STATISTICAL CONTROL OF A NORMAL PROCESS

BY

HOWARD M. TAYLOR

TECHNICAL REPORT NO. 82
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Abstract

The optimal statistical control of a simple production process which has only two underlying states, in control and out of control, is studied. The produced items are assumed to have a measurable quality characteristic which has a normal distribution with known variance. When the process leaves control the mean of this distribution shifts from $\mu_0$ to $\mu_1$, both assumed known. Costs are associated with the quality of an item and with repairing the process which returns it to a state of control. Other than immediately after repair, the process state is assumed unknown. This report gives a statistical control rule which, based on the quality history of produced items, tells when to repair the machine. The rule given minimizes the time average of the total cost of repairs plus quality.
Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. Approximation in Continuous Time</td>
<td>3</td>
</tr>
<tr>
<td>3. The Formula for the Control Variable</td>
<td>5</td>
</tr>
<tr>
<td>4. The Potential Cost Differential Equation</td>
<td>7</td>
</tr>
<tr>
<td>5. Solving the Differential Equation</td>
<td>9</td>
</tr>
<tr>
<td>6. Computing the Answer</td>
<td>13</td>
</tr>
<tr>
<td>7. The Optimal Procedure in Use</td>
<td>16</td>
</tr>
<tr>
<td>References</td>
<td>20</td>
</tr>
</tbody>
</table>
STATISTICAL CONTROL OF A NORMAL PROCESS

1. Introduction

This paper discusses the statistical control of a simple production process which is assumed always to be in one of only two states, a good state and a bad state, which correspond to the process being in control and out of control, respectively. Specifically production begins in the good state and while there a chance event occurs before each item is produced so that the probability of remaining in the good state is \(1 - \pi\) and the probability of a transition to the bad state is \(\pi\). Once in the bad state the process remains there until trouble is removed.

Associated with each item produced is a measurable characteristic or quality which is assumed to be a normally distributed random variable whose mean depends on the underlying state of the production process. The mean in the good state is \(\mu_0\), the mean in the bad state is \(\mu_1\) and the common variance is \(\sigma^2\), all assumed to be known parameters. Presumably a mean of \(\mu_1\) is worse in some sense than a mean of \(\mu_0\). The observed item quality at time \(t\) is denoted \(U_t\).

A statistical control rule is a rule which specifies when the system is to be brought from production to repair, which has the effect of placing the system in the good state. Other than immediately after repair, the true system state is assumed unknown at all time. Hence a control rule must be based on the quality history of produced items. Such a rule is statistical in the sense that from the quality history the underlying process state will be inferred, at least implicitly.
It costs $K$ units to repair the machine and a unit of quality $U_t = u$ costs $\hat{C}(u)$ units. It is assumed, of course, that $E_1[\hat{C}(U)] > E_0[\hat{C}(U)]$ where $E_0(E_1)$ is expectation when the process is in the good (bad) state. The objective is to give a control rule which will minimize the average cost per unit time. It has been shown that any rule which achieves this objective will be based on the posterior probability that the system will be in the bad state for the production of the next item, assuming no repair is made. This probability at time $t$ will be denoted $X_t$. In the present work it will often be convenient to discuss the monotone function of this probability given by

$$Z_t = X_t / (1 - X_t).$$  

(1.1)

If $p_0(u)$ and $p_1(u)$ are the densities of quality for processes in the good and bad states respectively and if $X_t = x$, $Z_t = z$, and $U_t = u$, one has

$$X_{t+1} = \frac{xp_1(u) + \pi(1 - x)p_0(u)}{xp_1(u) + (1 - x)p_0(u)}$$  

(1.2)

and

$$Z_{t+1} = \left( \frac{1}{1 - \pi} \right) \left( \pi + z \frac{p_1(u)}{p_0(u)} \right),$$  

(1.3)

where $X_0 = 0$ and $Z_0 = 0$.

This model of a production process was first introduced by Girshick and Rubin [4] in 1952. They show that the minimal cost rule is of the
form: "Stop and repair at time $t$ if and only if $X_t \geq \lambda$" for some critical value $\lambda$. In what follows, a means of choosing the optimal critical value $\lambda^*$ is given for the case where $\pi$, the probability of a process breakdown, is small.

2. Approximation in Continuous Time

The requirement that $\pi$, the probability of a process breakdown, be small was inserted so that the given process would be approximated by a continuous time process. It is the optimal control method for this continuous time analog which is found and then extrapolated to the discrete case as in Bather [1].

It is more convenient to examine the process of cumulative sums

$$Y_t = \sum_{k=0}^{t} \left( \frac{U_k - \mu_0}{\sigma} \right)^+ , \quad t = 0, 1, 2, \ldots \quad (2.1)$$

If $T$ is the first time at which the process turns bad, then $T$ has a geometric distribution with the probability law:

$$Pr(T = j) = \pi (1 - \pi)^{j-1} , \quad j = 1, 2, \ldots \quad (2.2)$$

One may write the new process of interest in terms of $T$ as

$$Y_t = \theta_t + W_t , \quad t = 0, 1, 2, \ldots \quad (2.3)$$

where $W_t = \sum_{k=0}^{t} \eta_k , \quad \{\eta_k\}_{k=0}^{\infty}$ is a sequence of independent normally distributed random variables with zero mean and unit variance, and
\[ \Theta_t = \begin{cases} 0 & \text{for } 0 \leq t < T \\ (t - T + 1)(\mu_1 - \mu_0)/\sigma & \text{for } T \leq t. \end{cases} \]

If one restricts attention to processes for which \( \pi \) is small, then a continuous time process may be studied as an approximation to the given process. Consider first the process

\[ Y_0(t_0) = \Theta_0(t_0) + W_0(t_0), \quad t_0 \geq 0 \]  
\[ \tag{2.4} \]

where:

- \( W_0(t_0) \) is a Weiner process with variance parameter one.
- \( \Theta_0(t_0) = \begin{cases} 0 & \text{for } 0 \leq t_0 \leq T_0 \\ (t_0 - T_0)(\mu_1 - \mu_0)/\sigma & \text{for } T_0 \leq t_0 \end{cases} \)

\( T_0 \) is an exponentially distributed random variable such that

\[ \Pr(\Theta_0(t_0 + \Delta t_0) > 0|\Theta_0(t_0) = 0) = \pi(\Delta t_0) + o(\Delta t_0^2). \]

Let \( \mu = (\mu_1 - \mu_0)/\sigma \). One may, in effect, assume \( \mu = 1 \) by examining the equivalent process

\[ Y(t) = \mu Y_0(t/\mu^2), \quad t \geq 0 \]  
\[ \tag{2.5} \]

where

\[ Y(t) = \Theta(t) + W(t), \quad t \geq 0 \]
and \( W(t) = \mu W_0(t/\mu^2) \) is a Weiner process with variance parameter one,
\[
\Theta(t) = \begin{cases} 
0 & \text{for } 0 \leq t \leq T \\
(t - T) & \text{for } T \leq t
\end{cases}
\]

\( T \) is an exponentially distributed random variable such that
\[
\Pr(\Theta(t + \Delta t) > 0 | \Theta(t) = 0) = \rho(\Delta t) + O(\Delta t)^2 \quad \text{with} \quad \rho = \pi/\mu^2.
\]

The time scale has been changed by \( t = t_0\mu^2 \).

3. The Formula for the Control Variable

The variable used in controlling the continuous time process is
\[
Z(t) = X(t)/[1 - X(t)] 
\]

where \( X(t) = \Pr(\Theta(t) > 0 | Y(s), s \leq t) \).

Suppose \( X(t) = x, \ \Delta Y = Y(t + \Delta t) - Y(t) = \Delta y \), and
\[
Z(t) = z = x/(1 - x).
\]

Needed are expressions for \( Z(t + \Delta t) \), \( \Delta Z = Z(t + \Delta t) - Z(t) \), \( E(\Delta Z) \), and \( E(\Delta Z)^2 \), where \( E(\cdot) \) is expectation at time \( t \) given \( Z(t) = z \) or \( X(t) = x \). These expressions have been derived by Bather [2]. Letting \( f_x(\Delta y) \) be the density of \( \Delta Y \) given that \( X(t) = x \), and
\[
q(x, \Delta y) = \Pr(\Theta(t) > 0 | X(t) = x, \Delta Y = \Delta y)
\]

(3.2)
one has:

\[
    f_x(\Delta y) = \frac{1}{\sqrt{2\pi} \Delta t} \left\{ xe^{-\frac{1}{2} \frac{(\Delta y - \Delta t)^2}{\Delta t}} + (1 - x)e^{-\frac{1}{2} \frac{(\Delta y)^2}{\Delta t}} \right\}, \quad (3.3)
\]

\[
    q(x, \Delta y) = \frac{xf_x(\Delta y)}{f_x(\Delta y)} , \quad (3.4)
\]

and

\[
    X(t + \Delta t) = \Pr(\Theta(t + \Delta t) > 0 | Y(s), s \leq t + \Delta t)
\]

\[
    = \Pr(\Theta(t + \Delta t) > 0 | X(t) = x, \Delta Y = \Delta y)
\]

\[
    = q(x, \Delta y) + [1 - q(x, \Delta y)] \rho(\Delta t) + O(\Delta t)^2. \quad (3.5)
\]

Then

\[
    Z(t + \Delta t) = \frac{X(t + \Delta t)}{1 - X(t + \Delta t)} \quad (3.6)
\]

Continuing the evaluation,

\[
    \frac{q(x, \Delta y)}{1 - q(x, \Delta y)} = \frac{xf_x(\Delta y)}{(1 - x)f_x(\Delta y)}
\]

\[
    = \exp \left( -\frac{1}{2(\Delta t)} \left[ (\Delta y)^2 + (\Delta t)^2 \right] \right)
\]

\[
    = \exp \left( (\Delta y) - \frac{1}{2} (\Delta t) \right)
\]

\[
    = z + z \left\{ (\Delta y) - \frac{1}{2} (\Delta t) \right\} + \frac{1}{2} (\Delta y)^2 + O(\Delta t)^2 \quad (3.7)
\]
Hence

\[ Z(t + \Delta t) = z + z \left[ \left( \Delta y \right) - \frac{1}{2} \left( \Delta t \right) + \frac{1}{2} \left( \Delta y \right)^2 \right] + (1 + z)\rho(\Delta t) + o(\Delta t)^2 \]  

(3.8)

and

\[ \Delta Z = z \left[ \left( \Delta y \right) - \frac{1}{2} \left( \Delta t \right) + \frac{1}{2} \left( \Delta y \right)^2 \right] + (1 + z)\rho(\Delta t) + o(\Delta t)^2 \]  

(3.9)

Since \( E(\Delta Y) = x(\Delta t) + o(\Delta t)^2 \) and \( E(\Delta Y)^2 = (\Delta t) + o(\Delta t)^2 \) one has

\[ E(\Delta Z) = \left[ z^2/(1 + z) \right](\Delta t) + \rho(1 + z)(\Delta t) + o(\Delta t)^2 \]  

(3.10)

and

\[ E(\Delta Z)^2 = z^2(\Delta t) + o(\Delta t)^2 \]  

(3.11)

Bather derived these expressions while studying a detection problem of Shiryaev [5].

4. The Potential Cost Differential Equation

It has been assumed to cost \( K \) units to repair the machine and to cost \( \hat{C}(u) \) units to produce an item of quality \( u \). For simplicity, item costs are written relative to a base of zero for a good process.

That is, if

\[ a_0 = E_1\hat{C}(U) - E_0\hat{C}(U) \]  

(4.1)

where \( E_1(E_0) \) represents expectation assuming a bad (good) process, then the average cost of continuing for an increment \( (\Delta t_0) \) is \( a_0 x(\Delta t_0) \) if
\( X(t) = x \). But since the time scale was changed to \( t = t_0/\mu^2 \) in order to assume a mean \( \mu \) of one, one must set \( a = a_0/\mu^2 \) to get an incremental cost of \( ax(\Delta t) \).

Letting \( \gamma \) represent an average cost per unit time, by using the general theory developed in Taylor [6] one may write down the functional equation which the optimal plan must satisfy:

\[
f(z) = \min \left\{ K, \left( \frac{az}{1 + z} - \gamma \right) \Delta t + E_z f(z + \Delta Z) \right\}.
\]  

(4.2)

One interprets \( f(z) \) as being the potential cost, relative to some base, of being in state \( Z(t) = z \).

Expanding in a Taylor series,

\[
E_z f(z + \Delta Z) = f(z) + f'(z)E(\Delta Z) + \frac{1}{2} f''(z)E(\Delta Z)^2 + \cdots
\]

\[
= f(z) + f'(z) \left[ (1 + z)p + \frac{z^2}{1 + z} \right] (\Delta t) + \frac{1}{2} f''(z)z^2(\Delta t) + o(\Delta t)^2
\]

(4.3)

For a critical value of \( \lambda \), Equation (4.2) yields,

\[
f(z) = \left( \frac{az}{1 + z} - \gamma \right) (\Delta t) + f(z) + \left[ p(1 + z) + \frac{z^2}{1 + z} \right] f'(z) (\Delta t)
\]

\[
+ \frac{1}{2} z^2 f''(z) (\Delta t) + o(\Delta t)^2
\]

(4.4)

for \( 0 \leq z \leq \lambda \),

or the second order linear differential equation.
\[ f''(z) + 2 \left[ \frac{p(1 + z)}{z^2} + \frac{1}{1 + z} \right] f'(z) = \frac{2}{z^2} \left[ \frac{\gamma}{z} - \frac{a}{1 + z} \right] \]

for \( 0 \leq z \leq \lambda \) \( (4.5) \)

5. Solving the Differential Equation

Let

\[ y(z) = \frac{df(z)}{dz} = f'(z) \]

\[ P(z) = 2 \left[ \frac{p(1 + z)}{z^2} + \frac{1}{1 + z} \right] \]

\[ Q(z) = 2 \left[ \frac{\gamma}{z^2} - \frac{a}{z(1 + z)} \right] \] \( (5.1) \)

\( (5.2) \)

Equation \( (4.5) \) becomes

\[ \frac{dy}{dz} + P(z)y = Q(z) \]

\( (5.3) \)

The solution may be obtained through \( u(z) \) and \( v(z) \) where

\[ \frac{dv(z)}{v(z)} = - P(z) \, dz \]

\[ du(z) = \frac{1}{v(z)} \, Q(z) \, dz \] .

Upon substituting and integrating one has
\[ \ln v(z) = -2 \int \left[ \frac{(1 + z) \rho}{z^2} + \frac{1}{1 + z} \right] dz \]

\[ = \frac{2\rho}{z} - 2 \ln \left[ z^\rho (1 + z) \right] \]

and

\[ v(z) = \frac{2\rho}{z^\rho (1 + z)^2}. \tag{5.4} \]

Now let \( \alpha = 2\rho \) and

\[ \varphi(z) = z^\alpha e^{-\alpha/z}. \tag{5.5} \]

Then

\[ v(z) = \frac{1}{(1 + z)^2 \varphi(z)} \]

and

\[ u(z) = 2 \int (1 + z)^2 \varphi(z) \left[ \frac{\gamma}{z^2} - \frac{a}{z(1 + z)} \right] dz \]

\[ = 2 \int \left\{ (\gamma - a) + (2\gamma - a)z^{-1} + \gamma z^2 \right\} \varphi(z) dz \tag{5.6} \]

Let

\[ \psi(z) = \int_0^z \varphi(s) ds. \tag{5.7} \]
Since
\[ \int z^{-1} \varphi(z) dz = \frac{1}{\alpha} \left\{ z \varphi(z) - (1 + \alpha) \psi(z) \right\} \]
and
\[ \int z^{-2} \varphi(z) dz = \frac{1}{\alpha} \left\{ (1 - z) \varphi(z) + (1 + \alpha) \psi(z) \right\} , \]

\[ u(z) = \frac{2}{\alpha} \left\{ (a - \gamma) \psi(z) + [\gamma - (a - \gamma)z] \varphi(z) \right\} \] \hspace{1cm} (5.8)

The general solution to the differential equation (5.3) is
\[ y = v(z)[u(z) + c] \] where \( c \) is a constant of integration. Hence
\[ y(z) = \frac{2}{\alpha} \left\{ \frac{(a - \gamma)}{(1 + z)^2} [\psi(z) - c] \right\} + \frac{\gamma}{(1 + z)^2} - \frac{(a - \gamma)z}{(1 + z)^2} \] \hspace{1cm} (5.9)

From Equation (4.4) and the condition \( f(0) = 0 \) one sees that \( y(0) = \gamma/\rho \) and hence \( c = \psi(0) = 0 \). Let

\[ A(z) = \psi(z)/\varphi(z) \] \hspace{1cm} (5.10)

and
\[ C(z) = \int_0^z \frac{A(t)}{(1 + t)^2} dt . \]

One has
\[ y(z) = \frac{2}{\alpha} \left\{ \frac{(a - \gamma)A(z)}{(1 + z)^2} - \frac{(a - \gamma)}{1 + z} + \frac{a}{(1 + z)^2} \right\} \] \hspace{1cm} (5.11)
and

\[ f(z) = \int_0^z y(t) \, dt \]
\[ = \frac{2}{\alpha} \left\{ (a - \gamma)C(z) - (a - \gamma)\ln(1 + z) + \frac{az}{1 + z} \right\}, \quad (5.12) \]

where \( f(0) = 0 \) has provided the lower limit for the integration. Since \( f(\lambda) = K, \) and \( \rho = \alpha/2 \)

\[ \rho K = (a - \gamma)[C(\lambda) - \ln(1 + \lambda)] + a \left( \frac{\lambda}{1 + \lambda} \right) \]

and

\[ \gamma = \left[ \frac{\alpha[C(\lambda) - \ln(1 + \lambda) + \frac{\lambda}{1 + \lambda}] - \rho K}{C(\lambda) - \ln(1 + \lambda)} \right] \quad (5.13) \]

Let \( r(\lambda) = C(\lambda) - \ln(1 + \lambda) + \lambda/(1 + \lambda) \) so that

\[ \gamma = \frac{ar(\lambda) - \rho K}{r(\lambda) - \lambda/(1 + \lambda)} \quad (5.14) \]

To minimize the average cost per unit time \( \gamma \) one takes the derivative with respect to \( \lambda \) in Equation (5.14) and equates it to zero. The resulting equality, which may be used to evaluate the optimal critical value \( \lambda^* \) is:

\[ \frac{\rho K}{a} = \frac{r(\lambda^*) - \lambda^*(1 + \lambda^*)r'(\lambda^*)}{1 - (1 + \lambda^*)^2r'(\lambda^*)} \quad (5.15) \]
where \( x'(\lambda) = \frac{dx(\lambda)}{d\lambda} = A(\lambda) - \frac{\lambda}{(1 + \lambda)^2} \). Hence

\[
\frac{\rho k}{a} = \frac{\lambda^* + C(\lambda^*) - \ln(1 + \lambda^*) - [\lambda^*/(1 + \lambda^*)]A(\lambda^*)}{1 + \lambda^* - A(\lambda^*)} \tag{5.16}
\]

Since \( \rho = \pi/\mu^2 \) and \( a = a_0/\mu^2 \) where \( a_0 = E_1 \hat{C}(U) - E_0 \hat{C}(U) \) one can write the left hand side of (5.16) as \( \pi K/a_0 \).

6. Computing the Answer

In order to solve Equation (5.16) it appears that at least some numerical integration will be required. One needs to evaluate

\[
A(z) = \psi(z)/\varphi(z) \tag{6.1}
\]

and

\[
C(z) = \int_0^z [A(t)/(1 + t)^2]dt \tag{6.2}
\]

where

\[
\varphi(z) = z^\alpha e^{-(\alpha/z)}
\]

\[
\psi(z) = \int_0^z \varphi(s)ds.
\]

In order to avoid two successive numerical integrations one may approximate \( A(z) \) using an inequality due to Gautschi [3]. Let \( u = \alpha/z \) so that \( dz = -(\alpha/u^2)du \) and \( z^\alpha = \alpha^\alpha u^{-\alpha} \). Then
\[ \psi(\lambda) = \int_0^\lambda z^{\alpha - 1} e^{-z} \, dz \]
\[ = \int_0^\infty \frac{\alpha^{1+\alpha}}{\lambda} u^{1+\alpha} - (1+\alpha) e^{-u} \, du \]
\[ = \alpha^{1+\alpha} \Gamma((-1+\alpha), \frac{\lambda}{\alpha}) \]
(6.3)

where \( \Gamma(m, x) \) is the incomplete gamma function,

\[ \Gamma(m, x) = \int_x^\infty u^{m-1} e^{-u} \, du . \]

In general

\[ \Gamma(m, x) = \frac{1}{m}[\Gamma(m + 1, x) - e^{-x} x^m] \]

which when applied twice yields

\[ \Gamma(-1+\alpha, \frac{\alpha}{z}) = \frac{1}{\alpha(1-\alpha)} \Gamma(1 - \alpha, \alpha/z) + (1 - 1/z) \frac{e^{-\alpha/z} (\alpha/z)^{-1+\alpha}}{1+\alpha} \]
(6.4)

Substituting Equation (6.4) into Equation (6.3) gives

\[ \psi(z) = \left( \frac{\alpha^\alpha}{1+\alpha} \right) \Gamma(1 - \alpha, \alpha/z) + \frac{e^{-\alpha/z} (\alpha/z)^{-1+\alpha}}{1+\alpha} \left( 1 - \frac{1}{z} \right) \]  
(6.5)

\[ A(z) = \left( \frac{1}{1+\alpha} \right) \left( \frac{\alpha^\alpha}{z} \right) e^{\alpha/z} \Gamma(1 - \alpha, \alpha/z) + \frac{z - 1}{1+\alpha} \]  
(6.6)
Gautschi's inequality is

\[ \frac{1}{2} p \left[ (x + 2)^p - x^p \right] < e^{x \Gamma \left( \frac{1}{p}, x \right)} \leq c_p \left[ \left( x + \frac{1}{c_p} \right)^p - x^p \right] \]  \hspace{1cm} (6.7)

where \( c_p = \left[ \Gamma \left( 1 + \frac{1}{p} \right) \right]^{p/(p-1)} \) for \( p > 1 \). Interpreting for this problem yields

\[ A(z) = \frac{z - 1}{1 + \alpha} + \frac{M}{(1 - \alpha^2)z} \left[ \left( 1 + \frac{z}{M} \right)^{1-\alpha} - 1 \right] \]  \hspace{1cm} (6.8)

where

\[ M = \begin{cases} 
\alpha/2 & \text{for a lower bound} \\
\alpha[\Gamma(2 - \alpha)]^{1/\alpha} & \text{for an upper bound}
\end{cases} \]

This approximation allows the right hand side of Equation (5.16) to be evaluated, resulting in Figure 1, where for several fixed values of \( \alpha \), the optimal critical value \( \lambda^* \) is plotted as a function of \( \rho K/a \).

In evaluating the right hand side of Equation (5.16) the value used for \( A(\lambda) \) was derived from the numerical integration of \( \varphi(t) \) as indicated by Equations (5.7) and (5.10). For small values of \( t \) and \( \alpha \), \( \varphi(t) = t^{\alpha} e^{-\alpha t} \) is difficult to compute accurately, thus limiting the values given for the optimal critical value \( \lambda^* \) to values exceeding 1.0, or a probability of being out of control of 0.50. The upper and lower limits for \( A(\lambda) \) were also computed and it was found that the upper limit gave values close to the numerical integration results.
Consequently, for the value of $C(\lambda)$ in Equation (5.16), the numerical integral of the upper bound for $A(t)$ was used as indicated in Equation (6.2). This allowed a finer mesh size to be used in this integration than if the numerical $A(t)$ function itself had been re-integrated. The results given in Figure 1 were finally checked against complete upper and lower bounds for the function, derived from appropriate use of the upper and lower bounds for $A(t)$. For the range of values in Figure 1, these bounds were sufficiently narrow to indicate that the numerical procedure used was satisfactory.

7. The Optimal Procedure in Use

This section summarizes and completes the description of the optimal procedure. Recall the symbols

- $\pi$ = the probability that the process goes out of control between the production of any item and the production of next,
- $U_t$ = the observed quality of $t^{th}$ item produced, assumed to be a normally distributed random variable,
- $\mu_0$ = the mean of $U_t$ when the process is in control,
- $\mu_1$ = the mean of $U_t$ when the process is out of control,
- $\sigma^2$ = the variance of $U_t$,
- $K$ = the cost to repair the process,
- $\hat{C}(u) =$ the cost of producing an item of quality $U_t = u$.

It is assumed that these quantities are all known. To estimate $\pi$ one might first estimate $1/\pi$, the average number of items produced before the process goes out of control. It is assumed that $\pi$ is small.
Figure 1. Each curve gives the optimal critical value $\lambda^*$ as a function of the parameter $nK/a_o$ for a particular value of $\alpha = 2\pi/\mu^2$. 
The rule used in controlling the process is: "Stop and repair at time $t$ if and only if $Z_t \geq \lambda^*"$. $Z_t$ is the control variable, computed recursively from $Z_{t-1}$ and the quality $U_t$ of the $t$th or last item. If

$$V_t = (U_t - \mu_0)/\sigma$$

and

$$\mu = (\mu_1 - \mu_0)/\sigma$$

then if $Z_t = z$ one computes $Z_{t+1}$ by $Z_0 = 0$ and

$$Z_{t+1} = \frac{1}{1 - \pi}\left[\pi + z \Lambda(V_t)\right]$$

where

$$\Lambda(v) = \exp\left\{-\frac{1}{2}[v^2 - (v - \mu)^2]\right\}.$$  

To find the critical value $\lambda^*$ compute

$$a = \frac{[E_1\tilde{C}(U) - E_0\tilde{C}(U)]/\mu^2}{\alpha},$$

the difference in expected costs between an out-of-control process and an in-control process.

and $\alpha = 2\pi/\mu^2$.

Then on Figure 1, using the line corresponding to $\alpha$, one may read $\lambda^*$ as the ordinate value corresponding to an abscissa value of $\frac{\pi K}{\alpha \mu^2}$.

As an example, consider the manufacture of some electronic device, perhaps transistors. The items are graded at the end of the manufacturing
process according to some quality characteristic and are assigned a value according to their nearness to some target value \( \mu_0 \), i.e., lower grade items are sold at a lower price. In particular, suppose that

\[
\hat{C}(u) = c \left( \frac{u - \mu_0}{\sigma} \right)^2 \quad \text{for} \quad c = .25 .
\]

When the process goes out of control a shift in the mean takes place from \( \mu_0 = .90 \) to \( \mu_1 = .95 \). The known standard deviation of the process is \( \sigma = .05 \). On the average the process stays in control for about 40 items so that one estimates \( \pi = 1/40 = 0.025 \). When the machine goes out of control repair is instituted at a cost of \( K = 1 \). The repair involves the replacing of a tool costing this amount, \( K \). One has

\[
\mu = (\mu_1 - \mu_0)/\sigma = 1 .
\]

\[
a = [E_1 \hat{C}(U) - E_0 \hat{C}(U)]/\mu^2 = c = .25
\]

\[
\alpha = 2\pi/\mu^2 = 0.05
\]

\[
\frac{\pi K}{\mu^2} = \frac{\pi K}{a_0} = \frac{\pi K}{c} = \frac{0.025}{0.25} = 0.1 .
\]

From Figure 1, one reads the optimal critical value \( \lambda^* = 3.1 \).
References


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The optimal statistical control of a simple production process which has only two underlying states, in control and out of control, is studied. The produced items are assumed to have a measurable quality characteristic which has a normal distribution with known variance. When the process leaves control the mean of this distribution shifts from $\mu_0$ to $\mu_1$, both assumed known. Costs are associated with the quality of an item and with repairing the process which returns it to a state of control. Other than immediately after repair, the process state is assumed unknown. This report gives a statistical control rule which, based on the quality history of produced items, tells when to repair the machine. The rule given minimizes the time average of the total cost of repairs plus quality.
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