} \int_0^1 \left[ P + \frac{x-E[X(t)]}{2} \right] [x-E[X(t)]] f_X(t)(x) dx
\]

\[
- \frac{C_D}{\theta \mu \Delta \rho} \int_0^1 \left[ \frac{[x-E[X(t)]]^2}{2} - \frac{[P-E[X(t)]]^2}{2} + (x-P)E[X(t)] \right] f_X(t)(x) dx
\]

\[
+ E_z \left\{ \sum_{n=1}^{t} d_n(z) [C_I - C_D z_n] | r(t)(d(t), z) \right\}
\]
\[
\frac{C_D}{2\theta\mu_{\Delta p}} \text{Var}(X(t)) + E_Z \left\{ \sum_{n=1}^{t} d_n(z)(C_1 - C_D z_n) r^{(t)}(d^{(t)}, z) \right\} \\
- \frac{C_D}{\theta\mu_{\Delta p}} \int_0^1 \left[ \frac{(x-E[X(t)])^2}{2} - \frac{[P-E[X(t)]]^2}{2} + (x-P)E[X(t)] \right] f_{X(t)}(x) \, dx,
\]

as was to be proved.

**Lemma 5.** Assume that \( t \in \{1, 2, \ldots \} \) and \( t < s^*(r(s^*)(d^{(t)}, z)) \) almost surely. Therefore,

\[
E_Z (N(d^{(0)}, s^*, z) - N(d^{(t)}, s^*, z)) = E_Z \left\{ \sum_{n=1}^{t} d_n(z) \right\}.
\]

**Proof:** Since \( t < s^*(r(s^*)(d^{(t)}, z)) \) almost surely, then \( t < s^*(r(s^*)(d^{(0)}, z)) \) almost surely. This is immediate since if the a posteriori mean \( E[X(u)] < P \) for \( 0 \leq u \leq t \) for all possible inspection results, then the a priori mean \( E[X(u)] < P \) for \( 0 \leq u \leq t \).

Therefore, it is clear that, for \( d' = d^{(0)}, d^{(t)} \),

\[
E_Z(s^*|d') = t + E_Z(s^* - t \mid d')
\]

\[
= t + E_Z(s^* - t \mid d', (s^*-t) > 0).
\]

It is a direct result of Lemma 3 and the definition of \( s^* \) that

\[
E_Z(s^* - t \mid (s^*-t) > 0, r^{(t)}(d, z), d_n=0 \text{ for } n > t) = \frac{P-E[X(t)]}{\theta\mu_{\Delta p}} \frac{r^{(t)}(d, z)}{\theta\mu_{\Delta p}}
\]
Therefore, for \( d' = d^{(o)} , d^{(t)} \),

\[
E_z(s^*-t|d') = E_z \left\{ \frac{P-E[X(t)|r^{(t)}(d',z)]}{\sigma^u_{\Delta P}} \right\} .
\]

But

\[
E_z \left\{ E[X(t)|r^{(t)}(d^{(o)},z)] \right\} = E_z \left\{ E[X(t)|r^{(t)}(d^{(t)},z)] \right\} .
\]

Hence,

\[
E_z \left\{ s^*-t|d^{(o)} \right\} = E_z \left\{ s^*-t|d^{(t)} \right\} .
\]

Therefore,

\[
E_z \left\{ s^*|d^{(o)} \right\} = E[s^*|d^{(t)}] .
\]

Thus, the desired result now follows directly from the definition of \( N(d,s,z) \).

To motivate how Lemmas 4 and 5 will be used, consider a sampling rule \( d \) and a small positive integer \( t \) such that \( t < s^*(r^{(s^*)}(d^{(t)},z)) \), almost surely. Compare the consequences if some inspections are conducted by using \( d^{(t)} \) and if no inspections are allowed by using \( d^{(o)} \). Then, from Lemma 4,

\[
E_z \left\{ C(d^{(o)},s^*,z) - C(d^{(t)},s^*,z) \right\} = \frac{C^D}{2\theta_{\Delta P}} \left\{ \text{Var}[X(t)|d^{(o)}] - \text{Var}[X(t)|d^{(t)}] \right\} \]

\[
- E_z \left\{ \sum_{n=1}^{t} d_n(z)[C_I - C^D z_n] \right\} .
\]
is an excellent approximation. The only error introduced is that the third term on the right side of the equation of Lemma 4 is ignored. However, this term would tend to cancel out in the above equation where it would enter twice with different signs. Furthermore, when \( t \) is small so that \( \Pr(X(t) > \mu) \) is small, this term is quite negligible.

From Lemma 5,

\[
E_z \left[ N(d^{(o)}, s^*, z) - N(d^{(t)}, s^*, z) \right] = E_z \left\{ \sum_{n=1}^{t} d_n(z) \right\}.
\]

Therefore, since it has been shown that

\[
A(d^{(o)}, s^*) = C_D^P,
\]

it is easily seen that

\[
A(d^{(o)}, s^*) > A(d^{(t)}, s^*)
\]

if and only if

\[
\frac{E_z \left[ C(d^{(o)}, s^*, z) - C(d^{(t)}, s^*, z) \right]}{E_z \left[ N(d^{(o)}, s^*, z) - N(d^{(t)}, s^*, z) \right]} > C_D^P,
\]

which reduces to

\[
\frac{C_D}{2 \Delta \mu} E_z \left[ \text{Var}(X(t)|d^{(o)}) - \text{Var}(X(t)|d^{(t)}) \right] > E_z \left\{ \sum_{n=1}^{t} d_n(z) \cdot \left[ C_D^P + (C_I^o - C_D^P) \right] \right\}.
\]
or, equivalently,

\[
\frac{C_D}{2\theta \mu \Delta P} E_z \left[ \text{Var}(X(t) | \text{d}^{(0)}) - \text{Var}(X(t) | \text{d}^{(t)}) \right]
\]

\[> C_D E_z \left\{ \sum_{n=1}^{t} d_n(z)(P-z_n) \right\} + C_I E_z \left\{ \sum_{n=1}^{t} d_n(z) \right\} .
\]

In words, the sampling rule \( \text{d}^{(t)} \) should be preferred to no inspection at all, \( \text{d}^{(0)} \), if and only if the expected reduction in the a posteriori variance of \( X(t) \) is sufficiently large relative to the expected cost of the inspections and the difference between the average cost per item, \( C_D \), and the expected savings from removing defective items from the process by inspecting them.

Furthermore, the very same analysis applies for comparing any two sampling rules, \( \text{d}' \) and \( \text{d}'' \), requiring no inspection after the \( t^{th} \) item, assuming only that the approximation given earlier that \( A(\text{d}, s^*) = C_D \) is accepted as a reference point. Thus, if \( \text{d}' \) involves the lesser expected number of inspections, \( \text{d}'' \) is preferred over \( \text{d}' \), i.e., \( A(\text{d}', s^*) > A(\text{d}'', s^*) \), if and only if

\[
\frac{C_D}{2\theta \mu \Delta P} E_z \left[ \text{Var}(X(t) | \text{d}') - \text{Var}(X(t) | \text{d}'') \right]
\]

\[> C_D E_z \left\{ \sum_{n=1}^{t} (d''(z) - d'(z))(P-z_n) \right\} + C_I E_z \left\{ \sum_{n=1}^{t} (d''(z) - d'(z)) \right\} .
\]
The next logical step is to drop the requirement on the smallness of $t$. An examination of Lemma 3 indicates that $[\text{Var}(X(t+\Delta t)) - \text{Var}(X(t))]$ is independent of $\text{Var}(X(t))$, assuming that $d_n = 0$ if $t \leq n \leq t + \Delta t$. Therefore, the reduction in $\text{Var}(X(t))$ achieved by an inspection is a permanent reduction. Thus, the expected savings to be achieved by a given sampling program is proportional to the expected total of the reductions in $\text{Var}(X(t))$, as $t$ increases, attributable to the inspections conducted. The expected savings would be compared to the expected cost of inspection plus the expected extra cost of replacing inspected items. Although this extension of the results of Lemma 4 and 5 is not exact, the approximation appears to be acceptable.

As a specific illustration, consider the basic case where no more than one additional inspection is allowed. Then the $t^{th}$ item should be inspected only if

$$\frac{C_D}{2\theta P_{\Delta p}} \left[ \text{Var}(X(t)) - \text{E}_{z_t} \left[ \text{Var}(X(t)|z_t) \right] \right]$$

$$> C_D [P - \text{E}(X(t))] + C_I.$$

The one question left unanswered is, if this condition holds for a number of items, which one should preferably be inspected. This question will be answered in an arbitrary manner for the following general case.

Assume that there are no restrictions on the sampling rule. Define $d^*$ as that sampling rule such that $d^* = 1$ is and only if
\[
\frac{C_D}{2\pi \mu L} \left[ \text{Var}(X(t) | r(t-1)(d^*, z)) - \mathbb{E}_{Z_t} [\text{Var}(X(t) | Z_t, r(t-1)(d^*, z))] \right] > C_D [P - \text{E}(X(t) | r(t-1)(d^*, z))] + C_I.
\]

Therefore, referring to Lemma 3 and to the analysis following the proof of Lemma 5, it is seen that

\[A(d^{(o)}, s^*) > A(d^{*}(t), s^*)\]

is at least a good approximation for any value of \( t \), so that

\[A(d^{(o)}, s^*) > A(d^*, s^*) .\]

Thus, \( d^* \) is a reasonable sampling rule which is at least as good as not inspecting at all, \( d^{(o)} \). It provides for inspecting as soon as it can be justified economically on the basis of the input information. As explained following the proof of Lemma 1, it would seem preferable, from an heuristic viewpoint, to obtain new information on the process early and proceed accordingly rather than postpone inspection. Furthermore, inspecting the maximum amount that the input information can justify gives some protection against inaccuracies in this information. Finally, if \( d^* \) specifies that no inspections should be conducted between process adjustments, it is then certain that this is the optimal policy. Therefore, granted that the analysis has not been sufficiently powerful to identify the optimum sampling rule for the general
case, $d^*$ appears to be the most reasonable choice. Not only does it have the appealing features mentioned above, but its expected average cost is even less than the expected average cost of not inspecting.

Since destructive testing makes inspection costly, the expected average cost of not inspecting, $C_D$, probably is itself close to the minimum attainable expected average cost per outgoing item, $\inf_{d \in D} \{A(d,s^*)\}$.

Before applying the procedure $(d^*, s^*)$, it is necessary to evaluate

$$\text{Var}(X(t) | r^{(t-1)}(d^*, z)) - \mathbb{E}_{\mathbb{Z}_t} \{\text{Var}(X(t) | r^{(t)}(d^*, z))\},$$

assuming that $d^*_t(z) = 1$. This is done in the following lemma.

**Lemma 6.** Assume that $d^*_t(z) = 1$. Let $\alpha$ and $\beta$ be such that

$$f_{X(t)}(x | r^{(t-1)}(d^*, z)) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1) \Gamma(\beta+1)} x^{\alpha}(1-x)^{\beta}, \quad 0 \leq x \leq 1,$$

so that

$$\mathbb{E}(X(t) | r^{(t-1)}(d^*, z)) = \frac{\alpha+1}{\alpha+\beta+2},$$

$$\text{Var}(X(t) | r^{(t-1)}(d^*, z)) = \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2 (\alpha+\beta+3)}.$$

Then
$$\text{Var}(X(t)|r(t-1)(d^*, z)) - \mathbb{E}_{z_t}[\text{Var}(X(t)|r(t)(d^*, z))]$$

$$= \frac{[\text{Var}(X(t)|r(t-1)(d^*, z))]^2}{\mathbb{E}(X(t)|r(t-1)(d^*, z))[1-\mathbb{E}(X(t)|r(t-1)(d^*, z))]}$$

**Proof:** Since

$$\mathbb{E}_{z_t}[\text{Var}(X(t)|r(t)(d^*, z))] = P(z_t=1) \text{Var}(X(t)|r(t-1)(d^*, z), z_t = 1)$$

$$+ P(z_t = 0) \text{Var}(X(t)|r(t-1)(d^*, z), z_t = 0)$$

$$= \mathbb{E}(X(t)|r(t-1)(d^*, z)) \text{Var}(X(t)|r(t-1)(d^*, z), z_t = 1)$$

$$+ \{1-\mathbb{E}(X(t)|r(t-1)(d^*, z))\} \text{Var}(X(t)|r(t-1)(d^*, z), z_t = 0),$$

Lemma 2 indicates that

$$\mathbb{E}_{z_t}[\text{Var}(X(t)|r(t)(d^*, z))] = \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)^2(\alpha+\beta+4)} \left[ (\alpha+2) + (\beta+2) \right]$$

$$= \frac{\alpha+\beta+2}{\alpha+\beta+3} \text{Var}(X(t)|r(t-1)(d^*, z)).$$

From the proof of Lemma 2, it is seen that
\[
\alpha + \beta + 3 - \frac{E(X(t)|r^{(t-1)}(d^*, z))}{\text{Var}(X(t)|r^{(t-1)}(d^*, z))} [1-E(X(t)|r^{(t-1)}(d^*, z))]
\]

Therefore,

\[
\text{Var}(X(t)|r^{(t-1)}(d^*, z)) - E_{z_t} \{\text{Var}(X(t)|r^{(t)}(d^*, z))\}
= \frac{1}{\alpha + \beta + 3} \text{Var}(X(t)|r^{(t-1)}(d^*, z))
\]

\[
= \frac{[\text{Var}(X(t)|r^{(t-1)}(d^*, z))]}{E(X(t)|r^{(t-1)}(d^*, z))[1-E(X(t)|r^{(t-1)}(d^*, z))]}^2
\]

as was to be proved.

In summary, the recommended sampling rule, \(d^*\), may be defined as that rule such that, for \(t \in \{1, 2, \ldots\}\), \(d^*_t = 1\) if and only if

\[
\frac{C_D}{20\mu \Delta p} \frac{[\text{Var}(X(t)|r^{(t-1)}(d^*, z))]}{E(X(t)|r^{(t-1)}(d^*, z))[1-E(X(t)|r^{(t-1)}(d^*, z))]}^2
\]

\[
> C_D [P-E(X(t)|r^{(t-1)}(d^*, z))] + C_I
\]

Thus, to apply \(d^*\), it would be sufficient to test the above inequality for \(t = 1, 2, \ldots, s^*\). However, in many cases, \(d^*_t = 0\) for \(t = 1, 2, \ldots, s^*\), or at least for \(t = 1, 2, \ldots, l\), where \(l\) is some large integer. Therefore, as a practical consideration, it would be very desirable to be able to identify these cases and avoid the
extensive calculations that would otherwise be required. This provides
the motivation for Lemma 7.

**Lemma 7.** Assume that \( \frac{C_D}{2 \theta \mu \Delta \mu} \left( \frac{(a_1^2)^2}{\mu_1(1-\mu_1)} \right) < C_I \). Let

\[
a = \frac{C_D}{2 \theta \mu \Delta \mu} \theta^2 (\sigma^2 + \mu_1^2)^2 + C_I \theta \mu (\sigma \Delta \mu)^2 ;
\]

\[
b = \frac{C_D \sigma_1^2}{\theta \mu \Delta \mu} \theta (\sigma^2 + \mu_1^2) - C_I \theta \mu (1-\mu_1) ;
\]

\[
c = \frac{C_D}{2 \theta \mu \Delta \mu} (\sigma_1^2)^2 - C_I \mu_1 (1-\mu_1) .
\]

Let

\[
n_1 = -\frac{b + \sqrt{b^2 - 4ac}}{2a}, \quad n_2 = -\frac{b - \sqrt{b^2 - 4ac}}{2a} .
\]

(1) Assume that \( b^2 \geq 4ac \) and that \( n_1 > 0 \) or \( n_2 > 0 \). Let

\[
n = \min \{ n_1, n_2 \} .
\]

Then,

\[
d_t^* = 0, \text{ if } t \leq n + 1 \text{ and } t \leq \sqrt{\frac{2C_A}{C_D \theta \mu \Delta \mu}} .
\]

(2) Assume that neither \( n_1 \) nor \( n_2 \) are positive real numbers. Then,

\[
d_t^* = 0, \text{ if } t \leq \sqrt{\frac{2C_A}{C_D \theta \mu \Delta \mu}} .
\]
Proof: Consider Part (1) first. Assume to the contrary, that there exists a smallest positive integer, $m$, such that

$$d^*_m = 1; \quad m \leq n + 1; \quad m \leq \sqrt{\frac{2C_A}{C_D \theta \mu \Delta p}}.$$ 

It was shown during the analysis of $a^{(0)}$ that, since $m \leq \sqrt{\frac{2C_A}{C_D \theta \mu \Delta p}}$, $E(X(m)) \leq P$.

Therefore,

$$\frac{C_D}{2\theta \mu \Delta p} \left[ \frac{[\text{Var}(X(m)|r^{(m-1)}(d^*,z))]^2}{E(X(m)|r^{(m-1)}(d^*,z))[1-E(X(m)|r^{(m-1)}(d^*,z))]} \right] > C_I.$$

Since $d^*_t = 0$ for $t = 1, 2, \ldots, m-1$,

$$\text{Var}(X(t)|r^{(t-1)}(d^*,z)) = \sigma^2 + (t-1) \theta (\sigma^2 + \mu^2),$$

$$E(X(t)|r^{(t-1)}(d^*,z)) = \mu_1 + (t-1) \theta \mu \Delta p,$$

for $t \leq m$. Thus,

$$g(t) = \frac{C_D}{2\theta \mu \Delta p} \left[ \frac{[\sigma_1^2 + t \theta (\sigma_1^2 + \mu_1^2)]^2}{[\mu_1 + t \theta \mu \Delta p][1-\mu_1 - t \theta \mu \Delta p]} \right]$$

is a continuous function of $t$ such that

$$g(0) < C_I,$$

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by hypothesis, and
\[ g(m-1) > c_{\overline{1}}. \]

Therefore, by the Intermediate Value Theorem, there exists a minimum value \( n^* \), such that \( 0 < n^* < m-1 \) and
\[ g(n^*) = c_{\overline{1}}. \]

With a little algebraic manipulation, this equation reduces to
\[ at^2 + bt + c = 0, \]
where
\[ a = \frac{c_D}{2\theta \mu} \varphi^2 (\varphi^2 + \mu^2 )^2 + c_{\overline{1}} (\theta \mu \Delta \varphi \varphi \varphi \varphi) ; \]
\[ b = \frac{c_D \overline{1}}{\theta \mu} \varphi (\varphi^2 + \mu^2 ) - c_{\overline{1}} \theta \mu \Delta \varphi (1-2\mu_1) ; \]
\[ c = \frac{c_D}{2\theta \mu} (\varphi^2_1)^2 - c_{\overline{1}} \mu_1 (1-\mu_1) . \]

Therefore, if
\[ n^*_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad n^*_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \]

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where $b^2 \geq 4ac$ and $n^*_1 > 0$ or $n^*_2 > 0$,

$$n^* = \min \{ n^*_1, n^*_2 \}.$$

But $n^* = n$, so that $n < m-1 \leq n$, which is, of course, impossible. This contradiction verifies Part (1).

To verify Part (2), it is sufficient to repeat the proof of Part (1), deleting the requirement that $m \leq n + 1$, through the verification that there exists a root, $n^*$, of the equation,

$$at^2 + bt + c = 0,$$

such that $0 < n^* < m-1$. But this is impossible, since the only roots of this equation, $n_1$ and $n_2$, are not positive real numbers. This contradiction verifies Part (2), and thereby completes the proof of the lemma.

In order to fully interpret Lemma 7, it is important to recall the implications if

$$d^*_t = 0 \text{ for } t \leq \sqrt{\frac{2cA}{C_D \theta u \Delta_p}}.$$

Assume that this does describe $d^*$. Then,

$$E[X \left( \sqrt{\frac{2cA}{C_D \theta u \Delta_p}} \right)] = P.$$
as was shown during the analysis of $d^{(0)}$. Therefore, the process would be stopped and adjusted at $t = \sqrt{\frac{2C_A}{\theta \mu_{\Delta p}}}$, so that no inspections would be conducted between adjustments. Hence, $(d^*, s^*) = (d^{(0)}, s^*)$.

The continuous sampling plan recommended by the above analysis will now be summarized in the following section.

4. **Summary of Recommended Procedure**

   Phase 1:
   
   (1) Determine $\mu_1$, $\sigma_1^2$, $\theta$, $\mu_{\Delta p}$, and $\sigma_{\Delta p}^2$, as defined in Section 2. Rather than determining them independently, keep in mind that the a priori distribution of the probability that the $t^{th}$ item since the last adjustment is defective is a Beta distribution with mean, $\mu_t$, and variance, $\sigma_t^2$, given by

   $\mu_t = \mu_1 + (t-1) \theta \mu_{\Delta p}$,

   $\sigma_t^2 = \sigma_1^2 + (t-1) \theta (\sigma_{\Delta p}^2 + \mu_{\Delta p}^2)$.

   (2) Determine $C_I$, $C_D$, and $C_A$, as defined in Section 2.

   (3) Calculate $P = \mu_1 + \left( \sqrt{\frac{2C_A}{C_D \theta \mu_{\Delta p}}} - \frac{1}{2} \right) \theta \mu_{\Delta p}$.
Phase 2:

Calculate \( U = \frac{C_D}{2\theta \mu} \left( \frac{c_1^2}{\mu_1(1-\mu_1)} \right) \).

1. If \( U \geq C_I \), set \([n] = -1\) and go to Phase 3.

2. If \( U < C_I \), calculate

\[
\begin{align*}
n_1 &= -\frac{b + \sqrt{b^2 - 4ac}}{2a}, \\
n_2 &= -\frac{b - \sqrt{b^2 - 4ac}}{2a}, \\
s &= \sqrt{\frac{2CA}{C_D \theta \mu_1}}
\end{align*}
\]

where

\[
\begin{align*}
a &= \frac{C_D}{2\theta \mu} \theta^2 \left( (\frac{c_1}{\Delta p})^2 + (\frac{\mu_1}{\Delta p})^2 \right) + C_I (\theta \mu_1)^2, \\
b &= \frac{C_D \sigma_1^2}{\theta \mu} \theta \left( (\frac{c_1}{\Delta p})^2 + (\frac{\mu_1}{\Delta p})^2 \right) - C_I \theta \mu_1 (1-2\mu_1), \\
c &= \frac{C_D}{2\theta \mu} \left( \frac{c_1}{\Delta p} \right)^2 - C_I \mu_1 (1-\mu_1).
\end{align*}
\]

(a) If neither \( n_1 \) nor \( n_2 \) is a positive real number, never inspect.

Adjust the production process after every \( s \) items. Do not go to Phase 3.

\( \)

Phase 2 is optional. Its purpose is to ascertain the minimum number of items after an adjustment before the first inspection, in order to possibly avoid many or all of the cycles of Phase 3. If desired, one may instead set \([n] = -1\) and go to Phase 3 immediately.
If \( n_1 \) or \( n_2 \) is a positive real number, set \( n = \min \{ n_1, n_2 \} \). Let \([n]\) be the largest integer not exceeding \( n \).

(i) If \([n] + 2 > s\), never inspect. Adjust the production process after every \( s \) items. Do not go to Phase 3.

(ii) If \([n] + 2 \leq s\), go to Phase 3.

Phase 3:

1/ Adjust the production process in an attempt to establish the quality level at an acceptable level, as defined by \( \mu_1 \) and \( \sigma_1^2 \). Do not inspect the first \([n] + 1\) items. Set \( t = [n] + 2 \). Calculate \( \mu_t \) and \( \sigma_t^2 \) as given in Phase 1, Step (1).

2/ Calculate \( V = \frac{(\sigma_t^2)^2}{\mu_t(1-\mu_t)} \cdot \frac{C_D}{2\theta \Delta p} + C_D(\mu_t - P) \).

(a) If \( V > C_1 \), inspect the \( t \)th item after the last process adjustment.

(i) If defective,

\[
\mu_{t+1} = \mu_t + \frac{\sigma_t^2}{\mu_t};
\]

\[
\sigma_{t+1} = \frac{\mu_{t+1}(1-\mu_{t+1})}{\mu_t(1-\mu_t) + \sigma_t^2} \sigma_t^2 + \delta(\sigma_t^2 + \mu_t^2 \Delta p) \Delta p; \]

increase \( \mu_{t+1} \) by \( \theta \mu \Delta p \).

1/ The two exceptions to the following instructions occur when they would set \( \mu_t > 1 \) or \( \sigma_t^2 > \frac{1}{4} \). For these cases, set \( \mu_t = 1 \) or \( \sigma_t^2 = \frac{1}{4} \) instead.

2/ If inspection were non-destructive, the same plan could be used. The one change would be that the \( C_D(\mu_t - P) \) term should be deleted from the definition of \( V \).
(ii) If not defective,

\[ \mu_{t+1} = \mu_t - \frac{\sigma_t^2}{1 - \mu_t} ; \]

\[ \sigma_{t+1}^2 = \frac{\mu_{t+1} (1 - \mu_{t+1})}{\mu_t (1 - \mu_t) + \sigma_t^2} \sigma_t^2 + \theta (\sigma_{\Delta p}^2 + \mu_{\Delta p}^2) ; \]

increase \( \mu_{t+1} \) by \( \theta \mu_{\Delta p} \).

(b) If \( V \leq C_I \), do not inspect the \( t^{th} \) item after the last production process.

Set \( \mu_{t+1} = \mu_t + \theta \mu_{\Delta p} \).

Set \( \sigma_{t+1}^2 = \sigma_t^2 + \theta (\sigma_{\Delta p}^2 + \mu_{\Delta p}^2) \).

(3) Compare \( \mu_{t+1} \) with \( P \).

(a) If \( \mu_{t+1} > P \), go to (1).

(b) If \( \mu_{t+1} \leq P \), increase \( t \) by one and go to (2).

5. Operating Characteristics

5.1 Average Cost:

It was shown in Section 3 that \( A(\hat{d}^*, s^*) \), the expected average cost per outgoing item, is bounded above by \( A(\hat{d}^{(o)}, s^*) = C_D P \). Thus,

\[ A(\hat{d}^*, s^*) \leq C_D P . \]
5.2 Average Interval between Process Adjustments:

It can be concluded from Lemma 5 that \( E_z[\sigma^*(s^*)(d^*, z)] \), the expected number of incoming items between process adjustments, can be approximated by \( E_z[N(d(t), s^*, z)] \). Thus,

\[
E_z[s^*] \approx \sqrt{\frac{2CA}{CD^2\mu_d}}.
\]

5.3 Average Outgoing Quality:

The a priori expected probability of being defective is \( \mu_1 \) for the first item after an adjustment. By Lemma 3, this a priori expected value increases as a linear function of the number of items since the last adjustment. The form of the stopping rule \( s^* \) is such that the a posteriori expected probability of being defective is \( P \) when the process is stopped to be adjusted. Therefore, although rigorous analysis would sometimes indicate that the true value should be a little smaller, it is clear that a rough approximation for AOQ, the asymptotic average outgoing quality (fraction defective) is

\[
AOQ = \frac{\mu_1 + P}{2}.
\]

5.4 Average Fraction Inspected:

The asymptotic average proportion of all incoming items which are inspected, hereafter denoted as AFI, is especially difficult to estimate. However, by making certain simplifications, a rough estimate can be obtained.

Referring to Section 4, it is seen that the amount of inspection is
determined in part by the number of items before the first inspection, and in part by how fast \( V \) increases between inspections relative to how much \( V \) is decreased, on the average, by an inspection. In order to simplify the analysis in investigating changes in \( V \), the item-by-item process will again be approximated by a strictly continuous process for purposes of defining \( V \) as a function of \( \mu_t \) and \( \sigma_t^2 \) for all values of \( t \geq 1 \).

By continuing through the procedure of Section 4 until an inspection is indicated, the number of items before the first inspection is easily determined. Let \( t^* \) denote this number. Next, an average value of \( \frac{dS}{dt} \), where \( \alpha_t^* = 0 \), needs to be determined. The simplification will be made that

\[
\left. \frac{dS}{dt} \right|_{t = x} = K_1, \quad \text{if} \quad t^* + 1 \leq x \leq \gamma^*, \quad \alpha_x^* = 0,
\]

where \( K_1 \) is set equal to the value that \( \frac{dS}{dt} \) would assume at \( t = t^* + 1 \) if inspections were postponed until after the \((t^* + 1)^{th}\) item. Thus,

\[
K_1 = \frac{d}{dy} \left( \frac{C_D}{2\theta\mu_{\Delta \mu}} \left[ \frac{\sigma_1^2 + \theta(\sigma_1^2 + \mu_{\Delta \mu}^2) y^2}{[\mu_1 + \theta\mu_{\Delta \mu} y][1 - \mu_1 - \theta\mu_{\Delta \mu} y]} + C_D^2 (\mu_1 + \theta\mu_{\Delta \mu} y) \right] \right) \bigg|_{y = t^*} = \frac{C_D}{\theta\mu_{\Delta \mu}} \left[ \frac{\sigma_1^2 + \theta(\sigma_1^2 + \mu_{\Delta \mu}^2) t^* x}{[\mu_1 + \theta\mu_{\Delta \mu} t^*][1 - \mu_1 - \theta\mu_{\Delta \mu} t^*]} \right]
\]

\[
- \frac{C_D}{2\theta\mu_{\Delta \mu}} \left[ \theta\mu_{\Delta \mu} - 2\mu_1 \mu_{\Delta \mu} - 2\theta^2 \mu_{\Delta \mu} \frac{t^*}{t^*} [\sigma_1^2 + \theta(\sigma_1^2 + \mu_{\Delta \mu}^2) t^*]^2 \right]
\]

\[
+ C_D \theta\mu_{\Delta \mu}.
\]
Let $V(x)$ denote the value of $V$ for $t = x$. The next simplification to be made is that

$$\lim_{\varepsilon \to 0^+} \frac{[V(x-\varepsilon) - E[V(x+\varepsilon)]]}{\varepsilon} = K_2,$$

if $t^* + 1 \leq x \leq s^*$, $\hat{r}_x = 1$,

where

$$K_2 = \lim_{\varepsilon \to 0^+} \frac{[V(t^* + 1 - \varepsilon) - E[V(t^* + 1 + \varepsilon)]]}{\varepsilon}$$

$$= \frac{C_D}{2\varepsilon \mu \Delta p} \left[ \frac{(\sigma^{2* + 1})^2}{\mu_{t^* + 1}(1 - \mu_{t^* + 1})} - \mu_{t^* + 1} \right]$$

$$\times \frac{\left[\mu_{t^* + 1} + \frac{\sigma_{t^* + 1}^2}{\mu_{t^* + 1}}\right] \left[1 - \mu_{t^* + 1} - \frac{\sigma_{t^* + 1}^2}{\mu_{t^* + 1}}\right]^2}{\left[\mu_{t^* + 1}(1 - \mu_{t^* + 1}) + \sigma_{t^* + 1}^2\right] \left[\mu_{t^* + 1} + \frac{\sigma_{t^* + 1}^2}{\mu_{t^* + 1}}\right] \left[1 - \mu_{t^* + 1} - \frac{\sigma_{t^* + 1}^2}{\mu_{t^* + 1}}\right]}$$

$$- \frac{C_D}{2\varepsilon \mu \Delta p} (1 - \mu_{t^* + 1})$$

$$\times \frac{\left[\mu_{t^* + 1} - \frac{\sigma_{t^* + 1}^2}{1 - \mu_{t^* + 1}}\right] \left[1 - \mu_{t^* + 1} + \frac{\sigma_{t^* + 1}^2}{1 - \mu_{t^* + 1}}\right]^2}{\left[\mu_{t^* + 1}(1 - \mu_{t^* + 1}) + \sigma_{t^* + 1}^2\right] \left[\mu_{t^* + 1} - \frac{\sigma_{t^* + 1}^2}{1 - \mu_{t^* + 1}}\right] \left[1 - \mu_{t^* + 1} + \frac{\sigma_{t^* + 1}^2}{1 - \mu_{t^* + 1}}\right]}.$$
where

\[
\mu_{t^*+1} = \mu_1 + \theta \mu_0 t^*,
\]

\[
\sigma^2_{t^*+1} = \sigma^2_1 + \theta (\sigma^2_0 + \mu_0^2 t^*).
\]

Thus, with the help of Section 5.2, \( t^* \), \( K_1 \), and \( K_2 \) provide the basis for a rough estimate of AFI such as the following,

\[
AFI = \max \left\{ \frac{K_1}{K_2} \left[ \sqrt{\frac{2C_A}{C_D \theta \mu_0 \Delta \rho}} - t^* \right] + \frac{1}{2} \right\}
\]

5.5 Concluding Remarks:

For most existing continuous sampling plans, the operating characteristics (especially AOQ and AFI) provide the basis for selecting the specific input parameters of the plan. It should be recognized that the operating characteristics play a distinctly less significant role for the continuous sampling plan presented here. The parameters of this plan are selected entirely on the basis of more fundamental considerations. The plan is then designed to approximate the procedure which would minimize total expected cost, given these parameters. The resulting operating characteristics are somewhat coincidental in the sense that they are the necessary by-products of a plan designed in this manner. Therefore, while they may be of interest to the user, the operating characteristics should not be viewed as variables to be assigned

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desired values as a basis for specifying the specific plan to be used.

Because of their reduced significance, the estimates of the operating characteristics given above have placed considerable more emphasis on analytical and computational simplicity than on precision. If more precise estimates are desired, it is suggested that simulation should be very suitable for this purpose.

6. Example

A certain company has received a contract from the government for the production of a large number of fuel packages of a certain type. The rockets are to be used for experimental purposes. The contract, which includes an incentive for good quality, stipulates that destructive testing inspections and process adjustments will be conducted according to the continuous sampling plan presented here. The input parameter values are specified in the contract, although the assigned values of the quality parameters are subject to change as data on the quality performance is gathered. The following discussion summarizes the rationale for each of the parameter values.

It was estimated that the incremental cost involved in producing and inspecting a fuel package is about $100,000. If a fuel package is defective, the result is the failure of the experiment involving the rocket. Such a failure creates an imputed cost for the resulting delay in the experimental program. The value of this imputed cost was assigned by deciding how large an expenditure would be justified in order to avoid this delay. The imputed cost of a delay caused by a defective, plus the cost of the rocket and the experiment, is estimated
to be $500,000. Because of the complexity of the production process and the difficulty of diagnosing causes of unacceptable quality, the cost of investigating and making the necessary adjustments to a process producing too many defectives is very high. A major component of this cost is the replacement of inadequate equipment and machinery. Also, an extremely high cost and imputed cost is incurred by the long delay involved in stopping production to investigate and make adjustments. It was therefore decided that the total cost and imputed cost incurred by such a process adjustment would average about $2,500,000. Actually, there was some doubt about the accuracy of the estimates of the absolute magnitude of these costs. However, it was recognized that only the relative magnitude of the costs is relevant for this plan, and there was more confidence in these quantities. Thus, using units of one hundred thousand dollars, the cost parameters were assigned the values, 
\[ C_I = 1, \quad C_D = 5, \quad \text{and} \quad C_A = 25. \]

The quality parameters were somewhat more difficult to estimate. The company has had no previous experience with this particular product. Furthermore, the complexity of the product made the analysis even more difficult. However, by studying the quality performance on similar products and by carefully identifying and evaluating potential causes of defective product, some notion of the potential quality behavior was

\[1/ \] Many continuous sampling situations requiring destructive testing would have an appropriate value of \( C_I \) which is much larger relative to \( C_D \) and \( C_A \) than in this example. The result would be that the recommended amount of inspection would tend to be completely eliminated, or at least considerably less than in this example.
gained. Thus, it was considered quite likely that the proportion defective among the first items after a process adjustment would be less than 0.10. However, it was recognized that there were many bugs which could cause a high proportion defective that might be retained or introduced during the process adjustment. Therefore, considering the relative likelihood of these circumstances, it was decided that the expected value of the probability that the first item is defective is about 0.10. The a priori variance of this probability was then obtained by drawing the probability density function of a Beta distribution (as given near the beginning of Section 2) with mean 0.10 and with various values for the variance. It was already agreed that, with the high degree of uncertainty involved, the variance should be high. It was finally decided that a variance of 0.0064 (standard deviation of 0.08) gave a curve whose spread and shape most accurately reflected the prevailing notion of the relative likelihood of the various probabilities. Thus, the parameter values, \( \mu_1 = 0.10 \), and \( \sigma_1^2 = 0.0064 \), were assigned.

The next job was to predict, in probabilistic terms, how the quality level would change over time. To begin, an attempt was made to predict the situation after ten items and after fifty items. It was decided that the quality level might have deteriorated slightly after ten items, and that the probability of an item being defective should, on the average, about double (increase by 0.10) by the 50th item. Therefore, it was decided that \( \theta \mu_{dp} = 0.002 \) would be appropriate. Recognizing that the split between \( \theta \) and \( \mu_{dp} \) is not extremely critical, the value, \( \theta = 0.10 \) (signifying that some change in the probability of items being defectives occurs, on the average, once every
ten items), was selected. As a result, \( \mu_{\Delta p} = 0.02 \). Finally, it remained to determine \( \sigma^{2}_{\Delta p} \). As an initial approximation, it was assumed that the probability distribution of each change in the probability of an item being defective is a normal distribution with mean \( \mu_{\Delta p} \) and standard deviation \( \sigma_{\Delta p} \). This approximation gave a concrete interpretation of \( \sigma_{\Delta p}^{2} \). Then, an attempt was made to decide upon an a priori value of \( \sigma_{t}^{2} \) for various values of \( t \) by the method used for \( \sigma_{1}^{2} \). The next step was to reconcile these values with the formula given in Section 4 for the a priori value of \( \sigma_{t}^{2} \) as a function of \( \sigma_{\Delta p}^{2} \). This led to a value of \( (\sigma_{t}^{2} - \sigma_{t-1}^{2}) \) of 0.0002, so that \( \sigma_{\Delta p}^{2} = 0.0016 \).

The continuous sampling plan was begun immediately after the start of production of the fuel packages. Following is a summary of the application of the plan, as described in Section 4, until the first process adjustment.

Phase 1:

1. \( \mu_{1} = 0.10, \ \sigma_{1}^{2} = 0.0064, \ \theta = 0.1, \ \mu_{\Delta p} = 0.02, \ \sigma_{\Delta p}^{2} = 0.0016. \)

2. \( C_{I} = 1, \ \ C_{D} = 5, \ \ C_{A} = 25. \)

3. \( F = 0.240. \)

Phase 2:

\( U = 0.569 \)

2. \( n_{1} = 15.812, \ n_{2} = -45.442, \ s = 70.711. \)

\( a = 0.000054, \ b = 0.0016, \ c = -0.0388. \)

2b. \( n = 15.812, \ [n] = 15. \)

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Phase 3:

(1) Set $t = 17$, $\mu_t = 0.132$, $\sigma^2_t = 0.0096$.

(2,3)

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7. Conclusions

The proposed continuous sampling scheme has a number of obvious limitations. It requires considerable a priori information, usually more than the user could supply with a high degree of confidence. Furthermore, this information must be supplied in a form which requires at least a limited degree of statistical knowledge and understanding. Also, the scheme involves calculations after each item which might be considered complex by an untrained inspector. The time required for these calculations necessitates that at least several minutes must intervene between consecutive items. Additionally, if the calculations indicate that the next item should be inspected, the results of this inspection must be available before the disposition of the following item can be considered. It should thus be evident that the proposed continuous sampling plan can not seriously be considered as an adequate substitute for many other available continuous sampling plans for perhaps a majority of the current applications.

On the other hand, in this space age with its astronomical costs for certain products, there are a significant and growing number of very important applications for which the above disadvantages fade into insignificance relative to the inadequacies of other available continuous sampling plans. The characteristics of these applications are those of the typical case where destructive testing is required, although destructive testing is neither a necessary nor sufficient condition to recommend the use of the proposed plan. These characteristics primarily are an extremely high cost of inspection, a high imputed cost of permitting too many defectives to be cleared for use, and a relatively low rate
of production. For such a situation, where the amount of inspection is stringently limited and yet a high degree of quality assurance must be attained, those continuous sampling plans now in use must be described as entirely inadequate. These plans guarantee a high quality level only by providing for the use of extensive and even 100% inspection under certain circumstances. In order to reduce total costs to a reasonable level by simultaneously achieving the objectives of minimal inspection and satisfactory quality assurance, it appears to be mandatory to use considerable a priori information and to increase both the number and caliber of the employees responsible for this phase of the quality assurance program. In short, the proposed continuous sampling plan promises to be eminently better suited for such a situation than other available continuous sampling plans.

There may be room for considerable argument, when considering specific applications, regarding the suitability of some of the details of the proposed model. However, this model should at least serve as a reasonable approximation, considering the inexactitude of the input data, for most of the relevant applications. Furthermore, it appears that the general approach used can be extended to many other models, as well as to applications outside the area of continuous sampling.
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