RANK TESTS FOR RANDOMIZED BLOCKS WHEN THE ALTERNATIVES HAVE AN A PRIORI ORDERING

BY

MYLES HOLLANDER

TECHNICAL REPORT NO. 9
June 30, 1965

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Chapter 1

Introduction

Let $X_{ij}$, $i = 1 \ldots n$, $j = 1 \ldots k$ be independent random variables, with $X_{ij}$ having the continuous distribution function

$$P(X_{ij} \leq x) = F_j(x - b_j)$$

where $b_j$ is the nuisance parameter corresponding to "block" $i$. This report is primarily concerned with procedures for testing the null hypothesis

$$H_0: F_j = F \text{ (unknown)}, \ j = 1 \ldots k, \quad (1.1)$$

which are sensitive to the ordered alternatives

$$H_a: F_1 \geq F_2 \geq \cdots \geq F_k, \quad (1.2)$$

where at least one of the inequalities is strict. In particular, in Chapter 2 we investigate the properties of various statistics that have been proposed for this problem, and we introduce a test which will be seen to compare favorably with respect to its competitors. In this chapter asymptotic distribution theory is developed, and necessary and sufficient conditions for consistency are obtained. Chapter 3 is
devoted to asymptotic comparisons of the procedures under consideration. These comparisons are made on the basis of Pitman and Bahadur efficiencies. Finally, in Chapter 4 we modify a multiple comparison method proposed by Nemenyi [10] (which is related to the test statistic introduced in Chapter 2) so that it has the property of being asymptotically distribution-free under the null hypothesis.
Chapter 2

Definitions, Asymptotic Distribution Theory, and Consistency of the Tests

2.1 Descriptions of some standard tests:

Let $X_{ij}, i = 1 \cdots n, j = 1 \cdots k$ be independent and let $F_j(x - b_j)$ denote the continuous distribution function of $X_{ij}$. To test

$H_0: F_j \equiv F$ (unknown), $j = 1 \cdots k$

vs.

$H_a: F_1 \geq F_2 \geq \cdots \geq F_k$,

(where at least one inequality is strict), the following statistics, among others, have been proposed.

(i) (Jonckheere [6]) Let $\tau_i$ denote Kendall's rank correlation coefficient between postulated order and observation order in the $i$th block. Jonckheere's procedure is to reject $H_0$ for large values of

$$\tau = \sum_{i=1}^{n} \tau_i \quad (2.1.1)$$

(ii) (Page [12]) This test is similar to Jonckheere's as Page suggests a rejection region consisting of large values of
\[ \rho = \sum_{i=1}^{n} \rho_i \]  \hspace{1cm} (2.1.2) 

where \( \rho_i \) is Spearman's rank correlation coefficient between postulated order and observation order in block \( i \).

(iii) Let \( X_{ij} = b_i + (j - 1)\theta + \epsilon_{ij} \) where the \( \epsilon_{ij} \) are i.i.d. according to \( N(0, \sigma^2) \). The likelihood ratio statistic of \( H_0: \theta = 0 \), is well known to be

\[ t = \frac{\hat{\theta}}{\hat{\sigma}_\theta} \]  \hspace{1cm} (2.1.3) 

where \( \hat{\theta} \) is the least squares estimate of \( \theta \) and \( \hat{\sigma}_\theta \) is the appropriate estimate of the standard deviation of \( \hat{\theta} \). Specifically, setting

\[ \frac{d}{d\theta} \sum_{i=1}^{n} \sum_{j=1}^{k} (X_{ij} - b_i - (j - 1)\theta)^2 = 0 \]

\[ \frac{d}{db_i} \sum_{i=1}^{n} \sum_{j=1}^{k} (X_{ij} - b_i - (j - 1)\theta)^2 = 0 \]

shows

\[ \hat{\theta} = 6 \cdot \sum_{i-1}^{n} \sum_{j-1}^{k} (2j - k - 1)X_{ij}/nk(k - 1)(k + 1) \]  \hspace{1cm} (2.1.4) 

Also,

\[ \hat{\sigma}_\theta^2 = 12 s^2 /k(k - 1)(k + 1)n \]  \hspace{1cm} (2.1.5)
where

\[ s^2 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{k} (X_{ij} - \hat{b}_i - (j - 1)\hat{\theta})^2}{n(k - 1)} - 1 \] (2.1.6)

and

\[ \hat{b}_i = \frac{\sum_{j=1}^{k} X_{ij}}{k} - \left( \frac{k - 1}{2} \right) \cdot \hat{\theta} \] (2.1.7)

We do not list standard tests of \( H_0 \) such as Friedman's rank test [2] and the usual normal theory \( F \) test as they guard against more general alternatives and do not take the prior ordering into account.

We now proceed to define the \( Y \) statistic.

2.2 **Definition of "Y" test:**

Let \( Y^{(1)}_{uv} = |X_{iu} - X_{iv}| \) and \( R^{(1)}_{uv} = \text{rank of } Y^{(1)}_{uv} \) in the ranking from least to greatest of \( \{Y^{(1)}_{uv}\}_{i=1}^{n} \). Furthermore, let

\[ T_{uv} = \sum_{i=1}^{n} R^{(1)}_{uv} \psi_{uv} \] (2.2.1)

where

\[ \psi_{uv}^{(1)} = \begin{cases} 1 & \text{if } X_{iu} < X_{iv} \\ 0 & \text{otherwise} \end{cases} \] (2.2.2)
The statistic

\[ Y \equiv \sum_{u < v}^{k} T_{uv} \]  \hspace{1cm} (2.2.3)

is proposed as one which will be sensitive to the ordered alternatives \( H_a \neq H_0 \) is to be rejected for large values of \( Y \). Here, \( T_{uv} \) is a measure of the difference between the \( u^{th} \) and \( v^{th} \) treatments, and the particular weighting \( \sum_{u < v}^{k} \) takes into account the prior ordering of the treatments. It should be remarked that in this respect, \( t \) (2.1.3), \( \rho \) (2.1.2) and \( \tau \) (2.1.1) are quite similar in character to \( Y \). This statement can be formalized by the following notes.

**Note 1:** The test based on \( \tau \) is equivalent to one based on the statistic

\[ Z_{\tau} = \sum_{u < v}^{k} N_{uv} \]  \hspace{1cm} (2.2.4)

where

\[ N_{uv} = \text{the number of blocks in which } X_{iu} < X_{iv} \]

**Proof:** (Note 1)

\[ Z_{\tau} = \sum_{u < v}^{k} N_{uv} \]

\[ = \sum_{u < v}^{k} \sum_{i=1}^{n} \psi_{uv}^{(1)} \]
Since, by definition,

\[ \tau_i = \left( 2 \sum_{u < v} \psi_{uv} (i) \right) \left( \frac{k(k - 1)}{2} \right) - 1, \]

we have,

\[ Z_r = \sum_{i=1}^{n} \frac{k(k - 1)}{4} (1 + \tau_i) \]

\[ = \frac{k(k - 1)}{4} (n + \tau) \quad \text{q.e.d.} \]

Note 2: The test based on \( \rho \) is equivalent to one based on the statistic

\[ Z_\rho = \sum_{u < v}^{k} (R_v - R_u) \quad \text{(2.2.5)} \]

where the \( \{R_i\}_{i=1}^{k} \) are defined as follows.

\[ R_u = \sum_{i=1}^{n} r_{iu} \]

where

\[ r_{iu} = \text{rank of } X_{iu} \text{ in the joint ranking of } \{X_{ij}\}_{j=1}^{k} \]
Proof: (Note 2)

\[ Z_p = \sum_{u<v}^k (R_v - R_u) = \sum_{i=1}^n \left( \sum_{u<v}^k (r_{iv} - r_{iu}) \right) \]

But,

\[ \sum_{u<v}^k (r_{iv} - r_{iu}) = \frac{k^3 - k}{6} - \sum_{j=1}^k (r_{ij} - j)^2 \]  \hspace{1cm} (2.2.6)

In fact,

\[ \frac{k^3 - k}{6} - \sum_{j=1}^k (r_{ij} - j)^2 \]

\[ = \frac{(k+1)^2}{2} + 2 \sum_{j=1}^k ji + (k+1)r_{ik} + (k-1)r_{ik} \]

Making the substitution,

\[ r_{ik} = \frac{k(k+1)}{2} - \sum_{j=1}^{k-1} r_{ij} \]

\[ = \sum_{j=1}^k (2j - k - 1)r_{ij} \]

\[ = \sum_{u<v}^k (r_{iv} - r_{iu}) \text{ as (2.2.6) asserts.} \]
Since,

\[ \rho_i = 1 - \frac{6 \sum_{j=1}^{k} (r_{ij} - j)^2}{k^3 - k} , \]

we obtain

\[ Z_\rho = \sum_{i=1}^{n} \left( \frac{k^3 - k}{6} \right) \rho_i \]
\[ = \left( \frac{k^3 - k}{6} \right) \rho \]  \hspace{1cm} (2.2.7)

q.e.d.

Hence, the tests \( \tau, \ Y, \ \rho, \ \text{and} \ t \) all employ the weighting \( \sum_{u<v}^{k} \) with \( N_{uv}, \ T_{uv}, \ (R_v - R_u), \ \text{and} \ (X_v - X_u) \), respectively, all being plausible measures of the difference between the \( u^{th} \) and \( v^{th} \) treatments.\(^1\)

2.3 Asymptotic Distribution Theory:

In this section we develop the asymptotic distribution of \( Y \) (2.2.3). For completeness we state two well known lemmas.

Lemma (2.3.1) (Tukey [14])

If \( S_\perp = \sum_{i=1}^{n} (\text{rank}|X_i|)^2 \psi_i \)

\(^1\)In the remainder of the text \( Z_\tau \) and \( \tau \) (\( Z_\rho \) and \( \rho \)) will be used interchangeably as convenience and clarity dictate.
where

$$\psi_i = \begin{cases} 
1 & \text{if } x_i < 0 \\
0 & \text{otherwise}
\end{cases}$$

and

$$S_2 = \sum_{i<j}^{n} \psi_{ij} + \sum_{i=1}^{n} \psi_i$$

where

$$\psi_{ij} = \begin{cases} 
1 & \text{if } x_i + x_j < 0 \\
0 & \text{otherwise}
\end{cases}$$

then, $S_1 = S_2$.

**Lemma 2.3.2** (Hoeffding [8]). Let $[X'_i]_{i=1}^{n}$ be $n$ independent, identically distributed random vectors. Let $\phi^{(i)}(x'_i, ..., x'_{m(i)})$, $i = 1, ..., g$ be $g$ real-valued functions not involving $n$, $\phi^{(i)}$ being symmetric in its $m(i)$ ($\leq n$) vector arguments $x'_i$. Define $U^{(i)} = (\frac{n}{m(i)})^{-1} \sum_{\alpha_{1} < \cdots < \alpha_{m(i)}} \phi^{(i)}(x'_{\alpha_{1}}, ..., x'_{\alpha_{m(i)}})$ where the summation extends over all subscripts such that $1 \leq \alpha_{1} < \cdots < \alpha_{m(i)} \leq n$. Then if $E[\phi^{(i)}(x'_1, ..., x'_{m(i)})]$ and $E[\phi^{(i)}(x'_1, ..., x'_{m(i)})]^2$ exist, the joint d.f. of the random variables $[\sqrt{n}(U^{(i)} - E(U^{(i)}))]_{i=1}^{g}$ is asymptotically normal.
Theorem 2.3.3

Let \( X_{ij} \) have the continuous distribution function \( F_j(x - b_i) \). Furthermore, assume \( 0 < \int F_i \, dF_j < 1 \) for each \( (i, j) \). Then \( Y \), suitably normed, is asymptotically normal.

Proof:

From (2.2.1) and lemma (2.3.1) we rewrite \( T_{uv} \) as

\[
T_{uv} = \sum_{i<j}^{n} \psi_{uv}^{(i,j)} + \sum_{i=1}^{n} \xi_{uv}^{(i)} \tag{2.3.1}
\]

where

\[
\psi_{uv}^{(i,j)} = \begin{cases} 
1 & \text{if } X_{iu} - X_{iv} + X_{ju} - X_{jv} < 0 \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\xi_{uv}^{(i)} = \begin{cases} 
1 & \text{if } X_{iu} < X_{iv} \\
0 & \text{otherwise}
\end{cases}
\]

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Since \(0 < \int_{uv} F \, dF < 1\) (if such were not the case \(T_{uv}\) would equal 0 or \(\frac{n(n+1)}{2}\), with probability 1), and writing

\[U(u,v) = \frac{T_{uv}}{n(n - 1)}\]

we apply lemma 2.3.2 and conclude the random variables

\[
\sqrt{n} \left( U(u,v) - E(U(u,v)) \right) \quad 1 \leq u < v \leq k
\]

have an asymptotic \(\frac{k(k - 1)}{2}\) - variate normal distribution. Since \(Y\) is a linear combination of the \([T_{uv}]\), the result is established.

q.e.i.

In order to characterize the asymptotic distribution of \(Y\), we need only derive its mean and variance. We postpone this calculation under the general \([F_j]\) alternative until Chapter 4 and proceed to the null situation when \(F_j = F, j = 1 \cdots k\). On the null hypothesis, \(T_{uv}\) has the same distribution as the random variable \(T\) where

\[T = \sum_{i=1}^{n} \delta_{uv}^{(i)} \quad (2.3.2)\]

where

\[
\delta_{uv}^{(i)} = \begin{cases} 
1 \text{ w.p. } 1/2 \\
0 \text{ w.p. } 1/2 
\end{cases}
\]

and the \([\delta_{uv}^{(i)}]_{i=1}^{n}\) are independent.
Hence,

$$E_o(Y) = \frac{k(k-1)}{2}E(T) = \frac{k(k-1)n(n+1)}{6} \quad (2.3.3)$$

Before deriving $\sigma_o^2(Y)$, the following comment is appropriate. The rank tests, $\rho^1$ and $\tau$, mentioned in Section 2.1 are distribution-free under the null hypothesis. One may ask if $Y$ also has this property. In fact, since each $T_{uv}$ is distribution-free, it is not unreasonable to expect this to also be true of $Y$. However, this is not the case as the null correlation coefficient $\rho_o^n(F)$ between $T_{uv}$ and $T_{uw}$, $u \neq v \neq w$, depends on $F$ (except for $n = 1$), and hence so does the null variance of $Y$. We now derive the expression for $\rho_o^n(F)$.

Theorem 2.3.4. Under $H_o$, the correlation coefficient $\rho_o^n(F)$ between $T_{uv}$ and $T_{uw}$, $u \neq v \neq w$, is,

$$\rho_o^n(F) = \frac{[(24\lambda(F) - 6)n^2 + (48\mu(F) - 72\lambda(F) + 7)n + (48\lambda(F) - 48\mu(F) + 1)]}{(n + 1)(2n + 1)} \quad (2.3.4)$$

where

$$\mu(F) = P(X_1 < X_2; X_1 < X_5 + X_6 - X_7) \quad (2.3.5)$$

and

$$\lambda(F) = P(X_1 < X_2 + X_3 - X_4; X_1 < X_5 + X_6 - X_7) \quad (2.3.6)$$

when $X_1, X_2, \ldots, X_7$ are i.i.d. according to $F$. 

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Proof:

We first find $E_o(T_{uv} \cdot T_{uw})$. From (2.3.1) we have,

$$T_{uv} \cdot T_{uw} = \left( \sum_{i < j} \psi_{uv}^{(i,j)} \right) \left( \sum_{i < j} \psi_{uw}^{(i,j)} \right) + \left( \sum_{i < j} \psi_{uv}^{(i,j)} \right) \left( \sum_{i=1}^{n} \xi_{uv}^{(i)} \right)$$

$$+ \left( \sum_{i < j} \psi_{uw}^{(i,j)} \right) \left( \sum_{i=1}^{n} \xi_{uw}^{(i)} \right) + \left( \sum_{i=1}^{n} \xi_{uv}^{(i)} \right) \left( \sum_{i=1}^{n} \xi_{uw}^{(i)} \right),$$

and

$$E_o(T_{uv} \cdot T_{uw}) = E_o \left( \sum_{i < j} \psi_{uv}^{(i,j)} \right) \left( \sum_{i < j} \psi_{uw}^{(i,j)} \right)$$

$$+ 2E_o \left( \sum_{i < j} \psi_{uv}^{(i,j)} \right) \left( \sum_{i=1}^{n} \xi_{uv}^{(i)} \right) + E_o \left( \sum_{i=1}^{n} \xi_{uv}^{(i)} \right) \left( \sum_{i=1}^{n} \xi_{uw}^{(i)} \right)$$

$$= E_o(A_1) + 2E_o(A_2) + E_o(A_3) \quad (2.3.7)$$

where $A_1$, $A_2$, and $A_3$ are the corresponding expressions in (2.3.7).

$$E_o(A_1) = \frac{n(n - 1)}{2} E_o \left[ \psi_{uv}^{(i,j)}, \psi_{uw}^{(i,j)} \right]$$

$$+ n(n - 1)(n - 2) E_o \left[ \psi_{uv}^{(i,j)}, \psi_{uw}^{(j,k)} \right]$$

$$+ \frac{n(n - 1)(n - 2)(n - 3)}{4} E_o \left[ \psi_{uv}^{(i,j)}, \psi_{uw}^{(k,l)} \right]$$

$$= \frac{n(n - 1)}{2} p_o (X_{iu} - X_{iv} + X_{ju} - X_{jv} < 0; X_{iu} - X_{iw} + X_{ju} - X_{jw} < 0)$$
\[ + n(n-1)(n-2)p_o(x_{iu} - x_{iv} + x_{ju} - x_{jv} < 0; \]
\[ x_{ju} - x_{jw} + x_{ku} - x_{kw} < 0) \]
\[ + \frac{n(n-1)(n-2)(n-3)}{4} p_o(x_{iu} - x_{iv} + x_{ju} - x_{jv} < 0; \]
\[ x_{ku} - x_{kw} + x_{iu} - x_{jw} < 0) \]
\[ E_o(A_1) = \frac{n(n-1)}{2} \cdot \frac{1}{2} + n(n-1)(n-2) \cdot \lambda(F) \]
\[ + \frac{n(n-1)(n-2)(n-3)}{4} \cdot \frac{1}{4} \quad (2.3.8) \]

Also,
\[ E_o(A_2) = n(n-1)E_o(\psi_{uv}^{(i,j)} \cdot \psi_{uw}^{(i)}) + \frac{n(n-1)(n-2)}{2} E_o(\psi_{uv}^{(i,j)} \cdot \psi_{uw}^{(k)}) \]
\[ = n(n-1)p_o(x_{iu} - x_{iv} + x_{ju} - x_{jv} < 0; x_{iu} < x_{iw}) \]
\[ + \frac{n(n-1)(n-2)}{2} p_o(x_{iu} - x_{iv} + x_{ju} - x_{jv} < 0; x_{ku} < x_{kw}) \]

Hence,
\[ E_o(A_2) = n(n-1) \cdot \mu(F) + \frac{n(n-1)(n-2)}{2} \cdot \frac{1}{4} \quad (2.3.9) \]

Finally,
\[ E_o(A_3) = n \left[ \psi_{uv}^{(i)} \cdot \psi_{uw}^{(i)} \right] + n(n-1)E_o\left[ \psi_{uv}^{(i)} \cdot \psi_{uw}^{(j)} \right] \]
\[ = n p_o(x_{iu} < x_{iv}; x_{iu} < x_{iw}) + n(n-1)p_o(x_{iu} < x_{iv}; x_{ju} < x_{jw}) \]
Thus,

\[ E_o(A_3) = n \cdot \frac{1}{3} + n(n - 1) \cdot \frac{1}{4} \quad (2.3.10) \]

Combining (2.3.7), (2.3.8), (2.3.9) and (2.3.10), we obtain

\[ E_o(T_{uv} \cdot T_{uw}) = \left[ 3n^4 + (48\lambda(F) - 6)n^3 + (96\mu(F) - 144\lambda(F) + 17)n^2 \\
+ (96\lambda(F) - 96\mu(F) + 2)n \right] / 48 \]

\[ (2.3.11) \]

From 2.3.1 (or 2.3.2) it is easily seen that (these are well known expressions)

\[ o_o^2(T_{uv}) = \frac{n(n + 1)(2n + 1)}{24} \quad (2.3.12) \]

and

\[ E_o(T_{uv}) = \frac{n(n + 1)}{4} \quad (2.3.13) \]

The result then follows from (2.3.11), (2.3.12) and (2.3.13).

q.e.d.

We now discuss the null correlation \( \rho_o^n(F) \) in more detail and state

**Corollary 2.3.5**

\[ 0 < \rho^*(F) \leq \frac{1}{2} \], where

\[ \rho^*(F) = \lim_{n} \rho_o^n(F) \]

**Proof:**

From (2.3.4) we immediately see that

\[ \rho^*(F) = 12\lambda(F) - 3 \quad (2.3.14) \]
In [9], Lehmann proves that

\[
\frac{6}{24} < \lambda(F) \leq \frac{7}{24}
\]

and the bounds on \( \rho^*(F) \) follow. \( \text{q.e.d.} \)

Lehmann gives the three values of \( \lambda(F) \):

<table>
<thead>
<tr>
<th>( \lambda(F) )</th>
<th>Normal</th>
<th>Rectangular</th>
<th>Cauchy</th>
</tr>
</thead>
<tbody>
<tr>
<td>.2902</td>
<td>.2909</td>
<td>.2879</td>
<td></td>
</tr>
</tbody>
</table>

When \( F(x) = 1 - e^{-x} \), \( \lambda(F) = .2894 \) and some values of \( \mu(F) \) are:

<table>
<thead>
<tr>
<th>( \mu(F) )</th>
<th>Normal</th>
<th>Rectangular</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>.3075</td>
<td>.3083</td>
<td>.3056</td>
<td></td>
</tr>
</tbody>
</table>

We illustrate the calculation of \( \mu(F) \) when \( F \) is normal, and \( \lambda(F) \) when \( F \) is exponential.

1. \( \mu(\phi) = P(X_1 < X_2; X_1 < X_5 + X_6 - X_7) \) where \( X_1, X_2, X_5, X_6, X_7 \) i.i.d. according to \( \text{N}(0, 1) \). If \( Z_1 = X_1 - X_2 \), and \( Z_2 = X_1 - X_5 - X_6 + X_7 \), then \( \mu(\phi) = P(Z_1 < 0; Z_2 < 0) \) where

\[
\rho_{Z_1, Z_2} = \frac{1}{2 \sqrt{2}}.
\]

Hence, using Stieltjes' result (see, e.g., [4]),

\[
\mu(\phi) = \frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{1}{2 \sqrt{2}} = .3075
\]

\(^1\)The lower bound follows from Schwarz's inequality, the strict part a consequence of the fact that a distribution function cannot be a constant.
2. \( \lambda(E) = P(X_1 < X_2 + X_3 - X_4; X_1 < X_5 + X_6 - X_7) \) where 
\( X_1, X_2, \cdots, X_7 \) i. i. d. according to \( F(x) = 1 - e^{-x} \). Equivalently,

\[
\lambda(E) = \int_{0}^{\infty} F(x) dx [1 - (1 - G(x))^2]
\]

(2.3.15)

where

\[
G(x) = P(X_2 + X_3 - X_4 \leq x)
\]

A simple calculation shows

\[
G(x) = 1 - \frac{3e^{-x}}{4} + \frac{xe^{-x}}{2} \quad \text{when } x > 0
\]

Then, evaluating (2.3.15) yields

\[
\lambda(E) = .2894
\]

Utilizing (2.3.14), we obtain these values for \( \rho^*(F) \)

<table>
<thead>
<tr>
<th>F</th>
<th>Normal</th>
<th>Rectangular</th>
<th>Exponential</th>
<th>Cauchy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho^*(F) )</td>
<td>.482</td>
<td>.491</td>
<td>.472</td>
<td>.455</td>
</tr>
</tbody>
</table>

Also, from (2.3.4) and the corresponding values of \( \mu(F) \) and \( \lambda(F) \), we form the following table.
Table 2.1

The entries are $\phi_o^n(F)$ for $F$ normal, rectangular and exponential.

<table>
<thead>
<tr>
<th>n</th>
<th>Normal</th>
<th>Rectangular</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
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<td>.333</td>
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<td>.456</td>
<td>.463</td>
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We are now ready to state the null variance of $Y$.

**Corollary 2.3.6:**

$$\phi_o^2(Y) = \frac{n(n + 1)(2n + 1)(k(k - 1)}{24} + k(k - 1)(k - 2) \cdot \phi_o^n(F)$$

where $\phi_o^n(F)$ is given by (2.3.4).  

(2.3.16)
Proof:

\[ Y = \sum_{u < v}^{k} T_{uv} \]

and hence,

\[ \sigma^2_{o}(Y) = \sum_{u < v}^{k} \sigma^2_{o}(T_{uv}) + 2 \sum_{u < v < w} \text{Cov}_{o}(T_{uv}, T_{uw}) \]

\[ + 2 \sum_{u < v < w} \text{Cov}_{o}(T_{uv}, T_{uw}) + 2 \sum_{u < v < w} \text{Cov}_{o}(T_{uw}, T_{vw}) \]  \hspace{5cm} (2.3.17)

Terms of the form \( \text{Cov}_{o}(T_{uv}, T_{wx}) \), \( u \neq v \neq w \neq x \) do not appear in (2.3.17) since they are equal to zero due to the independence of \( T_{uv} \) and \( T_{wx} \). Also, by employing the obvious symmetries we obtain

\[ \sigma^2_{o}(Y) = \sum_{u < v}^{k} \sigma^2_{o}(T_{uv}) + \frac{2k(k-1)(k-2)}{3} \text{Cov}_{o}(T_{uv}, T_{uw}) \]

\[ + \frac{k(k-1)(k-2)}{3} \text{Cov}_{o}(T_{uw}, T_{vw}) \]

Noting that

\[ \text{Cov}_{o}(T_{uv}, T_{uw}) = - \text{Cov}_{o}(T_{uv}, T_{vw}) \]

and making the substitution (2.3.12), yields expression (2.3.16).

q.e.d.

In view of (2.3.16), in order that the test based on \( Y \) be asymptotically distribution-free, we require a consistent estimate of \( \rho^{*}(F) \). One possible estimate can be obtained as follows. In [9],

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Lehmann proposed the unbiased and consistent estimate \( \hat{\lambda}(F) \) of \( \lambda(F) \), where
\[
\hat{\lambda}(F) = \sum_{\substack{\alpha \neq \beta \neq \gamma; u, v, w}} \frac{\beta_1(\alpha, \beta, \gamma; u, v, w)}{n(n - 1)(n - 2)k(k - 1)(k - 2)} 
\]
(2.3.18)
where
\[
\beta_1(\alpha, \beta, \gamma; u, v, w) = \begin{cases} 
1 & \text{if } X_{\alpha u} < X_{\beta u} + X_{\alpha v} - X_{\beta v} \\
X_{\alpha u} < X_{\gamma u} + X_{\alpha w} - X_{\gamma w} & \\
0 & \text{otherwise}
\end{cases}
\]
Similarly, one unbiased and consistent estimate of \( \mu(F) \) is
\[
\hat{\mu}(F) = \sum_{\substack{\alpha \neq \beta; u, v, w}} \frac{\beta_2(\alpha, \beta; u, v, w)}{n(n - 1)k(k - 1)(k - 2)}
\]
where
\[
\beta_2(\alpha, \beta; u, v, w) = \begin{cases} 
1 & \text{if } X_{\alpha u} < X_{\alpha v}; X_{\alpha u} < X_{\alpha w} + X_{\beta u} - X_{\beta w} \\
0 & \text{otherwise}
\end{cases}
\]
Now, if in expression (2.3.4) we replace \( \lambda(F) \) and \( \mu(F) \) by \( \hat{\lambda}(F) \) and \( \hat{\mu}(F) \) respectively, and call the resulting expression \( \hat{\rho}^n_o(F) \), it is clear that \( \hat{\rho}^n_o(F) \) is an unbiased estimate of \( \rho^n_o(F) \) and a consistent estimate of \( \rho^*(F) \). However, the computation of the estimate
\( \hat{\rho}^n_o(F) \) is very tedious, and since the approach to \( \rho^*(F) \) is rapid, the consistent estimate

\[
\hat{\rho} = -3 + 12\hat{\lambda}(F) \tag{2.3.19}
\]

would suffice. Also, as Lehmann mentions, \( \hat{\lambda}(F) \) itself is computationally tedious and in practice an estimate based on a small subset of the original number of six-tuples \((\alpha, \beta, \gamma; u, v, w)\) should be used. In fact, we can utilize the prior ordering in deciding what subset of inequalities should be checked to estimate \( \lambda(F) \). Specifically, we propose an estimate of a slightly different form, namely,

\[
\hat{\lambda}_1(F) = \sum_{(\alpha, \beta, \gamma) \neq (\alpha', \beta', \gamma')} \frac{\beta_3(\alpha, \beta, \gamma)}{n(\alpha, \beta, \gamma)}
\]

where

\[
\beta_3(\alpha, \beta, \gamma) = \begin{cases} 
1 & \text{if } x_{\alpha_k} < x_{\alpha_1} + x_{\beta_1} - x_{\beta_k}; x_{\alpha_k} < x_{\alpha_2} + x_{\gamma_1} - x_{\gamma_k} \\
0 & \text{otherwise}
\end{cases}
\]

Under \( H_a \), the above system of inequalities would tend to be satisfied less frequently than a set which is symmetric in the column subscript, and hence we would be increasing the power of the \( Y \) test against \( H_a \). Thus, the proposed test is to reject \( H_o \) at the \( \alpha \)-level if

\[
Y > F_o(Y) + z_{1-\alpha} \hat{\rho}_o(Y)
\]
where $\hat{\omega}_0^2(Y)$ is obtained by replacing $\hat{\rho}_0^n(F)$ by $\hat{\rho}_1$ in (2.3.16)

where

$$\hat{\rho}_1 = -3 + 12\hat{\lambda}_1(F)$$

and $z_{1-\alpha}$ is the $1-\alpha^{th}$ percentile point of a unit normal random variable.

With respect to the correlation coefficient tests, we repeat the well known facts that $\tau$ (2.1.1) and $\rho$ (2.1.2) are both distribution-free under $H_0$ as $\rho_i$ and $\tau_i$ enjoy this property. Also, the asymptotic normality of $\tau$ and $\rho$ under $H_0$ and $H_A$ is a simple consequence of the central limit theorem.

We now advance to Section 2.4 in which we investigate the conditions under which the tests are consistent.

2.4 Necessary and sufficient conditions for consistency:

In this section we derive consistency conditions for the various tests under the $H_A$ assumptions. The proof of each theorem utilizes the following standard argument. If $[T_n]$ is the sequence of test statistics in question, at the $\alpha$-level of significance we reject $H_0$ if $T_n > a_n$ where

$$P_0(T_n > a_n) = \alpha$$

---

1Here, and in the sequel, $H_A$ denotes the alternatives $P(X_{ij} \leq x) = \frac{F_j(x - b_i)}{\sum_{j=1}^{k} \frac{F_j(x - b_i)}{F_j(x - b_i)}}$ with the $\frac{F_j(x - b_i)}{\sum_{j=1}^{k} \frac{F_j(x - b_i)}{F_j(x - b_i)}}$ continuous. We exclude the case where $\int \int dF u dF v = 0$ or $1$ for every $(u, v)$ ($u \neq v$) pair, for then $Y$, $\rho$, and $\tau$ are constants with probability one.
Under $H_0$, we show
\[ p - \lim_{n \to \infty} \frac{T_n}{d_n} = C \]

where $d_n$ is a sequence of norming constants. Also,
\[ \lim_{n \to \infty} \frac{a_n}{d_n} = C \]

Under the alternative, we show
\[ p - \lim_{n \to \infty} \frac{T_n}{d_n} = h(F_1, F_2, \cdots, F_k) \]

Then a sufficient condition for
\[ \lim_{n \to \infty} P_A(T_n > a_n) = 1 \]

is
\[ h(F_1, F_2, \cdots, F_k) > C \quad (2.4.1) \]

The fact that $T_n$ is asymptotically normal under $H_0$ and $H_A$ is enough to insure that $(2.4.1)$ is also necessary. Having stated this argument, we turn to the consistency of $\tau$.

**Theorem 2.4.1:** $\tau$ is consistent if and only if
\[ \frac{1}{k(k-1)} \sum_{u < v} P(X_u < X_v) = \frac{1}{k(k-1)} \sum_{u < v} \int_u^v F_u dF_v > 1/4 \]

**Proof:** Using (2.2.4), we have
\[ Z_\tau = \sum_{u < v} N_{uv} \]

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\[ E_0(Z_1) = \frac{k(k - 1)}{4} \cdot n \]

\[ E_A(Z_1) = \sum_{u < v} \sum_{i=1}^{n} E_A(i_{uv}) \]

\[ = n \sum_{u < v} \int_{u}^{v} F_u dF_v \] (2.4.2)

Also,

\[ \sigma_A^2(Z_1) = \sum_{u < v} \sigma_A^2(N_{uv}) + 2 \sum_{u < v < w} \text{Cov}_A(N_{uv}, N_{uw}) \]

\[ + 2 \sum_{u < v < w} \text{Cov}_A(N_{uv}, N_{vw}) + 2 \sum_{u < v < w} \text{Cov}_A(N_{uw}, N_{vw}) \] (2.4.3)

Simple calculations show

(i) \[ \sigma_A^2(N_{uv}) = n p_{uv} \cdot (1 - p_{uv}) \] (2.4.4)

where \[ p_{uv} = \int F_u dF_v \]

(ii) \[ \text{Cov}_A(N_{uv}, N_{uw}) = n \left[ \int F_u dF_v + F_v - F_w \cdot F_v - p_{uv} \cdot p_{uw} \right] \] (2.4.5)

(iii) \[ \text{Cov}_A(N_{uv}, N_{vw}) = n \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_u(x_v) dF_v(x_v) dF_w(x_w) - p_{uv} \cdot p_{vw} \right] \] (2.4.6)

(iv) \[ \text{Cov}_A(N_{uw}, N_{vw}) = n \left[ \int F_u \cdot F_v dF_w - p_{uw} \cdot p_{vw} \right] \] (2.4.7)

It is obvious that

\[ \sigma_A^2(Z_1) = O(n) \]
and hence it follows from Chebychev's inequality that under $H_0$,

$$p - \lim \frac{Z_T}{nk(k - 1)} = \frac{1}{4}$$

while under $H_A$,

$$p - \lim \frac{Z_T}{nk(k - 1)} = \frac{\sum_{u<v} \int F_u dF_v}{k(k - 1)}$$

q.e.d.

**Theorem 2.4.2:** A necessary and sufficient condition for the consistency of the test based on $\rho$ is

$$\sum_{j=1}^{k} (2j - k - 1) \cdot \left( \sum_{j=1}^{k} \int F_{\alpha_j} dF_{\alpha_j} \right) > 0,$$

or equivalently,

$$\sum_{u<v} (v - u) \int F_u dF_v > \frac{k(k - 1)(k + 1)}{12}$$

**Proof:** From (2.2.5)

$$Z_\rho = \sum_{u<v} (R_u - R_v)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{k} (2j - k - 1)r_{ij}$$

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Hence,

\[ E_A(Z_p) = n \cdot \sum_{j=1}^{k} (2j - k - 1)E_A(r_j) \]  

(2.4.8)

Since,

\[ r_j = \frac{k + 1}{2} + \frac{1}{2} \sum_{i=1}^{k} s(X_j - X_i) \]  

(2.4.9)

where

\[ s(x) = \begin{cases}  
-1 & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
1 & \text{if } x > 0 
\end{cases} \]

we have

\[ E_A(r_j) = \frac{k + 1}{2} + \frac{1}{2} \sum_{i=1}^{k} \left(2 \int f_i dF_j - 1\right) \]

Then, upon substitution into (2.4.8), we find

\[ E_A(Z_p) = n \cdot \sum_{j=1}^{k} (2j - k - 1) \cdot \left(\sum_{\alpha=1}^{k} \int f_\alpha dF_j\right) \]  

(2.4.10)

Also,

\[ E_o(Z_p) = 0 \]

Whenever the meaning is clear, we drop the block subscript; e.g., \( r_{ij} \) becomes \( r_j \), \( X_{iu} \) becomes \( X_{u} \), etc.
Now,

\[ \sigma_A^2(Z_p) = n \left[ \sum_{j=1}^{k} (2j - k - 1)^2 \sigma_A^2(r_j) + \sum_{j \neq j'}^{k} (2j - k - 1)(2j' - k - 1) \text{Cov}_A(r_j, r_{j'}) \right] \] (2.4.11)

From (2.4.9), we see that

\[ \sigma_A^2(r_j) = \frac{\sigma_A^2 \left( \sum_{i=1}^{k} s(X_j - X_i) \right)}{4} \]

Now,

\[ E_A \left( \sum_{i=1}^{k} s(X_j - X_i) \right)^2 = \sum_{i=1}^{k} E_A(s(X_j - X_i))^2 \]

\[ + \sum_{i \neq i'}^{k} E_A(s(X_j - X_i) \cdot s(X_j - X_{i'})) \]

\[ = (k - 1) + \sum_{i \neq i'}^{k} P_j:(i,i') \]

where

\[ P_j:(i,i') = P(X_i < X_j; X_{i'} < X_j) + P(X_i > X_j; X_{i'} > X_j) \]

\[ - P(X_i < X_j; X_{i'} < X_i) - P(X_i > X_j; X_{i'} < X_i) \]

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Also,

\[ E_A \left( \sum_{i=1}^{k} s(X_j - X_i) \right) = \sum_{i=1}^{k} \left( 2 \int_{i} F_j dF_j - 1 \right) \]

Hence,

\[ \sigma_A^2(r_j) = \frac{k - 1}{4} + \sum_{i \neq i'} P_{j:(i,i')} - \left( \sum_{i=1}^{k} \int_{i} F_j dF_j \right)^2 + k \sum_{i=1}^{k} \int_{i} F_j dF_j - k^2/4 \]

(2.4.12)

Also,

\[ \text{Cov}_A(r_j, r_{j'}) = 1/4 \text{Cov}_A \left( \sum_{i=1}^{k} s(X_j - X_i), \sum_{i=1}^{k} s(X_{j'}, - X_i) \right) \]

\[ E_A \left( \sum_{i=1}^{k} s(X_j - X_i) \cdot \sum_{i=1}^{k} s(X_{j'}, - X_i) \right) \]

\[ = \sum_{i=1}^{k} E_A(s(X_j - X_i) \cdot s(X_{j'}, - X_i)) \]

\[ + \sum_{i \neq i'} E_A(s(X_j - X_i)) \cdot E_A(s(X_{j'}, - X_i')) \]

\[ = \sum_{i=1}^{k} P_{i:(j,j')} + \sum_{i \neq i'} \left( 2 \int_{i} F_j dF_j - 1 \right) \cdot \left( 2 \int_{i} F_{j'} dF_{j'} - 1 \right) \]
Hence,

\[
\text{Cov}_A(r_j, r_{j'}) = \frac{1}{4} \sum_{i=1}^{k} \left[ F_i: (j, j') - 4 \left( \int F_i dP_j \right) \left( \int F_i dP_{j'} \right) \right. \\
\left. + 2 \int F_i dF_j + 2 \int F_i dF_{j'}, - 1 \right] \quad (2.4.13)
\]

Combining (2.4.11 - 2.4.13) we obtain \( \sigma^2_A(Z_0) \), and note that

\[
\sigma^2_A(Z_0) = O(n)
\]

Hence, by Chebychev's inequality, under \( H_0 \)

\[
p - \lim \frac{Z_0}{n} = 0
\]

while under \( H_A \),

\[
p - \lim \frac{Z_0}{n} = \sum_{j=1}^{k} (2j - k - 1) \cdot \left( \sum_{a=1}^{k} \int F_a dF_j \right)
\]

q.e.d.

Theorems 2.4.1 and 2.4.2 imply that \( \rho \) and \( \tau \) are not consistent against the same class of alternatives. In fact, to illustrate this we need only exhibit orderings of the integers \( 1, \ldots, k \) for which Kendall's and Spearman's correlation coefficients (with respect to the natural ordering) have opposite signs. Then, if we choose the distribution functions \( [F_j] \) so that the corresponding random variables achieve these orderings with sufficiently high probability, we will have produced alternatives where \( \rho(\tau) \) is consistent and \( \tau(\rho) \) is not. As
a simple example, consider the ordering

\[ 6 \ 3 \ 1 \ 2 \ 4 \ 5 \]

The \( \tau \) correlation between \((6 \ 3 \ 1 \ 2 \ 4 \ 5)\) and \((1 \ 2 \ 3 \ 4 \ 5 \ 6)\) is \( +\frac{1}{15} \) while the corresponding \( \rho \) measure is \( -\frac{1}{35} \). If we then act (with \( \theta > 0 \))

\[
\begin{align*}
F_6 &= N(\theta, \sigma^2) \\
F_3 &= N(2\theta, \sigma^2) \\
F_1 &= N(3\theta, \sigma^2) \\
F_2 &= N(4\theta, \sigma^2) \\
F_4 &= N(5\theta, \sigma^2) \\
F_5 &= N(6\theta, \sigma^2)
\end{align*}
\]

and choose \( \sigma \) sufficiently small,

\[
\frac{6}{30} \sum_{i<j} F_{i}(x)dF_{j}(x) > \frac{1}{4}
\]

but

\[
\sum_{j=1}^{6} (2j - 6 - 1) \cdot \left( \frac{6}{3} \sum_{\alpha=1}^{6} F_{\alpha}(x)dF_{j}(x) \right) < 0
\]

and hence the test based on \( \tau \) will be consistent against the above \([F_j]\) alternative, but the test based on \( \rho \) will not. If instead, we considered the ordering

\[ 3 \ 2 \ 6 \ 5 \ 1 \ 4 \]

for which \( \tau = -1/15 \) and \( \rho = +1/35 \) we could, in the same manner, construct examples where the opposite consistency conclusion holds.
The correlation coefficients between postulated order and natural order are readily interpretable and sometimes confidence intervals for these unknown parameters are desirable. By estimating $\sigma_A^2(\rho)$ and $\sigma_A^2(\tau)$, we can develop asymptotic confidence intervals for

$$\tau' = \frac{1}{k(k-1)} \sum_{u < v} \int \frac{u}{v} dF_v$$

and

$$\rho' = \frac{6}{(k^2 - k)} \sum_{j=1}^{k} (2j - k - 1) \cdot \sum_{i=1}^{k} \int F_i dF_j$$

First let us define the random variables

$$e_{uvw} = \frac{1}{n} \sum_{i=1}^{n} e_i(u, v, w)$$

$$f_{uvw} = \frac{1}{n} \sum_{i=1}^{n} f_i(u, v, w)$$

$$g_{uvw} = \frac{1}{n} \sum_{i=1}^{n} g_i(u, v, w)$$

where $e_i(u, v, w)$, $f_i(u, v, w)$, and $g_i(u, v, w)$ are, respectively, the indicator functions of the events
\[ E^{(i)}_{uvw} = [X_{iu} < X_{iv}; X_{iu} < X_{iw}] \]
\[ F^{(i)}_{uvw} = [X_{iu} < X_{iv} < X_{iw}] \]
\[ G^{(i)}_{uvw} = [X_{iu} < X_{iw}; X_{iv} < X_{iw}] \], \quad i = 1 \cdots n

Recollecting that

\[ \bar{\tau} = \frac{\sum_{i=1}^{n} \tau_i}{n} = \frac{4 \cdot Z}{nk(k - 1)} - 1, \]

it follows from (2.4.3) that

\[
\hat{\sigma}^2 = \frac{16}{k^2(k - 1)^2} \left[ \sum_{u < v}^{k} \frac{N_{uv}}{n} \left( 1 - \frac{N_{uv}}{n} \right) + 2 \sum_{u < v < w} \left( e_{uvw} - \frac{N_{uv}}{n} \cdot \frac{N_{uw}}{n} \right) \right]
+ 2 \sum_{u < v < w} \left( f_{uvw} - \frac{N_{uv}}{n} \cdot \frac{N_{vw}}{n} \right) + 2 \sum_{u < v < w} \left( g_{uvw} - \frac{N_{uv}}{n} \cdot \frac{N_{vw}}{n} \right)
\]

is a consistent estimate of \( \hat{\sigma}^2(\tau_i) \). From the asymptotic normality of \( \bar{\tau} \) and the fact that \( E_{\lambda}(\tau) = \tau' \), we see that

\[ I_1 = \left( -z_{1-\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}} + \bar{\tau}, z_{1-\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}} + \bar{\tau} \right) \]

is an asymptotic \( 1 - \alpha \) confidence interval for \( \tau' \). For \( \rho' \), we proceed in the same manner. We define \( \hat{\sigma}^2(r_j) \) and \( \text{Cov}(r_j, r_j') \) by replacing \( \int \beta_{ij} dF_j(i 
eq j) \) and \( P_j[(i, i')] \) by their estimates, \( \frac{N_{ij}}{n} \)
and \((e_{ii'} + e_{ii''} - f_{i'i''} - f_{i'i'})\) respectively, in expressions (2.4.12) and (2.4.13). Then, since

\[
\hat{\rho} = 6 \cdot \hat{z}_\rho / (k^3 - k) n,
\]

\[
\hat{\sigma}^2 = \frac{36}{(k \cdot 3 - k)^2} \left[ \sum_{j=1}^{k} (2j - k - 1)^2 \hat{\sigma}^2(r_j) \right. \\
+ \left. \sum_{j \neq j'} (2j - k - 1)(2j' - k - 1) \text{Cov}(r_j, r_{j'}) \right]
\]

is a consistent estimate of \(\sigma^2_A(\rho_1)\) and

\[
I_2 = \left( -z^{1-\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}} + \hat{\rho} , z^{1-\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}} + \hat{\rho} \right)
\]

is an asymptotic 1 - \(\alpha\) confidence interval for \(\rho'\).

We now complete Section 2.4 with a necessary and sufficient condition for the consistency of \(Y\).

**Theorem 2.4.4:** The test based on \(Y\) is consistent if and only if

\[
\frac{1}{k(k - 1)} \sum_{u < v} \int_{H_{u}} dH_{v} > 1/4
\]

where

\[
H_{u} = \mathbb{P}^* \mathbb{P}
\]

\[
\mathbb{P}_{u} \mathbb{P}_{u}
\]
Proof: From (2.3.1), we have

$$E_A(T_{uv}) = \frac{n(n-1)}{2} P_A(X_{1u} + X_{2u} < X_{1v} + X_{2v}) + n P_A(X_{1u} < X_{1v})$$

where $[X_{1u}, X_{2u}]_{u=1}^k$ are i.i.d. $\sim F_u$. Hence

$$\lim_{n \to \infty} \frac{E_A(Y)}{n(n-1)} = \sum_{u < v} \frac{P_A(X_{1u} + X_{2u} < X_{1v} + X_{2v})}{2}$$

Also, in Section 4.1 we derive $\sigma_A^2(T_{uv})$ and it is seen that,

$$\sigma_A^2(T_{uv}) = O(n^3)$$

Hence,

$$\sigma_A^2(Y) = O(n^3)$$

and by Chebychev's inequality, we have under $H_0$

$$p - \lim_{n \to \infty} \frac{2Y}{n(n-1)} = \frac{k(k-1)}{4}$$

whereas under $H_A'$,

$$p - \lim_{n \to \infty} \frac{2Y}{n(n-1)} = \sum_{u < v} \int_{H_u}^H \frac{dH}{H_v}$$

q.e.d.

Theorems 2.4.4 and 2.4.1 show that the consistency conditions of the $\tau$ and $Y$ tests are quite similar. The consistency parameter for $\tau$
is \( \sum_{u<v}^k \int_u^v F_u dF_v \) while for \( \rho \) it is \( \sum_{u<v}^k \int_u^v F_u d(F^*_v) \). However, it is easy to produce examples that show \( Y \) and \( \tau \) are not consistent against the same class of alternatives. For instance, take \( k = 2 \), and let

\[
f_1(x) = \begin{cases} 
1 & \text{if } 4 \leq x \leq 5 \\
0 & \text{otherwise}
\end{cases}
\]

\[
f_2(x) = \begin{cases} 
.6 & \text{if } 1 \leq x \leq 2 \\
.4 & \text{if } 10 \leq x \leq 11 \\
0 & \text{otherwise}
\end{cases}
\]

Then if \( X_1 \) and \( X_2 \) are i. i. d. according to \( f_1 \), and \( Y_1 \) and \( Y_2 \) are i. i. d. according to \( f_2 \), we have,

\[
P(X_1 < Y_1) = P(Y_1 \in [10, 11]) = .4
\]

but

\[
P(X_1 + X_2 < Y_1 + Y_2) = 1 - P(Y_1 \in [1, 2], Y_2 \in [1, 2]) = .64
\]

and thus \( Y \) will be consistent and \( \tau \) will not. Of course, reversing the roles of \( f_1 \) and \( f_2 \) we get the opposite conclusion. Thus this example shows that \( Y \) and \( \tau \) are not consistent against the same alternatives and also illustrates the fact that Wilcoxon's paired signed-rank test and Wilcoxon's two-sample test are not consistent against the same class of alternatives.
Chapter 3

Efficiency Comparisons

In this chapter we develop expressions for the asymptotic relative efficiencies of the tests under consideration. We investigate different types of alternatives, including shift, contamination, and unequal block variances.

3.1 Shift alternatives:

Let us begin by defining the S alternatives as

\[ S: F_{ij}(x) = F(x - b_i - (j - 1)\theta) \quad i = 1, \ldots, n, \quad j = 1, \ldots, k \quad \text{and} \quad \theta > 0. \]

From [11] it follows that if \( T_1(n) \) and \( T_2(n) \) are two test sequences (from the group of \( \tau_1(n), \tau_2(n) \), \( \rho_1(n), \rho_2(n) \), and \( Y(n) \)), then the Pitman efficiency \( (\theta \to 0)^{\dagger} \) of \( T_1(n) \) to \( T_2(n) \) (hereafter called \( E(T_1, T_2) \)) is

\[
\lim_{n} \left[ \frac{\frac{d}{d\theta} E_{\theta}^{2}(T_1(n))}{\frac{d}{d\theta} E_{\theta}^{2}(T_2(n))} \right] \bigg| \theta = 0 = \frac{\sigma_{0}^{2}(T_2(n))}{\sigma_{0}^{2}(T_1(n))} \quad (3.1.1)
\]

\[ \dagger \text{Specifically, we are computing the efficiency for the alternatives} \]

\[ S^{(n)}: F_{ij}^{(n)}(x) = F(x - b_i - \frac{(j - 1)c}{\sqrt{n}}) \]

but we simply say "\( \theta \to 0. \)"
We are now ready to state the first result of this section.

**Theorem 3.1.1:** If $F$ is continuous, with variance $\sigma^2$ and density $f$, then for the $S$ alternatives ($\theta \to 0$),

$$E(\rho, t) = \frac{k}{k+1} \left( 12\sigma^2 \left[ \int f^2 \right]^2 \right) \quad (3.1.2)$$

**Proof:** As in (2.4.10)

$$E_\theta(Z_\rho) = n \sum_{j=1}^{k} (2j - k - 1) \left( \sum_{i=1}^{k} \int F(x - (i - 1)\theta) dF(x - (j - 1)\theta) \right)$$

Hence,

$$\left. \frac{d}{d\theta} E_\theta(Z_\rho) \right|_{\theta=0} = n \sum_{j=1}^{k} (2j - k - 1) \cdot \left( \sum_{i=1}^{k} (j - i) \int f^2 \right)$$

$$= \left( \frac{n k^2 (k - 1)(k + 1) \int f^2}{6} \right) \quad (3.1.3)$$

From (2.1.3), and the fact that

$$\lim_{n \to \infty} s^2 = \sigma^2$$

and

$$\sigma^2_{\theta} = \frac{12\sigma^2}{nk(k - 1)(k + 1)}$$

\text{1Whenever we differentiate under the integral, we assume sufficient regularity.}\n
\text{2We assume } E(x^4) < \infty \cdot
we have
\[ E_\theta(t) \sim \theta \left( \frac{12\sigma^2}{n(k-1)(k+1)} \right)^{-1/2} \]
and thus
\[ \frac{d}{d\theta} E_\theta(t) \bigg|_{\theta=0} \sim \left( \frac{12\sigma^2}{nk(k-1)(k+1)} \right)^{-1/2} \]  \hspace{1cm} (3.1.4)
and
\[ \sigma_o^2(t) \sim 1 \]  \hspace{1cm} (3.1.5)
Since (see e.g., [7])
\[ \sigma_o^2(\rho_1) = \frac{1}{k-1} \]
by (2.2.7)
\[ \sigma_o^2(\rho_1) = n \left( \frac{k^3-k}{6} \right)^2 \cdot \frac{1}{k-1} \]  \hspace{1cm} (3.1.6)
From 3.1.3 - 3.1.6 and 3.1.1, the result follows.
\textit{q.e.d.}
Hodges and Lehmann [5] have shown that
\[ .864 \leq 12\sigma^2[f^2]^2 \leq \infty \]
with the lower bound achieved for
\[ f_1(x) = \frac{3}{20\sqrt{5}} (5 - x^2) - \sqrt{5} \leq x \leq \sqrt{5} \]  \hspace{1cm} (3.1.7)
We can thus state the obvious corollary

**Corollary 3.1.2:** For the $S$ alternatives,

$$0.576 \leq E(\rho, t) \leq \infty$$

with the lower bound achieved with $f$ as in (3.1.7) and $k = 2$.

We also note that when $k = 2$, $\tau$ and $\rho$ are identical procedures and are equivalent to the paired sign test.

Pitman and others have evaluated $12\sigma^2/[f^2]^2$ for various $F$, and we thus state

<table>
<thead>
<tr>
<th>$F$</th>
<th>$E(\rho, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>$\frac{k}{k+1} \cdot \frac{3}{\pi}$</td>
</tr>
<tr>
<td>Rectangular</td>
<td>$\frac{k}{k+1}$</td>
</tr>
<tr>
<td>$\Gamma(5)$</td>
<td>$\frac{k}{k+1} \cdot \frac{81}{64}$</td>
</tr>
<tr>
<td>Cauchy</td>
<td>$+\infty$</td>
</tr>
</tbody>
</table>

We also give

**Table 3.1**

<table>
<thead>
<tr>
<th>$F$-Normal</th>
<th>$k$ 2 3 4 5 6 7 10 20 50 $\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(\rho, t)$</td>
<td>$\frac{2}{\pi}$ .716 .764 .796 .819 .836 .868 .909 .936 $\frac{3}{\pi}$</td>
</tr>
</tbody>
</table>

We remark that Noether and van Elteren [1] found expression (3.1.2) to be the Pitman efficiency (translation alternatives) of Friedman's rank test [2] with respect to the normal theory $F$-test.
We now consider \( \tau \), and state a result similar to (3.1.2), namely

**Theorem 3.1.2:** If \( F \) is continuous, with density \( f \) and variance \( \sigma^2 \), then for the \( S \) alternatives \( (\theta \to 0) \),

\[
E(\tau, \theta) = \frac{24(k + 1)}{(2k + 5)} \sigma^2 [f^2]^2
\]  
(3.1.8)

**Proof:**

\[
E_\theta(Z_\theta) = n \sum_{u < v} \int F(x - (u - 1)\theta) dF(x - (v - 1)\theta)
\]

Hence,

\[
\frac{d}{d\theta} E_\theta(Z_\theta)\bigg|_{\theta=0} = n \sum_{j=1}^{k-1} (k - j) \cdot j [f^2]
\]

\[
= nf^2 \cdot \frac{[k(k - 1)(k + 1)/6]}{f^2}
\]  
(3.1.9)

Since (see e.g., [7])

\[
\varphi^2(\tau_1) = \frac{2(2k + 5)}{9k(k - 1)}
\]

it follows that

\[
\sigma^2_o(Z_\theta) = \frac{nk(k - 1)(2k + 5)}{72}
\]  
(3.1.10)

The result now follows from (3.1.4), (3.1.5), (3.1.9), (3.1.10) and (3.1.1).

q.e.d.
Two obvious corollaries to Theorem 3.1.2 are

**Corollary 3.1.3:** For the $S$ alternatives, $(\theta \to 0)$

$$0.576 \leq E(\tau, t) \leq \infty$$

**Corollary 3.1.4:** For the $S$ alternatives $(\theta \to 0)$,

$$E(\rho, \tau) = \frac{k(2k + 5)}{2(k + 1)^2} \quad (3.1.11)$$

independent of $F$.

We note that $E(\rho, \tau) = 1$ when $k = 2$ since, as we have already remarked, in this case the tests are equivalent. Also, $E(\rho, \tau)$ increases until it reaches its maximum value of 1.0417 at $k = 5$ and then decreases to its limiting value $(k \to \infty)$ of 1.

The values of $E(\rho, t)$ and $E(\tau, t)$ for small $k$, especially when $F$ is normal, are somewhat discouraging and one would like to be able to improve on these. The $Y$ test, as we will now see, yields such an improvement.

**Theorem 3.1.5:** For the $S$ alternatives, $(\theta \to 0)$

$$E(Y, t) = \frac{24(k + 1)s^2 + \left[\left(\theta_1 - \theta_2\right)^2\right]}{3 + 2(k - 2)\rho^*(F)} \quad (3.1.12)$$

where, if $X_1, X_2$ are i. i. d. according to $F$, we define $G$ to be the distribution function of $X_1 - X_2$, with corresponding density $g$, and

$$s^2 = \text{Var}(F), \quad \rho^*(F)$$

is given by $(2.3.14)$. 

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Proof: From (2.3.1) and the definition of the S alternatives, we have

$$E_\theta(Y) = \sum_{u < v} \left[ \frac{n(n-1)}{2} \int G(x - (u-v)\theta)dG(x - (v-u)\theta) ight. \\
\left. + n \int F(x - (u-1)\theta)dF(x - (v-1)\theta) \right]$$

It follows that

$$\left. \frac{\frac{d}{d\theta} E_\theta(Y)}{n} \right|_{\theta=0} = \sum_{u < v} (v-u) \int g^2$$

$$= \frac{k(k-1)(k+1)}{6} \cdot \int g^2 \quad (3.1.13)$$

Expression (3.1.12) then follows by recalling (2.3.16), together with (3.1.13), (3.1.4), (3.1.5) and (3.1.1). 

q.e.d.

Corollary 3.1.6: For the S alternatives,

$$.864 < E(Y, t) \leq \infty$$

Proof: Since $\rho^*(F) \leq 1/2$, it follows from (3.1.12) that for each fixed $F$ ($\sigma^2$ finite), $E(Y, t)$ is an increasing function of $k$, unless $\rho^*(F) = 1/2$ in which case $E(Y, t)$ equals $24\sigma^2 \cdot [f g^2]^2$ for all $k$. For $k = 2$, $E(Y, t)$ is $24\sigma^2 \cdot [f g^2]^2$ and since $12\sigma^2[f^2]^2$ achieves its minimum for the $f_1$ given by (3.1.7), we conclude that $E(Y, t) > .864$. The last inequality is strict since there does not exist any density $h$ such that $X_1 - X_2$ is distributed according to $f_1$ when $X_1, X_2$ are i. i. d. $\sim h$. This can be seen as follows. For
\[ f_1(x) = b(a^2 - x^2) \quad -a < x < a, \]

\[ \varphi_{f_1}(t) = E(e^{itX}) = \int_{-a}^{a} \cos(tx)b(a^2 - x^2)dx = \frac{b^2}{t^3} (\sin(at) - (at)\cos(at)) \quad (3.1.14) \]

Thus, there does not exist a characteristic function \( \varphi_h \) such that

\[ \varphi_{f_1}(t) = \varphi_h(t) \cdot \overline{\varphi_h(t)} \quad (3.1.15) \]

since by taking \( t \) sufficiently large we see that \( \varphi_{f_1}(t) \) (3.1.14) will be negative, contradicting (3.1.15).

q.e.d.

We remark that for \( k = 2 \) and \( F \) uniform, \( E(Y, t) = .889 \). When \( k = 2, Y \) reduces to Wilcoxon's signed-rank test [15] and \( E(Y, t) \) reduces to \( 24\sigma^2[\tilde{Y}^2]^2 \), the efficiency of Wilcoxon's signed-rank test with respect to the \( t \)-test. When \( F \) is normal, expression (3.1.12) becomes

\[ E(Y, t)_{\text{Normal}} = \frac{3(k + 1)}{\pi(3 + 2(k - 2)(.482))} \quad (3.1.16) \]

Comparing (3.1.16) with (3.1.2) we see that

\[ E(Y, t)_{\text{Normal}} > E(\rho, t)_{\text{Normal}}, \quad \text{for every } k. \]

For reference, we give Table 3.2.
Table 3.2
Entries are $E(Y, t)_{\text{Normal}}$

<table>
<thead>
<tr>
<th>k</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{3}{\pi}$</td>
<td>.963</td>
<td>.969</td>
<td>.972</td>
<td>.975</td>
<td>.976</td>
<td>.980</td>
<td>.985</td>
<td>.988</td>
<td>.990</td>
</tr>
</tbody>
</table>

As in the case of $\rho$ and $\tau$, $E(Y, t) = +\infty$ when $F$ is cauchy, and when $F$ is uniform (3.1.12) reduces to

$$E(Y, t)_{\text{Uniform}} = \frac{8(k + 1)}{9[3 + 2(k - 2)(.491)]}$$

Some values are given in Table 3.3.

Table 3.3
Entries are $E(Y, t)_{\text{Uniform}}$

<table>
<thead>
<tr>
<th>k</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.889</td>
<td>.893</td>
<td>.895</td>
<td>.897</td>
<td>.898</td>
<td>.899</td>
<td>.901</td>
<td>.903</td>
<td>.905</td>
<td>.906</td>
</tr>
</tbody>
</table>

For small values of $k$, these values compare favorably with the corresponding ones of $E(\rho, t)_{\text{Uniform}}$ (see Table 3.4).

Table 3.4
Entries are $E(\rho, t)_{\text{Uniform}}$

<table>
<thead>
<tr>
<th>k</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E(\rho, t)$</td>
<td>.667</td>
<td>.750</td>
<td>.800</td>
<td>.833</td>
<td>.857</td>
<td>.875</td>
<td>.909</td>
<td>.952</td>
<td>.980</td>
</tr>
</tbody>
</table>

We should also point out that the Pitman efficiency results of this section are valid for the more general alternatives
S': \( F_{ij}(x) = F(x - b_i - (\alpha_j)\theta) \)

where we require that the sequence of constants \( \{\alpha_j\}_{j=1}^{K} \) are such that the tests under consideration are still consistent.

Up to this point, we have investigated the properties of the various tests with respect to the \( S \) alternatives by means of Pitman efficiency. If, instead of an asymptotic power comparison in the neighborhood of the null hypothesis \( (\theta = 0) \), we were interested in an efficiency measure for each fixed \( \theta > 0 \), we could consider, among others, the respective Bahadur efficiencies. In the notation of Gleser [p. 1542 of [3]] it is easy to verify, from the development already carried out in Sections 2.3 and 2.4, that his assumptions 4.1 - 4.3 are satisfied for the procedures under discussion with \( F^{(i)}(x) = \phi(x) \), \( a_i = 1 \), \( t_i = 2 \) and \( b_i = \sqrt{n} \).

Hence, if we denote \( B_\theta(T_1, T_2) \) as the Bahadur efficiency of \( [T_{ln}] \) with respect to \( [T_{2n}] \), we have

\[
B_\theta(T_1, T_2) = \frac{C_1(\theta)}{C_2(\theta)}^2
\]

(3.1.17)

where

\[
C_i(\theta) = p - \lim_{\theta} \frac{T_{ln}}{\sqrt{n}} \quad i = 1, 2 \quad (3.1.18)
\]

Since we have already evaluated the probability limit of (3.1.18) for the different tests, we may state
Theorem 3.1.8:

\[ B_\theta(\rho, t) = \frac{36 \cdot (12\sigma^2) \cdot \left[ \sum_{j=1}^{k} (2j - k - 1) \cdot \frac{1}{\theta} \int F(x - (1 - l)\theta) dF(x - (j - 1)\theta)^2 \right]}{(k^2 - k)^2 \cdot k(k + 1) \cdot \theta^2} \]  

(3.1.19)

\[ B_\theta(\tau, t) = \frac{72 \cdot (12\sigma^2) \cdot \left[ \sum_{u < v} \frac{1}{\theta} \int F(y + (v - u)\theta) dF(y) - \frac{k(k - 1)}{4} \right]^2}{k^2(k - 1)^2(k + 1)(2k + 5)\theta^2} \]  

(3.1.20)

\[ B_\theta(Y, t) = \frac{18 \cdot (12\sigma^2) \left[ \sum_{u < v} \frac{1}{\theta} \int G(x + 2(v - u)\theta) dG(x) - \frac{k(k - 1)}{4} \right]^2}{k^2(k - 1)^2(k + 1)(3 + 2(k - 2)\rho*(F))\theta^2} \]  

(3.1.21)

Corollary 3.1.9: For \( \sigma^2 \) finite,

\[ \lim_{\theta \to \infty} B_\theta(\rho, t) = \lim_{\theta \to \infty} B_\theta(\tau, t) = \lim_{\theta \to \infty} B_\theta(Y, t) = 0 \]

Corollary 3.1.9 is to be expected. It is a reflection of the familiar fact that rank tests are insensitive to a difference between two large deviations. We also have,

Corollary 3.1.10: For \( \sigma^2 \) finite, if \( G \) is unimodal, then \( B_\theta(\rho, t) \), \( B_\theta(\tau, t) \), and \( B_\theta(Y, t) \) are decreasing in \( \theta \).
Proof: We treat only $B_k(\tau, t)$, the cases for $Y$ and $\rho$ are analogous.

In (3.1.20),

$$
\frac{\left( \int F(x + (v - u)\theta)dF(x) - \frac{k(k - 1)}{4} \right)}{\theta} = \frac{P \left( 0 < X_2 - X_1 < (v - u)\theta \right)}{\theta}
$$

where $X_1, X_2$ are i.i.d. according to $F$,

$$
\frac{(v - u)\theta}{\theta} = \int_0^1 g(x)dx
$$

Hence, taking the derivative of (3.1.20) with respect to $\theta$, and applying the unimodality of $G$ yields the result. We note that for the case of $Y$, using the above argument, we need the unimodality of $G^*G$. Since $G$ is symmetric, $G$ unimodal implies that $G^*G$ is also unimodal.\textsuperscript{1}

q.e.d.

We remark that Hodges and Lehmann have shown (Matching in Paired Comparisons, A.M.S., 1954, p. 787-791) that $F$ unimodal implies $G$ is unimodal, and thus the condition of Corollary 3.1.10 (and Corollary 3.2.4) is satisfied when $F$ is unimodal.

\textsuperscript{1}K. L. Chung (p. 254 Gnedenko-Kolmogorov Limit distributions for sums of independent random variables) gives a counter-example which show that $F$ unimodal does not necessarily imply that $F^*F$ is unimodal, but A. Winter (p. 30 of Asymptotic distributions and infinite convolutions) shows that such is the case when $F$ is also symmetric.
Corollary 3.1.11:

\[ \lim_{\theta \to \infty} B_\theta(\tau, \rho) = \frac{9k}{2(2k + 5)} \quad (3.1.22) \]

Proof: We note that,

\[ \lim_{\theta \to \infty} \sum_{u < v} \int F(x + (v - u)\theta) dF(x) = \frac{k(k - 1)}{2} \]

and

\[ \lim_{\theta \to \infty} \sum_{j=1}^{k} \left[ (2j - k - 1) \cdot \sum_{i=1}^{k} \int F(y + (j - i)\theta) dF(y) \right] \]

\[ = \sum_{j=1}^{k} (2j - k - 1) \cdot (j - 1) = \frac{k(k + 1)(k - 1)}{6} \]

(3.1.22) now follows from (3.1.19) and (3.1.20).

q.e.d.

Corollary 3.1.11 is a reasonable result (we note that \( \frac{\sigma^2_0(\tau_1)}{\sigma^2_0(\rho_1)} = \frac{2(2k + 5)}{9k} \)). As \( \theta \to \infty \), we have "perfect" ordering in all the blocks with very high probability and the extra sensitivity usually obtained by \( \rho \) (since \( \rho \) actually uses the difference in ranks rather than just the sign of the difference) is not needed, and we pay for this since \( \rho \) has a greater variance than \( \tau \).
In the same manner as Corollary 3.1.11, we also have

**Corollary 3.1.12:**

\[
\lim_{\theta \to \infty} B_0(Y, \tau) = \frac{(2k + 5)}{4[2(k - 2)\rho\ast(F) + 3]}
\]

We remark that

\[
\lim_{k \to \infty} \lim_{\theta \to \infty} B_0(Y, \tau) \geq 1/2
\]

since \( \rho\ast(F) \leq 1/2 \).

3.2 **Contamination Alternatives:**

We first define the contamination (C) alternatives as

\[
C: F_{ij}(x) = F(x - b_i - (j - 1)\theta)
\]

in a fixed fraction \( p \) of the blocks and

\[
F_{ij}(x) = F(x - b_i)
\]

in the remaining fraction \( 1 - p \) of the blocks. Contamination of the above type can occur, for example, when the treatments "take" on only a proportion of the blocks. We shall proceed to investigate the Pitman efficiency of the various tests for the C alternatives \( (p \to 0) \) with \( \theta \) fixed.

**Theorem 3.2.1:** For the C alternatives \( (p \to 0) \), \( E(\tau, t) \) is given by (3.1.20), and \( E(\rho, t) \) is given by (3.1.19).
Proof: This equivalence to the Bahadur efficiency occurs because of the following simple relationship. When the statistic $T$ can be written as $\sum_{i=1}^{n} t_i$ where the $t_i$ are i. i. d. (as is true in the case of $\rho$, $\tau$, and $t$), then

$$E_0(T) = npE_0(t_i) + n(1-p)E_0(t_i)$$

and thus

$$\frac{d}{dp} E_0(T) \bigg|_{p=0} = n(E_0(t_i) - E_0(t_i))$$

with $E_0(t_i) - E_0(t_i)$ being the same quantity that entered into the Bahadur efficiency expression.

q.e.d.

Theorem 3.2.2: For the $C$ alternatives ($p \to 0$),

$$E(Y, t) = \frac{72 \cdot 12c^2 \left[ \sum_{u \leq v} \int G(x - (u - v)\theta)dG(x) - \frac{k(k - 1)}{4} \right]^2}{k^2(k - 1)^2(k + 1)[3 + 2(k - 2)p^*(F)]\theta^2}$$

(3.2.1)

Proof: Let $X_n$ = number of "active" blocks (i.e., those blocks which exhibit a treatment effect). As in (2.3.1) we write,

$$T_{uv} = \sum_{i \leq j} \psi_{ij} + \sum_{i=1}^{n} \xi_{uv}(i)$$

Now, we break the summation $\sum_{i \leq j} \psi_{ij}$ into the $\frac{x_n(x_n - 1)}{2}$ terms where both blocks "i" and "j" are active, the
\[
\frac{(n - x_n)(n - x_n - 1)}{2} \quad \text{terms where both "i" and "j" are inactive,}
\]
and the remaining \((n - x_n) \cdot (x_n)\) terms where one block is active
while the other is inactive. We then obtain,

\[
E(T_{uv}) = \frac{x_n(x_n - 1)}{2} \int G(x - (u - v)\theta)dG(x - (v - u)\theta)
+ \frac{(n - x_n)(n - x_n - 1)}{2} \int G(x)dG(x)
+ (n - x_n)(x_n) \cdot \int G(x - (u - v)\theta)dG(x) + o(n^2)
\]

Since

\[
\frac{x_n}{n} \approx p
\]

it follows that

\[
\lim_{n \to \infty} \frac{d}{dp} \frac{E(y)}{n(n - 1)} \Bigg|_{p=0} = \left[ \sum_{u<v}^k \int G(x - (u - v)\theta)dG(x) - \frac{k(k - 1)}{4} \right]
\]

(3.2.3)

Also,

\[
E_\theta(t) \sim \frac{p^\theta}{\sqrt{\frac{12\sigma^2}{nk(k - 1)(k + 1)}}}
\]

and

\[
\frac{d}{dp} E_\theta(t) \Bigg|_{p=0} \sim \frac{\theta}{\sqrt{\frac{12\sigma^2}{nk(k - 1)(k + 1)}}}
\]

(3.2.4)
The expression (3.2.1) now follows from (3.2.3), (3.2.4), (3.1.5) and (2.3.16).

\[ q.e.d. \]

Thus, in the same manner as the Bahadur efficiency for the S alternatives, we may state

**Corollary 3.2.3:** For the C alternatives \((p \to 0)\) with \(\sigma^2\) finite,

\[ \lim_{\theta \to \infty} E(p, t) = \lim_{\theta \to \infty} E(\tau, t) = \lim_{\theta \to \infty} E(Y, t) = 0 \]

**Corollary 3.2.4:** For the C alternatives \((p \to 0)\), \(\sigma^2\) finite, and \(G\) unimodal, \(E(p, t)\), \(E(\tau, t)\) and \(E(Y, t)\) are decreasing functions of \(\theta\).

The above corollaries follow in the same way as Corollaries 3.1.9 and 3.1.10 concerning the S alternatives. Corollary 3.2.4 shows that the rank tests are often more efficient (w.r.t. the t-test) against a pure shift than a mixture of inactive and active blocks (letting \(\theta \to 0\) gives the Pitman efficiency for the S alternatives). This, as we would expect, agrees with what Lehmann and Hodges found for the two sample problem [5], in the case of Wilcoxon's test and the t-test.

### 3.3 Unequal block-variance alternatives:

In this section we will investigate the performance of the tests when there is different variability in different blocks. Specifically, we will be interested in the V alternatives.

\[ V: F_{ij}(x) = F((x - b_1 - (j - 1)\theta)/\sqrt{\sigma_1}) \quad (3.3.1) \]
where
\[ c_i = 1 \] in a fixed fraction \( p \) of the blocks

and
\[ c_i = c \] in the remaining fraction \( 1 - p \) of the blocks.

We first state Theorem 3.3.1.

**Theorem 3.3.1:** Under the \( V \) alternatives \( (\theta \to 0) \),
\[
E(\rho, t) = \frac{k}{k+1} \cdot 12\sigma^2 (f_x^2)^2 \cdot \left( \frac{\sqrt{c}p + (1 - p)}{\sqrt{c}} \right)^2 (p + c(1 - p))
\]
(3.3.2)

**Proof:** Since \( \text{Var}(X_{ij}) = c_i \sigma^2, \quad j = 1 \cdots k \) then,
\[
\lim_{p \to 1} \text{Var}(s) = \sigma^2 (p + (1 - p)c)
\]

Hence, from (2.1.3) and (2.1.5), we have that under the \( V \) alternatives
\[
E(t) \sim \theta \left( \frac{12(p + (1 - p)c)\sigma^2}{nk(k - 1)(k + 1)} \right)^{-1/2}
\]

and thus,
\[
\frac{d}{d\theta} E(t) \bigg|_{\theta=0} \sim \left[ \frac{12(p + (1 - p)c)\sigma^2}{nk(k - 1)(k + 1)} \right]^{-1/2}
\]
(3.3.3)

\[ ^{1} \text{We assume } E(X^4) < \infty . \]
Also,

\[ \sigma_o^2(t) \sim 1 \quad (3.3.4) \]

Now,

\[ E_\theta(z_\rho) = \sum_{i=1}^{n} \sum_{j=1}^{k} (2j - k - 1) \]

\[ \times \left[ \sum_{\alpha=1}^{k} \int F((x - b_{i\alpha} - (\alpha - 1)\theta)/\sqrt{c_i})dF((x - b_{ij} - (j - 1)\theta)/\sqrt{c_i}) \right] \]

Hence,

\[ \frac{d}{d\theta} E_\theta(z_\rho) \bigg|_{\theta=0} = \left[ \sum_{i=1}^{n} \frac{1}{\sqrt{c_i}} \right] \frac{k^2(k - 1)(k + 1)}{6} \cdot \int f^2 \quad (3.3.5) \]

Also, \( \sigma_o^2(\theta_0) \) (the subscript \( o \) relating to \( \theta = 0 \)) is the same for the \( V \) alternatives as it was for the \( S \) alternatives, as the null distribution of \( \rho_i \) will not depend on the block variance. Thus (3.3.2) is a consequence of (3.3.3), (3.3.4), (3.3.5), and (3.1.6).

\[ \text{q.e.d.} \]

For the \( \tau \) test we have a similar result, namely

\textbf{Theorem 3.3.2:} For the \( V \) alternatives \((\theta \to 0)\),

\[ E(\tau, t) = \frac{2(k + 1)}{2k + 5} \left[ 12\sigma^2[f^2]^2 \right] \cdot \left[ p + c(1 - p) \right] \left[ \frac{c_{\rho} + (1 - p)}{\sqrt{c}} \right]^2 \]

\[ (3.3.6) \]

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Proof: We simply refer to the proof of Theorem 3.3.1, noting that

\[ \frac{d}{d\theta} E_0(z_{1r}) \bigg|_{\theta=0} = \frac{k(k-1)(k+1)}{6} \left( \frac{1}{f^2} \right) \left( np + n(1-p) \right) \sqrt{v_c} \]

and that \( c_0^2(z_{1r}) \) is unaffected by the block variance and is the same as for the \( S \) alternatives (3.1.10).

q.e.d.

Some immediate consequences of Theorems 3.3.1 and 3.3.2 are

Corollary 3.3.3: For the \( V \) alternatives \( (\theta \to 0) \)

\[ E(\rho, \tau) = \frac{k(2k+5)}{2(k+1)^2} \]

(the same expression as for the \( S \) alternatives)

Corollary 3.3.4: For any fixed \( p, \ 0 < p < 1 \), for the \( V \) alternatives,

\[ \lim_{c \to \infty} E(\rho, t) = \lim_{c \to \infty} E(\tau, t) = \infty \]

(3.3.7)

Corollary 3.3.4 expresses a logical result. As \( c \) gets very large, the main non-centrality contribution to both the \( \rho(\tau) \) and \( t \) statistics comes from the "\( c_1 = 1 \)" blocks. However, in the \( t \)-test we are normalizing by a standard error estimate which is blowing up due to the large variance contribution of the "\( c \)" blocks, whereas in the rank tests the norming standard deviations remain constant.

Property (3.3.7) provides further evidence of the robustness of these rank procedures. It would be desirable for the test based on \( Y \) to have this property. Theorem 3.3.5 shows this to be the case.
Theorem 3.3.5: Under the V alternatives \( (\theta \to 0) \)

\[
E(Y, t) = \frac{p + (1 - p)c}{3 + 2(k - 2)\rho^*_{V}(F)} \cdot \frac{F(c, g) \cdot (k + 1)}{
\sqrt{c}}
\]

where

\[
F(c, g) = \left[ p \int g^2(x)dx + \frac{(1 - p)^2}{\sqrt{c}} \int g^2(x)dx
\right]
+ \frac{2p(1 - p)}{\sqrt{c}} \cdot \int g \left( \frac{x}{\sqrt{c}} \right) \cdot g(x)dx
\]

and

\[
\rho^*_{V}(F) = \lim_{n\to\infty} \rho^{(n)}_{V}(F)
\]

where \( \rho^{(n)}_{V}(F) \) is the correlation coefficient between \( T_{uv} \) and \( T_{uw} \) under the V alternatives with \( \theta = 0 \).

Proof: If we break the summation \( \sum_{1 \leq j \leq n} \psi_{uv}^{(i,j)} \) in the same manner as in the proof of Theorem 3.2.2, we readily find that under the V alternatives,

\[
E(T_{uv}) = \frac{x_n(x_n - 1)}{2} \int \left[ G(x + 2(v - u)\theta) - 2 \int G \left( x + \frac{2(v - u)\theta}{\sqrt{c}} \right) dx \right]
\]

\[
+ \frac{(n - x_n)(n - x_n - 1)}{2} \int G \left( x + \frac{2(v - u)\theta}{\sqrt{c}} \right) dx
\]

\[
+ (n - x_n)(x_n) \cdot \int G \left( \frac{x + 2(v - u)\theta}{\sqrt{c}} \right) dx + o(n^2)
\]

\((x_n = \text{number of } "c_1 = 1" \text{ blocks})\)
Hence,

\[
\lim_{n} \left. \frac{1}{n(n-1)} \frac{\mathbb{E}(T_{uv}^{2})}{\theta-0} \right| = p^2(v-u) \cdot \int g^2(x)dx + \frac{(1-p)^2(v-u)}{\sqrt{c}} \int g^2(x)dx
\]

\[+ \frac{2p(1-p)(v-u)}{\sqrt{c}} \int g \left( \frac{x}{\sqrt{c}} \right) g(x)dx \quad (3.3.10)\]

Taking the \[\sum_{u<v}^{n} (3.3.10)\] yields \(F(c, g)\).

Now, under the \(V\) alternatives, when \(\theta = 0\), \(T_{uv}\) retains its equivalence to the random variable \(T\) (2.3.2), and hence \(c_{0}^2(T_{uv})\) is given by (2.3.12). Thus, \(c_{0}^2(Y)\) (under the \(V\) alternatives), is given by (2.3.16) with \(p_{o,n}(F)\), replaced by \(p_{v_{o}}(F)\).

The result now follows from (3.3.10), (2.3.16), (3.3.2), and (3.3.3).

q.e.d.

**Corollary 3.3.6:** Under the \(V\) alternatives \((\theta \to 0)\), for any fixed \(p, 0 < p < 1,\)

\[
\lim_{c \to \infty} \mathbb{E}(Y, t) = +\infty
\]
Chapter 4

Some Notions Concerning a Multiple-Comparison Method

Let $X_{i0}$ and $X_{ij}$ ($i = 1, \ldots, n$, $j = 1 \ldots k$) be the independent measurements on the control and $j^{th}$ treatments in the $i^{th}$ block. Nemenyi [10] suggests treatment-control comparisons based on the statistic

$$U = \text{Max}_j T_{0j} \quad (4.1.1)$$

where the $[T_{0j}]^k_{j=1}$ are defined by (2.2.1). For example, for the problem of selecting (without regard to order) which treatments are better than the control, select treatment $j$ if

$$T_{0j} \geq \frac{n(n+1)}{4} + d^n_{(\alpha,k)} \sqrt{\frac{n(n+1)(2n+1)}{24}} \quad j = 1 \ldots k \quad (4.1.2)$$

where $d^n_{(\alpha,k)}$ is chosen so that

$$P_o(T_{0j} \leq \frac{n(n+1)}{4} + d^n_{(\alpha,k)} \sqrt{\frac{n(n+1)(2n+1)}{24}}, j = 1 \ldots k) \approx 1 - \alpha \quad (4.1.3)$$

the subscript $o$ referring to treatment-control equivalence. Based on monte carlo evidence that the null correlation between $T_{0u}$ and $T_{0v}$ was close to $1/2$, and the assumption that $U$ was distribution-free under treatment-control equivalence, Nemenyi proposed obtaining $d^n_{(\alpha,k)}$
from Dunnett's multivariate t tables \((n = \infty)\). As seen in Theorem 2.3.4, \(p_0^n(F)\) depends on \(F\) (the common distribution of treatment and control) and hence so does the null distribution of \(U\). We thus propose a slightly different comparison method so that it has the property of being asymptotically distribution-free. Namely:

For each \(j\), select treatment \(j\) as being better than the control if

\[
T_{0j} > \frac{n(n + 1)}{4} + c^n_{(\alpha, k)} \sqrt{\frac{n(n + 1)(2n + 1)}{24}} \quad (4.1.4)
\]

where \(c^n_{(\alpha, k)}\) is obtained from Gupta's [4] tables of the equi-correlated multivariate normal distribution, the tables being entered at \(\rho = \hat{\rho}_1\) (2.3.19) (or at the value of some other consistent estimate of \(\rho(F)\) based perhaps on one of the many simpler, but less efficient, estimates of \(\lambda(F)\)).

Nemenyi and Steel have pointed out that the dependence, of the distribution of the maximum of several equi-correlated unit normal random variables, on the common correlation is slight. In view of this fact, Nemenyi's procedure (even though not exactly distribution-free), should be of value for comparisons of this nature.

We close Chapter 4 with a large sample approximation to the probability of making the "correct" decision for a set of treatment-control alternatives. To do this, we first give expressions for \(E_A(T_{uv})\), \(\sigma_A^2(T_{uv})\) and \(\rho_A(T_{uv}, T_{uw})\) under the \(A\) alternatives. Let us make the following definitions:
(i) If \( X_1, X_2 \) are i.i.d. according to \( F_u \), then the distribution function of \( Z = X_1 + X_2 \) is \( H_u \).

(ii) If \( X_1 \) has the distribution function \( F_u \) and the distribution function of \( X_2 \) is \( F_v \), with \( X_1 \) and \( X_2 \) independent, then the distribution function of \( W = X_1 - X_2 \) is \( I_{uv} \).

Now, from (2.3.1), we have

\[
E_A(T_{uv}) = \frac{n(n - 1)}{2} \int H_u dH_v + n \int F_u dF_v \quad (4.1.5)
\]

Also,

\[
E_A(T_{uv}^2) = E_A\left( \sum_{i < j} \psi_{uv}(i,j) \right)^2 + 2E_A\left( \sum_{i < j} \psi_{uv}(i,j) \right) \left( \sum_{i=1}^n \xi_{uv}(i) \right) + E_A\left( \sum_{i=1}^n \xi_{uv}(i) \right)^2
\]

By calculations similar to those in Theorem 2.3.4, we obtain

\[
E_A(T_{uv}^2) = \frac{n(n - 1)}{2} \int H_u dH_v
\]

\[
+ n(n - 1)(n - 2) \int I_{uv} d(1 - (1 - I_{vu})^2)
\]

\[
+ \frac{n(n - 1)(n - 2)(n - 3)}{4} (\int H_u dH_v)^2
\]

\[
+ 2n(n - 1) \int (1 - I_{vu})^* dI_{uv} + n(n - 1)(n - 2)(\int H_u dH_v)(\int F_u dF_v)
\]

\[
+ n(\int F_u dF_v) + n(n - 1)(\int F_u dF_v)^2 \quad (4.1.6)
\]

where \( I_{vu}^* \) is the distribution function of \( \text{Min}[0, X_v - X_u] \).
From (4.1.5) and (4.1.6) we obtain

\[ \sigma_A^2(T_{uv}) = \frac{n(n-1)}{2} \int H_u \, dH_v + n(n-1)(n-2) \int I_{uv} \, d(1 - (1 - I_{uv})^2) \]

\[ + \frac{10n^2 - 6n - 4n}{4} (\int H_u \, dH_v)^2 + 2n(n-1) \int (1 - I_{uv}^*) \, dI_{uv} \]

\[ + (2n - 2n^2)(\int H_u \, dH_v)(\int F_u \, dF_v) - n(\int F_u \, dF_v)^2 + nF_u \, dF_v \]  \hspace{1cm} (4.1.7)

To find \( \rho_A(T_{uv}, T_{uw}) \) we need only evaluate \( E_A(T_{uv} \cdot T_{uw}) \). In the same manner as the calculation of \( E_\iota(T_{uv} \cdot T_{uw}) \) we obtain

\[ E_A(T_{uv} \cdot T_{uw}) = E_A(A_1) + E_A(A_2) + E_A(A_2') + E_A(A_3) \] \hspace{1cm} (4.1.8)

where

\[ E_A(A_1) = \frac{n(n-1)}{2} \, P_A(X_{iu} + X_{ju} < X_{iv} + X_{jw}; X_{iu} + X_{ju} < X_{jw} + X_{iw}) \]

\[ + n(n-1)(n-2) \, P_A(X_{iu} < X_{jv} + X_{iw} - X_{ju}; X_{iu} < X_{jw} + X_{kw} - X_{ku}) \]

\[ + \frac{n(n-1)(n-2)(n-3)}{4} \, P_A(X_{iu} + X_{ju} < X_{iv} + X_{jv}) \]

\[ \times P_A(X_{ku} + X_{lw} < X_{kw} + X_{lw}) \] \hspace{1cm} (4.1.9)

\[ E_A(A_2) = \frac{n(n-1)(n-2)}{2} \, P_A(X_{iu} + X_{jw} < X_{iv} + X_{jv}) \cdot P_A(X_{ku} < X_{kw}) \]

\[ + n(n-1) \, P_A(X_{ju} < X_{jv} + X_{iw} - X_{iu}; X_{ju} < X_{jv}) \] \hspace{1cm} (4.1.10)

\[ E_A(A_2') \] is obtained from \( E_A(A_2) \) by interchanging \( v \) and \( w \), and
\[ E_{A}(A_2) = nP_{A_1}(X_{iu} < X_{iv}; X_{iu} < X_{iw}) + n(n - 1)P_{A_1}(X_{iu} < X_{iv})P_{A_1}(X_{iu} < X_{iw}) \]

\[(4.1.11)\]

For simplicity, we have left expressions (4.1.9 - 4.1.11) in terms of the alternative probabilities rather than the corresponding integrals. Now, suppose that \( k = 2 \), and let us define the \( A_1 \) alternatives by

\[ F_0(x) = \Phi(x), \quad F_1(x) = \phi(x), \quad F_2(x) = \phi(x - \theta), \quad \theta > 0 \]

where \( \Phi \) is the normal c. d. f.

If we use the signed-rank procedure, the probability of making the correct decision is

\[ P_{A_1} = P_{A_1} \left( \frac{T_{01} - \frac{n(n + 1)}{4}}{\sqrt{\frac{n(n + 1)(2n + 1)}{24}}} < c(n) ; \frac{T_{02} - \frac{n(n + 1)}{4}}{\sqrt{\frac{n(n + 1)(2n + 1)}{24}}} > c(n) \right) \]

Under the \( A_1 \) alternatives, expressions (4.1.5), (4.1.7), and (4.1.8) are easily evaluated and utilizing the asymptotic joint normality of \( (T_{01}, T_{02}) \), we obtain approximate values of \( P_{A_1} \) from tables of the bivariate normal. Table 4.1 gives the approximate values of \( P_{A_1} \) for \( \alpha = .05 \), \( n = 10, 15, 20, 25, 50 \) and \( \frac{\theta}{\sigma} = .2, .5, 1 \).
Table 4.1

\[ P_{A_1} (\alpha = .05) \]

<table>
<thead>
<tr>
<th>n</th>
<th>( \theta/\sigma )</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
</tr>
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<tr>
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<td>.06</td>
<td>.15</td>
<td>.42</td>
<td></td>
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<td>15</td>
<td>.07</td>
<td>.22</td>
<td>.68</td>
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<td>50</td>
<td>.15</td>
<td>.65</td>
<td>.97</td>
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</table>

For a comparison, we consider a multiple comparison method proposed by Steel [13]. Specifically, say treatment \( j \) is better than the control if

\[ N_{0j} \geq \frac{n}{2} + e_{(\alpha,k)}^n \cdot \sqrt{\frac{n}{4}} \]

where

\[ N_{0j} = \text{number of blocks in which } X_{i0} < X_{ij} \text{ and } e_{(\alpha,k)}^n \text{ can be obtained from Gupta's equi-correlated normal tables (entered at } \rho = 1/3) \]

such that

\[ P_{0} \left( N_{0j} \leq \frac{n}{2} + e_{(\alpha,k)}^n \sqrt{\frac{n}{4}} , \ j = 1 \cdots k \right) \approx 1 - \alpha \]

From (2.4.4), (2.4.5), and the asymptotic joint normality of \((N_{01}, N_{02})\), we can calculate approximate values of
\[
Q_{A_1} = P_{A_1} \left( \frac{N_{01} - \frac{n}{2}}{\sqrt{\frac{n}{4}}} < e^{\frac{n}{n}}; \frac{N_{02} - \frac{n}{2}}{\sqrt{\frac{n}{4}}} > e^{\frac{n}{n}} \right)
\]

Table 4.2 gives the approximate values of \( Q_{A_1} \) for \( \alpha = .05 \), \( n = 10, 15, 20, 25, 50 \), and \( \frac{\theta}{\sigma} = .2, .5, 1 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \frac{\theta}{\sigma} )</th>
<th>.2</th>
<th>.5</th>
<th>1</th>
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<td>.48</td>
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</table>

From Tables 4.1 and 4.2, we see that \( P_{A_1} \) compares favorably with \( Q_{A_1} \). This, of course, is not an unexpected result in view of what we know about the sign test and signed-rank test for detecting normal translation.
Chapter 5

Summary

We have been interested in procedures for testing $H_0$ (1.1), which are also sensitive to the ordered alternatives $H_a$ (1.2). The tests under consideration were:

(i) Jonckheere's $\tau$ test (2.1.1) based on Kendall's rank correlation coefficient between postulated order and observation order in each block.

(ii) Page's $\rho$ test (2.1.2) based on Spearman's rank correlation coefficient between postulated order and observation order in each block.

(iii) The normal theory $t$ statistic for testing $\theta = 0$ in the model $X_{ij} = b_i + (j - 1)\theta + \epsilon_{ij}$ where the $\epsilon_{ij}$ are i. i. d. according to $N(0, \sigma^2)$.

(iv) The $Y$ test, introduced here (2.2.3), based on a weighted sum of signed-rank statistics.

In Chapter 2 we found that, unlike $\tau$ and $\rho$, $Y$ is not distribution-free under $H_0$, but using the asymptotic normality of $Y$ we defined a procedure which is asymptotically distribution-free. In this chapter we also determined the consistency parameters of the rank
tests. For \( \tau, \rho, \) and \( Y, \) these parameters were \( \sum_{u < v}^{k} \int_{u}^{v} F_{u} dF_{v}, \sum_{u < v}^{k} (v - u) \int_{u}^{v} F_{u} dF_{v}, \) and \( \sum_{u < v}^{k} \int_{u}^{v} (F_{u} F_{v}) d(F_{u} F_{v}), \) respectively. We thus saw that the tests are consistent against large classes of alternatives which of course include \( H_{a}. \) The consistency conditions also show that if we postulate an ordering which is "close to" but not exactly the correct ordering, we do not destroy the desired consistency of the tests.

In Chapter 3 we compared the tests on the basis of asymptotic efficiencies. In particular, for the shift (S) alternatives, we found that \( Y \) compares quite favorably with respect to its competitors. For example, \( E(Y, t) > .864 \) for all \( F \) and all \( k. \) Also, when \( F \) is normal, \( E(Y, t) = .963 \) for \( k = 3, \) \( E(Y, t) \rightarrow .990 \) as \( k \rightarrow \infty, \) and \( E(Y, \rho) > 1 \ (E(Y, \tau) > 1) \) for every \( k. \) For the contamination (C) alternatives, with a fraction of the blocks having no treatment effect, we saw that the rank tests are frequently more (Pitman) efficient against a pure shift than a mixture of active and inactive blocks. In Section 3.3 we mentioned another desirable property of the rank tests as compared to the t-test. Namely, for the \( V \) alternatives (3.3.1),

\[
\lim_{c \rightarrow \infty} E(Y, t) = \lim_{c \rightarrow \infty} E(\rho, t) = \lim_{c \rightarrow \infty} E(\tau, t) = +\infty
\]

In Chapter 4 we modified Nemenyi's signed-rank multiple comparison method so that it would be asymptotically distribution-free. For a group of normal translation alternatives, we found that the approximate probabilities of making the correct decision with the signed-rank statistic compare favorably with the corresponding ones of a multiple sign procedure proposed by Steel.
References


