A SOLUTION TO A COUNTABLE SYSTEM OF EQUATIONS ARISING IN MARKOVIAN DECISION PROCESSES

BY

CYRUS DERMAN and ARTHUR F. VEINOTT, JR.

TECHNICAL REPORT NO. 89

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Nontechnical Summary

Let \( X_0, X_1, \ldots \) be a sequence of non-negative integer valued random variables with the property that

\[
\Pr(X_{n+1} = j | X_n = x_n, \ldots, X_{n-1} = x_{n-1}, X_n = x_n) = p_{ij}
\]

for all \( i, j, x_0, \ldots, x_n, n \). The collection of random variables \( \{X_n\} \) is called a Markov chain and the \( p_{ij} \) are called transition probabilities. We refer to \( X_n \) as the state of the process at time \( n \).

Let \( v_i \) be the cost incurred at time \( n \) if the process is in state \( i \) at that time. Consider the system of equations

\[
(1) \quad g + v_i = v_i + \sum_{j=0}^{\infty} p_{ij} v_j, \quad i = 0, 1, \ldots
\]

in the unknown variables \( g, v_0, v_1, \ldots \). Such a system arises in connection with constructing optimal rules for controlling Markovian decision processes. Also the numbers \( g, v_0, v_1, \ldots \) are of interest in their own right. Often \( g \) is the long run expected average cost and \( v_i - v_j \) is the limit, as \( n \to \infty \), of the difference between expected total cost during times \( 0, 1, \ldots, n \) given that the process starts in states \( i \) and \( j \) respectively.

We show in this paper that one solution to the system (1) is given by

\[
(2) \quad g = \frac{c_{oo}}{m_{oo}} \quad \text{and} \quad v_i = c_{i0} - gm_{i0}, \quad i = 0, 1, \ldots
\]
provided that the expected time \( m_{10} \) required to go from state \( i \) to state \( 0 \) is finite and that the expected cost \( c_{10} \) incurred during that time is also finite, \( i = 0, 1, \ldots \). Notice that \( v_0 = 0 \).

As an illustration of the above ideas, consider a single item inventory model in which the demands in periods \( 1, 2, \ldots \) are independent. A demand of size one occurs with probability \( p, 0 < p < 1 \), and a demand of size zero occurs with probability \( 1 - p \). Let \( X_n \) denote the stock on hand at the beginning of period \( n \). An order for one unit is placed in period \( n \) with immediate delivery if \( X_n = 0 \); otherwise, no order is placed in period \( n \). There is a unit cost \( h \) for each unit of stock on hand after ordering in a period. There is a cost \( K \) for placing an order in a period. Under these assumptions the nonzero transition probabilities are \( P_{00} = p, P_{01} = 1 - p, P_{ii} = 1 - p, \) and \( P_{i, i-1} = p, i = 1, 2, \ldots \). Also \( w_0 = K + h \) and \( w_i = hi, i = 1, 2, \ldots \). Thus the system (1) becomes

\[
g + v_0 = K + h + pv_0 + (1 - p)v_1
\]

\[
g + v_i = ih + pv_{i-1} + (1 - p)v_i, \quad i = 1, 2, \ldots
\]

The solution given in (2) is

\[
g = pK + h,
\]

\[
v_i = \frac{hi(i-1)}{2p} - Ki, \quad i = 0, 1, \ldots
\]
Thus $g$ is here the long run expected average cost under the indicated ordering policy. Also $v_i$ is the limit, as $n \to \infty$, of the amount by which the expected cost in periods $0, 1, \ldots, n$ starting with $i$ units of stock on hand exceeds that starting with no stock on hand.
A solution to a countable system of equations arising in Markovian decision processes

by

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Let \{X_n\}, \(n = 0, 1, \ldots\), be a Markov chain having a state space consisting of the non-negative integers and having stationary transition probabilities \(\{p_{ij}\}\). Let \(\{w_i\}, i = 0, 1, \ldots\), be a sequence of real numbers. Consider the system of equations

\[
(1) \quad g + v_i = w_i + \sum_{j=0}^{\infty} p_{ij} v_j, \quad i = 0, 1, \ldots,
\]

in the unknown variables \(\{g, v_0, v_1, \ldots\}\). In [2], the system (1) arises in connection with conditions for the existence and construction of optimal rules for controlling a Markovian decision process. For a finite state space existence of solutions to (1) is guaranteed by the condition that the Markov chain have at most one ergodic class of states. (See [3].) In this note we give conditions ensuring the existence (Theorem 1) and uniqueness (Theorem 2) of solutions to (1).

Let

\[
Z_n(j) = \begin{cases} 
1, & \text{if } X_n = j \text{ and if } X_m \neq 0 \text{ for } 0 < m < n \\
0, & \text{otherwise}
\end{cases},
\]

\[
j, n = 0, 1, \ldots,
\]

\[
\sigma^{*}_{ij} = E \left( \sum_{n=0}^{\infty} Z_n(j) | X_0 = i \right), \quad i, j = 0, 1, \ldots
\]
and

\[ m_{i0} = \sum_{j=0}^{\infty} o_{ij}^*, \quad i = 0, 1, \ldots. \]

If the last series converges absolutely, then \( m_{i0} \) is the mean first passage time from \( i \) to 0 and we say \( m_{i0} \) is finite. If the \( m_{i0} \) are all finite, as we assume throughout, then state 0 is positive recurrent and there is only one recurrent class.

Let \( Y_n = \sum_{j=0}^{\infty} w_j Z_n(j) \) and \( c_{i0} = E \left( \sum_{n=0}^{\infty} Y_n | X_0 = i \right) \).

By an obvious generalization of Theorem 5 in [1, p. 81] we get

\[ c_{i0} = \sum_{j=0}^{\infty} o_{ij}^* w_j \]

provided the series is absolutely convergent. If the series is absolutely convergent we say \( c_{i0} \) is finite. In applications \( w_i \) is often the cost incurred when in state \( i \) so \( c_{i0} \) is then the expected cost during a first passage from \( i \) to 0.

**Theorem 1 (Existence)**

If the numbers \( m_{i0} \) and \( c_{i0} \), \( i = 0, 1, \ldots, \) are finite, then the numbers

\[ g = \frac{c_{00}}{m_{00}} \quad \text{and} \quad v_i = c_{i0} - g m_{i0}, \quad i = 0, 1, \ldots \]

satisfy (1) and \( \sum_{j=0}^{\infty} p_{ij} v_j \) converges absolutely, \( i = 0, 1, \ldots. \)

**Proof:**

Let \( w_i^* = v_i - g \) and \( Y_n^* = \sum_{j=0}^{\infty} w_j^* Z_n(j) \). Then for \( i = 0, 1, \ldots \)
\[ v_1 = E \left( \sum_{n=0}^{\infty} Y_n^* | X_0 = i \right) \]
\[ = w_1^* + \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} E(Y_n^* | X_0 = i, X_1 = j) p_{ij} \]
\[ = w_1^* + \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} E(Y_n^* | X_0 = i, X_1 = j) p_{ij} \]
\[ = w_1^* + \sum_{j=0}^{\infty} p_{ij} v_j \]

so (1) holds. The interchange of expectation and summation is justified since the finiteness of the \( m_{i0} \) and \( c_{i0} \) imply that \[ \sum_{n=0}^{\infty} E(|Y_n^*| | X_0 = i) < \infty. \] This in turn implies that the series above are absolutely convergent so the interchange of summations is also justified.

**Theorem 2** (Uniqueness)

If the numbers \( m_{i0} \) and \( c_{i0}, i = 0, 1, \ldots, \) are finite, if
\[ \sum_{j=0}^{\infty} o_{ij}^* \left( c_{i0} - \frac{c_{i0}}{m_{i0}} \right), i = 0, 1, \ldots \]
converges absolutely, and if (\( g, v_0, v_1, \ldots \)) is a sequence with \[ \sum_{j=0}^{\infty} o_{ij}^* v_j, i = 0, 1, \ldots, \]
converging absolutely, then (\( g, v_0, v_1, \ldots \)) satisfies (1) if and only if there is a real number \( r \) such that

\[ g = \frac{c_{i0}}{m_{i0}} \] and \( v_i = c_{i0} - g m_{i0} + r, i = 0, 1, \ldots. \)

**Proof:**

It is immediate from the hypotheses and Theorem 1 that (\( g, v_0, v_1, \ldots \)) defined in (3) satisfies (1) and \[ \sum_{j=0}^{\infty} o_{ij}^* v_j \]
converges absolutely as well as \[ \sum_{j=0}^{\infty} p_{ij} v_j. \] Let (\( g', v_0', v_1', \ldots \)) be
any other solution to (1) with \( \sum_{j=0}^{\infty} p_{ij}^* v_j^i \) converging absolutely for \( i = 0, 1, \ldots \). Hence \( \sum_{k=0}^{\infty} p_{ik} v_k^i \) is absolutely convergent. Now pre-multiplying both sides of (1) by \( \pi_i = \frac{\omega_{i1}}{m_{o1}} \), summing over \( i = 0, 1, \ldots \), using the relations \( \sum_{i=0}^{\infty} \pi_i = 1 \) and \( \pi_j = \sum_{k=0}^{\infty} p_{kj} \pi_k, j = 0, 1, \ldots \), and the fact that the interchange of summations is justified, we get

\[
g^i = \sum_{i=0}^{\infty} \pi_i w_i^i \text{ which is independent of } (v_o^1, v_1^1, \ldots). \text{ Thus since } \\
\{g, v_o^1, v_1^1, \ldots\} \text{ satisfies (1) we must have } g = g^i.
\]

Letting \( \Delta_i = v_i^1 - v_i^1, i = 0, 1, \ldots \), we get from (1) on subtracting one system from the other that

\[
(4) \quad \Delta_i = \sum_{j=0}^{\infty} p_{ij} \Delta_j, \quad i = 0, 1, \ldots.
\]

Let \( p_{ij}^n = \Pr(X_n = j | X_o = i) \). Evidently for \( N = 1, 2, \ldots \),

\[
\sum_{n=1}^{N} p_{ij}^n \leq \omega_{ij}^* + (N-1) \omega_{o,j}^* \quad j = 0, 1, \ldots
\]

so

\[
(5) \quad \frac{1}{N} \sum_{n=1}^{N} p_{ij}^n |\Delta_j| \leq \left[ \omega_{ij}^* + \omega_{o,j}^* \right] |\Delta_j|, \quad j = 0, 1, \ldots.
\]

Since the series on the right side of (5) converges absolutely by hypothesis, and \( \lim \frac{1}{N} \sum_{n=1}^{\infty} p_{ij}^n = \pi_j \), we get from the dominated convergence theorem that
(6) \[ \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} p_{ij}^n A_j = \sum_{j=0}^{\infty} \pi_j A_j. \]

Since from (5), \( \sum_{j=0}^{\infty} p_{ij}^n A_j \) converges absolutely we can iterate (4), yielding

(7) \[ \Delta_i = \sum_{j=0}^{\infty} p_{ij}^n A_j, \quad i = 0, 1, \ldots ; \quad n = 1, 2, \ldots . \]

Hence on substituting (7) into (6)

\[ \Delta_i = \sum_{j=0}^{\infty} \pi_j A_j, \quad i = 0, 1, \ldots . \]

Thus \( \Delta_i \) is independent of \( i \), which completes the proof.

**Example:**

If the sequences \( \{m_i\} \) and \( \{w_i\}, \quad i = 0, 1, \ldots \), are bounded, then so is the sequence \( \{c_{i0}\}, \quad i = 0, 1, \ldots \), since

\[ |c_{i0}| \leq \sup_{k,j} m_{k0} |w_j|. \]

Thus Theorem 1 applies and in addition the solution to (1) given in (2) is bounded. This result is used in [2].

We remark that since

\[ \sum_{j=0}^{\infty} \varphi_{k,j}^* |u_j| \geq \omega_{ok} \sum_{j=0}^{\infty} \varphi_{k,j}^* |u_j|, \]

where

\[ \omega_{ok} = \text{Pr} \left( \sum_{n=0}^{\infty} z_n(k) > 0 | x_0 = 0 \right) > 0 , \]
\sum_{j=0}^{\infty} \sigma \pi_{xj}^k |u_j| \text{ is absolutely convergent for every recurrent state } k

provided that \sum_{j=0}^{\infty} \sigma \pi_{oj}^k |u_j| \text{ is absolutely convergent. Thus the hypotheses of Theorems 1 and 2 could have been stated only for state 0 and the transient states.}
References


A Solution to a Countable System of Equations Arising in Markovian Decision Processes.

A countable system of equations arising in Markovian decision processes is studied. Conditions are given ensuring the existence and uniqueness of an explicit solution to the system.
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