DURATION OF AN INTERRUPTED COUNTDOWN

BY

SIDNEY I. FIRSTMAN

TECHNICAL REPORT NO. 65
SEPTEMBER 27, 1962

SUPPORTED BY THE ARMY, NAVY AND AIR FORCE UNDER
CONTRACT Nonr-225(53) (NR-042-002)
WITH THE OFFICE OF NAVAL RESEARCH
DURATION OF AN INTERRUPTED COUNTDOWN

by

Sidney I. Firstman

TECHNICAL REPORT NO. 65

September 27, 1962

Supported by the Army, Navy and Air Force under

Contract Nonr-225(53) (NR-042-002)

with the Office of Naval Research

Gerald J. Lieberman, Project Director

Reproduction in Whole or in Part is Permitted for
any Purpose of the United States Government

APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
DURATION OF AN INTERRUPTED COUNTPDOWN

by

Sidney I. Firstman
Stanford University and the RAND Corporation

ABSTRACT

A countdown could be interrupted by several kinds of problems, and, following each interruption, it could continue according to one of several policies; seven such policies are considered in this paper. The problem addressed is that of accounting for the randomly occurring problems and the time required to deal with them when estimating the distribution of interrupted countdown durations. The countdown is considered as an interrupted service process, and expressions are derived from which the moments of the interrupted countdown duration distribution can be obtained. The first two moments are derived explicitly.

1. Introduction

Immediately prior to being launched, a space vehicle (booster and payload) undergoes a sequence of actions, typically called a countdown, during which the numerous vehicle parts and subsystems are energized and checked. The countdown is intended to bring the vehicle from a quiescent condition to a launch-ready condition. During the course of the countdown, problems could arise; e.g., vehicle or ground equipment could fail or be found inoperative, errors could be made by launch system personnel,
or range difficulties could occur. Each of these problems could require
time, either to repair or correct an equipment condition, to undo the
results of an error, or just to wait until the range problem has ended.

For several types of space missions, the time of launch must be
restricted to within several minutes or several hours of a target or
preferred time. For these situations some knowledge of the distribution
of possible (not-time-constrained) countdown durations would be useful
for planning the launch exercise, so as to achieve a high likelihood of
completing the countdown in a timely manner. Once sufficient experience
has been gained with a launch vehicle this experience can be used to
plan the starting time for launch exercises. Prior to that time, pre-
dictions would be needed, based largely on countdown performance data
from other vehicles. These same problems of countdown duration esti-
mation and prediction could arise when planning the use of range facili-
ties, or when estimating how long a new vehicle's countdown would require
while evaluating various countdown procedures for compatibility with the
vehicle's mission. Typical questions that could be posed during such
planning exercises are: What is the expected countdown duration? What
is the probability of completing the countdown in less than a given
length of time? Or, what is the probability of completing the countdown
within a given interval of time?

If no problems were to arise, the countdown would be followed, as
designed, and would take some (statistically) predictable amount of time.
Problems can occur, however, and depending on what shall be called the
recovery policy, there are numerous ways a process could proceed follow-
ing the occurrence of a problem. At one extreme the policy could call for
the process to resume (where it left off) after the problem is cleared. At the other extreme the policy could require a return to the beginning of the sequence after the problem had been cleared. In general, each policy could result in a different measure of launch effectiveness; but that problem is beyond the scope of this paper. What is of interest here is that each policy could also result in a different distribution of countdown duration. The problem addressed in this paper is that of making allowance for the randomly occurring problems and the time required to deal with them* when estimating the countdown duration, so that questions such as those posed above can be answered.

In general, the countdown is treated as a service process that is interrupted in a random manner. The uninterrupted countdown duration is taken as given--by a distribution function, and expressions are derived from which the moments of the interrupted countdown duration distribution can be obtained. The first two moments are derived explicitly. Seven plausible recovery policies are examined, and they will be described in the context of the analysis, as will the data needed to use the models developed.** For two of the basic policies, results which are almost identical to some of Gaver [1] are re-derived in a launch process context. (using essentially the same method as Gaver)

* Some problems could force the countdown to terminate, but this analysis is conditioned upon the event that these problems do not occur. While a terminated countdown would probably be repeated, this would usually not be done for several days, and it would then be a separate activity. The interest of this paper is on contiguous countdown activities.

** The models are designed to use mainly data that is readily available from countdown records for vehicles with some launch experience; these must, of course, be projected for new vehicles.
Although expressed in terms of a launch process, several of the results appear applicable to other problems, such as that of estimating the completion time of a primary job when it is subject to priority interruptions by secondary jobs that are generated by, or during, the primary job. That was a problem addressed by Gaver [1], and this paper extends and generalizes some of his results in this problem area.

2. **Launch Duration Models**

The types of problems that can arise during the countdown are arbitrarily divided into two major groups: those from which the process can recover and continue, and those which force the countdown to terminate. The processes that generate the two kinds of problems are assumed to be statistically independent, and the following analysis is conditioned upon the event that the termination-causing problems do not occur.

**Case 1: Always Resume Process.** This basic case represents a lower bound for the interrupted process time. The process to be investigated is this: The countdown begins and progresses normally until a problem occurs. If the process can recover from the problem, then, following the interruption, regardless of what caused it and what actions it entailed, the recovery policy calls for the process to resume where it left off (i.e., from the exact point the process had reached prior to the interruption). The activities are continued until the process culminates in a launch attempt.

The interrupted process appears schematically as

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
x_1 & d_1 & x_2 & d_2 & \ldots & d_N & x_{N+1} \\
\hline
\text{start} & \quad & \quad & \quad & \quad & \quad & \text{launch} \\
\hline
\end{array}
\]

\[\text{time,}\]
where the notation means:

\( x_i \) is the \( i \)-th interval of the process,
\( d_i \) is the \( i \)-th delay (interruption).

According to the process description, \( \sum_{i=1}^{N+1} x_i = X \), the overall required (uninterrupted) countdown time. Note that \( X \) is a random variable because the process is a collection of man-machine activities which have inherent stochastic variability.

The lengths of the delays, \( d_i, \ i = 1,2,\ldots,N \), are considered to be random variables which are non-negative, independent, and identically distributed according to some arbitrary distribution function, \( D(t) \). This assumption is based on the observation that many kinds of interruptions can occur (for this case, all delays are grouped), and many of the causes would necessitate manual activity with its inherent variability.

For the process as described, given that a catastrophic problem does not occur, a countdown completion time, \( C \), is given by

\[
C = \sum_{i=1}^{N+1} x_i + \sum_{i=1}^N d_i
\]

\[= X + \sum_{i=1}^N d_i \quad .\]

(1)

Following the method of Gaver [1], taking conditional expectations gives

\[
E(e^{-sC|A,X,N}) = E(e^{-s\{X + \sum_{i=1}^N d_i\} } )
\]

\[= e^{-sX} E(e^{-s \sum_{i=1}^N d_i} ) \]

\[= e^{-sX} [d(s)]^N \quad ,\]

(2)

where \( d(s) \) is the Laplace-Stieltjes Transform (LST) of \( D(t) \), the delay
distribution function, and \( A \) is the event "no catastrophic problem occurs."

To remove the condition on \( N \), the number of delays, observe that the process is one that can be delayed by numerous causes, and that the type and length of each delay and the time in the process that it occurs are considered to be independent of each other. Moreover, the delays typically occur randomly during the process, such that the probability of a delay occurring in a small interval of time appears largely independent of the elapsed process time; this probability is assumed constant over the entire process. If the probability of more than one delay-causing incident occurring in the same small interval of time is assumed to be negligibly small, and this appears reasonable, then the Poisson process is the appropriate delay-arrival model. This means that

\[
E(e^{-sC} | A, X) = e^{-sX} \sum_{n=0}^{\infty} \frac{e^{-\lambda X} \lambda^n X^n}{n!} [d(s)]^n,
\]

where \( \lambda \) is the rate at which delays "arrive."*

The summation term is seen to be the generating function of the compound Poisson distribution which is equal to

\[
e^{-\lambda X + \lambda X d(s)},
\]

so that

\[
E(e^{-sC} | A, X) = e^{-sX} e^{-\lambda X (1 - d(s))}
\]

\[
= e^{-X[s + \lambda (1 - d(s))]}.
\]

* One can lessen some restrictions imposed by these assumptions and others to be made during subsequent cases by defining countdown "milestones," as is usually done. Then, if a different \( \lambda \) appears appropriate for different parts of the countdown, the uninterrupted process time can be divided into the appropriate segments and the analysis would be identical for each segment. This topic will be considered further in the concluding section.

Note, too, that the delay arrival distribution is based on the uninterrupted process time \( X \), and not "running" clock time.
Removing the condition on $X$ yields

\[ E(e^{-sC}|A) = \int_0^\infty e^{-X[s + \lambda(1-d(s))]} dV(X|A), \]

where $V(X|A)$ is the distribution function for $X$ conditional upon $A$, i.e., given that the countdown is not forced to terminate. This conditional distribution is the one that could normally be obtained from engineering estimates, controlled test programs or records of previous countdowns. Continuity conditions are satisfied, so that differentiating with respect to $s$ and evaluating the resulting function at $s = 0$ will yield $E(-C|A)$ and an expression for the expectation which contains $d'(0) = E(-d)$.

\[ E(-Ce^{-sC}|A) = \int_0^\infty -X[1 - \lambda d'(s)] e^{-X[s + \lambda(1-d(s))]} dV(X|A) \]

\[ E(-C|A) = \int_0^\infty -X[1 - \lambda d'(0)] dV(X|A), \]

so that

\[ E(C|A) = [1 + \lambda E(d)] E(X|A), \]

where $E(X|A)$ is the expected length of an uninterrupted, unterminated countdown.

Taking $\frac{d^2}{ds^2}$ of Eq. 5 and evaluating the result at $s = 0$ gives

\[ E(C^2|A) = \lambda E(d^2) E(X|A) + [1 + \lambda E(d)]^2 E(X^2|A). \]

These are the desired results and, except for the conditioning on $A$, they are identical to some results of Gaver [1]. Higher moments follow directly.
Case 1a: Test before Resume. Prior to continuing the countdown after a delay, and especially one that involved some repair activity, it is often appropriate to check the condition of the vehicle and/or ground system, and it is sometimes necessary to repeat some previous countdown steps, i.e., to "backtrack." A recovery policy could dictate a check of the equipment functionally related to that which was worked on, or a cursory check of all subsystems that had been previously checked and approved. As before, while the nature and extent of the testing or backtracking done to recover from a delay could affect the launch success such that the recovery policy used should reflect that problem, the interest of this paper is on the impact of the recovery policy on the countdown duration. For this case, the recovery policy dictates some recovery action of an arbitrary nature, all recovery actions are grouped, and the recovery time required after the i-th delay is called $r_i$, which for $i = 1, 2, \ldots, N$ are assumed to be non-negative, independent and identically distributed random variables. This assumption follows from the general nature of recovery activities, which are primarily tests performed by man-machine combinations. The $d_i$ and $r_i$ are also assumed to be independent, and this follows from the observation that the acts of repairing a subsystem and testing after the repairs are completed are most often physically unrelated in terms of effort and technique. If a recovery time, $r_i$, where some of the $r_i$ could be of zero duration, follows each delay $d_i$, and then the process resumes where it had left off prior to the delay, the process appears schematically as
Given that it is not forced to terminate, a countdown completion time is given by:

\[
C = \sum_{i=1}^{N+1} x_i + \sum_{i=1}^{N} d_i + \sum_{i=1}^{N} r_i
\]

\[
= X + \sum_{i=1}^{N} (d_i + r_i)
\]

For ease of derivation, let \( z_i = d_i + r_i \), \( i = 1, 2, \ldots, N \). Then the \( z_i \) are non-negative, independent and identically distributed random variables with the properties:

a) \( E(z) = E(d) + E(r) \)

b) \( z(s) = d(s) \cdot r(s) \), where \( z(s) \) is the LST of the distribution of \( z \).

Equation 10 then becomes

\[
C = X + \sum_{i=1}^{N} z_i
\]

which is seen to be identical in form and properties with Eq. 1, so that results can be written directly. From Eq. 8,

\[
E(C|A) = (1 + \lambda E(z)) E(X|A)
\]

so that

* For application to the problem of estimating the completion time of a job subject to priority interruptions from secondary or generated jobs, the \( r_i \) could be the time required to set up again after completing the secondary job in time \( d_i \).
(13) \[ E(C|A) = [1 + \lambda \left( E(d) + E(r) \right)] E(X|A), \]

which is the expected process time when a recovery action (e.g., backtracking and testing) follows each delay. From Eq. 9,

\[ E(C^2|A) = \lambda \left( E(z^2) \right) E(X|A) + [1 + \lambda E(z)]^2 E(X^2|A), \]

so that

\[ E(C^2|A) = \lambda \left( E(d^2) + 2E(d)E(r) + E(r^2) \right) E(X|A) + [1 + \lambda \left( E(d) + E(r) \right)]^2 E(X^2|A). \]

**Case 1b: Test before Resume; Separated Causes.** Instead of grouping all the problems, delays and recovery acts, they can also be treated separately, if the available data allow such a detailed breakdown. For this case the process and the recovery policy are the same as for Case la, except that the several kinds of problems (e.g., vehicle equipment failures, range problems) and their effects are considered separately. Let

\[ d_{ij} = i\text{-th delay due to cause } j; \quad j = 1, 2, \ldots, M \]
\[ r_{ij} = i\text{-th recovery time due to cause } j \]
\[ \lambda_j = \text{rate at which type } j \text{ delays arrive} \]

where the assumption is made that each type of delay (problem) arrives independently according to a Poisson process so that \( \sum_{j=1}^{n} \lambda_j = \lambda \), the parameter of the grouped (Poisson) arrival distribution. \(^*\) For equipment-centered problems, this assumption is justified by observing that the preponderance of vehicle and ground equipment is electronic in nature, and this equipment has failure characteristics that have exponential or quasi-exponential properties.

If it is not forced to terminate, a countdown completion time is given by

\(^*\) Note that each of these processes is also assumed to be independent of the process by which termination-causing problems arrive.
\[ C = \sum_{j=1}^{N^J} x_j + \sum_{j=1}^{N^J} \sum_{i=1}^{M} d_{ij} + \sum_{j=1}^{N^J} \sum_{i=1}^{M} r_{ij} \]

(16)

\[ = X + \sum_{j=1}^{N^J} \sum_{i=1}^{M} (d_{ij} + r_{ij}) \cdot \]

Taking conditional expectations,

\[ \mathbb{E}(e^{-sC} | A, X, N_1, N_2, \ldots, N_M) = e^{-sX} \mathbb{E}(e^{-s\sum_{j=1}^{N^J} \sum_{i=1}^{M} (d_{ij} + r_{ij})}) \]

(17)

\[ = e^{-sX} e^{-s\sum_{j=1}^{N^J} \sum_{i=1}^{M} (d_{ij} + r_{ij})} \]

\[ = e^{-sX} \prod_{j=1}^{N^J} [d_j(s) r_j(s)]^{N^J} \]

where the conditions can be removed sequentially.

A case of interest is for \( M = 2 \), i.e., (1) launch and vehicle system problems and (2) range and other problems. For this case,

(18) \[ \mathbb{E}(e^{-sC} | A, X, N_1, N_2) = e^{-sX} [d_1(s) r_1(s)]^{N_1} [d_2(s) r_2(s)]^{N_2}. \]

Since the delays arrive independently according to Poisson distributions, the conditions on \( N_1 \) and \( N_2 \) can be removed as before (Eqs. 3 and 4), using Poisson distributions with parameters \( \lambda_1 X \) and \( \lambda_2 X \), respectively. This reduction yields, successively,

(19) \[ \mathbb{E}(e^{-sC} | A, X, N_2) = e^{-sX} [d_2(s) r_2(s)]^{N_2} e^{-\lambda_1 X + \lambda_2 X d_1(s) r_1(s)} \]

(20) \[ \mathbb{E}(e^{-sC} | A, X) = e^{-sX} e^{-\lambda_1 X + \lambda_1 X d_1(s) r_1(s)} e^{-\lambda_2 X + \lambda_2 X d_2(s) r_2(s)} \]
\[ = \exp\left[-X\{s + \lambda_1[1-d_1(s) r_1(s)] + \lambda_2[1-d_2(s) r_2(s)]\}\right].\]

Using \(V(X|A)\) to remove the condition on \(X\) gives the expression from which the moments can be obtained:

(21) \[E(e^{-SC}|A) = \int_0^\infty \exp\left[-X\{s+\lambda_1[1-d_1(s) r_1(s)]+\lambda_2[1-d_2(s) r_2(s)]\}\right]dV(X|A).\]

Taking \(\partial/\partial s\) and \(\partial^2/\partial s^2\), and evaluating the results at \(s = 0\), gives

(22) \[E(C|A) = \left[1 + \lambda_1[E(d_1) + E(r_1)] + \lambda_2[E(d_2) + E(r_2)]\right] E(X|A)\]

and

(23) \[E(C^2|A) = \left[\lambda_1 E(z_1^2) + \lambda_2 E(z_2^2)\right] E(X|A) + \left[1 + \lambda_1 E(z_1) + \lambda_2 E(z_2)\right]^2 E(X^2|A)\]

where

\[E(z_j) = E(d_j) + E(r_j)\]

(24) \[E(z_j^2) = E(d_j^2) + 2E(d_j) E(r_j) + E(r_j^2).\]

**Case 2: Always Repeat Process.** The recovery policy upon which this case is based appears unrealistic for a general pre-launch process; possibly it is descriptive of certain special-case processes. Its use to describe part of a process will be discussed in the concluding section. The process to be investigated is one in which the countdown begins and progresses normally until the first interruption. Following the interruption, which for this case includes the time required to return the vehicle to its starting condition, the recovery policy dictates that the countdown must start over again. The process ends when the entire countdown is completed without a delay occurring. As before, the delay times are assumed to be independent, identically and arbitrarily distributed random variables. As
a consequence of the Poisson arrival process, the time between delays is exponentially distributed with parameter $\lambda$, which means that each interrupted process has an exponentially distributed length which is independent of other partial-process lengths.

The interrupted sequence appears schematically as

\[
\begin{array}{cccccccc}
\text{start} & x_1^i & d_1 & x_2^i & d_2 & \cdots & x_N^i & d_N & X \\
\text{launch} & & & & & & & & \\
\end{array}
\Rightarrow \text{time },
\]

so that, given that it is not forced to terminate, a process completion time $C$ is given by

\[
(25) \quad C = X + \sum_{i=1}^{N} x_i^i + \sum_{i=1}^{N} d_i,
\]

where $x_i^i$ is the length of the process completed prior to an interruption.

As before, following Gaver [1], taking conditional expectations gives

\[
E(e^{-sC|A,X,N}) = E(e^{-s\left\{X + \sum_{i=1}^{N} x_i^i + \sum_{i=1}^{N} d_i\right\}})
\]

\[
(26) \quad = e^{-sX} E(e^{-s \left\{ \sum_{i=1}^{N} x_i^i \right\}}) E(e^{-s \left\{ \sum_{i=1}^{N} d_i \right\}})
\]

as the $\Sigma x_i^i$ and $\Sigma d_i$ are independent.

As before,

\[
(27) \quad E(e^{-s \Sigma d_i}) = [d(s)]^N .
\]

To evaluate $E(e^{-s \Sigma x_i^i})$, note that for fixed $X$, and the $x_i^i$ assumed to be truncations of the given $X$,

\[
(28) \quad P(x_i^i) = P(x_i^i \leq t| x_i^i < X) = \frac{1-e^{-\lambda t}}{1-e^{-\lambda X}}, \quad \text{for} \quad 0 \leq t \leq X ,
\]

13
so that

\[ E(e^{-sx_1'}) = \int_0^X e^{-sx_1'} \, dP(x_1') \]

(29)

\[ = \int_0^X e^{-st} \, d\left[ \frac{1-e^{-\lambda t}}{1-e^{-\lambda X}} \right] , \]

which gives

\[ E(e^{-sx_1'}) = \frac{\lambda}{1-e^{-\lambda X}} \int_0^X e^{-t(\lambda+s)} \, dt \]

(30)

\[ = \frac{\lambda}{\lambda+s} \left[ \frac{1-e^{-X(\lambda+s)}}{1-e^{-\lambda X}} \right] . \]

Using this expression for the expectation, and noting that the \( x_1' \) are independent, identically distributed random variables, gives

(31)

\[ E(e^{-SC|A,X,N}) = e^{-sX} \left[ \frac{\lambda}{\lambda+s} \right]^N \left[ \frac{1-e^{-X(\lambda+s)}}{1-e^{-\lambda X}} \right]^N \left[ d(s) \right]^N . \]

Conditional upon a fixed \( X \), the distribution of the number of partial-processes (interruptions) that occur before a process is completed is given by a geometric distribution:

(32)

\[ P(N=n|X) = (1-e^{-\lambda X})^n e^{-\lambda X} . \]

Using this distribution to remove the condition on \( N \) yields

\[ E(e^{-SC|A,X}) = \sum_{n=0}^{\infty} e^{-sX} \left[ \frac{\lambda}{\lambda+s} \right]^n \left[ \frac{1-e^{-X(\lambda+s)}}{1-e^{-\lambda X}} \right]^n [d(s)]^n [e^{-\lambda X} (1-e^{-\lambda X})^n] \]

(33)

\[ = e^{-X(s+\lambda)} \sum_{n=0}^{\infty} \left[ \frac{\lambda}{\lambda+s} d(s) (1-e^{-X(\lambda+s)}) \right]^n . \]

Note that the term being summed is \( > 0 \) and \( < 1 \), so that the series is a geometric series, and therefore
\[(34) \quad E(e^{-SC}|A, X) = \frac{e^{-X(s+\lambda)}}{1 - \frac{\lambda}{\lambda+s} d(s)(1-e^{-X(\lambda+s)})} . \]

Removing the condition on \( X \) gives

\[(35) \quad E(e^{-SC}|A) = \int_0^\infty \frac{e^{-X(s+\lambda)}}{1 - \frac{\lambda}{\lambda+s} d(s)(1-e^{-X(\lambda+s)})} \, dV(X|A) . \]

The function to be integrated is continuous and, as will be seen, its partial derivative with respect to \( s \) is also continuous, so that differentiation with respect to \( s \) yields

\[
E(-Ce^{-SC}|A) = \int_0^\infty \left\{ \begin{array}{l}
-Xe^{-X(s+\lambda)}(1 - \frac{\lambda}{\lambda+s} d(s)(1-e^{-X(\lambda+s)})) \\
+ (e^{-X(s+\lambda)}) \left[ - \frac{\lambda}{(\lambda+s)^2} \{d(s)(1-e^{-X(\lambda+s)})\} \right] \\
+ d(s) \left\{ \frac{\lambda}{\lambda+s} (1-e^{-X(\lambda+s)}) \right\} \\
+ \frac{\lambda}{\lambda+s} d(s) \{Xe^{-X(\lambda+s)}\} \\
\end{array} \right. \, dV(X|A) . \]

Evaluation at \( s = 0 \) and simplification yields

\[(37) \quad E(C|A) = (E(\dot{d}) + \frac{1}{\lambda}) \int_0^\infty (e^{\lambda X} - 1) \, dV(X|A) , \]

so that

\[(38) \quad E(C|A) = [E(\dot{d}) + \frac{1}{\lambda}] \left[ E(e^{\lambda X}|A) - 1 \right] , \]

which is the expected countdown duration when the process operates.
according to the always-repeat policy. Similarly, the second derivative
of Eq. 35 evaluated at \( s = 0 \) yields

\[
E(d^2 | A) = 2[E(d) + \frac{1}{\lambda}]^2 \cdot E(e^{\lambda X} - 1)^2 | A) + [E(d^2) + \frac{2E(d)}{\lambda} + \frac{2}{\lambda^2}][E(e^{\lambda X} | A) - 1]
\]

(39)

\[-2[E(d) + \frac{1}{\lambda}] \cdot E(e^{\lambda X} | A).\]

This is the second set of results that, except for the conditioning
on \( A \), are identical to some previously derived by Gaver [1].

**Case 3: Repeat Once if Necessary; Resume Otherwise.** For this case,
both classes of interruptions from which the process can recover are
considered together; the "normal" problems, failures, and delays, and
extraordinary problems or failures (i.e., highly disruptive or exceptionally
troublesome problems) which are arbitrarily singled out for special treat-
ment during the process. The recovery policy for this process is that it
is to resume where it left off after each normal delay until the first
extraordinary problem occurs. Then the process is to repeat, i.e., start
over again, but just once (this single-return restriction is removed in
Case 4). After the first (if any) repeat, the process will continue by
resumption after each delay. The \( d_i \) are the normal delay times, and for
this policy no special actions, such as testing, follow each delay; hence,
there are no \( r_i \) terms (this will be added in Case 3a). In addition, it
is assumed that the extraordinary problem "arrives" according to an
exponential distribution with rate \( \mu \), and the length of the extraordinary
delay is obtained from an arbitrary distribution.

If an extraordinary problem does not occur, the process is the same as
the one considered in Case 1. If an extraordinary problem does occur and
one that requires a termination does not occur, the process appears as

\[
\begin{array}{cccccccccc}
x_1 & d_1 & - & - & - & - & - & - & - & - & d_M & x_{N+1} \\
\text{start} & & & & & & & & & & \text{launch} & \text{time}
\end{array}
\]

and a single countdown time, \( C_b \), is given by

\[
C_b = \sum_{i=1}^{N+1} x_i + \sum_{i=1}^{N} d_i + D + \sum_{j=1}^{M} \tilde{x}_j + \sum_{j=1}^{M} d_j .
\]

Following the delay \( D \) (which includes the time to return the process to the beginning), the process is identical to the one investigated as Case 1 where all problems were handled in the same manner. Let (cf., Eq. 1)

\[
C_a = \sum_{j=1}^{M+1} \tilde{x}_j + \sum_{j=1}^{M} d_j .
\]

Then a single process time, given that an extraordinary problem occurs, is

\[
C_b = C_a + D + \sum_{i=1}^{N+1} x_i + \sum_{i=1}^{N} d_i .
\]

The \( E(C_b) \) will be derived under the condition that an extraordinary problem occurs, and then the expected process time will be derived. For simplicity of derivation, let

\[
\hat{C}_b = \sum_{i=1}^{N+1} x_i + \sum_{i=1}^{N} d_i .
\]

Taking conditional expectations gives

\[
E(e^{-s\hat{C}_b} | A, N, X, Y) = E(e^{-s(Y + \sum_{i=1}^{N} d_i)})
\]

\[
= s^{-sy} [d(s)]^N ,
\]

where
\[ y = \sum_{i=1}^{N+1} x_i . \]

The distribution of \( y \) is given by

\[ P(y) = P[y \leq t | y < x] = \frac{1 - e^{-\mu t}}{1 - e^{-\mu X}}; \quad 0 \leq t \leq x , \]

which follows from the assumption of an exponential arrival distribution for the extraordinary problem.

For a fixed \( y \), the distribution of the number of normal delays is, as before,

\[ P[N = n | y] = e^{-\lambda' y} (\lambda' y)^n \frac{1}{n!} , \]

where \( \lambda' = \lambda - \mu \), if the extraordinary problems had been grouped with the normal problems when estimating \( \lambda \).

Using Eq. 46, the condition on \( N \) can be removed, and this yields

\[ E(e^{-S^A_b} | A, X, y) = e^{-Sy} \sum_{n=0}^{\infty} e^{-\lambda' y} (\lambda' y)^n \frac{1}{n!} [d(s)]^n , \]

Noting that the summation term is the generating function of the compound Poisson distribution gives

\[ E(e^{-S^A_b} | A, X, y) = e^{-Sy} e^{-\lambda' y(1-d(s))} , \]

\[ = e^{-y[s + \lambda'(1-d(s))]} . \]

Equation 45 can now be used to remove the condition on \( y \).

\[ E(e^{-S^A_b} | A, X) = \int_0^X e^{-y[s + \lambda'(1-d(s))]} dP(y) \]

\[ = \int_0^X e^{-y[s + \lambda'(1-d(s))]} \frac{\mu e^{-\mu y}}{1 - e^{-\mu X}} dy \]

18
(49, cont'd)
\[
\begin{align*}
&= \frac{\mu}{1-e^{-\mu X}} \int_0^X (s + \mu + \lambda' (1-d(s))) \, dy \\
&= \frac{\mu}{1-e^{-\mu X}} \cdot \frac{1-e^{-X(s + \mu + \lambda'(1-d(s)))}}{s + \mu + \lambda'(1-d(s))}. 
\end{align*}
\]

Taking the partial derivative with respect to \( s \), and evaluating the result at \( s = 0 \), yields
\[
(50) \quad E(\hat{C}_b | A, X) = [1 + \lambda' E(d)] \cdot \frac{1-e^{-\mu X} \cdot \mu X e^{-\mu X}}{\mu (1-e^{-\mu X})}.
\]

From Eq. 4, it is seen that
\[
(51) \quad E(e^{-sC_a} | A, X) = e^{-X(s + \lambda(1-d(s)))},
\]
where \( \lambda \) is the appropriate parameter because, according to the recovery policy, the process is to resume after each delay. Taking \( \partial / \partial s \) yields
\[
(52) \quad E(C_a e^{-sC_a} | A, X) = e^{-X(s + \lambda(1-d(s)))} \cdot (-X(1-\lambda d'(s))).
\]

Evaluation at \( s = 0 \) gives
\[
(53) \quad E(C_a | A, X) = X(1 + \lambda E(d)).
\]

Under the assumption that when the process is repeated it begins the identical realization of the random variable \( X \) (i.e., the uninterrupted process time), the \( E(C_b | A, X) \) is obtained from Eqs. 42, 50, and 53 as
\[
E(C_b | A, X) = E(C_a | A, X) + E(D) + E(\hat{C}_b | A, X)
(54) \quad = X(1+\lambda E(d)) + E(D) + (1+\lambda' E(d)) \cdot \frac{1-e^{-\mu X} \cdot \mu X e^{-\mu X}}{\mu (1-e^{-\mu X})}.
\]

Equation 54 has been derived under the condition that an extraordinary problem occurs. The probability of this event is, for a given \( X \),
$1-e^{-\mu X}$. The probability that the process is not interrupted by an extraordinary problem is given by $e^{-\mu X}$ for a given $X$. Using these probabilities yields

$$E(C|A,X) = E(C_a|A,X) \cdot e^{-\mu X} + E(C_b|A,X) \cdot (1-e^{-\mu X})$$

$$= X(1+\lambda'\ E(d)) \ e^{-\mu X} + X(1+\lambda\ E(d)) \ (1-e^{-\mu X})$$

$$+ E[D] \cdot (1-e^{-\mu X}) + (1+\lambda'\ E(d)) \ \frac{1-e^{-\mu X}}{\mu} e^{-\mu X}$$

(55)

where $\lambda'$ is the appropriate parameter for the process that is not interrupted by an extraordinary problem, which otherwise is the same as the one investigated in Case 1.

Removing the condition on $X$ by using $V(X|A)$ as the integrator gives the desired result:

$$E(C|A) = E(D) \cdot \left\{1-E(e^{-\mu X}|A)\right\} + \left\{1+\lambda\ E(d)\right\} \ \left\{E(X|A) - E(x e^{-\mu X}|A)\right\}$$

$$+ \left\{1+\lambda'\ E(d)\right\} \ \left\{\frac{1-E(e^{-\mu X}|A)}{\mu}\right\}.$$  

(56)

Higher moments are obtained in a similar manner; for example, for the second moment,

$$E(C^2|A,X) = E(C^2_a|A,X) \cdot e^{-\mu X} + E(C^2_b|A,X) \cdot (1-e^{-\mu X})$$

(57)

Taking $\partial^2/\partial s^2$ of Eq. 4 and evaluating the result at $s = 0$ gives

$$E(C^2_a|A,X) = X \lambda'\ E(d^2) + X^2(1+\lambda'\ E(d))^2$$

(58)

From Eq. 42,

$$C_b = C_a + D + \hat{C}_b$$

(59)

so that

$$E(C^2_b|A,X) = E(C^2_a|A,X) + 2E(D) \cdot E(C_a|A,X) + E(d^2) + 2E(D) \cdot E(C_a|A,X) \cdot E(\hat{C}_b|A,X)$$

$$+ 2E(d) \cdot E(C_b|A,X) + E(C^2_b|A,X).$$

(60)
Each term of Eq. 60 has been derived previously, with the exception of
\[ E(\theta^2|A,X) \] which can be obtained by taking \( \frac{\partial^2}{\partial s^2} \) of Eq. 49 and
evaluating the result at \( s = 0 \).

Substitution of the appropriate expressions into Eq. 51, simplification
of the result, and integration with respect to \( V(X|A) \) yields the
desired result:

\[
E(C^2|A) = \begin{cases}
2\{1+\lambda' \ E(d)\}^2 \cdot \left\{ \frac{1}{\mu} - \frac{E(e^{-\mu X}|A)}{\mu} \right\} - \frac{E(Xe^{-\mu X}|A)}{\mu} \\
+ \left\{ \frac{1}{\mu} - \frac{E(e^{-\mu X}|A)}{\mu} \right\} \\
+ \lambda' \ E(d^2) \cdot \left\{ \frac{E(e^{-\mu X}|A)}{\mu} \right\} \\
+ \lambda \ E(d^2) \cdot \left\{ E(X|A) - E(Xe^{-\mu X}|A) \right\} \\
\end{cases}
\]

(61)

\[
E(C|A) = \begin{cases}
2\{1+\lambda' \ E(d)\} \cdot \left\{ \frac{1}{\mu} - \frac{E(e^{-\mu X}|A)}{\mu} \right\} - \frac{E(Xe^{-\mu X}|A)}{\mu} \\
+ \left\{ \frac{1}{\mu} - \frac{E(e^{-\mu X}|A)}{\mu} \right\} \\
+ \lambda' \ E(d^2) \cdot \left\{ \frac{E(e^{-\mu X}|A)}{\mu} \right\} \\
+ \lambda \ E(d^2) \cdot \left\{ E(X|A) - E(Xe^{-\mu X}|A) \right\} \\
\end{cases}
\]

(62)

\[
E(C|A) = E(D) \left\{ 1 - E(e^{-\mu X}|A) \right\} + \left\{ \frac{1+\lambda \ E(d)\} \frac{E(X|A)}{\mu} - \frac{1-E(e^{-\mu X}|A)}{\mu} - \frac{E(Xe^{-\mu X}|A)}{\mu} \right\}
\]

* Note that, as \( \mu \to 0, \lambda' \to \lambda \) and Eqs. 56 and 61 reduce to Eqs. 8 and 9,
the Case 1 results.

Observe, too, that if the extraordinary problem is considered as one
that can be controlled, i.e., one that can be allowed to happen or be
suppressed according to a recovery or range policy, then \( \mu \) and \( \lambda \) could
be considered as separate in the sense that \( \lambda' = \lambda \) and \( \mu \) is something
superimposed on the process. In this case, the above results can be
reduced. For example, Eq. 56 reduces to

(62)

This latter treatment of the \( \mu \)-type problems could be a close descrip-
tion of some situations.
Case 3a: Repeat Once if Necessary; Test, then Resume Otherwise.

In the same manner as for Case 1a, the recovery policy could call for test or other recovery actions following each "normal" delay. For this case, then, the recovery policy under which the process operates is to test (or otherwise recover) after each delay caused by "normal" problems, and to return to the beginning of the countdown, but only once, if an extraordinary problem occurs.

If an extraordinary problem occurs, the process appears as

\[
\begin{array}{ccccccccc}
x_1 & d_1 & r_1 & \cdots & d_N & r_N & x_{N+1} & \xi_1 & d_1 & r_1 & \cdots & \xi_{M+1} \\
\text{start} & & & & & & & \text{time} & & & & \text{launch}
\end{array}
\]

and, if the process is not forced to terminate, a countdown duration is given by

\[
C_b = \sum_{i=1}^{N} x_i + \sum_{i=1}^{N} d_i + \sum_{i=1}^{N} r_i + D + \sum_{j=1}^{M} \xi_j + \sum_{j=1}^{M} d_j + \sum_{j=1}^{M} r_j .
\]

Let

\[
C_a = \sum_{j=1}^{M+1} \xi_j + \sum_{j=1}^{M} d_j + \sum_{j=1}^{M} r_j
\]

\[
= \sum_{j=1}^{M+1} \xi_j + \sum_{j=1}^{M} z_j ,
\]

and let

\[
\hat{C}_b = \sum_{i=1}^{N+1} x_i + \sum_{i=1}^{N} (d_i + r_i)
\]

\[
= \sum_{i=1}^{N+1} x_i + \sum_{i=1}^{N} z_i ,
\]

where \( z_i = d_i + r_i \), with properties as discussed in Case 1a. Then

\[
C_b = \hat{C}_b + D + C_a .
\]
Under the condition that an extraordinary problem does not occur, the process is the same as that of Case Ia, with parameter \( \lambda' \).

In this form, using \( z_i \), the results can be written down directly from those of Case 3, as the process model is identical in form and properties to that of Case 3. The expressions \( E(z) = E(d) + E(r) \) and \( E(z^2) = E(d^2) + 2E(d)E(r) + E(r^2) \) should be substituted for \( E(d) \) and \( E(d^2) \), respectively, in Eqs. 56 and 61.

**Case 3b:** Repeat Once if Necessary; Test, then Resume Otherwise;

**Separated Causes.** A natural generalization of Case 3a results if, as in Case 1b, the normal problems, delays and recovery acts can be treated according to separate causes. Using the notation introduced in Case 1b, a process completion time (given that an extraordinary problem occurs) is given by

\[
C_b = \sum_{i=1}^{\sum N_j + 1} x_i + \sum_{j=1}^M \sum_{i=1}^{N_j} d_{ij} + \sum_{j=1}^M \sum_{i=1}^{N_j} r_{ij} + D + C_a
\]

(67)

As before, let

\[
\hat{C}_b = y + \sum_{j=1}^M \sum_{i=1}^{N_j} (d_{ij} + r_{ij})
\]

(68)

Taking conditional expectations gives

\[
E(e^{-s\hat{C}_b | A, X, Y, N_1, N_2, \ldots, N_M}) = e^{-sy} E(e^{-s \sum_{j=1}^M \sum_{i=1}^{N_j} (d_{ij} + r_{ij})})
\]

(69)

\[
e^{-sy} \prod_{j=1}^M [d_j(s) \cdot r_j(s)]^{N_j}
\]

where the conditions can be removed sequentially.

For the case \( M = 2 \), previously considered,
\[
E(e^{-s\hat{C}_b} | A, X, y, N_1, N_2) = e^{-sy}[d_1(s)\cdot r_1(s)]^{N_1}[d_2(s)\cdot r_2(s)]^{N_2}.
\]

To remove the condition on \( N_1 \), the Poisson distribution with parameter \( \lambda_1' y \) is used; this gives

\[
E(e^{-s\hat{C}_b} | A, X, y, N_2) = e^{-sy} e^{-\lambda_1'|y[1-d_1(s)\cdot r_1(s)]} [d_2(s)\cdot r_2(s)]^{N_2}.
\]

Similarly, using the Poisson distribution with parameter \( \lambda_2' y \) gives

\[
E(e^{-s\hat{C}_b} | A, X, y) = e^{-sy} e^{-\lambda_1'|y[1-d_1(s)\cdot r_1(s)]} -\lambda_1'|y[1-d_2(s)\cdot r_2(s)]}
\]

\[
= \exp[-y[s+\lambda_1'[1-d_1(s)\cdot r_1(s)] + \lambda_2'[1-d_2(s)\cdot r_2(s)]].
\]

Note that \( \lambda_1' + \lambda_2' + \mu = \lambda \), the aggregated problem-arrival parameter.

The condition on \( y \) is removed, using its distribution as given by Eq. 45. Following Eq. 49,

\[
E(e^{-s\hat{C}_b} | A, X) = \int_0^X \exp[-y[s+\lambda_1'[1-d_1(s)\cdot r_1(s)] + \lambda_2'[1-d_2(s)\cdot r_2(s)]] dP(y)
\]

\[
= \frac{\mu}{1-e^{-\mu X}} \cdot \frac{1-\exp[-x[s+\lambda_1'[1-d_1(s)\cdot r_1(s)] + \lambda_2'[1-d_2(s)\cdot r_2(s)]]]}{\mu+s[1-d_1(s)\cdot r_1(s)] + \lambda_2'[1-d_2(s)\cdot r_2(s)]}
\]

Taking \( \partial/\partial s \) and evaluating the result at \( s = 0 \) gives

\[
E(\hat{C}_b | A, X) = [1+\lambda_1'[E(d_1)+E(r_1)] + \lambda_2'[E(d_2)+E(r_2)]]. \frac{1-e^{-\mu X}}{(1-e^{-\mu X})}. \frac{-\mu X e^{-\mu X}}{\mu(1-e^{-\mu X})}.
\]

To find the \( E(C_a | X) \), given that an extraordinary problem has occurred, three classes of problems should be considered: those associated with \( \lambda_1' \), \( \lambda_2' \), and \( \mu \). The recovery policy dictates, however, that all problems will be treated similarly following the first extraordinary problem.

This means that the problems that could otherwise cause a return to the
beginning are grouped with the other problems into the parameters \( \lambda_1 \) and \( \lambda_2 \). The recovery policy of Case 4 will remove this single-return restriction and treat all extraordinary problems alike.

Using Eq. 17 from Case 1b, which is

\[
E(e^{-\lambda X} \mid A, X) = \exp[-X \{s + \lambda_1(1-d_1(s) \cdot r_1(s)) + \lambda_2(1-d_2(s) \cdot r_2(s)) \}],
\]

taking \( \partial/\partial s \), and then evaluating the result at \( s = 0 \), gives

\[
E(C_a \mid A, X) = X(1+\lambda_1[E(d_1)+E(r_1)] + \lambda_2[E(d_2)+E(r_2)]).
\]

Combining terms, given that an extraordinary problem occurs, gives

\[
E(C_b \mid A, X) = E(C_b \mid A, X) + E(D) + E(C_a \mid A, X)
\]

\[
= \{1+\lambda_1[E(d_1)+E(r_1)] + \lambda_2[E(d_2)+E(r_2)]\} \left\{ \frac{1-e^{-\mu X} \cdot \mu X - e^{-\mu X}}{\mu (1-e^{-\mu X})} \right\}
\]

\[
+ E(D) + X \{1+\lambda_1[E(d_1)+E(r_1)] + \lambda_2[E(d_2)+E(r_2)]\}.
\]

To find \( E(C) \), the probabilities \( e^{-\mu X} \) and \( 1-e^{-\mu X} \) are employed as before in

\[
E(C \mid A, X) = E(C_a \mid A, X) \cdot e^{-\mu X} + E(C_b \mid A, X) \cdot (1-e^{-\mu X}).
\]

Using \( \lambda_1 \) and \( \lambda_2 \) in the \( E(C_a \mid A, X) \), and the expressions derived previously, yields, after substitution and simplification,

\[
E(C \mid A, X) = E(D) \cdot (1-e^{-\mu X}) + \{1+\lambda_1[E(d_1)+E(r_1)] + \lambda_2[E(d_2)+E(r_2)]\} X(1-e^{-\mu X})
\]

\[
+ \{1+\lambda_1[E(d_1)+E(r_1)] + \lambda_2[E(d_2)+E(r_2)]\} \left\{ \frac{1-e^{-\mu X}}{\mu} \right\}.
\]

Removing the condition on \( X \), using \( V(X \mid A) \) as the integrator yields the desired results:
\[ E(C|A) = E(D) \{ 1 - E(e^{-\mu X}|A) \} + \{ 1 + \lambda_1 E(d_1) + E(r_1) \} + \lambda_2 \{ E(d_2) + E(r_2) \} \cdot \{ E(X|A) - E(Xe^{-\mu X}|A) \} \]

\[ + \{ 1 + \lambda_1' [E(d_1) + E(r_1)] + \lambda_2' [E(d_2) + E(r_2)] \} \left\{ \frac{1 - E(e^{-\mu X}|A)}{\mu} \right\}, \]

which can be reduced to the \( E(C|A) \) with \( \lambda_j' = \lambda_j \) under the condition that the extraordinary problems are something superimposed upon the normal process, as may be the case for many situations.

Higher moments are obtained in a similar manner; for example, for the second moment,

\[ E(C^2|A,X) = E(c_a^2|A,X) \cdot e^{-\mu X} + E(c_b^2|A,X)(1 - e^{-\mu X}) \]

where, as before (cf., Eq. 60),

\[ E(c_a^2|A,X) = E(c_a^2|A,X) + 2E(D) \cdot E(c_a|A,X) + E(D^2) \]

\[ + 2E(c_a|A,X) \cdot E(c_b|A,X) + 2E(D) \cdot E(c_b|A,X) + E(c_b|A,X). \]

The \( E(c_a^2|A,X) \) is obtained by taking \( \frac{\partial^2}{\partial s^2} \) of Eq. 75, and \( E(c_b^2|A,X) \) is similarly obtained from Eq. 73. The other terms have been derived previously. Substitution of the appropriate expressions into Eq. 82, simplification of the result, and integration with respect to \( V(X|A) \) yields the desired result:

\[ E(C^2|A) = \frac{1}{2} \left\{ 1 + \lambda_1 E(z_1) + \lambda_2 E(z_2) \right\} \left\{ \frac{1}{\mu^2} - \frac{E(e^{-\mu X}|A)}{\mu^2} - \frac{E(Xe^{-\mu X}|A)}{\mu} \right\} \]

\[ + \left\{ 1 + \lambda_1 E(z_1) + \lambda_2 E(z_2) \right\} \left\{ E(X^2|A) - E(X^2e^{-\mu X}|A) \right\} \]

\[ + \left\{ \lambda_1 E(z_1^2) + \lambda_2 E(z_2^2) \right\} \left\{ \frac{1}{\mu} - \frac{E(e^{-\mu X}|A)}{\mu} \right\} + \]

26
\[
+ \left\{ \lambda_1 E(z_1^2) + \lambda_2 E(z_2^2) \right\} \left\{ E(X|A) - E(Xe^{-\mu X}|A) \right\} \\
+ 2 \left\{ 1 + \lambda_1 E(z_1^2) + \lambda_2 E(z_2^2) \right\} E(D) \left\{ \frac{1}{\mu} - \frac{E(e^{-\mu X}|A)}{\mu} - E(Xe^{-\mu X}|A) \right\} \\
+ 2 \left\{ 1 + \lambda_1 E(z_1^2) + \lambda_2 E(z_2^2) \right\} \left\{ 1 + \lambda_1 E(z_1^2) + \lambda_2 E(z_2^2) \right\} \left\{ \frac{E(X|A)}{\mu} - \frac{E(Xe^{-\mu X}|A)}{\mu} \right\} \\
- E(X^2 e^{-\mu X}|A) \\
+ E(D) \left\{ 1 - E(e^{-\mu X}|A) \right\} ,
\]

where

\[ E(z_j) = E(\delta_j) + E(r_j) \]

and

\[ E(z_j^2) = E(\delta_j^2) + 2E(\delta_j)E(r_j) + E(r_j^2) . \]

**Case 4: Repeat as Often as Necessary; Otherwise Resume.** This case is a generalization of the preceding case. The generalization consists in allowing the first part of the Case 3 process to be repeated as often as an extraordinary problem (fault, situation) arises. The recovery policy dictates that after each extraordinary problem, the process starts from scratch, and it resumes after each normal-type delay (until an extraordinary problem arises, if at all). The process culminates in a launch attempt when an entire process is completed without an extraordinary problem.

If it is not forced to terminate, the process appears as:

\[
\begin{array}{cccccccc}
\bar{C}_b & \bar{C}_b & \bar{C}_b & \bar{C}_b & C_a \\
x_1 & d_1 & \cdots & D & x_1 & d_1 & \cdots & D & x_1 & \cdots & d_M & x_{M+1} \\
\text{start} & & & & & & & & & & \text{time} & \text{launch}
\end{array}
\]
where $C_a$ is the same variable as that investigated in Case 1, and

\begin{equation}
\tilde{C}_b = \sum_{i=1}^{N+1} x_i + \sum_{i=1}^{N} d_i + D,
\end{equation}

the time expired prior to the extraordinary problem, plus the delay it causes. Using previously derived results (cf., Eqs. 43 and 53) gives

\begin{equation}
E(\tilde{C}_b | A, X) = E(D) + E(\tilde{C}_b | A, X)
\end{equation}

\begin{equation}
= E(D) + (1+\lambda'E(d)) \frac{1-e^{-\mu X}}{\mu(1-e^{-\mu X})}
\end{equation}

and

\begin{equation}
E(C_a | A, X) = X(1+\lambda' E(d)).
\end{equation}

If now,

\begin{align*}
p &= P \text{ (a process is completed without an extraordinary problem/}X) \\
&= e^{-\mu X} \\
q &= P \text{ (a process must be stopped due to an extraordinary problem/}X) \\
&= 1-e^{-\mu X},
\end{align*}

then, using the same realization of $X$ throughout,

\begin{equation}
E(C | A, X) = \sum_{n=0}^{\infty} [E(C_a | A, X) + n E(\tilde{C}_b | A, X)] q^n p
\end{equation}

\begin{equation}
= E(C_a | A, X) + E(\tilde{C}_b | A, X) \sum_{n=0}^{\infty} n q^n p,
\end{equation}

where $\sum_{n=0}^{\infty} n q^n p$ is seen to be the expression for the expectation of a random variable given by the geometric distribution; this expectation is equal to $q/p$. This gives

\begin{equation}
E(C | A, X) = E(C_a | A, X) + \left(\frac{1-e^{-\mu X}}{e^{-\mu X}}\right) E(\tilde{C}_b | A, X) =
\end{equation}
\[ = X(1+\lambda') E(d) + \left(1-e^{-\mu X}\right) \left\{ E(D) + (1+\lambda') E(d) \right\} \frac{1-e^{-\mu X} - \mu X}{\mu(1-e^{-\mu X})} \]

\[ = \left( \frac{1}{\mu} + \frac{\lambda'}{\mu} E(d) + E(D) \right) (e^{\mu X} - 1). \]

Removing the condition on \( X \), using \( V(X|A) \) as the integrator, as before, gives the desired result.

(89) \[ E(C|A) = \left\{ \frac{1}{\mu} + \frac{\lambda'}{\mu} E(d) + E(D) \right\} \left\{ E(e^{\mu X}|A)-1 \right\} . \]

As before, the reduction to Eq. 8 results by letting \( \mu \to 0 \).

Higher moments are obtained in a similar manner. For example, the second moment is obtained as follows: Let \( C(n,A,X) \) be a countdown duration conditioned on \( n \) extraordinary interruptions, as well as the event \( A \) and the random variable \( X \). Then

(90) \[ C(n,A,X) = n \tilde{C}_b(A,X) + C_a(A,X) \]

and

(91) \[ C^2(n,A,X) = C^2_a(A,X) + 2n C_a(A,X) \tilde{C}_b(A,X) + n^2 \tilde{C}^2_b(A,X). \]

Taking the expectation and removing the condition on \( n \) yields

\[ E(C^2|A,X) = E(C^2_a|A,X) + 2E(C_a|A,X) E(\tilde{C}_b|A,X) \sum_{n=0}^{\infty} n q^n p \]

\[ + E(\tilde{C}^2_b|A,X) \sum_{n=0}^{\infty} n^2 q^n p \]

(92) \[ = E(C^2_a|A,X) + 2E(C_a|A,X) E(\tilde{C}_b|A,X) \frac{q}{p} \]

\[ + E(\tilde{C}^2_b|A,X) \frac{q^2 + q^2}{p^2} , \]

where \( \frac{q^2 + q^2}{p^2} \) is the second moment of the geometric distribution.

Each of the terms in Eq. 92 has been derived previously, with the exception of \( E(\tilde{C}^2|A,X) \), which is given by
(93) \[ E(\tilde{C}_b|A,X) = E(\tilde{C}_b^2|A,X) + 2E(D)E(\tilde{C}_b) + E(D^2) \]

since \( \tilde{C}_b = \tilde{C}_b + D \). Each term in Eq. 93 has been derived previously. Therefore, the \( E(C^2|A,X) \) is obtained by straightforward substitution. Simplification of the resulting expression and integration with respect to \( V(X|A) \) yields the desired result, which reduces to the Case 1 result as \( \mu \to 0 \).

\[
E(C^2|A) = \\
\lambda \cdot E(d^2) \left\{ 2E(X|A) - 2E(Xe^{\mu X}|A) + \frac{2E(e^{2\mu X}|A) - 3E(e^{\mu X}|A) + 1}{\mu} \right\} \\
+ 2\left\{ 1 + \lambda \cdot E(d) \right\} E(D) \left\{ \frac{2E(e^{2\mu X}|A) - E(e^{\mu X}|A) - 2E(e^{\mu X}|A) + 1}{\mu} \right\}
\]

(94)

\[
+ \left\{ 1 + \lambda \cdot E(d) \right\}^2 \left\{ \frac{1}{\mu^2} \left( \frac{4E(e^{2\mu X}|A) - 6E(e^{\mu X}|A) + 1}{\mu} \right) - \frac{2E(Xe^{\mu X}|A)}{\mu} \right\} \\
+ E(D^2) \left\{ 2E(e^{2\mu X}|A) - 3E(e^{\mu X}|A) + 1 \right\} .
\]

Case 4a: Repeat as Often as Necessary; Otherwise Test and Resume.

For this case, the recovery policy is to test (or perform whatever recovery actions that are appropriate) after each normal-type delay and return to the beginning of the pre-launch sequence after each extraordinary problem arises. All normal-type delays and recovery actions are grouped.

For this process, if it is not forced to terminate,

(95) \[ \tilde{C}_b = \sum_{i=1}^{N+1} x_i + \sum_{i=1}^{N} d_i + \sum_{i=1}^{N} r_i + D = \]
\[ \sum_{i=1}^{N+1} x_i + \sum_{i=1}^{N} z_i + D \]

and

\[ C_a = \sum_{j=1}^{M+1} x_j + \sum_{j=1}^{M} d_j + \sum_{j=1}^{M} r_j \]

(96)

\[ = \sum_{j=1}^{M+1} x_j + \sum_{j=1}^{M} z_j \]

so that results can be written down directly from the results of Case 4 by substituting \( E(z) = E(d) + E(r) \) for \( E(d) \) and \( E(z^2) = E(d^2) + 2E(d)E(r) + E(r^2) \) for \( E(d^2) \).

Case 4b: Repeat as Often as Necessary; Otherwise Test and Resume;

Separated Causes. This is the most general case to be treated. The recovery policy is identical to the Case 4a policy, but the several causes of problems are treated separately.

For this process,

(97)

\[ \hat{C}_b = \sum_{i=1}^{N+1} x_i + \sum_{j=1}^{M} \sum_{i=1}^{N} (d_{1,j} + r_{1,j}) + D \]

and

(98)

\[ C_a = \sum_{k=1}^{N+1} x_k + \sum_{j=1}^{M} \sum_{k=1}^{N} (d_{k,j} + r_{k,j}) \]

Using Case 3b results for the situation with \( M = 2 \),

\[ E(C_b | A, X) = \{1 + \lambda_1 [E(d_1) + E(r_1)] + \lambda_2 [E(d_2) + E(r_2)]\} \left\{ \frac{1 - e^{-\mu X} - \mu X e^{-\mu X}}{\mu (1 - e^{-\mu X})} \right\} + E(D) \]

(99)

and

\[ E(C_a | A, X) = X \{1 + \lambda_1 [E(d_1) + E(r_1)] + \lambda_2 [E(d_2) + E(r_2)]\} \]

(100)

Combining these into \( E(C | A, X) \) gives
\[ E(C|A,X) = E(C_0|A,X) + \left( \frac{1-e^{-\mu X}}{e^{-\mu X}} \right) E(C_1|A,X) \]

\[ \{ \frac{1}{\mu} + \frac{\lambda_1}{\mu} [E(d_1)+E(r_1)] + \frac{\lambda_2}{\mu} [E(d_2)+E(r_2)] + E(D) \} \{ e^{\mu X} - 1 \}, \]

so that removing the condition on \( X \) yields the desired result.*

\[ E(C|A) = \{ \frac{1}{\mu} + \frac{\lambda_1}{\mu} [E(d_1)+E(r_1)] + \frac{\lambda_2}{\mu} [E(d_2)+E(r_2)] + E(D) \} \{ e^{\mu X | A} - 1 \}. \]

The higher moments are obtained in a similar manner. Proceeding as in Case 4, the second moment is found to be

\[ E(C^2|A) = \]

\[ \{ \lambda_1 E(z_1^2) + \lambda_2 E(z_2^2) \} \{ 2E(X|A) - 2E(Xe^{\mu X | A}) + \frac{2E(e^{2\mu X | A}) - 3E(e^{\mu X | A}) + 1}{\mu} \}
\]

\[ + 2 \{ 1 + \lambda_1 E(z_1^2) + \lambda_2 E(z_2^2) \} E(D) \{ \frac{2E(e^{2\mu X | A}) - E(e^{\mu X | A}) - 2E(e^{\mu X | A}) + 1}{\mu} \}
\]

\[ - E(Xe^{\mu X | A}) \}

\[ + \{ 1 + \lambda_1 E(z_1^2) + \lambda_2 E(z_2^2) \} \left\{ \frac{4E(e^{2\mu X | A}) - 6E(e^{\mu X | A}) + 1}{\mu^2} - \frac{2E(Xe^{\mu X | A}) - 2E(2e^{\mu X | A}) + 1}{\mu} \right\}
\]

\[ + E(D^2) \{ 2E(e^{2\mu X | A}) - 3E(e^{\mu X | A}) + 1 \}, \]

using the \( z \) notation introduced previously.

* This result also corresponds to a situation in which a primary job which requires a long period of time \( X \) is interrupted by secondary jobs which are generated by (or during) the primary one, according to rates \( \lambda_1 \), \( \lambda_2 \), and \( \mu \). In order to handle the \( \lambda_1 \) and \( \lambda_2 \) type of jobs (in times \( d_{11} \) and \( d_{12} \)), the primary job can be just set aside and then returned to, with some re-set up time given by \( r_{11} \) and \( r_{12} \). When a \( \mu \)-type job is generated, however, the primary job must be torn down and started anew after completing the \( \mu \)-type job; these interruptive actions require time \( D \).
3. Conclusion

The countdown duration is seen to depend upon the uninterrupted countdown time, the kind, length and frequency of interruptions that can occur, and the recovery policy. During the analysis, a countdown was considered as an entity and moments of its distribution estimated. For some vehicles, the countdown is divided into two parts, called, for example, the countdown and the terminal countdown. It is likely that, in such situations, both the recovery policy and some of the operational parameters will differ for the two parts of the process. In many instances, countdown "milestones" are defined, and here again policy and parameters could vary between intra-milestone intervals. In these situations the countdown could be broken into the appropriate intervals, for purposes of analysis, and moments estimated for each of the segments. These segment moments could then be combined into total countdown moments.

REFERENCE

STANFORD UNIVERSITY

TECHNICAL REPORT DISTRIBUTION LIST

CONTRACT No-225(53)
(NR-042-002)

Chief of Ordnance
U. S. Army
Research and Development Division
Washington 25, D. C.,
Attn: GORDTB

Chief
Office of Ordnance Research
Duke University, Duke Station
Durham, North Carolina

Chief, Surveillance Branch
Ballistics Research Lab.
Allentown Proving Ground, Maryland
Attn: Mr. Bruno

Chief of Naval Material
Code MS33, Room 2236 Main Navy
Department of the Navy
Washington 25, D. C.

Chief of Naval Operations
Operations Evaluation Group
(OP-036G)
The Pentagon
Washington 25, D. C.

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.,
Attn: AC Division

Chief, Bureau of Yards & Docks
Material Division
Department of the Navy
Washington 25, D. C.,
Attn: M. W. Wolman, Code A660

Chief, Bureau of Yards & Docks
Department of the Navy
Washington 25, D. C.,
Attn: Code M400

Chief, Bureau of Ordnance
Department of the Navy (A43)
Washington 25, D. C.,

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.,
Attn: Code Renly-3

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.,
Attn: W. S. Koontz (tcf)

Chief, Bureau of Ordnance
Quality Control Division (QCC)
Department of the Navy
Washington 25, D. C.,
Attn: Dr. W. R. Fisher, Jr.

Chief
Arizona Air Procurement Dist.
Attn: Quality Control Div.
2975 Sky Harbor Blvd.
Phoenix, Arizona

Chief
San Diego Air Procurement Dist.
San Bernadino Air Materiel Area
Attn: Quality Control Division
P. O. Box 1548
Old San Diego Station
4325 Pacific Highway
San Diego 10, California

Chief, Los Angeles Air Proc. Dist.
San Bernardino Air Materiel Area
Attn: Quality Control Division
Bendix Bldg.
1206 S. Maple Street
Los Angeles, California

Chief
San Francisco Air Procurement Dist.
Sacramento Air Materiel Area
Attn: Quality Control Division
1515 Clay Street
Oakland 12, California

Chief
Atlanta Air Procurement Dist.
Warner Robins Air Materiel Area
Attn: Quality Control Division
441 West Peachtree Street N. E.
Atlanta, Georgia

Boston Air Force Contract
Management District
Attn: Chief, Quality Control Div. AF Systems Command
Boston Army Base
Boston 10, Massachusetts

Chief
St. Louis Air Procurement Dist.
Oklahoma City Air Materiel Area
Attn: Quality Control Division
1114 Market Street
St. Louis 1, Missouri

Chief Indianapolis Air Procurement Dist.
Mobile Air Materiel Area
Attn: Quality Control Division
54 Monument Circle
Indianapolis 6, Indiana

Chief
Rochester Air Procurement Dist.
Middletown Air Materiel Area
Attn: Quality Control Division
20 Simsbury Place, P. O. Box 1669
Rochester 3, New York

Chief
Cleveland Air Procurement Dist.
Mobile Air Materiel Area
Attn: Quality Control Division
1279 W. Third Street
Cleveland 13, Ohio

Chief
Dallas Air Procurement Dist.
San Antonio Air Materiel Area
Attn: Quality Control Division
William Building - Room 336
Dallas 1, Texas

Chief
Statistical Engineering Lab.
National Bureau of Standards
Washington 25, D. C.,

Commanding Officer
Office of Naval Research Branch Office
New York
New York 13, New York
Attn: Dr. J. Laidner

Commanding Officer
Office of Naval Research Branch Office
Navy No. 100
Fleet Post Office
New York, New York

Contract No-225(53)
September 1962
Commander
Rome Air Force Depot
Attn: Directorate of Supply and Services
Griffiss AFB, New York 1

Commander
Air Material Command
Attn: Quality Control Office
APD 10
New York, New York 1

Commander
Air Technical Intelligence Center
Attn: (AFIN-10)
Wright-Patterson AFB, Ohio 1

Commander
Air Materiel Command
Attn: Quality Control Office, MCQ
Wright-Patterson AFB, Ohio 10

Commander
Middletown Materiel Area
Attn: Assistant for Quality Directorate of Procurement
and Production
Olmsted Air Force Base
Middletown, Pennsylvania 1

Commander
Middletown Materiel Area
Attn: Quality Control Division
Olmsted Air Force Base
Middletown, Pennsylvania 1

Commander
Memphis Air Force Depot
Mail: Air Force Station
Attn: Directorate of Supply and Services
3500 Jackson Avenue
Memphis 1, Tennessee 1

Commander
San Antonio Air Materiel Area
Attn: Quality Control Office, SHQ
Norton AFB, California 2

Quality Control Office (SAQ)
Hq, San Antonio Air Materiel Area
Kelly AF Base, Texas 1

Commander
San Antonio Air Materiel Area
Attn: Quality Control Division
Kelly AF Base, Texas 1

Commander
Oklahoma Air Materiel Area
Attn: Quality Control Division
Dir, Procurement and Production
Hill AFB, Utah 1

David Taylor Model Basin
Applied Mechanics Lab. (Code B20)
Washington 7, 0, C.
Attn: Dr. Julius Lieblin 1

Detroit Air Procurement Dist.
Attn: Quality Control Div., MHQD
W. Warren Avenue & Loryo Blvd.
Detroit 32, Michigan
Attn: MHQD 1

Director
National Security Agency
Attn: REMP-2
Fort George G. Meade
Maryland 1

Director, Development Division
Air Force Special Weapons Project
Albuquerque, New Mexico 1

Eastern Contract Management Region
Attn: Directorate, Quality Control
Olmsted AFB, Pennsylvania 1

Engineering Statistics Group
Research Division
New York University
New York 3, New York 1

Engineering Statistics Unit
Chemical Corps Engineering Agency
Army Chemical Center, Maryland
Attn: Mr. T. M. Viining, Chief 1

Federal Telephone and Radio Co.
100 Kingsland Road
Clifton, New Jersey 1

Military Clothing & Textile Supply Agency
Philadelphia Quartermaster Center
U. S. Army
2800 So. 20th Street
Philadelphia 1, Pa.
Attn: Policy Office, Technical Division 1

Headquarters
San Bernardino Air Materiel Area
U. S. Air Force
Norton AFB, California
Attn: Chief, Planning & Control Offices 1

Headquarters
U. S. Army Signal Equipment Support Agency
Fort Monmouth, New Jersey
Attn: SIGFM/ES-PFE 1

Headquarters, Defense Supply Agency
DSAH-PI
Inspection and Quality Control Division
Washington 25, D. C. 1

Industrial Division
Office, Chief of Ordnance
Department of the Army
Washington 25, D. C.
Attn: Mr. John J. Riordan
Mr. Irving A. Altman 1

Inspection and Quality Control Division
Office, Asst. Secretary of Defense
(SL)
Washington 25, D. C.
Attn: Mr. John T. Quirk 1

Library
Institute for Defense Analyses
Communications Research Division
Van Neumann Hall
Princeton, New Jersey 1

Library
Department of Industrial Engineering
and Operations Research
College of Engineering
New York University
University Heights
New York 3, New York 1

Library
Remal Analysis Research
University of California
Los Angeles 24, California 1

Logistics Research Division
Attn: MCFR
Wright-Patterson AFB, Ohio 1

Logistics Research Project
George Washington University
707 22nd Street, N. W.
Washington 7, D. C. 1

Milwaukee Air Procurement Dist.
Attn: Quality Control Div.
770 N. Plankinton Avenue
Milwaukee 3, Wisconsin 1

Military Medical Supply Agency
3rd Avenue and 29th Street
Brooklyn 32, New York
Attn: Mr. Joseph Zador 1

N. A. C. A.
1512 H. Street, N. W.
Washington 25, D. C.
Attn: Div. of Res. Information 1

Newark Air Procurement Dist.,
Attn: Quality Control Division
218 Market Street
Newark, New Jersey 1

New York Air Procurement Dist.,
Attn: Quality Control Division
111 East 85th Street
New York 3, New York 1

Office, Asst. Secretary of Defense
(R&D)
Room 351065, The Pentagon
Washington 25, D. C.
Attn: Technical Library 1

Operations Analysis Office
Headquarters, Pacific Air Forces
U. S. Air Force, APO 955
Attn: C. E. Thompson
Senior Analyst
San Francisco, California 1

Office, Chief of Engineers
Department of the Army
Washington 25, D. C.
Attn: Procurement Division
Military Supply 2

Office of Operations Analysis
DCS/Operations
Elgin AFB, Florida 1

Office of the Chief
WDL/DF
Arlington Hall Station
Arlington, Virginia
Attn: Dr. J. R. Hershner, Jr. 1

Office in Charge
U. S. Navy Central Tuzdoff Office
Navy Yard Island
Attn: Mr. G. B. Habicht 1

Ordnance Corps
Industrial Engineering Div.,
Diamond Ordnance Fuel Lab.
Washington 25, D. C. 1

Ordnance Mission
White Sands Proving Ground
Las Cruces, New Mexico
Attn: Mr. Paul G. Cox 1

Philadelphia Air Procurement Dist.
Attn: Quality Control Division
1411 Walnut Street
Philadelphia 6, Pennsylvania 1

Physical Research Branch
Evans Signal Lab., SCIL
Belmar, New Jersey
Attn: Mr. Joseph Weinstein 1

The RAND Corporation
1700 Main Street
Santa Monica, California
Attn: Library 1

Report Library
University of California
Los Alamos Scientific Lab.
P. O. Box 1683
Los Alamos, New Mexico 1

Contract Non-225(53)
September 1962 1
Rocket Development Group
Redstone Arsenal
Huntsville, Alabama
Attn: Lt. E. L. Bonbarra

San Francisco Air Procurement Dist.
Attn: Quality Control Division
Goldman Army Terminal, Bldg. 1
West Grand & Montana
Oakland 34, California

Scranton Ordnance Plant
196 Cedar Avenue
Scranton, Pennsylvania
Attn: Capt. D. L. Larsen
Chief Inspector

Standards Branch
Procurement Division
DOS Logistics, U. S. Army
Washington 25, D. C.
Attn: Mr. Silas Williams, Jr.

Statistical Laboratory
University of California
Berkeley 4, California

Superintendent
U. S. Naval Postgraduate School
Monterey, California
Attn: Library

Technical Information Officer
Naval Research Laboratory
Washington 25, D. C.

Technical Operator, Inc.
and C. O. R. G.
HQ. Continental Army Command
Fort Monroe, Virginia

U. S. Naval Aviation Facilities
Indianapolis 18, Indiana
Attn: Library

U. S. Naval Engineering Experiment Station
Annapolis, Maryland
Attn: Mr. F. R. Delfs

U. S. Naval Inspector of Ordinance
400 S. Beiger Street
Mitsubishi, Indiana

U. S. Naval Inspector of Ordinance
Eastman Kodak Company
50 W. Main Street
Rochester 24, New York

Western Contract Management Region
Attn: Director, Quality Control
Miny Loma AF Station, California

Dr. Adam Alwazi
Dept. of Economics and Engineering
Stevens Institute of Technology
 Hoboken, New Jersey

Dr. Max Astachan
Department of Logistic
The RAND Corporation
1700 Main Street
Santa Monica, California

Professor T. W. Anderson
Department of Mathematical Statistics
Columbia University
New York 27, New York

Professor Fred C. Andrews
Mathematics Department
University of Oregon
Eugene, Oregon

Professor Robert Bechtel
Sibley School of Mech. Engineering
Cornell University
Ithaca, New York

Professor R. E. Beckwith
Graduate School of Business Administration
University of Southern California
Los Angeles 7, California

Professor J. N. Benettini
Western Reserve University
Cleveland, Ohio

Mr. Carlton M. Beyer
Office of Guided Missiles
Office of Assi. Secretary of Defense (R&D)
Washington, D. C.

Professor Z. W. Birnbaum
Laboratory of Statistical Research
Department of Mathematics
University of Washington
Seattle 5, Washington

Professor Russell Bradt
Department of Mathematics
University of Kansas
Lawrence, Kansas

Professor Irving W. Burr
Department of Mathematics
Purdue University
Lafayette, Indiana

Mr. G. Burrows
Kohns Atomic Power Lab.
Schenectady, New York

Dr. A. Cannean
The Technological Institute
Northwestern University
Evanston, Illinois

Mr. W. H. Clarkworthy
Delphi Plan, Westinghouse
Electric Corporation
Box 1404-9
Pittsburgh 30, Pennsylvania

Professor Paul Clifford
New Jersey State Teachers College
Montclair, New Jersey

Professor W. G. Cochran
Department of Statistics
Harvard University
Cambridge, Massachusetts

Professor C. C. Cockerm
Institute of Statistics
North Carolina State College
Raleigh, North Carolina

Professor Edward P. Coleman
Engineering Department
University of California
Los Angeles 24, California

Miss Gertrude M. Cox
Institute of Statistics
North Carolina State College
Raleigh, North Carolina

Dr. Joseph Daly
U. S. Census Bureau
Washington 25, D. C.

Professor Cyrus Derman
Department of Industrial Engineering
Columbia University
New York 27, New York

Mr. H. F. Dodge
Rutgers University
New Brunswick, New Jersey

Dr. Francis Dresch
Stanford Research Institute
333 Ravenswood Avenue
Menlo Park, California

Professor Archeson J. Duncan
Department of Industrial Engineering
Johns Hopkins University
Baltimore 18, Maryland

Professor Meyer Dwass
Department of Mathematics
Northwestern University
Evanston, Illinois

Professor D. A. S. Fraser
Department of Mathematics
University of Toronto
Toronto 5, Canada

Mr. William E. Gilbert, Chief
Mathematical Statistics Branch
Atomic Energy Commission
Washington 25, D. C.

Mr. Leon Gifford
Operations Research Inc.
8605 Cameron Street
Silver Springs, Maryland

Mr. Bernard P. Golden
Associate Professor
Northeastern University
360 Huntington Avenue
Boston 15, Massachusetts

Professor Lee A. Goodman
Statistical Research Center
University of Chicago
Chicago 37, Illinois

Dr. J. Greenwood
430 Great Falls Street
Fall Church, Virginia

Professor Donald Gaver
Westinghouse Research Labs.
Euclid Road
Churchill Bros.
Pittsburgh 35, Pennsylvania

Professor Frank M. Gryna, Jr.
University College
Rutgers University
New Brunswick, New Jersey

Dr. Donald Guthrie
Stanford Research Institute
333 Ravenswood Avenue
Menlo Park, California

Dr. Theodore E. Harris
The RAND Corporation
1700 Main Street
Santa Monica, California

Dr. Leon H. Herback
Dept. of Industrial Eng. & Operations
Research
College of Engineering
New York University
New York 53, New York

Professor W. Hirsch
Institute of Mathematical Sciences
New York University
New York 3, New York

Mr. Eugene Hixson
Cape Kilo 1
GSFC, NASA
Greenbelt, Maryland

Contract No. 2-25(53)
September 1962