NONPARAMETRIC TESTS AND ESTIMATION OF SCALE IN THE TWO SAMPLE PROBLEM

BY
GALEN R. SHORACK

TECHNICAL REPORT NO. 10
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I. INTRODUCTION

1. Statement of the problem.

Let \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \) be independent random samples from populations having continuous c.d.f.'s \( F_X(t) = \psi \left( \frac{t - \mu}{\sigma} \right) \) and \( F_Y(t) = \psi \left( \frac{t - \nu}{\tau} \right) \). Thus \( (X-\mu)/\sigma \) and \( (Y-\nu)/\tau \) both have c.d.f. \( \psi(\cdot) \). We define \( \Delta = \tau/\sigma \) to be the ratio of the scale parameters; and we assume that the location parameters (medians) \( \mu \) and \( \nu \) are unknown.

This paper will examine a class of rank-like statistics, proposed by Moses [14], for testing hypotheses concerning \( \Delta^2 \). We will also examine how these statistics may be used to provide point and confidence interval estimates for \( \Delta^2 \). In addition we will compare these tests and estimates with others which have been proposed for this problem.

2. The limitations of the classical F-test.

If \( \psi(\cdot) \) were known to be the unit normal c.d.f., then our answers to the above problem would be based on the statistic

\[
F = \frac{\sum_{j=1}^{n} (Y_j - \bar{Y})^2}{\frac{m-1}{n-1} \sum_{i=1}^{m} (X_i - \bar{X})^2}
\]

which is distributed as \( \Delta^2 F_{n-1,m-1} \); and for which

\[
\frac{1}{\sqrt{2} \Delta^2} \sqrt{\frac{mn}{m+n}} (F - \Delta^2)
\]

is asymptotically \( N(0,1) \).
Suppose now we know only that the c.d.f. \( \psi(\cdot) \) has variance \( \mu_2 \) and fourth central moment \( \mu_4 \). Then by page 353 of Cramér [6], we know that

\[
(2.1) \quad \frac{1}{\sqrt{2 \Delta^2}} \sqrt{\frac{mn}{m+n}} (F - \Delta^2)
\]
is asymptotically \( N(0, 1 + \frac{\gamma_2}{2}) \) where \( \gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 \).

Thus, if we use the classical F-test when \( \mu_1/\mu_2^2 \neq 3 \) \((\mu_1/\mu_2^2 = 3 \) if \( \psi(\cdot) \) is a normal c.d.f.), then even asymptotically a test which is supposed to be of size \( \alpha \) will not be. This error can be serious; Box [3] gives examples when the true level is as large as 0.166 or as small as 0.0056 when the level is supposed to be 0.050. The classical F-test is seen to be very non-robust.

3. History of suggested remedies.

In this section we will present a very brief summary of six proposals for overcoming the difficulties associated with the use of the classical F-test. These are:

1. Approximation to a permutation theory test
2. Reducing the problem to an ANOVA test about means
   (Levene [13])
3. Rank tests
4. Rank-like tests
5. Reducing the problem to an ANOVA test about means
   (Box [3])

2
(6) Rank tests adjusted by centering at sample medians.

(1) Box and Anderson [4]:

The Box and Anderson proposal is to compare the two variances by means of a permutation theory test. Thus they regard the observed values $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ of the random variables as fixed numbers constituting the sample space. If the two populations are identical, then each of the $\binom{N}{m}$ permutations associating $m$ of these numbers with the $X$ population would be equally likely. However, under the alternative that the $Y$ population has greater dispersion then the $X$ population certain of these permutations would be more likely to occur than others. The critical region of the permutation test is that $100\alpha\%$ of the permutations judged beforehand to be most likely to occur if the alternative is true. Box and Anderson suggest using for the critical region those permutations for which the classical $F$-ratio is large.

Under normal theory $B = (1 + \left(\frac{n-1}{m-1} F\right)\frac{1}{2})$ has a Beta($\frac{n-1}{2}, \frac{m-1}{2}$) distribution; and hence it has mean $E_N = \frac{n-1}{N-2}$ and variance $V_N = \frac{2(n-1)(m-1)}{(N-2)^2 N}$ and range $(0,1)$, where $N = m + n$. Under permutation theory $B$ has mean $E_P = \frac{n-1}{N-2}$ and variance $V_P = \frac{2(n-1)(m-1)}{(N-2)^2 N} \{1 + \frac{1}{2} \frac{N-2}{N-3} (b_2-3)\}$ and range $(0,1)$, where

$$b_2 = N \left( \sum_{j=1}^{n} (Y_j - \overline{Y})^4 + \sum_{i=1}^{m} (X_i - \overline{X})^4 \right) + \left( \sum_{j=1}^{n} (Y_j - \overline{Y})^2 \right) \frac{m}{N} + \left( \sum_{i=1}^{m} (X_i - \overline{X})^2 \right) \frac{n}{N}.$$

Box and Anderson thus suggest approximating the permutation distribution of $B$ by a beta distribution having mean $E_P$ and variance $V_P$; i.e., by a Beta $\left( \frac{d(n-1)}{2}, \frac{d(m-1)}{2} \right)$ distribution where
\[ d = \left[ 1 + \frac{1}{2} \left( \frac{N}{N-b_2} \right)(b_2-3) \right]^{-1} = \left[ 1 + \frac{b_2-3}{2} \right]^{-1} + O\left( \frac{1}{N} \right) \]. Monte Carlo sampling by them showed that their proposal maintained level quite well, had little power loss under normality, and had good power for the uniform and double exponential distributions. (See Figures 4, 5 and 6.)

(2) Levene [13]:

If \( F_X(\cdot) \) and \( F_Y(\cdot) \) differ only in location and scale, then the distributions of \( |X_i - \mu| \) and \( |Y_j - \nu| \) differ only by a scale factor.

Levene thus suggests applying an ANOVA tests for means to one of the sets of variables \( Z_{ij} = \frac{X_i - \bar{X}}{S_{ij}}, \log_{10} Z_{ij}, \text{ or } \sqrt{Z_{ij}} \) where \( Z_{ij} = |X_j - \bar{X}| \) and \( Z_{2j} = |Y_j - \bar{Y}| \). Now ANOVA tests for means are based on assumptions of (a) normality, (b) equality of variance, (c) independence of the observations. Except for (b) in the case of \( \log_{10} Z_{ij} \), all three of these basic assumptions are violated in the present case. However, (a) tests of means are robust for non-normality, (b) tests of means are robust for inequality of variances provided we insist that \( m = n \), and (c) correlations are of order \( \frac{1}{N^2} \) and so we have reason to hope that the effect of dependence is small. Monte Carlo sampling by him led him to prefer using the \( Z_{ij} \)'s. The corresponding ANOVA test maintained level well, had small power loss under normality, and had good power for the uniform and double exponential distributions. See Figures 4b, 5 and 6. (This author feels the results show the Box-Anderson method to be superior to that of Levene. Also the Box-Anderson method is applicable even if \( m \neq n \).)

These first two proposals have reasonable power; but their level
is only approximate. Proposals (3) and (4) were made with this inadequacy in mind.

(3) Rank tests:

It is imperative that some information about $\mu$ and $\nu$ be assumed before rank tests of dispersion can reasonably be applied; see for example Moses [14]. The usual assumption made is that $\mu = \nu$. The following idea provides the basis for the proposed tests. Knowing that the medians are equal, if $\tau > \sigma$ then more of the ranks in the tails will tend to be $Y$ ranks.

Statistics of the form $\sum_{i=1}^{N} E_{N,i} Z_{N,i}$, where the $E_{N,i}$'s are real numbers (scores) and where $Z_{N,i}$ equals 1 if the $i$th smallest observation is an $X$ and 0 if not, are distribution free when $\sigma = \tau$ and are asymptotically normal under broad hypothesis in both the null and non-null cases (by the Chernoff-Savage theorem [5]). The dispersion tests of Barton-David, Ansari-Bradley-Freund, Siegel-Tukey, Mood, Klotz and Capon are all of this form; see [11, 2] for a description of them. Sukhatme [20] proposed an additional rank statistic, easily seen to be a $U$-statistic, which require the stronger assumption that $\mu = \nu = 0$.

The assumption that $\mu = \nu$ is not a reasonable one for many practical applications. Proposal (4) does not suffer from this deficiency.

(4) Moses [14]:

Randomly divide the observations from the $X$ and $Y$ populations into $m'$ and $n'$ subgroups of size $k$ respectively; any extra observations
are discarded. From these groups of size \( k \) we obtain \( m' \) and \( n' \) independent estimates of the dispersion in the \( X \) and \( Y \) samples, respectively. Denote these by \( w_1^k, \ldots, w_m^k \) and \( \omega_1^k, \ldots, \omega_n^k \). The test statistic proposed is

\[
W_k = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(w_i^k, \omega_j^k)
\]

where \( \chi(w, \omega) = \begin{cases} 1 & \text{if } w < \omega \\ 0 & \text{if } w \geq \omega \end{cases} \)

This is clearly a statistic of the Mann-Whitney type. Large values of the statistic imply \( \tau^2 > \sigma^2 \) (\( \Delta^2 > 1 \)) while small values imply \( \tau^2 < \sigma^2 \) (\( \Delta^2 < 1 \)).

The specific members of this class with which we shall concern ourselves are:

\[
V_k = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(v_i^k, v_j^k)
\]

\[
R_k = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(r_i^k, \rho_j^k)
\]

\[
G_k = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(g_i^k, \gamma_j^k)
\]

\[
D_k = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(d_i^k, \delta_j^k)
\]

where \( v_i^k, r_i^k, g_i^k \) and \( d_i^k \) are the sample variance times \( (k-1) \), the
sample range, Gini's average difference and the mean deviation about the sample mean computed for the $i^{th}$ subgroup of size $k$ in the $X$-sample; and $v_j^k$, $p_j^k$, $\gamma_j^k$ and $\delta_j^k$ are the corresponding statistics computed for the $j^{th}$ subgroups of size $k$ in the $Y$-sample. Later we will let $v_k$, $v_k'$, $r_k'$, etc., denote generic random variables distributed as are the $v_i^k$, $v_j^k$, $r_i^k$, etc. Also, let $N' = m' + n'$.

Moses pointed out the following properties of tests of the hypothesis $H_0: \Delta^2 \leq 1$ based of these statistics.

(a) They are of exact size when $\sigma = 1$.

(b) $\mu$ and $\nu$ can be completely unspecified.

(c) They are consistent against any alternative such that 

$$P\{w_k < w_k'\} > \frac{1}{2};$$

and are appropriate when $P\{w_k < w_k'\}$ is a "natural non-parametric measure of dispersion".

(d) When $\psi(\cdot)$ is the unit normal c.d.f. the asymptotic relative efficiency of the $V_j$-test with respect to the classical $F$-test is $\frac{1}{2}$.

Moses refers to these tests as rank-like tests; we will also.

(5) Box [3]:

This proposal is to apply an ANOVA test for means to the random variables $\log v^k_i, i = 1, \ldots, m'$ and $\log v^k_j, j = 1, \ldots, n'$. This test has only approximately correct size $\alpha$. (The rank-like tests are simply an adaptation of Box's test that have exact size.)

(6) Sukhatme [19]:

In another attempt to avoid the assumption that $\mu = \nu$ Sukhatme suggested
applying the usual rank tests to the random variables $X_1 - \tilde{X}$ and $Y_j - \tilde{Y}$; where $\tilde{X}$ and $\tilde{Y}$ are the sample medians. He gave conditions for these statistics being asymptotically distribution free under $H_0$; the asymptotic distribution being that of the same rank test applied to the $(X_1 - \mu)'s$ and $(Y_j - \nu)'s$. However, such a test would only have an approximately correct size $\alpha$.

It is now our purpose to investigate more fully the class of tests proposed by Moses [14] as described in (14). We will compare these tests with those of (1), (2), (3), (5) and (6) whenever possible. Specifically, we first present a convenient graphical procedure for obtaining point and confidence interval estimates for $\Delta^2$ using Moses' rank-like statistics; properties of the point estimate are discussed. Then Pitman efficiency of the rank-like tests with respect to the classical F-test for testing $H_0: \Delta^2 \leq 1$ is discussed; a general formula is presented and is evaluated in special cases. An expression for the exact small sample power of the best rank-like test is given for the special cases of $\Psi(\cdot)$ being a normal c.d.f. and $\Psi(\cdot)$ being a uniform c.d.f. Power functions, under normality, of a particular rank test of dispersion due to B.V. Sukhatme are given for the cases of (i) known medians and (ii) medians estimated by sample means. Graphs reflecting the results of previous investigations of the problem are presented for comparison in Figures 4, 5 and 6. Conclusions are presented in section 10.
II. CONFIDENCE INTERVAL AND POINT ESTIMATION OF $\Delta^2$

4. Confidence intervals for $\Delta^2$ using the rank-like statistic $W_k$.

Suppose now we wish to obtain one- or two-sided confidence intervals for $\Delta^2 = \tau^2 / \sigma^2$. The usual method of obtaining a confidence interval is to invert a test of significance. Thus if $\psi(\cdot)$ were known to be a normal c.d.f. we could invert the statement $\alpha = P(F \leq \Delta^2 F_{n-1,m-1}^{(\alpha)})$ to obtain the one-sided confidence statement $1 - \alpha = P(\Delta^2 \leq F_{n-1,m-1}^{(\alpha)})$.

We will now outline a simple method of obtaining confidence intervals for $\Delta^2$ by inverting rank-like tests. Note first that if $\int x^2 d\psi(x) < + \infty$, then $\Delta^2$ is the ratio of the variances of the two populations. In what follows we will write $U \sim Z$ to denote that $U$ is identically distributed as $Z$. We now state two obvious propositions and then follow them with a statement of the result which provides the key to our confidence interval procedure.

Proposition 1: If $\Delta$ is the true value of $\tau / \sigma$, then

(i) $\Delta^2 v_k \sim v_k$

(ii) $\Delta r_k \sim c_k$

(iii) $\Delta \varepsilon_k \sim \gamma_k$

(iv) $\Delta d_k \sim \delta_k$.

Proposition 2: If $\Delta$ is the true value of $\tau / \sigma$, then each of the following has the null distribution of the Mann-Whitney statistic based on samples of size $m'$ and $n'$.
(i) $V_k(\Delta) = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(\Delta v_{i,j}^k, \nu_j^k)$

(ii) $R_k(\Delta) = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(\Delta x_{i,j}^k, \beta_j^k)$

(iii) $G_k(\Delta) = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(\Delta g_{i,j}^k, \gamma_j^k)$

(iv) $D_k(\Delta) = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(\Delta d_{i,j}^k, \delta_j^k)$

Throughout the rest of this paper $W_k(\Delta)$ will be used to denote one of the quantities $V_k(\Delta), R_k(\Delta), G_k(\Delta)$ or $D_k(\Delta)$. Also $h(\Delta)$ will denote that function of $\Delta$ (either $\Delta$ or $\Delta^2$) for which $h(\Delta)W_k \sim \omega_k$.

Theorem 1: Let $C_{m',n'}(\alpha)$ denote the $\alpha$ percent point of the null distribution of the Mann-Whitney statistic. Then

$$[\Delta^2: C_{m',n'}(\alpha^2) < W_k(\Delta) < m'n' - C_{m',n'}(\alpha^2)]$$

is a two-sided confidence interval for $\Delta^2$ having confidence coefficient $1 - \alpha$. One-sided intervals are similarly obtained.

Proof: See Theorem 4(i) in Chapter 3 of Lehmann [12]. q.e.d.

Once $w_1, \ldots, w_{m'}$ and $a_1, \ldots, a_n$, have been computed, two-sided confidence intervals are quickly and simply obtained by the following graphical procedure.
Graphical Procedure to Obtain a 90% Confidence Interval for $\Delta^2$ when $m = n = 3$.

Figure 1

Graphical procedure:

(i) Consider a $w, \omega$ coordinate system.

(ii) Draw the lines $w = w_1, \ldots, w = w_m$ and also the lines $\omega = \omega_1, \ldots, \omega = \omega_n$, so that a lattice of $m'n'$ points is formed.

(iii) Determine the value of $C_{m',n'}^{(\frac{\alpha}{2})}$ from the Mann-Whitney tables [16]; or from the normal approximation.

(iv) Draw a line through the origin and one of the lattice points so that exactly $C_{m',n'}^{(\frac{\alpha}{2})}$ lattice points are strictly below the line. Let $\Delta_1$ denote the slope of this line.

(v) Draw a line through the origin and one of the lattice points
so that exactly \( c'_{m',n'}(\frac{\alpha}{2}) \) lattice points are strictly above the line. Let \( \Delta_2 \) denote the slope of this line.

(vi) If \( W_k = V_k \) is used, then \((\Delta_1, \Delta_2)\) is a \(100(1-\alpha)\%\) confidence interval for \( \Delta^2 \).

(vi') If \( W_k = R_k, U_k \) or \( D_k \) is used, then \((\Delta_1^2, \Delta_2^2)\) is a \(100(1-\alpha)\%\) confidence interval for \( \Delta^2 \).

One sided intervals are similarly obtained.

We now justify that if we are using \( R_k \) then \((0, \Delta_2^2)\) is a \(100(1-\alpha)\%\) confidence interval for \( \Delta^2 \). Justification in the other cases is completely analogous.

\[
\{ \Delta^2 : c'_{m',n'}(\frac{\alpha}{2}) < R_k(\Delta) \} = \{ \Delta^2 : c'_{m',n'}(\frac{\alpha}{2}) < \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(\Delta x_i^k, \rho_j^k) \}
\]

\[
= \{ \Delta^2 : \text{there are more than } c'_{m',n'}(\frac{\alpha}{2}) \text{ pairs } i,j \text{ for which } \Delta x_i^k < \rho_j^k \}
\]

\[
= \{ \Delta^2 : \text{there are more than } c'_{m',n'}(\frac{\alpha}{2}) \text{ lattice points strictly above the line } \rho = \Delta x \}
\]

\[
= (0, \Delta_2^2)
\]

Note that \( R_k(\Delta) \) is the number of lattice points strictly above the line \( \rho = \Delta x \), and is thus a decreasing function of \( \Delta \).

5. Confidence intervals for \( \Delta^2 \) using proposals (1), (2), (3), (5) and (6).

In the general situation of arbitrary \( \psi(\cdot) \) the Box-Anderson
approximation to the permutation theory test is easily inverted in theory to obtain confidence intervals of approximately correct size; the results are

\[
1 - \alpha \approx P\left\{ \Delta^2 \leq F \cdot \left( F_{d_\Delta(n-1),d_\Delta(m-1)}(\alpha) \right)^{-1} \right\}
\]

\[
(5.1) \quad 1 - \alpha \approx P\left\{ F \cdot \left( F_{d_\Delta(n-1),d_\Delta(m-1)}(1-\alpha) \right)^{-1} \leq \Delta^2 \leq F \cdot \left( F_{d_\Delta(n-1),d_\Delta(m-1)}(\frac{\alpha}{2}) \right)^{-1} \right\}
\]

\[
1 - \alpha \approx P\left\{ F \cdot \left( F_{d_\Delta(n-1),d_\Delta(m-1)}(1-\alpha) \right)^{-1} \leq \Delta^2 \right\}
\]

where \( d_\Delta = \left( 1 + \frac{b_2(\Delta)-3}{2} \right)^{-1} + O(\frac{1}{N}) \)

and \( b_2(\Delta) = N \left[ \frac{1}{\Delta^2} \sum_{j=1}^{n} (Y_j - \overline{Y})^4 + \sum_{i=1}^{m} (X_i - \overline{X})^4 \right] / \left[ \frac{1}{\Delta^2} \sum_{j=1}^{n} (Y_j - \overline{Y})^2 + \sum_{i=1}^{m} (X_i - \overline{X})^2 \right]^2 \).

Actually performing this inversion for a given set of data would appear to be quite tedious.

Levene does not present a method whereby his test based on \( Z_{ij} \)'s, \( Z_{ij}^2 \)'s or \( \sqrt{Z_{ij}} \)'s can be inverted to give confidence intervals for \( \Delta^2 \); nor is one obvious. His findings showed the power of the test based on the \( \log_{10}Z_{ij} \)'s to be so poor that we ignore the fact that a test based on these variables is easily inverted.

Rank tests of dispersion can not be inverted to give confidence intervals unless \( \mu \) and \( \nu \) are both known; since if we follow the usual procedure, a value \( \Delta^2_{\chi} \) is included in the confidence interval for \( \Delta^2 \) if and only if when the test of hypothesis is applied to the random variables \( \Delta X_1, \ldots, \Delta X_m \) and \( Y_1, \ldots, Y_n \) it does not lead to
rejection. However the $\Delta X$'s have median $\Delta \mu$ and the Y's have median $\nu$; and since the difference in location parameters $\Delta \mu-\nu$ is unknown the rank test of dispersion is inappropriate. Note that $\Delta L-\nu$ is known only if both $\mu$ and $\nu$ are known; knowing the difference $\mu-\nu$ is not sufficient.

The method of inverting Box's [3] procedure to obtain confidence intervals for $\Delta^2$ is obvious.

If we try to use proposal (6) to obtain approximate confidence intervals then we essentially have a case in which $\mu = \nu = 0$. But even if both medians are known to be zero, it appears to this author that, except for Sukhatme's [20] test, discussed below, none of the tests listed above in Section 3 (3) could be inverted in a simple fashion as is done for the tests based on $W_k$ by the graphical procedure. They could of course be inverted by trial and error methods.

Sukhatme's [20] test is based on the statistic

(5.2) $S = \sum_{i=1}^{m} \sum_{j=1}^{n} \Lambda(X_1, Y_1)$

where $\Lambda(x, y) = \begin{cases} 
1 & \text{if } y < x < 0 \text{ or } 0 < x < y \\
0 & \text{if not}
\end{cases}$

A simple graphical procedure could be used to invert tests based on this statistic to obtain confidence intervals for $\Delta^2$. It would be based on drawing the lines $X = X_1, \ldots, X = X_m$ and $Y = Y_1, \ldots, Y = Y_n$ on an $X,Y$ coordinate system and varying the slopes of two lines through
the origin until certain regions each contained a specific number of the \( mn \) lattice points.

![Graphical Estimation of \( \Delta^2 \) Using Sukhatme's Statistic](image)

**Figure 2**

This is illustrated in Figure 2 where the union of region A and region B contain the number of lattice points appropriate to define a two-sided critical region. The resulting confidence interval would be \((\Delta^2, \Delta^2)\); see Figure 2 for definition of \( \Delta \) and \( \Delta \). The problem in actually using this is that the null distribution of \( S \) has not yet been tabulated.

A test differing only slightly from Sukhatme's [20] test is the test based on the statistic

\[
(5.3) \quad \sum_{i=1}^{m} \sum_{j=1}^{n} x(|X_i|, |Y_j|) ;
\]
and it could clearly be inverted using the graphical procedure described in section 4.

6a. Point estimates using the rank-like statistic $W_k$. 

Now when $\Delta$ is the true value of $\tau/\sigma$ the random variable $W_k(\Delta)$ has the null Mann-Whitney distribution. Moreover $E[W_k(\Delta)] = \frac{m'n'}{2}$. It is thus reasonable to propose as an estimate of $\Delta^2$ a value $\hat{\Delta}^2$ for which $W_k(\hat{\Delta})$ is as near as possible to $\frac{m'n'}{2}$. The estimate we propose can in fact be described as

\[
\hat{\Delta}^2 = \begin{cases} 
\frac{\inf \{\Delta^2 : W_k(\Delta) = \frac{m'n' + 1}{2}\}}{2} & \text{if } m'n' \text{ is odd} \\
\text{geometric mean } \{\Delta^2 : W_k(\Delta) = \frac{m'n'}{2}\} & \text{if } m'n' \text{ is even.}
\end{cases}
\]

This estimate may also be obtained by the graphical procedure of section 4.

(i) Draw the lines $w = v_1, \ldots, w = v_m$, and $\omega = \omega_1, \ldots, \omega = \omega_n$, on a $w, \omega$ coordinate system so that a lattice of $m'n'$ points is formed.

(ii) If $m'n'$ is odd: Draw a line through the origin and a lattice point so that exactly $\frac{m'n' + 1}{2}$ lattice points lie on or below the line. Call its slope $\Delta_3$. If $W_k = V_k$, then $\hat{\Delta}^2 = \Delta_3^2$. If $W_k = R_k$, $G_k$ or $D_k$, then $\hat{\Delta}^2 = \Delta_3^2$.

(ii') If $m'n'$ is even: Draw a line through the origin and a lattice point so that exactly $\frac{m'n'}{2}$ lattice points lie on or below the line. Call its slope $\Delta_4$. Draw a line through
the origin and a lattice point so that exactly $\frac{m'n'}{2}$ lattice points lie on or above the line. Call its slope $\Delta y$. If $W_k = V_k$, then $\Delta^2 = \sqrt{\Delta y \Delta y}$. If $W_k = R_k, G_k$ or $D_k$, then $\Delta^2 = \Delta y \Delta y$.

Theorem 2: The estimate $\hat{\Delta}^2$ has the following properties:

(i) $\hat{\Delta}^2$ is a consistent estimate of $\Delta^2$.

(ii) $\sqrt{N'} (\hat{\Delta}^2 - \Delta^2)$ is asymptotically

$$
N\left(0, \frac{\Delta^4}{3 [t'(1)]^2 \lambda (1-\lambda) \int_0^\infty \frac{t^2}{\int W_k} dt} \right),
$$
where $\lambda = \lim_{N \to \infty} \frac{m}{m+n}$.

(iii) $\hat{\Delta}^2$ is median unbiased if $m = n$; in fact if $F_{\hat{\Delta}^2}(\cdot)$ denotes the c.d.f. of our estimate, then for all $1 < a < b$ we have

$$
F_{\hat{\Delta}^2}(\Delta^2 b) - F_{\hat{\Delta}^2}(\Delta^2 a) = F_{\hat{\Delta}^2}(\frac{\Delta^2 b}{a}) - F_{\hat{\Delta}^2}(\frac{\Delta^2 a}{b}).
$$

Aid to Understanding Theorem 2 (iii)

Figure 5

Proof: (i) and (ii) are direct applications of a result given by Sen [18]; the proof of which uses U-statistics. We prove (iii) first for the case $m'n'(= n'n')$ is odd. It suffices to show that if the graphical procedure is applied to the random variables $\Delta X_1, \ldots, \Delta X_n$ and $Y_1, \ldots, Y_n$ then $F_{\hat{\Delta}^2 1}(b) - F_{\hat{\Delta}^2 1}(a) = F_{\hat{\Delta}^2 1}(\frac{1}{b}) - F_{\hat{\Delta}^2 1}(\frac{1}{a})$; where $F_{\hat{\Delta}^2 1}(\cdot)$ denotes
the c.d.f. of our estimator in the case of two identically distributed samples. But this is clear since if the sample \( \Delta X_1 = \Delta x_1, \ldots, \Delta X_n = \Delta x_n, \\
Y_1 = y_1, \ldots, Y_n = y_n \) leads to the estimate \( \epsilon \) then the second sample \( \Delta X_1 = y_1, \ldots, \Delta X_n = y_n, \ Y_1 = \Delta x_1, \ldots, Y_n = \Delta x_n \) leads to the estimate \( \frac{1}{\epsilon} \) (since the graphical procedure for the second sample can be accomplished merely by interchanging the labels of the \( w \) and \( \omega \) coordinate axes); and since the \( \Delta X \)'s and \( Y \)'s are identically distributed the two samples have the same likelihood. If \( m'n' \) is even then in order that the result will hold it is necessary to choose a method of compromising between \( \Delta_4 \) and \( \Delta_5 \) such that the statement in parentheses in the preceding sentence remains true; but the geometric mean is an appropriate function. q.e.d.

6b. Point estimates using the Box-Anderson procedure.

The natural point estimate of \( \Delta^2 \) based on the Box-Anderson procedure is the value \( \hat{\Delta}^2 \) which satisfies

\[
(6.2) \quad \hat{\Delta}^2 = F \times \left( \frac{1}{F} \right)^{-1} \left( d_{\Delta}^{-1} (n-1), d_{\Delta}^{-1} (m-1) \right).
\]

where \( d_\Delta \) is defined in (5.1).

Now \( \frac{1}{F}, \ell = 1 \) for all \( \ell \). Hence in the case \( m = n \) (6.2) reduces to

\[ \hat{\Delta}^2 = F. \]
Thus by (2.1), when \( m = n \)

\[ \sqrt{N} \left( \hat{\Delta}^2 - \Delta^2 \right) \] is asymptotically

\[ N \left( 0, \frac{\Delta^4 \left( \beta_2 - 1 \right)}{\lambda(1-\lambda)} \right) . \]

Thus from Theorem 2(ii) we see that the asymptotic efficiency of \( \hat{\Delta}^2 \) with respect to \( \Delta^2 \) is

\[
\frac{\Delta^4 \left( \beta_2 - 1 \right)}{\lambda(1-\lambda)} \cdot \frac{k \Delta^4}{3[h'(1)]^2 \lambda(1-\lambda) \left[ \int_0^\infty t f_{W_k}^2(t) dt \right]^2}
\]

\[ = \frac{3[h'(1)]^2 \left( \beta_2 - 1 \right)}{k} \left[ \int_0^\infty t f_{W_k}^2(t) dt \right]^2 . \]

We will see in (7.1) that this expression is exactly equal to the asymptotic relative efficiency of the \( W_k \)-test with respect to the \( F \)-test for testing \( H_0: \Delta^2 \leq 1 \).

If \( m \neq n \), then (6.2) appears to be rather tedious to solve precisely because of the need to find the percent point of an \( F \) distribution having fractional degrees of freedom.
III. TESTS OF HYPOTHESIS

We now investigate the power of tests of hypothesis about $\Delta^2$ based on the statistics $W_k$. In sections 7 and 7a-7e power will be investigated in terms of asymptotic relative efficiency (a.r.e.) as defined by Pitman. In sections 8, 8a and 8b we derive expressions for exact small sample power in two particular special cases. These results form the basis of some conclusions drawn in section 10 about the appropriateness of using the statistics $W_k$ in the present problem.

In section 9 we examine the power of the test (5.3) applied to data that has first been centered at the sample means. These results form the basis for conclusions drawn in section 10 about the appropriateness of rank tests applied to centered data.

7. General expression for asymptotic relative efficiency (a.r.e.)

Suppose now we wish to test the hypothesis $H_0: \Delta^2 \leq 1$ vs. the alternative $K: \Delta^2 > 1$.

The efficacy of the classical $F$-test based on $m$ and $n$ observations is well known (see [11]) to be

$$
\epsilon_F = \frac{\beta_2}{(m+n)(\beta_2-1)}
$$

where $\beta_2 = \text{E}((X-E(X))^4) / \text{E}^2((X-E(X))^2)$.

We now derive an expression for the efficacy of the $W_k$-test based on $m$ and $n$ observations. Now
\[ E_\Delta(W_k) = P_\Delta(w_k < \omega_k) = \int_0^\infty F_{w_k}(t) dF_{\omega_k}(t) \]

\[ = \int_0^\infty F_{w_k}(t) dF_{\omega_k}(\frac{t}{h(\Delta)}) = \int_0^\infty F_{w_k}(h(\Delta)t) dF_{\omega_k}(t) \]

We now assume that \( w_k \) has a density function \( f_{w_k}(t) \); and that we can differentiate under the integral sign (this is true if \( \frac{\partial}{\partial \Delta} F_{w_k}(h(\Delta)t) f_{w_k}(t) \) is dominated by an integrable function independent of \( \Delta \)). These two conditions will need to be verified in any particular application. Thus

\[ \frac{d}{d\Delta} E_\Delta(W_k) |_{\Delta=1} = \int_0^\infty h'(l) f_{w_k}(h(l)t) f_{w_k}(t) dt \]

\[ = h'(1) \int_0^\infty t f_{w_k}^2(t) dt \]

Also

\[ \text{Var}_{H_0}[W_k] = \frac{m'+n'+1}{12} \frac{kN}{m' n'} \sim \frac{kn}{12 mn} \]

Thus the efficacy of the \( W_k \)-test based on \( m \) and \( n \) observations is

\[ \epsilon_{W_k} \sim \frac{12 mn}{k N} \left[ h'(1) \int_0^\infty t f_{w_k}^2(t) dt \right]^2 \]

Hence the asymptotic relative efficiency of the \( W_k \)-test with respect to the \( F \)-test is

\[ (7.1) \quad \epsilon_{W_k,F} = \frac{3[h'(1)]^2(b_2-1)}{k \left[ \int_0^\infty t f_{w_k}^2(t) dt \right]^2} \]
In any particular situation evaluation of $e_{W_k}^F$ will hinge on being able to derive an expression for $f_{W_k}(\cdot)$. Because of the difficulties encountered in attempting such derivations, it is found convenient at this point to introduce two additional test statistics of the form $W_k$. These are

$$B_k = \sum_{i=1}^{m'} \sum_{j=1}^{n'} X(b^k_i, \beta^k_j)$$

$$Z_k = \sum_{i=1}^{m'} \sum_{j=1}^{n'} X(z^k_i, \xi^k_j)$$

where $b^k_i$ and $z^k_i$ are the sums of the absolute values of the observations and the .93 sample quantile minus the .07 sample quantile respectively for the $i^{th}$ subgroup of $k$ $X$'s and $\beta^k_i$ and $\xi^k_j$ are the corresponding quantities for the $j^{th}$ subgroup of $k$ $Y$'s (Note that $Z_k = R_k$ if $k \leq 20$. Note that $\Delta b^k \sim \beta^k$ and $\Delta z^k \sim \xi^k$.) Some of the tabulated values below refer to these two statistics.

Below are tabulated various values of $\lim_{k \to \infty} e_{W_k}^F$. These are obtained by direct application of the following lemma.

Lemma 1: Suppose $\sqrt{k} \left( \frac{W_k - a}{b} \right)$ is asymptotically $N(0,1)$. Then

$$\int_0^\infty \frac{t}{\sqrt{k}} \chi^2_{W_k} (t) dt = \frac{a}{b} \int_0^\infty \frac{r^2}{\sqrt{k} a} \sqrt{k(w_k - a)} (t) dt + \frac{1}{\sqrt{k}} \int_0^\infty \frac{t r^2}{\sqrt{k} a} \sqrt{k(w_k - a)} (t) dt$$

(7.2) $\quad \to \frac{a}{2 \sqrt{\pi} b}$ as $k \to \infty$. 

22
Proof: Substitute in
\[ f_{\kappa}^{\prime} (t) dt = \frac{\sqrt{k}}{b} \sqrt{\kappa \left( \frac{t-a}{b} \right) dt} \]
and make the change of variable \( t' = \frac{\sqrt{k(a-t)}}{b} \), q.e.d.

7a. A.R.E. when \( \psi(\cdot) \) is the normal c.d.f.

Suppose \( \psi(\cdot) \) is known to be a normal c.d.f. Suppose also that we have agreed to break the samples up into \( m' \) and \( n' \) subgroups of size \( k \) and to confine our information on dispersion within each subgroup to the use of a single statistic. In this case the best parametric test of \( H_0: \Delta^2 \leq 1 \) vs. \( K: \Delta^2 > 1 \) is to reject if

\[ F^* = \frac{\sum_{j=1}^{n'} \sum_{i=1}^{m'} \kappa_j}{\sum_{j=1}^{n'} \sum_{i=1}^{m'} \nu_{ij}} \]

is "too large". Now \( F^* \) is distributed as \( \Delta^2 F_{n'(k-1),m'(k-1)} \). Let us consider a fixed alternative \( \Delta \). The power of the \( F^* \) test at level \( \alpha \) against this alternative is

\[ \beta_{F^*}^\alpha(\Delta) = P \left\{ F_{n'(k-1),m'(k-1)} > \frac{F_{(1-\alpha)}}{\Delta^2} \right\} . \]

The power of the classical \( F \)-test at level \( \alpha \) against this same alternative is
\[ \beta^\alpha_F(\Delta) = P\left\{ \frac{F_{n-l, m-l}^{l-\alpha}}{\Delta^2} > \frac{F_{n-l, m-l}^{l-\alpha}}{\Delta^2} \right\} \]

In comparing the F-test with the \( F^* \)-test we are concerned with two statistics having the F-distribution; one with \( n-l \) and \( m-l \) degrees of freedom and the other with \( (k-l)n^* = (k-l)[\frac{m}{k}] \) and \( (k-l)m^* = (k-l)[\frac{m}{k}] \) degrees of freedom; where \( [ \ ] \) is the greatest integer function. Let us now denote the sample sizes associated with the \( F^* \)-test by \( m^* \) and \( n^* \) and let us reserve \( m \) and \( n \) for the sample sizes associated with the F-test. Let us now let \( m, n, m^* \) and \( n^* \) satisfy \( \frac{m}{m+n} = \frac{m^*}{m^*+n^*} \). Then requiring the F-test and the \( F^* \)-test to have equal power at equal alternatives requires that \( n \) and \( n^* \) satisfy \( n - 1 = (k-l)[\frac{n^*}{k}] \). Hence the exact small sample efficiency of the \( F^* \)-test with respect to the F-test is

\[ \frac{n}{n^*} = \frac{k-l}{k} \left[ \frac{n^*}{k} \right] + \frac{1}{n} ; \]

and this is independent of \( \Delta \). It is a trivial consequence that

\[ e_{F^*,F} = \frac{k-l}{k} \]

Since the \( F^* \)-test is the best test under the conditions stated above we see that \( \frac{k-l}{k} \) is an upper bound on \( e_{W_k, F^*} \) (See Table 1).

Let us now consider the \( V_k \)-tests. It would be highly surprising
if the $V_k$-test is not the most powerful test of the class of $W_k$-tests. Now $V_k$ has the $Y_{k-1}^2$ distribution. Hence, it is straightforward to calculate that

$$(7.3) \quad e_{V_k, F} = \frac{6}{kn} \left[ \frac{\Gamma(k)}{\Gamma(k-1)} \right]^2.$$

By page 168 of [1], we see that this expression converges monotonically to $3/\pi$ as $k \to \infty$. Various values of $e_{V_k, F}$ are given in Table 1.

Tables of the distribution of the range $R_k$ and of the distribution of the mean deviation $d_k$ are available (see [9] and [8] respectively). These would make it possible to calculate $e_{R_k, F}$ and $e_{D_k, F}$ by means of numerical integration. See Table 1 for $e_{R_{20}, F}$ and $e_{R_{20}, F}$.

We remind the reader that if $\mu = \nu$ then the rank tests of dispersion of Mood (M), Ansari-Bradley (A), and Klotz (K) are appropriate and have a.r.e.'s as given at the end of Table 1.

Following an idea of Box [3], it would be possible to recover some additional information by forming estimates of $\sigma^2$ and $\tau^2$ from the sample means of the $m'$ and $n'$ different subgroups. These estimates would be independent of our previous ones and could be scaled so as to have the same distribution. Hence they could easily be combined into our Mann-Whitney statistic. However, since this is true only when $\psi(\cdot)$ is a normal c.d.f., we have chosen to ignore this possibility.
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<th>$e_{R,F}^*$</th>
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$e_{M,F} = .76$  $e_{A,F} = .61$  $e_{K,F} = 1.00$

Values of a.r.e. Under Normal Theory.

**TABLE 1**

Note also that the $V_2^*, R_2^*, G_2^*, M_2^*$ and $Z_2$-tests are all equivalent. So are the $R_3^*, G_3^*$ and $Z_3$-tests. So are the $R_k^*$ and $Z_k$-tests for $k \leq 20$. 

26
7b. A.R.E. when $\psi(\cdot)$ is the uniform c.d.f.

If $\psi(\cdot)$ is known to be a uniform c.d.f. then it is highly likely that the $R_k$-test is the most powerful test of the class of $W_k$-tests. Now $R_k$ has a $\text{Beta}(k-1,2)$ distribution. We can thus calculate that

\[(7.4) \quad e_{R_k,F} = \frac{3}{5} \frac{k(k-1)^2}{(2k-1)^2} \cdot\]

This expression increases monotonically toward $\infty$. See Table 2.

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$e_{M,F} = 1.00$ \quad $e_{A,F} = 0.60$ \quad $e_{K,F} = \infty$

Values of a.r.e. Under Uniform Distribution Theory.

**TABLE 2**

27
The A.R.E. when $\psi(\cdot)$ is the double exponential c.d.f.

If $\psi(\cdot)$ is known to be the double exponential c.d.f. then it is highly likely that $D_k$ is the most powerful of the $W_k$-tests.

However evaluation of $f_{d_k}(\cdot)$ seems too difficult. Hence in order to obtain an upper bound on $e^{W_k,F}$ we propose to assume $\mu = \nu = 0$ and apply the $B_k$-test; since given $\mu = \nu = 0$ the $B_k$-test is almost sure to be the most powerful test of the class of $W_k$-tests.

Now when $\mu = 0$ the statistic $\frac{1}{2} b_k$ has the $\chi^2_{2k}$ distribution. Hence we can easily calculate

$$e^{B_k,F} = \frac{15}{k} \left( \frac{\Gamma(2k)}{2^{2k} \Gamma^2(k)} \right)^2 = \frac{15}{4\pi k} \left( \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)} \right)^2$$

This converges monotonically to $\frac{15}{4\pi k}$ as $k \to \infty$.

However, it would seem more appropriate to compare the $B_k$-test to a parametric test which made use of the information $\mu = \nu = 0$. The test which rejects when

$$T = \frac{\sum_{j=1}^{n} |Y_j|}{m \sum_{i=1}^{m} |X_{i1}|}$$

is "too large" is uniformly most powerful unbiased and uniformly most powerful scale invariant in this case. (The joint density function of the $X$'s and $Y$'s may be written as follows

$$f(X_1,\ldots,X_m,Y_1,\ldots,Y_n) = \frac{1}{2^{m+n} \sigma^m \tau^n} e^{-\left( \frac{1}{\sigma^2} \sum_{j=1}^{n} |Y_j| + \left( \frac{1}{\tau^2} \sum_{i=1}^{m} |X_{i1}| + \sum_{j=1}^{n} |Y_j| \right) \right) \left( \frac{1}{\tau^2} \sum_{i=1}^{m} |X_{i1}| + \sum_{j=1}^{n} |Y_j| \right) \left( \frac{1}{\sigma^2} \sum_{j=1}^{n} |Y_j| + \left( \frac{1}{\tau^2} \sum_{i=1}^{m} |X_{i1}| + \sum_{j=1}^{n} |Y_j| \right) \right)}.$$
Further when \( r = \sigma \) the distribution of the statistic \( T \) does not depend on \( \sigma \); and hence \( T \) is independent of \( \sum_{i=1}^{m} |X_i| + \sum_{j=1}^{n} |Y_j| \). That the UMFU claim now follows is analogous to page 169 in Lehmann [12]. The reasoning that shows the test UMPI for common changes in scale in the two populations is analogous to that of Ex. 6 pg. 219 in [12].) Now \( T \) is distributed as \( \Delta F_{2n,2m} \). It is easy to calculate \( e_{F,T} = \frac{4}{5} \). Thus

\[
(7.5) \quad e_{k,T} = \frac{12}{k} \left[ \frac{\Gamma(2k)}{2^{2k}\Gamma^2(k)} \right]^2 \rightarrow \frac{3}{\pi} \text{ as } k \rightarrow \infty.
\]

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<tr>
<th>( k )</th>
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<th>( e_{G_k,F} )</th>
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\[
\begin{align*}
   e_{M,F} &= 1.08 \\
   e_{A,F} &= 0.94 \\
   e_{K,F} &= 1.20
\end{align*}
\]

Values of a.r.e. Under Double Exponential Distribution Theory.

<box>
 \[ e_{M,F} = 1.08 \quad e_{A,F} = 0.94 \quad e_{K,F} = 1.20 \]
</box>

TABLE 3
7d. A.R.E. when \( \psi(\cdot) \) is a contaminated normal c.d.f.

Suppose now that \( \psi(\cdot) \) is a contaminated normal c.d.f.; i.e., 
\[
\psi(t) = (1-p) \Phi(t) + p\Phi(t/S) \quad \text{when} \quad 0 \leq p \leq 1, \quad S \geq 1 \quad \text{and} \quad \Phi(\cdot) \quad \text{is the unit normal c.d.f.} \]
For this distribution the ratio \( \beta_2 \) of the fourth central moment to the square of the variance is \( 3(1-p+pS^4)/(1-p+pS^2)^2 \).
To evaluate \( \int_0^\infty t^2 f_{\tau_k}(t) dt \) we may assume without loss of generality that \( \sigma = 1 \). Now

\[
f_{\tau_k}(t) = k(k-1) \int_{-\infty}^{\infty} \psi'(r+t) \{\psi(r+t)-\psi(t)\}^{k-2} \psi'(t) dt .
\]

Also

\[
(7.6) \quad e_{R_k} = \frac{2}{k} \left[ \frac{3(1-p+pS^4)}{(1-p+pS^2)^2} - 1 \right] \left[ \int_0^{\infty} t^2 f_{\tau_k}(t) dt \right]^2 .
\]

In case \( k = 2 \) the expression for \( f_{\tau_2}(\cdot) \) is easily evaluated as

\[
f_{\tau_2}(t) = (1-p)^2 \frac{1}{\sqrt{\pi}} \exp\left(-\frac{t^2}{4}\right) + p(1-p) \frac{4}{\sqrt{2\pi} \sqrt{1+S^2}} \exp\left(-\frac{r^2}{2(1+S^2)}\right) \]
\[
\quad + \frac{p^2}{\sqrt{\pi} S} \exp\left(-\frac{r^2}{4S^2}\right) .
\]

From this we find that

\[
\int_0^{\infty} t^2 f_{\tau_2}(t) dt = \frac{(1-p)^4}{\pi} + (1-p)^3 p \frac{8\sqrt{2} \sqrt{1+S^2}}{\pi(3+S^2)} + (1-p)^2 p^2 \frac{4}{\pi} \frac{(1+S+S^2)}{1+S^2} \]
\[
\quad + (1-p)^3 p \frac{8\sqrt{2} S \sqrt{1+S^2}}{\pi(1+3S^2)} + \frac{p^4}{\pi} .
\]

30
Plugging this into (7.6) gives an expression for $e_{R_2,F}$. Values of $e_{R_2,F}$ are given in Table 1a for various choices of $p$ and $S$.

For estimating the scale parameter of a normal distribution by means of the mean of the ranges of subgroups of size $k$, the efficiency increases with subgroup size up to the most efficient subgroup size of 8, and then falls off. For a contaminated normal distribution we would expect the efficiency to increase with subgroup size up to a most efficient subgroup size that is larger than 8 before it falls off. Moreover, from Tukey [21] we would expect the mean deviation to be more appropriate than the range. Thus we would expect the values $e_{D_6,F}, e_{D_{12},F}$ to substantially exceed the values of $e_{R_2,F}$ given in Table 1a. We chose to table the values of $e_{R_2,F}$, even though the $R_2$-test is probably one of the least appropriate of all the $W_k$-tests, because it was the only one we could evaluate without resorting to rather heavy numerical integration.

<table>
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<td>5</td>
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Values of a.r.e. under Contaminated Normal Distribution Theory.

TABLE 1a
7e. A.re.e. comparison of $V_k$-tests to Box's [3] tests when $\psi(\cdot)$ is a normal c.d.f.

Let $\psi(\cdot)$ be a normal c.d.f. Letting the $L_k$-test stand for the Box [3] test applied to subgroups of size $k$ we now seek an expression for $e_{V_k,L_k}$. Since the logarithmic function is monotone we see that

$$V_k = \sum_{i=1}^{m'} \sum_{j=1}^{n'} \chi(\log v_i^k, \log v_j^k) .$$

Recall that the $L_k$-test is just the student $t$-test applied to the random variables $\log v_i^k$, $i = 1, \ldots, m'$ and $\log v_j^k$, $j = 1, \ldots, n'$.

Now the distribution of $\log v_i^k$ and $\log v_j^k$ differ only by the location factor $\log \Delta^2$. Hence

$$e_{V_k,L_k} = 12 \text{Var} [\log V_k] \left[ \int_{-\infty}^{\infty} f_{\log V_k}(t) dt \right]^2$$

by page 496 of Kendall and Stuart [10], where

$$f_{\log V_k}(t) = \frac{1}{\sqrt{2\pi(k-1)}} \exp \left\{ -\frac{1}{2} \left[ (k-1) t - e^t \right] \right\}$$

is the density function of the $\log v_i^k$'s.

Now by letting $z = e^t$ we see that

$$\int_{-\infty}^{\infty} f_{\log V_k}^2(t) dt = \int_0^{\infty} z^{k-2} e^{-z} dz$$

$$= \frac{\Gamma(k-1)}{2^{k-1}} \frac{\Gamma(k-1/2)}{\Gamma(k-1/2)} = \frac{\Gamma(k/2)}{2\sqrt{\pi}}$$
Also
\[ E\{\log V_k\} = \int_0^\infty (\log z) z^{(2-k)/2} \exp \left(-\frac{z}{2}\right) \frac{k-1}{2} e^{-z/\Gamma(k-1/2)} \]
\[ = \int_0^\infty (\log 2 + \log z) z^{(2-k)/2} \exp \left(-\frac{z}{2}\right) \frac{k-1}{2} e^{-z/\Gamma(k-1/2)} \]
\[ = \log 2 + \frac{\Gamma'(k-1/2)}{\Gamma(k-1/2)} \]

Similarly
\[ E\{(\log V_k)^2\} = \int_0^\infty (\log 2 + \log z)^2 z^{(2-k)/2} \exp \left(-\frac{z}{2}\right) \frac{k-1}{2} e^{-z/\Gamma(k-1/2)} \]
\[ = (\log 2)^2 + 2(\log 2) \frac{\Gamma'(k-1/2)}{\Gamma(k-1/2)} + \frac{\Gamma''(k-1/2)}{\Gamma(k-1/2)} \]

Thus
\[ \text{Var}[\log V_k] = \frac{\Gamma''(k-1/2) \Gamma(k-1/2) - \left[\Gamma'(k-1/2)\right]^2}{\Gamma^2(k-1/2)} \]

or letting \( \text{Trigamma}(z) \equiv \frac{d^2}{dt^2} \log \Gamma(t) \bigg|_{t=z} \) denote the well known trigamma function we see that

\[ \text{Var}[\log V_k] = \text{Trigamma} \left(\frac{k-1}{2}\right) \]

Thus

\[ (7.7) \quad e_{V_k} = \frac{3}{\pi} \frac{\Gamma^2(k/2)}{\Gamma^2(k-1/2)} \text{Trigamma} \left(\frac{k-1}{2}\right) \]
<table>
<thead>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>$\cdots$</th>
<th>21 $\cdots$</th>
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<tbody>
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<td>1.500</td>
<td>1.233</td>
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<td>1.042</td>
<td>1.028</td>
<td>$\cdots$</td>
<td>.980 $\cdots$</td>
<td></td>
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</tbody>
</table>

A.R.E. of the $V_k$-test with respect to Box's $L_k$-test when $\psi(\cdot)$ is a normal c.d.f.

**TABLE 1b**

From the standpoint of ar.e. there seems to be very little difference between the Box [3] $L_k$-tests and the $V_k$-tests when $\psi(\cdot)$ is in fact normal. However, the small sample power results available do show a considerable difference. In section 8a we will derive the exact power of the $V_2$-test with $m = n = 20$ and the exact power of the $V_7$-test with $m = n = 21$. These are plotted in Figure 4a. Monte Carlo work by Olds, Branson and Odeh [15] gives the approximate power function of the $L_6$-test with $m = n = 24$; their work also indicates that $k = 6$ is the best subgroup size when $m = n = 24$. The power function of the $L_6$-test is plotted in Figure 4b. A comparison of Figure 4a with Figure 4b clearly favors the $V_k$-tests. Remember also that the $V_k$-tests have exact level, while the level of the $L_k$-tests is only approximate.
8. Exact small sample power of the $W_k$-test.

Define

$$S_i = \begin{cases} 
1 & \text{if the } i^{th} \text{ smallest of } w_1^k, \ldots, w_m^k \text{ and } \omega_1^k, \ldots, \omega_n^k, \text{ is} \\
\omega & \text{an } \omega \\
0 & \text{if not} 
\end{cases}$$

for $i = 1, \ldots, N'$. Then it is well known (see pg. 237 and Problem 22 on page 254 in Lehmann [12]) that

$$P(S_1, \ldots, S_{N'}) = (s_1, \ldots, s_{N'})$$

$$= \frac{1}{(N')^m} \cdot \mathbb{E} \left\{ \prod_{i:s_i = 0} \frac{f_{w_k}(V(i))}{f_{\omega_k}(V(i))} \right\}$$

where $V(1) < \ldots < V(N')$ is an ordered sample from the distribution $F_{\omega_k}(\cdot)$. But the probability density function of these $N'$ ordered statistics is $N'! f_{\omega_k}(V(1)) \cdots f_{\omega_k}(V(N'))$ for $V(1) < \ldots < V(N')$.

Hence

$$P((S_1, \ldots, S_{N'}) = (s_1, \ldots, s_{N'}))$$

(8.1)

$$= m'! n'! \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_{i:s_i = 0} f_{w_k}(t_i) \prod_{i:s_i = 1} f_{\omega_k}(t_i) dt_{N'} \cdots dt_1.$$

Also recall that $f_{\omega_k}(t) = \frac{1}{h(\Delta)} f_{w_k}(\frac{t}{h(\Delta)})$. The power of the $W_k$-test is found by summing the above expression over all rank sets in the critical region.
8a. Exact small sample power of the $V_k$-test when $\psi(\cdot)$ is a normal c.d.f.

We now indicate how the exact small sample power of the $V_k$-test can be calculated when $\psi(\cdot)$ is a normal c.d.f.

In the present case the expression (8.1) can be written as

$$P\{ (S_1, \ldots, S_N) = (s_1, \ldots, s_N) \}$$

(8.2)

$$= m' \cdot n' \int_0^\infty \int_{t_1}^\infty \int_{t_2}^\infty \cdots \int_{t_{N'-1}}^\infty \prod_{l=1}^{N'} \left[ \frac{-25_1, s_1}{\Delta} f_{V_k} \left( \frac{t_l}{25_1, s_1} \right) \right] dt_N \cdots dt_1$$

where $\delta$ is the Kronecker delta. In our case

$$f_{V_k}(t) = \frac{1}{k-3} \left( \frac{t}{k-1} \right)^{k-3} e^{-}\frac{t}{2} \frac{2^{k-2} \Gamma(k-2)}{\Gamma(k-1)}$$

The necessary $N'$-fold integration (for $k$ an odd integer) was performed in the following fashion. Let $((A,I,a))$ denote the function $A t^I e^{-at}$. The following operations are easily described in this notation.

(i) $\int_0^\infty ((A, I, a)) dt = ((A/a, I, a)) + ((A/I a^2, I-1, a))$

+ $(A/I-1/a^2, I-2, a)) + \cdots + ((A/I a^{I+1}, 0, a))$

(ii) $((A_1, I_1, a_1)) \times ((A_2, I_2, a_2)) = ((A_1 A_2, I_1+I_2, a_1+a_2))$

(iii) $((A_1, I, a)) + ((A_2, I, a)) = ((A_1+A_2, I, a))$

(iv) $\int_0^\infty ((A, I, a)) dt = AI/a^{I+1}$

36
Using these four operations a computer program that evaluates \( P((S_1, \ldots, S_N) = (s_1, \ldots, s_N)) \) exactly is easily written. When this expression is summed over the rank sets in the critical region it gives the power of the \( V_k \)-test.

The power of the \( V_p \)-test when \( m' = n' = 4 \) (m=n=20) at any level \( 0 < \alpha \leq .10 \) and at any alternative \( \Delta = 1, 1.2, \sqrt{2}, 1.7, 2, \sqrt{6}, 3, 4 \) (note that \( \Delta = 1 \) is the null hypothesis) can be obtained from Table 4. The power of the \( V_l \)-test when \( m' = n' = 3 \) (m=n=21) at any level \( 0 < \alpha \leq .10 \) and at alternatives \( \Delta = 1, 1.2, \sqrt{2}, 1.7, 2, \sqrt{6}, 3, 4 \) can be obtained from Table 5. These results for \( \alpha = .05 \) are plotted in Figure 4a along with the power functions of other tests.

8b. Exact small sample power of the \( R_k \)-test when \( \psi(\cdot) \) is a uniform c.d.f.

In this section we give an expression for \( P((S_1, \ldots, S_N) = (s_1, \ldots, s_N)) \) when \( \psi(\cdot) \) is in fact the c.d.f. of a uniform distribution. We now assume without loss of generality that

\[
\psi(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\frac{t}{1} & \text{if } 0 \leq t \leq 1 \\
1 & \text{if } 1 \leq t
\end{cases}
\]

We further assume without loss of generality that the scale factor \( \sigma \) has value 1. Thus \( r_1, \ldots, r_m \) take values between 0 and 1; while \( \rho_1, \ldots, \rho_n \) take values between 0 and \( \Delta \) with \( \Delta \geq 1 \).
Let $L_I$ denote the event that the $I$ largest members of the sample $w_1, \ldots, w_m$, and $\omega_1, \ldots, \omega_n$, are $\omega$'s and the next smallest observation is a $w$. Let $M_j$ denote the event that exactly $J$ members of the sample $w_1, \ldots, w_m$ and $\omega_1, \ldots, \omega_n$ exceed 1. Let 

$$((S_1, \ldots, S_{N'}) = (s_1, \ldots, s_{N'}))$$

denote the event that $(S_1, \ldots, S_{N'}) = (s_1, \ldots, s_{N'})$. Then if the event $((S_1, \ldots, S_{N'}) = (s_1, \ldots, s_{N'})) \cap L_I$ is non-void we have 

$$P(((S_1, \ldots, S_{N'}) = (s_1, \ldots, s_{N'})) = \sum_{J=0}^{I} P((S_1, \ldots, S_{N'}) = (s_1, \ldots, s_{N'})) \cap L_I \cap M_j;$$

and furthermore, by making a minor adjustment in (8.1) we obtain 

$$P(((S_1, \ldots, S_{N'}) = (s_1, \ldots, s_{N'})) \cap L_I \cap M_j)$$

$$= m'!n'! \int_0^1 \int_0^{t_{N'-J}} \int_0^{t_{N'-J-1}} \cdots \int_0^{t_2} \prod_{i=1}^{N'-J} \left[ \frac{-s_i}{s_i} \right] \int_1^{\Delta} f_{k, \frac{t_1}{\Delta}} \left( \frac{t_i}{\Delta} \right) dt_1 \cdots dt_{N'-J}$$

$$\cdot \int_1^{\Delta} \int_1^{t_1} \int_1^{t_2} \cdots \int_1^{t_{N'-J+2}} \prod_{i=N'-J+1}^{N'} \left[ \frac{1}{\Delta} f_{k, \frac{t_1}{\Delta}} \right] dt_{N'-J+1} \cdots dt_N'$$

$$= m'!n'! \left[ \frac{k(k-1)}{\Delta^{N'}} \right] \int_1^{\Delta} \left[ \frac{\Delta - \Delta k + k - 1}{k(k-1)\Delta^{k-1}} \right]^{J}$$

$$\cdot \int_0^1 \int_0^{t_{N'-J}} \int_0^{t_{N'-J+1}} \cdots \int_0^{t_2} \prod_{i=1}^{N'-J} \left[ \frac{t_i}{\Delta} \right]^{k-2} \left( \frac{t_1}{\Delta} \right)^{k-1} dt_1 \cdots dt_{N'-J}$$

(8.3)
since \( f_k(t) = k(k-1)t^{k-2}(1-t) \) for \( 0 < t < 1 \).

The value of the integral in (8.3) can be determined from the following recursion relationship. Set \( A_{i,0} = 0 \) for \( i = 1,2,\ldots,N'-J \). Set \( A_{i,1+2} = 0 \) for \( i = 1,\ldots,N'-J \). Set

\[
A_{1,1} = \frac{1}{(k-2)\Delta l_1 s_1} \quad \text{and} \quad A_{1,2} = \frac{1}{(k-1)\Delta l_1 s_1}.
\]

For \( i = 2,\ldots,N'-J \) and \( j = 1,\ldots,i+2 \) set

\[
A_{i,j} = \left[ \frac{A_{i-1,j-1}}{(k-1)\Delta l_1 s_1} + \frac{A_{i-1,j}}{(k-2)\Delta l_1 s_1} \right] \cdot \frac{1}{i(k-1)+j-1}.
\]

Then \( P((\mathbf{s}_1,\ldots,\mathbf{s}_{N'})) = (s_1,\ldots,s_{N'},(I;J)) \cap L_i \cap M_j \)

\[
= m'! n'! \left[ \frac{k(k-1)]^{N'}}{\Delta n'} \right] \frac{1}{J'} \left[ \frac{\Delta - \Delta k + k - 1}{k(k-1)\Delta k - 1} \right]^J \cdot \left[ A_{N'-J,1} - A_{N'-J,2} + A_{N'-J,3} - \cdots + (-1)^{N'-J} A_{N'-J,N'-J+1} \right]
\]

(8.4)

Rather than presenting a proof of this, it is suggested that the reader try the integration in (8.3). He will soon convince himself.

A computer program was written which determined the value of \( I \) for each rank set in the critical region and then summed the expression (8.4) for the values \( 0 \leq J \leq I \). The results are given in Tables 6 and 7. The power of the \( R_j \)-test when \( m' = n' = 4 \) (m=n=20) at any level

39
$0 < \alpha \leq .10$ and at alternatives $\Delta = 1, 1.2, \sqrt{2}, 1.7, 2, \sqrt{6}, 3, 4$ can be obtained from Table 6. The power of the $R_\gamma$-test when $m' = n' = 3$ ($m=n=21$) at any level $0 < \alpha \leq .10$ and at alternatives $\Delta = 1, 1.2, \sqrt{2}, 1.7, 2, \sqrt{6}, 3, 4$ can be obtained from Table 7. These results when $\alpha = .05$ are plotted in Figure 5 along with the power function of other tests.


In this section we seek information on how sensitive the level of a particular rank test of dispersion is to centering the observations at an estimate of the location parameter; we also seek information on the power of such a test. The particular test we choose to investigate is Sukhatme's test of $(5,3)$. More specifically we consider the statistic

$$U_N^* = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X(|x_i - \bar{x}|, |y_j - \bar{y}|).$$

If $\psi(\cdot)$ is a normal distribution, then it is a consequence of Theorem 3 of the Appendix that if we let $\lambda = \lim_{N \rightarrow \infty} \frac{m}{N}$ then

(i) $\sqrt{N} (U_N^* - E[U_N^*])$ is asymptotically normal.

(ii) If further $\sigma = \tau$, then $\sqrt{N} (U_N^* - \frac{1}{2})$ is asymptotically $N(0, \frac{1}{12\lambda(1-\lambda^2)})$; and hence $U_N^*$ is asymptotically distribution free under the hypothesis that $\Delta^2 = 1$.

For the case $m = n$, Theorem 4 of the appendix gives an exact expression for $E[U_N^*]$ and gives an expression for $\text{Var}[\sqrt{N} (U_N^* - E[U_N^*])$ valid.
to order $O\left(\frac{1}{N^3}\right)$. These expressions are functions of the true value of $\Delta$.

Since $\psi(\cdot)$ is a normal c.d.f. the quantities $E[X]$ and $E[Y]$ exist; denote them by $\xi$ and $\eta$ respectively. (Note that $\xi = \mu$ and $\eta = \nu$.) If the parameters $\xi$ and $\eta$ were known, then

$$U_N = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} x(|X_i - \xi|, |Y_j - \eta|)$$

would be a more appropriate test statistic than $U_N^*$. Since $U_N$ is just a Mann-Whitney statistic it is asymptotically normal. For the case $m = n$, Theorem 5 of the Appendix gives exact expressions for $E[U_N]$ and $\text{Var}[\sqrt{N}(U_N - E[U_N])]$; these expressions are functions of the true value of $\Delta$.

Consider the hypothesis $H_0: \Delta^2 \leq 1$. By the level $\alpha$ $U_N^*$-test of $H_0$ we mean that test which rejects $H_0$ if $U_N^*$ exceeds the $1 - \alpha$ percent point of the distribution of $U_N^*$. The level $\alpha$ $U_N$-test of $H_0$ is similarly defined. Now the $U_N$-test could be applied exactly; even if we knew nothing of the form of $\psi(\cdot)$. However, in order to apply the $U_N^*$-test we must know the $100(1-\alpha)$ percent point of the distribution of $U_N^*$; and in order to know this we must know $\psi(\cdot)$. It is thus of interest to also consider the test of $H_0$ which rejects $H_0$ if $U_N^*$ exceeds the $100(1-\alpha)$ percent point of the distribution of $U_N$; we call this test the $\bar{U}_N^*$-test. The $\bar{U}_N^*$-test can be applied without knowledge of the form of $\psi(\cdot)$. However, its true level, as well as its power, depends on $\psi(\cdot)$. The results of the Appendix cited above allow us to examine the power of the $U_N^*$, $\bar{U}_N^*$ and $\tilde{U}_N^*$-tests for the case
that $\psi(\cdot)$ is a normal c.d.f. and $m = n$.

We can now mention several reasons why we choose to work with the particular statistic $U_N^*$. (1) Sukhatme's [19] theorem on when statistics based on data adjusted by an estimate of the unknown location parameter are asymptotically distribution free under the null hypothesis required that $\psi(\cdot)$ be a symmetric function. Thus his statistic (5.3) seems more appropriate than (5.2). (11) Also since $\psi(\cdot)$ is symmetric the mean and the median are equal. If the normal distribution is in some sense "central" to the type of distributions we expect to encounter in practice, then the sample mean is a more appropriate estimate of location than is the sample median. (iii) In the particular case now under consideration (i.e., when $\psi(\cdot)$ is a normal c.d.f.) we know the mean of $U_N^*$ and can give an expression for the variance of $\sqrt{N}(U_N^* - E[U_N^*])$ valid to order $O(\frac{1}{N^3})$. The author was unable to do this for centering at the sample median.

The reader is now referred to Table 8. For $m = n = 7, 10, 11, 12, 13, 14, 15, 17, 20, 25, 30, 35, 40, 50$ the table contains the following information. The entry in column 1 is $\Delta$ ($\Delta = 1, 1.2, 1.4, 1.7, 2.0$ or $2.4$). For $i = 2, 3, \ldots, 10$ we will denote the entry in the $i^{th}$ column of Table 8 corresponding to the value $\Delta$ by $C_i(\Delta)$. Then

\[
C_2(\Delta) = C(\Delta) = E[U_N^*] = E[U_N]
\]

\[
C_3(\Delta) = \text{Var}[\sqrt{2n}(U_N^* - C(\Delta))] \quad \text{accurate to order } O\left(\frac{1}{n(n-2)^2}\right)
\]

\[
C_4(\Delta) = 2[A(\Delta) + B(\frac{1}{\Delta}) - 2C^2(\Delta)] = \lim_{n \to \infty} \text{Var}[\sqrt{2n}(U_N^* - C(\Delta))]
\]

\[
= \lim_{n \to \infty} \text{Var}[\sqrt{2n}(U_N^* - C(\Delta))]
\]

42
\[ C_5(\Delta) = \frac{2}{n} (A(\Delta) + B(\frac{1}{\Delta}) - C^2(\Delta) - C(\Delta)) \]

\[ C_6(\Delta) = \frac{2}{n-2} [G(\frac{1}{\Delta}) H(\frac{1}{\Delta}) - E(\frac{1}{\Delta}) F(\frac{1}{\Delta})] \]

\[ C_7(\Delta) = \frac{2}{n^2} [n(\Delta) + n(\frac{1}{\Delta}) + 2F(\frac{1}{\Delta}) F(\frac{1}{\Delta}) - 2G(\frac{1}{\Delta}) H(\frac{1}{\Delta})] \]

\[ C_8(\Delta) = C_4(\Delta) - C_5(\Delta) + C_6(\Delta) + C_7(\Delta) \] (see Th. 4)

\[ C_8(\Delta) \] is characterized by the fact that if \( n \) is large enough so that the normal approximation to the distribution of \( U_N^* \) is accurate, then

\[ \beta_{U_N}(\Delta) = P(N(0,1) > C_8(\Delta)) \]

where \( \beta_{U_N}(\Delta) \) denotes the power of the level \( \alpha = .05 \) \( U_N^* \)-test against the alternative \( \Delta \). If \( n \) is large enough so that the normal approximation to the distribution of \( U_N \) is accurate, then

\[ \beta_{U_N}(\Delta) = P(N(0,1) > C_9(\Delta)) \]

where \( \beta_{U_N}(\Delta) \) denotes the power of the level \( \alpha = .05 \) \( U_N \)-test against the alternative \( \Delta \). The \( U_N \)-test is appropriate if \( \mu \) and \( \sigma \) are known. If \( n \) is large enough so that the normal approximation to the distribution of \( U_N^* \) is accurate, then

\[ \beta_{U_N^*}(\Delta) = P(N(0,1) > C_{10}(\Delta)) \]

where \( \beta_{U_N^*}(\Delta) \) denotes the power at the alternative \( \Delta \) of the \( U_N^* \)-test
which rejects if \( U_N^* \) exceeds the 95\% point of the null distribution of \( U_N^* \). \( \beta_{\sim}(1) \) is the true level of this test.

Examination of the error terms \( C_5(\Delta), C_6(\Delta) \) and \( C_7(\Delta) \) of Table 8 leads us to believe that for \( m = n > 15 \) approximating \( \text{Var}[\sqrt{2n} (U_N^* - C(\Delta))] \) by \( C_3(\Delta) \) should have no appreciable effect on power calculations. It is hoped that \( m = n = 20 \) is sufficiently large for the normal approximation to the distributions of \( U_N \) and \( U_N^* \) to be accurate. On this basis Table 9 presents the values of \( \beta_{\sim}(\Delta), \beta_{U_N}(\Delta) \) and \( \beta_{\sim}(\Delta) \) for the case \( m = n = 20 \). The results for \( \beta_{U_N}(\Delta) \) and \( \beta_{\sim}(\Delta) \) are graphed in Figure 4c.

<table>
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<th>( \Delta )</th>
<th>( \beta_{\sim}(\Delta) )</th>
<th>( \beta_{U_N}(\Delta) )</th>
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<td>1.7</td>
<td>.532</td>
<td>.543</td>
<td>.543</td>
</tr>
<tr>
<td>2.0</td>
<td>.727</td>
<td>.737</td>
<td>.738</td>
</tr>
<tr>
<td>2.4</td>
<td>.886</td>
<td>.891</td>
<td>.892</td>
</tr>
</tbody>
</table>

\[ m = n = 20 \]

Power of \( U_N^* \), \( U_N \), and \( \tilde{U}_N \)-tests

**TABLE 9**

44
Though of limited scope this result might provide some encouragement to one inclined to use rank tests applied to centered data. However, the power function of the $U^*_N$-test given in Figure 4c is considerably inferior to the power function of the Box-Anderson test. It might be hoped that using the test of Mood or the one of Klotz applied to centered data might improve the power enough to make it comparable to the power of the Box-Anderson test. We remark that the proof that Mood's test is asymptotically distribution free under $H: \Delta^2 = 1$ requires that $\psi'(\cdot)$ be symmetric. This condition should also be necessary for Klotz' test to be asymptotically distribution free under $H: \Delta^2 = 1$; though I don't know of the result being established even in this case. Thus before we are willing to use one of these two tests, we should be convinced of its robustness in case $\psi'(\cdot)$ is not symmetric (assuming $\psi'(\cdot)$ symmetric in most practical application is not very appealing). Thus this author prefers the Box-Anderson test to rank tests applied to centered data.
10. On the usefulness of the various procedures.

We first consider the hypothesis testing situation; we will compare each of the procedures (2), (3), (4), (5) and (6) of section 3 to the Box-Anderson procedure (1).

The Levene procedure (2) requires that \( m = n \); the Box-Anderson procedure suffers from no such restriction. Moreover, the Monte Carlo power calculations carried out in [14] and [13] show that even in the case \( m = n \) the Box-Anderson procedure has better power; the reader is referred to Figures 4b, 5 and 6.

Rank tests of dispersion (3) are too limited in applicability because of the necessity to assume that \( \mu = 1 \).

The small sample power calculations of sections 8a and 8b (see Figures 4a and 5) and many of the a.r.e. values tabled in Tables 1, 2 and 3 reflect how well the best rank-like test (4) can do in a particular situation. That is, we have assumed \( \Psi(\cdot) \) to be of a specified form and have then chosen the rank-like test that seems most suited to detecting differences in scale for that \( \Psi(\cdot) \). Clearly, in a practical situation the form of \( \Psi(\cdot) \) would be unknown to us; and hence we could not expect to do so well. Figures 4a and 5 demonstrate two cases in which the power of the Box-Anderson approximation to the permutation theory test substantially exceeds the power of the best rank-like test. Also the a.r.e.'s computed when \( \Psi(\cdot) \) is a double exponential c.d.f. and when \( \Psi(\cdot) \) is a contaminated normal c.d.f. with \( \beta_2 \leq 6.5 \) indicate a superiority of the Box-Anderson procedure. Thus the Box-Anderson
test is the more powerful test in most of the cases studied. These cases are probably most typical of practical applications. However, it is noted (in the cases considered) that the \( W_k \)-tests do better with respect to the Box-Anderson test when the distribution \( \psi(\cdot) \) has heavier tails. In fact, we saw in section 7d that if we make \( \beta_2 \) large enough we find that \( e_{W_k} > 1 \). This is to be expected since heavy tails in the distribution \( \psi(\cdot) \) not only cause large values of \( \beta_2 \) but also cause the density function \( f_{W_k}(\cdot) \) to have heavy tails; and hence cause \( \int_0^\infty t f_{W_k}^2(t) \) to be large. But the formula for \( e_{W_k} \) given in (7.1) is just a constant times the product of such terms; and hence heavy tails on the c.d.f. \( \psi(\cdot) \) cause \( e_{W_k} \) to increase. If \( \psi(\cdot) \) does not have fourth moments then \( e_{W_k} \) would be infinite. Thus when heavy tails are suspected it would be appropriate to use a rank-like test; such a test will also have exact level. As to which \( W_k \)-test is appropriate we make only a suggestion. The subgroup size \( k \) must be small enough so that the natural levels of the Mann-Whitney test are appropriate as levels of significance. (Thus \( m' = n' = 2 \) would not be appropriate; since in order to test at a level \( \alpha \) less than \( 1/6 \) we would have to use a randomized test.) Otherwise choose \( k \) as large as possible without exceeding 10. The \( C_k \)- and \( D_k \)-tests seem to be the best all around statistics to use; though of course, the data analysis would proceed extremely fast if the \( R_k \)-test were used.

For the case that \( \psi(\cdot) \) is a normal c.d.f., approximate power functions for the test (5) proposed by Box [3] are given by Olad, Branson and Odeh [15]. Figure 4b shows the best of these approximate
power functions for the case $m = n = 24$. The approximate level is 0.06. The Box-Anderson test with $m = n = 20$ and approximate level 0.04 has its power function graphed in Figure 4b also. Despite the disadvantages of smaller level and smaller sample sizes, the Box-Anderson test has much better power.

Work by Sukhatme [19] and Crouse [7] show respectively that the rank test of dispersion (5.3) and the test of Mood [11] applied to observations centered at sample medians (6) are asymptotically distribution free under the null hypotheses provided certain conditions (one of which is that $\psi'(\cdot)$ be symmetric) hold. Nothing is known of the robustness of these tests in the case that $\psi'(\cdot)$ is not symmetric. Moreover section 9 of this paper shows the superiority of the power of the Box-Anderson test to the power of the test (5.3) applied to data centered at sample means; the reader is referred to Figure 4c.

Comparing Figures 4a, 4b and 4c to Figure 4d which gives the power of the classical F-test in the cases $m = n = 13$ and $m = n = 11$ indicates small sample efficiencies for tests based on procedures (2), (3), (4) and (6) of about 55 to 65 percent. Procedure (5) does somewhat worse.

On the basis of these remarks we recommend the use of the Box-Anderson approximation to the permutation theory test in the hypothesis testing situation; though a $W_k$-test would be appropriate if it were suspected that $\psi(\cdot)$ had heavy tails. The only one of these procedures discussed in Scheffe's [17] standard textbook is that of Box [3]. However, we saw that the Box procedure was substantially inferior to the Box-Anderson procedure. Thus, this author feels that the Box-Anderson
procedure should become more widely adopted as the appropriate hypothesis testing procedure.

We now consider the problem of nonparametric estimation of \( \Delta^2 \). Of the procedures listed in section 3 only procedures (1), (4), (5) and (6) seem capable of yielding point and confidence interval estimates of \( \Delta^2 \). Because of the poor power of procedure (5) we dismiss it. We reject procedure (6) estimates because procedure (4) estimates are better. They are better for the following reasons. (i) Procedure (6) has only approximately correct level. (ii) The distribution theory of procedure (6) requires \( \psi'(\cdot) \) to be symmetric. (iii) Procedure (6) does not seem capable of easy inversion for confidence intervals; procedure (4) is easily inverted. (iv) In the important special case when \( \psi(\cdot) \) is a normal c.d.f. procedure (6) does not have substantially higher power than procedure (4).

Comparisons of interval estimates based on the Box-Anderson procedure (1) with those based on the rank-like procedure (4) parallel the comparison made in the hypothesis testing situation. However, the Box-Anderson estimates appear to be much more difficult to obtain; because of the dependence of \( d \) on \( \Delta \) in equation (5.1). The necessity to compute fourth moments is also time consuming.

When \( m \neq n \) point estimates based on the Box-Anderson procedure are time consuming to obtain; see equation (6.2). Also the Box-Anderson point estimate is likely to be quite biased in small samples. From Theorem 2 (iii) we see that the point estimates based on rank-like statistics enjoy small sample median unbiasedness if \( m = n \).
This author feels it appropriate to use rank-like procedures for confidence interval estimation unless one feels that $\psi(\cdot)$ does not have heavy tails and is willing to pay the price of heavy calculations; in which case the interval estimates (5.1) should be used.
V. APPENDIX


Theorem 3: Let $\psi(*)$ have a symmetric, bounded continuous derivative $\psi'(\cdot)$. Let $F_X(x) = \psi(\frac{x - \xi}{\sigma})$ and $F_Y(y) = \psi(\frac{y - \eta}{\tau})$ and let $E[X]$ exist (hence $\xi = E(X)$ and $\eta = E(Y)$). Let $\lambda = \lim_{N \to \infty} \frac{m}{N}$. Then

(i) $\sqrt{N}(U_N^* - E(U_N^*))$ is asymptotically normal.

(ii) If further $\sigma = \tau$, then $\sqrt{N}(U_N^* - \frac{1}{2})$ is asymptotically $N(0, \frac{1}{12\lambda(1-\lambda)})$; and hence $U_N^*$ is asymptotically distribution free under the hypothesis $H: \Delta^2 = 1$.

Proof: We define

$$U_N = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \chi(|X_i - \xi|, |Y_j - \eta|).$$

Now $U_N$ is a Mann-Whitney statistic; and hence is asymptotically normal. Furthermore, when $\sigma = \tau$, $U_N$ has the null Mann-Whitney distribution; and hence $\sqrt{N}(U_N - \frac{1}{2})$ is asymptotically $N(0, \frac{1}{12\lambda(1-\lambda)})$. Theorem 3 merely states that these conclusions remain valid if $\xi$ and $\eta$ are replaced by $\overline{X}$ and $\overline{Y}$ respectively. That this is indeed the case is a consequence of Theorem 4.2 of Sukhatme [19]. We will first state and then verify the hypothesis of Sukhatme's theorem. (Note that we may assume without loss of generality that $\sigma = 1$ and $\Delta = \tau \geq 1$.)

We define

$$A(s_1, s_2) = E(\chi(|X - \xi - s_1|, |Y - \eta - s_2|)).$$
and we define
\[ W(x,y;s_1,s_2) = X(|x-s_1|,|y-s_2|) - E(X(|X-s_1|,|Y-s_2|)) \]

The hypotheses we must verify are the following.

(a) There exist constants \( M_{11}, M_{21}, M_{22} \) such that
\[ |W(x,y;s_1,s_2)| \leq M_{11} \]
and \( E[|W(X,Y;s_1+h,s_2) - W(X,Y;s_1,s_2)|] \leq M_{21} h \)
and \( E[|W(X,Y;s_1,s_2+h) - W(X,Y;s_1,s_2)|] \leq M_{22} h \)

(b) There exist sequences \( \{ s_j \} \) and \( \{ l_j \} \) such that for every set of \( x \)'s and \( y \)'s
\[
\sup_{0 \leq s_j \leq h_1} |W(x,y;s_j,l_j) - W(x,y;0,0)| = \sup_{0 \leq s \leq h_1} |W(x,y;s,l) - W(x,y;0,0)|
\]
\[
\sup_{0 \leq l_j \leq h_2} |W(x,y;s_1,l_j) - W(x,y;0,0)| = \sup_{0 \leq l \leq h_2} |W(x,y;s_1,l) - W(x,y;0,0)|
\]

(c) \( \sqrt{N(X-\bar{X})} \) and \( \sqrt{N(Y-\bar{Y})} \) have limiting distributions.

(d) \( A(s_1,s_2) \) possesses first order partial derivatives continuous in a neighborhood of the origin and
\[
\frac{\partial}{\partial s_1} A(s_1,s_2) \bigg|_{s_1=s_2=0} = \frac{\partial}{\partial s_2} A(s_1,s_2) \bigg|_{s_1=s_2=0} = 0.
\]

We now verify (a).

Now \[ |W(x,y;s_1,s_2)| \leq 2 \]
Also \( E[|W(X,Y;s_1+h,s_2) - W(X,Y;s_1,s_2)|] \)

\[ \leq E[|X(|X-s_1-h|,|Y-s_2|) - X(|X-s_1|,|Y-s_2|)|] \\
+ |E[X(|X-s_1-h|,|Y-s_2|)] - E[X(|X-s_1|,|Y-s_2|)])| \]

Now \( |X(|X-s_1-h|,|Y-s_2|) - X(|X-s_1|,|Y-s_2|)| \) is always equal to either 0 or 1; and it equals 1 if and only if \(|Y-s_2|\) lies between \(|X-s_1-h|\) and \(|X-s_1|\). Thus

\[ E[|W(X,Y;s_1+h,s_2) - W(X,Y;s_1,s_2)|] \]

\[ \leq 2P(|Y-s_2| \text{ is between } |X-s_1-h| \text{ and } |X-s_1|) \]

\[ = 2 \ E_{|Y-s_2|} [P(z \text{ is between } |X-s_1-h| \text{ and } |X-s_1| \mid |Y-s_2|=z)] \]

\[ \leq 2 \ E_{|Y-s_2|} (2h \cdot \sup_t f_x(t)) \]

\[ = h \cdot 4 \sup_t f_x(t) \]

\[ = h \cdot 4 \sup_t \psi'(t) \]

Similarly \( E[|W(X,Y;s_1,s_2+2h) - W(X,Y;s_1,s_2)|] \)

\[ \leq 2P(|X-s_1| \text{ is between } |Y-s_2-h| \text{ and } |Y-s_2|) \]

\[ \leq h \cdot 4 \sup_t \psi'(t) . \]

Thus (a) holds with \( M_{11} = 2 \) and \( M_{21} = M_{22} = 4 \sup_t \psi'(t) \).
We now verify (b) and (c).

Condition (b) is clear; since we will show below that \( A(s_1, s_2) \) is a continuous function. Condition (c) is clear since \( \sqrt{N} (\bar{X} - \bar{x}) \) and \( \sqrt{N} (\bar{Y} - \eta) \) are asymptotically normal.

We now verify (d).

Now \( A(s_1, s_2) = E[|X - \bar{x} - s_1|, |Y - \eta - s_2|] \)

\[
= P(|X - \bar{x} - s_1| < |Y - \eta - s_2|)
\]

\[
= E_Y\{P(|X - \bar{x} - s_1| < |Y - \eta - s_2| \mid Y = y + \eta + s_2)\}
\]

\[
= \int_0^\infty P(|X - \bar{x} - s_1| < y) \ dF_Y(y + \eta + s_2)
\]

\[+ \int_{-\infty}^0 P(|X - \bar{x} - s_1| < -y) \ dF_Y(y + \eta + s_2)\]

\[
= \int_0^\infty [F_X(\bar{x} + s_1 + y) - F_X(\bar{x} + s_1 - y)] \ dF(y + \eta + s_2)
\]

\[+ \int_{-\infty}^0 [F_X(\bar{x} + s_1 - y) - F_X(\bar{x} + s_1 + y)] \ dF(y + \eta + s_2)\]

Thus \( \frac{\partial}{\partial s_1} A(s_1, s_2) = \int_0^\infty [f_X(\bar{x} + s_1 + y) - f_X(\bar{x} + s_1 - y)] \ dF(y + \eta + s_2) \)

\[+ \int_{-\infty}^0 [f_X(\bar{x} + s_1 - y) - f_X(\bar{x} + s_1 + y)] \ dF(y + \eta + s_2)\]

where differentiation inside the integral sign is permissible since \( f_X(\bar{x} + s_1 + y) - f_X(\bar{x} + s_1 - y) \) is dominated by the integrable function \( g(y) = 2 \sup \psi(t) \). Further, this partial derivative is continuous.
in a neighborhood of the origin since \( \psi(\cdot) \) is continuous. (See page 67 in Cramér [6].)

Integrating by parts we find that

\[
A(s_1, s_2) = 1 - \int_0^\infty F_Y(y+\eta+s_2) \, d[F_X(\xi+s_1+y) - F_X(\xi+s_1-y)] \\
- \int_{-\infty}^0 F_Y(y+\eta+s_2) \, d[F_X(\xi+s_1-y) - F_X(\xi+s_1+y)]
\]

Thus

\[
\frac{\partial}{\partial s_2} A(s_1, s_2) = \int_0^\infty [f_X(\xi+s_1-y) - f_X(\xi+s_1+y)] \, dF_Y(y+\eta+s_2) \\
+ \int_{-\infty}^0 [f_X(\xi+s_1+y) - f_X(\xi+s_1-y)] \, dF_Y(y+\eta+s_2);
\]

and this partial derivative is also continuous in a neighborhood of the origin.

It is obvious that if \( f_X(\cdot) \) is symmetric about \( \xi \) (i.e., if \( \psi(\cdot) \) is symmetric about the origin), then

\[
\left. \frac{\partial}{\partial s_1} A(s_1, s_2) \right|_{s_1 = s_2 = 0} = 0 = \left. \frac{\partial}{\partial s_2} A(s_1, s_2) \right|_{s_1 = s_2 = 0}.
\]

Thus condition (d) holds.

Having verified these conditions we can now claim the conclusion of Sukhatme's [19] Theorem 4.2; namely, that \( \sqrt{N} \left( U_N^* - E[U_N^*] \right) \) and \( \sqrt{N} \left( U_N - E[U_N] \right) \) have the same limiting distribution. q.e.d.
Theorem 4: Let $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ be random samples of size $n$ from $N(\mu,\sigma^2)$ and $N(\nu,\tau^2)$ distributions respectively. Let

$$U_N^* = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \chi(|X_i - \bar{X}|,|Y_j - \bar{Y}|) .$$

Then $\sqrt{2n} (U_N^* - E[U_N^*]) / \sqrt{\text{Var}[\sqrt{2n} (U_N^* - E[U_N^*])]}$ is asymptotically $N(0,1)$. Moreover

$$E[U_N^*] = C(\Delta)$$

and

$$\text{Var}[\sqrt{2n} (U_N^* - C(\Delta))] = 2[(A(\Delta) + B(\Delta)) - 2C^2(\Delta)] - \frac{1}{n} (A(\Delta) + B(\Delta) + C^2(\Delta) - C(\Delta))]$$

$$+ \frac{2}{n-2} \left[G\left(\frac{1}{\Delta}\right)H\left(\frac{1}{\Delta}\right) + F\left(\frac{1}{\Delta}\right) - 2G\left(\frac{1}{\Delta}\right)H\left(\frac{1}{\Delta}\right)\right]$$

$$+ o\left(\frac{1}{n(n-2)^2}\right)$$

where

$$A(\Delta) = \int_0^\infty (2\Phi(\omega) - 1)^2 \frac{2}{\Delta} \phi\left(\frac{\omega}{\Delta}\right) d\omega = 8Z(\Delta) - 8Y(\Delta) + 1 = \frac{2}{\pi} \sin^{-1}\left(\frac{\Delta^2}{1+\Delta^2}\right)$$

$$B(\frac{1}{\Delta}) = \int_0^\infty (1 - \Phi(\omega))^2 2\Delta \phi(\Delta\omega) d\omega = 4 - 16Y(\frac{1}{\Delta}) + 8Z(\frac{1}{\Delta})$$

$$C(\Delta) = \int_0^\infty (2\Phi(\omega) - 1) \frac{2}{\Delta} \phi\left(\frac{\omega}{\Delta}\right) d\omega = 4Y(\Delta) - 1$$

$$D(\Delta) = 4 \int_0^\infty \omega^2 \phi^2(\omega) \frac{1}{\Delta} \phi\left(\frac{\omega}{\Delta}\right) d\omega = \frac{\Delta^2}{\pi(2\Delta^2 + 1)^{3/2}}$$

$$E(\frac{1}{\Delta}) = 8 \int_0^\infty \Delta^2 \omega \phi(\Delta\omega) \phi(\omega) d\omega = \frac{4\Delta^2}{\pi(1+\Delta^2)}$$

$$F(\frac{1}{\Delta}) = \int_0^\infty (\Phi(\Delta\omega) - \Delta\omega \phi(\Delta\omega) - \frac{1}{2}) \phi(\omega) d\omega = Y(\Delta) - \frac{\Delta}{2\pi(1+\Delta^2)} - \frac{1}{4}$$
\[ G(\frac{1}{\Delta}) = \int_0^\infty 4(2 \Phi(\Delta \omega) - 1) \phi^{(2)}(\omega) d\omega = \frac{4\Delta}{\pi(1+\Delta^2)} \]

\[ H(\frac{1}{\Delta}) = \int_0^\infty (\Phi(\Delta \omega) - \frac{1}{2})\omega^2 \phi(\omega) d\omega = Y(\Delta) + \frac{\Delta}{2\pi(1+\Delta^2)} - \frac{1}{4} \]

and

\[ Z(\Delta) = \int_0^\infty \phi^2(\omega) \frac{1}{\Delta} \phi(\omega) d\omega = Y(\Delta) - \frac{1}{4} + \frac{1}{8} \left(1 + \frac{2}{\pi} \sin^{-1}\left(\frac{\Delta^2}{1+\Delta^2}\right)\right) \]

\[ Y(\Delta) = \int_0^\infty \Phi(\omega) \frac{1}{\Delta} \phi(\omega) d\omega = \frac{1}{4} \left(1 + \frac{2}{\pi} \sin^{-1}\left(\frac{\Delta}{\sqrt{1+\Delta^2}}\right)\right) \]

Proof: The asymptotic normality is an immediate consequence of Theorem 3. It remains only to verify the expression for the mean and variance.

Now \( \text{Var}[n^2 U_N] = \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^n \chi(|X_i - \overline{X}|, |Y_j - \overline{Y}|)\right]^2 - \mathbb{E}^2\left(\sum_{i=1}^n \sum_{j=1}^n \chi(|X_i - \overline{X}|, |Y_j - \overline{Y}|)\right) \)

\[ = n^2 \varepsilon(\Delta) + n^2(n-1)(\delta(\Delta) + \gamma(\Delta)) + n^2(n-1)^2 \theta(\Delta) - n^4 \varepsilon^2(\Delta) \]

where \( \varepsilon(\Delta) = \mathbb{P}(|X_1 - \overline{X}| < |Y_1 - \overline{Y}|) = \mathbb{P}(|W_1| < |Z_1|) \)

\( \delta(\Delta) = \mathbb{P}(|X_1 - \overline{X}| < |Y_1 - \overline{Y}| \text{ and } |X_2 - \overline{X}| < |Y_1 - \overline{Y}|) \)

\[ \equiv \mathbb{P}(|W_1| < |Z_1| \text{ and } |W_2| < |Z_1|) \]

\( \gamma(\Delta) = \mathbb{P}(|X_1 - \overline{X}| < |Y_1 - \overline{Y}| \text{ and } |X_1 - \overline{X}| < |Y_2 - \overline{Y}|) \)

\[ \equiv \mathbb{P}(|W_1| < |Z_1| \text{ and } |W_1| < |Z_2|) \]

\( \theta(\Delta) = \mathbb{P}(|X_1 - \overline{X}| < |Y_1 - \overline{Y}| \text{ and } |X_2 - \overline{X}| < |Y_2 - \overline{Y}|) \)

\[ \equiv \mathbb{P}(|W_1| < |Z_1| \text{ and } |W_2| < |Z_2|) \]

where we may assume without loss of generality that
\[
\begin{pmatrix}
  W_1 \\
  W_2
\end{pmatrix}
\text{ is distributed as } \mathcal{N}
\begin{pmatrix}
  \begin{pmatrix}
  0 \\
  0
\end{pmatrix},
  \begin{pmatrix}
  \frac{n-1}{n} & -\frac{1}{n} \\
  -\frac{1}{n} & \frac{n-1}{n}
\end{pmatrix}
\end{pmatrix}
\]

and
\[
\begin{pmatrix}
  Z_1 \\
  Z_2
\end{pmatrix}
\text{ is distributed as } \mathcal{N}
\begin{pmatrix}
  \begin{pmatrix}
  0 \\
  0
\end{pmatrix},
  \begin{pmatrix}
  \frac{\Delta^2}{n} & -\frac{\Delta^2}{n} \\
  -\frac{\Delta^2}{n} & \frac{\Delta^2}{n}
\end{pmatrix}
\end{pmatrix}
\]

and \( \begin{pmatrix}
  W_1 \\
  W_2
\end{pmatrix} \) is independent of \( \begin{pmatrix}
  Z_1 \\
  Z_2
\end{pmatrix} \).

We will now derive expressions for \( \varepsilon(\Delta) \), \( \delta(\Delta) \), \( \gamma(\Delta) \) and \( \theta(\Delta) \).

Note first that
\[
E_\Delta(U_N^*) = \varepsilon(\Delta) = P(|W_1| < |Z_1|) = \int_0^\infty (2 \Phi(\omega) - 1) \frac{\Delta^2}{\Delta} \phi(\omega) d\omega.
\]

Now
\[
\delta(\Delta) = P(|W_1| < |Z_1| \text{ and } |W_2| < |Z_2|)
\]
\[
= E|Z_1| \left[ P(|W_1| < |Z_1| \text{ and } |W_2| < |Z_2|) \right]
\]
\[
= E|Z_1| \left\{ \int_{-|Z_1|}^{Z_1} \int_{-|Z_1|}^{Z_1} \phi\left(\frac{\omega_1 + \omega_2}{n-1}\right) d\omega_1 \phi\left(\frac{\omega_2}{\sqrt{n-1}}\right) d\omega_2 \right\}
\]

since the conditional distribution of \( W_1 \) given \( W_2 \) is \( \mathcal{N}(\frac{W_2}{n-1}, \frac{n-2}{n-1}) \).

Thus
\[ \delta(\Delta) = E \left\{ \int_{-\left| Z_1 \right|}^{\left| Z_1 \right|} \left[ \Phi \left( \frac{\alpha_2}{\sqrt{n-1}} \right) - \Phi \left( \frac{\alpha_2}{\sqrt{n-1}} | Z_1 | \right) \right] \frac{1}{\sqrt{n-1}} \phi \left( \frac{\alpha_2}{\sqrt{n-1}} \right) d\alpha_2 \right\} 
\]

\[ = E \left\{ \int_{-\left| Z_1 \right|}^{\left| Z_1 \right|} \left[ \Phi \left( \frac{u}{\sqrt{n(n-2)}} + \sqrt{\frac{n-1}{n-2}} | Z_1 | \right) - \Phi \left( \frac{u}{\sqrt{n(n-2)}} - \sqrt{\frac{n-1}{n-2}} | Z_1 | \right) \right] \phi(u) du \right\} 
\]

\[ = \int_0^\infty \left\{ \int_{-\left| Z_1 \right|}^{\left| Z_1 \right|} \left[ \Phi \left( \frac{u}{\sqrt{n(n-2)}} + \sqrt{\frac{n-1}{n-2}} z \right) - \Phi \left( \frac{u}{\sqrt{n(n-2)}} - \sqrt{\frac{n-1}{n-2}} z \right) \right] \phi(u) du \right\} \frac{2}{\sqrt{n-1}} \phi \left( \frac{z}{\sqrt{n-1}} \right) dz 
\]

\[ = 2 \int_0^\infty \int_{-v}^v \left[ \Phi \left( \frac{u+(n-1)v}{\sqrt{n(n-2)}} \right) - \Phi \left( \frac{u-(n-1)v}{\sqrt{n(n-2)}} \right) \right] \phi(u) du \frac{1}{\Delta} \phi(\frac{v}{\Delta}) dv 
\]

\[ = 2 \int_0^\infty \int_{-v}^v \left[ \Phi \left( \frac{n-1}{\sqrt{n(n-2)}} v \right) + \frac{u}{\sqrt{n(n-2)}} \Phi \left( \frac{n-1}{\sqrt{n(n-2)}} v \right) + \frac{u^2}{2n(n-2)} \Phi \left( \frac{n-1}{\sqrt{n(n-2)}} v \right) 
\]

\[ + \frac{u^3}{6(n(n-2))^{3/2}} \Phi(2) \left( \frac{n-1}{\sqrt{n(n-2)}} v \right) + O \left( \frac{1}{n^2(n-2)^2} \right) \]

\[ - \Phi \left( \frac{n-1}{\sqrt{n(n-2)}} v \right) - \frac{u}{\sqrt{n(n-2)}} \Phi \left( \frac{n-1}{\sqrt{n(n-2)}} v \right) + \frac{u^2}{2n(n-2)} \Phi \left( \frac{n-1}{\sqrt{n(n-2)}} v \right) 
\]

\[ - \frac{u^3}{6(n(n-2))^{3/2}} \Phi(2) \left( \frac{n-1}{\sqrt{n(n-2)}} v \right) + O \left( \frac{1}{n^2(n-2)^2} \right) \right\} \phi(u) du \frac{1}{\Delta} \phi(\frac{v}{\Delta}) dv 
\]
The terms \( O\left( \frac{1}{n^2(n-2)^2} \right) \) are really of the form \( (u^4/24 \, n^2(n-2)^2) \phi(3)(\nu^*) \) for some point \( \nu^* \). Now \( \phi(3)(\cdot) \) is a bounded function. Also
\[
\int_0^\infty \int_{-V}^V u^4 \phi(u) \frac{1}{\Delta} \phi(\frac{V}{\Delta}) du dv
\]
is a finite number. Hence we can pull the term \( O\left( \frac{1}{n^2(n-2)^2} \right) \) out from under the integral sign. We do this repeatedly in what follows without repeating the justification. Thus

\[
8(\Delta) = 2 \int_0^\infty \int_{-V}^V \left[ 2\Phi\left( \sqrt{1+\frac{1}{n(n-2)}} \right) - 1 + \frac{u^2}{n(n-2)} \phi(\sqrt{1+\frac{1}{n(n-2)}} \nu) \right]
\]

\[
\phi(u) du \frac{1}{\Delta} \phi(\frac{V}{\Delta}) dv + O\left( \frac{1}{n^2(n-2)^2} \right)
\]

\[
= 2 \int_0^\infty \left[ \int_{-V}^V \phi(u) du \right] \left[ 2\Phi\left( \left( 1+ \frac{1}{2n(n-2)} + \frac{1}{n^2(n-2)^2} \right) \nu \right) - 1 \right] \frac{1}{\Delta} \phi(\frac{V}{\Delta}) dv
\]

\[
+ \frac{2}{n(n-2)} \int_0^\infty \left[ \int_{-V}^V u^2 \phi(u) du \right] \left[ \phi\left( \left( 1+ 0 \left( \frac{1}{n(n-2)} \right) \nu \right) \right) \frac{1}{\Delta} \phi(\frac{V}{\Delta}) dv + O\left( \frac{1}{n^2(n-2)^2} \right)
\]

since \( \sqrt{1+x} = 1 + \frac{x}{2} + O(x^2) \). Thus

\[
8(\Delta) = 2 \int_0^\infty \left[ 2\Phi(\nu) - 1 \right] \left[ 2\Phi(\nu) - 1 + \frac{\nu}{n(n-2)} \phi(\nu) + O\left( \frac{1}{n^2(n-2)^2} \right) \right] \frac{1}{\Delta} \phi(\frac{V}{\Delta}) dv
\]

\[- \frac{2}{n(n-2)} \int_0^\infty \left[ 2\nu \phi(\nu) - (2\Phi(\nu) - 1) \right] \phi\left( \left( 1+ 0 \left( \frac{1}{n(n-2)} \right) \nu \right) \right) \frac{1}{\Delta} \phi(\frac{V}{\Delta}) dv
\]

\[+ O\left( \frac{1}{n^2(n-2)^2} \right)\]

since \( \int_{-V}^V u^2 \phi(u) du = -u\phi(u) + \phi(u) \bigg|_{-V}^V = -2\nu \phi(\nu) + 2\Phi(\nu) - 1 \). Thus
\[
\delta(\Delta) = 2 \int_0^\infty [2\Phi(v)-1]^2 \frac{1}{\Delta} \phi\left(\frac{v}{\Delta}\right)dv \\
+ \frac{1}{n(n-2)} \int_0^\infty [2\Phi(v)-1] 2v\phi(v) \frac{1}{\Delta} \phi\left(\frac{v}{\Delta}\right)dv \\
- \frac{1}{n(n-2)} \int_0^\infty [2v\phi(v)-(2\Phi(v)-1)]2\phi'(v) \frac{1}{\Delta} \phi\left(\frac{v}{\Delta}\right)dv + o\left(\frac{1}{n^2(n-2)^2}\right) \\
= 2 \int_0^\infty (2\Phi(v)-1)^2 \frac{1}{\Delta} \phi\left(\frac{v}{\Delta}\right)dv + \frac{4}{n(n-2)} \int_0^\infty v^2 \phi^2(v) \frac{1}{\Delta} \phi\left(\frac{v}{\Delta}\right)dv + o\left(\frac{1}{n^2(n-2)^2}\right)
\]

Also

\[
\gamma(\Delta) = P(|W_1| < |Z_1| \text{ and } |W_1| < |Z_2|)
\]

\[
= 1 - P(|W_1| \geq |Z_1|) - P(|W_1| \geq |Z_2|) + P(|W_1| \geq |Z_1| \text{ and } |W_1| \geq |Z_2|)
\]

\[
= 1 - 2 \epsilon \left(\frac{1}{\Delta}\right) + 8\left(\frac{1}{\Delta}\right)
\]

\[
= 8\left(\frac{1}{\Delta}\right) - \int_0^\infty \Phi(\omega) \Delta \phi(\Delta \omega)d\omega + 3
\]

\[
= 2 \int_0^\infty 4(1-\Phi(\omega))^2 \Delta \phi(\Delta \omega)d\omega + \frac{4}{n(n-2)} \int_0^\infty v^2 \phi^2(v) \Delta \phi(\Delta v)dv + o\left(\frac{1}{n^2(n-2)^2}\right)
\]

Also

\[
\theta(\Delta) = P(|W_1| < |Z_1| \text{ and } |W_2| < |Z_2|)
\]

\[
= \int_0^\infty \int_0^{z_2} P\left|N\left(-\frac{\omega_2}{n-1}, \frac{n-2}{n-1}\right) < N\left(-\frac{z_2}{n-1}, \frac{n-2}{n-1}\Delta^2\right)\right| \frac{2}{\sqrt{n-1}}
\]

\[
\phi\left(\frac{\omega_2}{\sqrt{n-1}}\right) \frac{2}{\sqrt{n-1}} \phi\left(\frac{z_2}{\sqrt{n-1}}\Delta\right) d\omega_2 dz_2
\]

61
\[
\int_0^\infty \int_0^z \left\{ \int_0^\infty \left[ \int_{-z_1}^{z_1} \frac{1}{\sqrt{n-2}} \phi \left( \frac{\omega_1 + \frac{\omega_2}{n-1}}{\sqrt{n-2}} \right) \frac{1}{\sqrt{n-2}} \phi \left( \frac{z_1 + \frac{z_2}{n-1}}{\sqrt{n-2}} \right) dz_1 \right] \frac{2}{\sqrt{n-1}} \right\} \frac{2}{\sqrt{n-1}} \phi \left( \frac{\omega_2}{\sqrt{n-1}} \right) \frac{2}{\sqrt{n-1}} \phi \left( \frac{z_2}{\sqrt{n-1}} \right) \phi \left( \frac{\omega_2}{\sqrt{n-1}} \right) \frac{2}{\sqrt{n-1}} \phi \left( \frac{z_2}{\sqrt{n-1}} \right) d\omega_2 dz_2 \\
+ \int_0^\infty \int_0^z \left\{ \int_0^\infty \left[ \int_{-z_1}^{z_1} \frac{1}{\sqrt{n-2}} \phi \left( \frac{\omega_1 + \frac{\omega_2}{n-1}}{\sqrt{n-2}} \right) \frac{1}{\sqrt{n-2}} \phi \left( \frac{z_1 + \frac{z_2}{n-1}}{\sqrt{n-2}} \right) dz_1 \right] \frac{2}{\sqrt{n-1}} \right\} \frac{2}{\sqrt{n-1}} \phi \left( \frac{\omega_2}{\sqrt{n-1}} \right) \frac{2}{\sqrt{n-1}} \phi \left( \frac{z_2}{\sqrt{n-1}} \right) \phi \left( \frac{\omega_2}{\sqrt{n-1}} \right) \frac{2}{\sqrt{n-1}} \phi \left( \frac{z_2}{\sqrt{n-1}} \right) d\omega_2 dz_2 \\
+ \int_0^\infty \int_0^z \left\{ \int_0^\infty \left[ \int_{-z_1}^{z_1} \phi (u) du \right] \frac{1}{\sqrt{n-2}} \phi \left( \frac{z_1 + \frac{z_2}{n-1}}{\sqrt{n-2}} \right) \frac{2}{\sqrt{n-1}} \right\} \frac{2}{\sqrt{n-1}} \phi \left( \frac{\omega_2}{\sqrt{n-1}} \right) \frac{2}{\sqrt{n-1}} \phi \left( \frac{z_2}{\sqrt{n-1}} \right) d\omega_2 dz_2
\]
\[
+ \int_{0}^{\infty} \int_{0}^{\infty} \{ \int_{-\infty}^{0} \left[ \int_{-\infty}^{\infty} \phi(u) du \right] \frac{1}{\sqrt{n-2}} \frac{2}{\sqrt{n-1} \Delta} \frac{1}{\sqrt{n-2}} \frac{2}{\sqrt{n-1} \Delta} \frac{1}{\sqrt{n-1} \Delta} \phi \left( \frac{z_2}{\sqrt{n-1}} \Delta \right) \} \frac{2}{\sqrt{n-1} n} dz_1 dz_2
\]

\[
\phi \left( \frac{\omega_2}{\sqrt{n-1} n} \right) \frac{2}{\sqrt{n-1} \Delta} \phi \left( \frac{z_2}{\sqrt{n-1} n} \Delta \right) \frac{2}{\sqrt{n-1} \Delta} \phi \left( \frac{z_2}{\sqrt{n-1} n} \Delta \right) d\omega_2 \right] dz_2
\]

\[
= \int_{0}^{\infty} \int_{0}^{\infty} \{ \int_{0}^{\infty} \left[ \phi \left( \frac{\omega_2}{\sqrt{n-1} n} \Delta \right) - \phi \left( \frac{\omega_2}{\sqrt{n-1} n} \Delta \right) \right] \frac{1}{\sqrt{n-2}} \frac{2}{\sqrt{n-1} \Delta} \frac{1}{\sqrt{n-2}} \frac{2}{\sqrt{n-1} \Delta} \frac{1}{\sqrt{n-1} \Delta} \phi \left( \frac{z_2}{\sqrt{n-1} n} \Delta \right) \} \frac{2}{\sqrt{n-1} n} dz_1 dz_2
\]

\[
\phi \left( \frac{\omega_2}{\sqrt{n-1} n} \right) \frac{2}{\sqrt{n-1} \Delta} \phi \left( \frac{z_2}{\sqrt{n-1} n} \Delta \right) \frac{z_2}{\sqrt{n-1} \Delta} \phi \left( \frac{z_2}{\sqrt{n-1} n} \Delta \right) d\omega_2 \right] dz_2
\]

\[
\phi \left( \frac{\omega_2}{\sqrt{n-1} n} \right) \frac{2}{\sqrt{n-1} \Delta} \phi \left( \frac{z_2}{\sqrt{n-1} n} \Delta \right) \frac{z_2}{\sqrt{n-1} \Delta} \phi \left( \frac{z_2}{\sqrt{n-1} n} \Delta \right) d\omega_2 \right] dz_2
\]

\[
= \int_{0}^{\infty} \int_{0}^{\infty} \{ \int_{0}^{\infty} \left[ \phi \left( \frac{\omega_2 - z_2}{\Delta \sqrt{(n-1)(n-2)}} \right) + \phi \left( \frac{\omega_2 + z_2}{\Delta \sqrt{(n-1)(n-2)}} \right) \right] \frac{2}{\sqrt{n-1} n} \frac{2}{\sqrt{n-1} n} \frac{2}{\sqrt{n-1} n} \phi \left( \frac{z_2}{\sqrt{n-1} n} \Delta \right) \} d\omega_2 dz_2
\]

\[
\phi(v) dv \frac{2}{\sqrt{n-1} n} \phi \left( \frac{\omega_2}{\sqrt{n-1} n} \Delta \right) \phi \left( \frac{z_2}{\sqrt{n-1} n} \Delta \right) d\omega_2 dz_2
\]
\[ + \int_{0}^{\infty} \int_{0}^{\infty} \frac{z_2}{\Delta \sqrt{(n-1)(n-2)}} \left[ \Phi\left( \frac{\omega_2 + z_2}{\sqrt{(n-1)(n-2)}} - \Delta \nu \right) - \Phi\left( \frac{\omega_2 - z_2}{\sqrt{(n-1)(n-2)}} + \Delta \nu \right) \right] \varphi(\nu) d\nu \left\{ \int_{-\infty}^{\infty} \varphi\left( \frac{\omega_2}{\sqrt{(n-1)(n-2)}} \right) d\omega_2 d\nu \right\} \]

\[ = \frac{2}{\sqrt{n-1}} \int_{0}^{\infty} \Phi\left( \frac{\omega_2}{\sqrt{n-1}} \right) \varphi\left( \frac{2 \omega_2}{\sqrt{n-1}} \right) d\omega_2 \]

\[ = 4 \int_{0}^{\infty} \int_{0}^{\infty} \left\{ \int_{-\infty}^{\infty} \Phi\left( \frac{\omega + \Delta \nu}{\sqrt{n(n-2)}} \right) - \Phi\left( \frac{\omega - \Delta \nu}{\sqrt{n(n-2)}} \right) \nu \varphi(\nu) d\nu \right\} \varphi(s) ds dr ds \]

\[ + 4 \int_{0}^{\infty} \int_{0}^{\infty} \left\{ \int_{-\infty}^{\infty} \Phi\left( \frac{\omega + \Delta \nu}{\sqrt{n(n-2)}} \right) - \Phi\left( \frac{\omega - \Delta \nu}{\sqrt{n(n-2)}} \right) \nu \varphi(\nu) d\nu \right\} \varphi(s) ds dr ds \]

\[ = 4 \int_{0}^{\infty} \int_{0}^{\infty} \left\{ \int_{-\infty}^{\infty} \Phi\left( \frac{\omega + \Delta \nu}{\sqrt{n(n-2)}} \right) - \Phi\left( \frac{\omega - \Delta \nu}{\sqrt{n(n-2)}} \right) \nu \varphi(\nu) d\nu \right\} \varphi(s) ds dr ds \]

\[ = \left\{ \int_{0}^{\infty} \Phi\left( \frac{\omega + \Delta \nu}{\sqrt{n(n-2)}} \right) - \Phi\left( \frac{\omega - \Delta \nu}{\sqrt{n(n-2)}} \right) \nu \varphi(\nu) d\nu \right\} \varphi(s) ds dr ds \]
\[
\begin{align*}
= 4 \int_0^\infty \int_0^{\Delta s} \left\{ \int_0^\infty \left[ 2(\Phi(\Delta \omega) - 1) + \frac{r^2}{n(n-2)} \Phi'(\Delta \omega) + O \left( \frac{1}{n^2(n-2)^2} \right) \right] \right. \\
\left. \left[ 2\Phi(\omega) + \frac{s^2}{n(n-2)} \Phi(2)(\omega) + O \left( \frac{1}{n^2(n-2)^2} \right) \right] \Phi(r) \Phi(s) dr ds \right\} \\
= 4 \int_0^\infty \int_0^{\Delta s} \left\{ \int_0^\infty \left( 2(\Phi(\Delta \omega) - 1) \right) \Phi(\omega) d\omega + \frac{r^2}{n(n-2)} \int_0^\infty \Phi'(\Delta \omega) \Phi(\omega) d\omega \\
\right. \\
\left. + \frac{s^2}{n(n-2)} \int_0^\infty (2(\Phi(\Delta \omega) - 1) \Phi(2)(\omega) d\omega \right\} \\
\left. \Phi(r) \Phi(s) dr ds + O \left( \frac{1}{n^2(n-2)^2} \right) \right\} \\
= \int_0^\infty (2\Phi(\Delta \omega) - 1) \Phi(\omega) d\omega \cdot 4 \int_0^\infty \int_0^{\Delta s} \Phi(r) \Phi(s) dr ds \\
\left. \right. \\
\left. + 2 \int_0^\infty \Phi'(\Delta \omega) \Phi(\omega) d\omega \cdot \frac{4}{n(n-2)} \int_0^\infty \left[ \left[ \int_0^{\Delta s} r^2 \Phi(r) dr \right] \Phi(s) ds \right. \\
\left. \left. + \int_0^\infty (2\Phi(\Delta \omega) - 1) \Phi(2)(\omega) d\omega \cdot \frac{4}{n(n-2)} \int_0^\infty \left[ \left[ \int_0^{\Delta s} \Phi(r) dr \right] s^2 \Phi(s) ds \right. \\
\left. \left. + 0 \left( \frac{1}{n^2(n-2)^2} \right) \right\] \right\} \\
= \left[ \int_0^\infty 2(2\Phi(\Delta \omega) - 1) \Phi(\omega) d\omega \right]^2
\end{align*}
\]
\[ + 2 \int_0^\infty \dot{\phi}^2(\Delta \omega) \phi(\omega) d\omega \cdot \frac{4}{n(n-2)} \int_0^\infty \left[ \phi(\Delta s) - \Delta s \phi(\Delta s) - \frac{1}{2} \right] \phi(s) ds \]

\[ + \int_0^\infty (2\phi(\Delta \omega) - 1) \phi^{(2)}(\omega) d\omega \cdot \frac{4}{n(n-2)} \int_0^\infty \left( \phi(\Delta s) - \frac{1}{2} \right) s^2 \phi(s) ds \]

\[ + o \left( \frac{1}{n^2(n-2)^2} \right) \]

Thus \( \text{Var}[n^2 U_N] = n^2 \epsilon(\Delta) + n^2(n-1)(\delta(\Delta) + \gamma(\Delta)) + n^2(n-1) \phi(\Delta) - n^4 \epsilon^2(\Delta) \]

\[ = n^2 \int_0^\infty (2\phi(\omega) - 1) \frac{2}{\Delta} \phi(\frac{\omega}{\Delta}) d\omega \]

\[ + (n^3 - n^2) \left\{ \int_0^\infty \left[ 2\phi(\omega) - 1 \right]^2 \frac{2}{\Delta} \phi(\frac{\omega}{\Delta}) d\omega \right\} \]

\[ + \int_0^\infty 4 \left[ 1 - \phi(\omega) \right]^2 2\Delta \phi(\Delta \omega) d\omega \]

\[ + \frac{4}{n(n-2)} \int_0^\infty v^2 \phi^2(v) \frac{1}{\Delta} \phi(\frac{v}{\Delta}) dv \]

\[ + \frac{4}{n(n-2)} \int_0^\infty v^2 \phi^2(v) \Delta \phi(\Delta v) dv \]

\[ + o \left( \frac{1}{n^2(n-2)^2} \right) \]

\[ + (n^4 - 2n^3 + n^2) \left\{ \left[ \int_0^\infty 2(2\phi(\Delta \omega) - 1) \phi(\omega) d\omega \right]^2 \right\} \]

\[ - \frac{4}{n(n-2)} \int_0^\infty 2\Delta^2 \omega \phi(\omega) \phi(\Delta \omega) d\omega \cdot \int_0^\infty \left[ \phi(\Delta \omega) - \Delta \omega \phi(\Delta \omega) - \frac{1}{2} \right] \phi(\omega) d\omega \]

66
\[ + \frac{4}{n(n-2)} \int_0^\infty (2\Phi(\omega)-1)\phi(2\omega)\omega d\omega \cdot \int_0^\infty (\Phi(\omega)-1)\frac{\omega^2}{2} \phi(\omega) d\omega + o\left(\frac{1}{n^2(n-2)^2}\right) \]

\[ -n^3 \left[ A(\Delta) + B\left(\frac{1}{\Delta}\right) - 2c^2(\Delta) \right] = n^3 \left[ A(\Delta) + B\left(\frac{1}{\Delta}\right) - c^2(\Delta) - C(\Delta) \right] \]

\[ + \frac{n^3}{n-2} \left[ C\left(\frac{1}{\Delta}\right)H\left(\frac{1}{\Delta}\right) - E\left(\frac{1}{\Delta}\right)F\left(\frac{1}{\Delta}\right) \right] \]

\[ + 3 \left( \frac{n}{n-2} \right) + o\left(\frac{1}{n(n-2)^2}\right) \]

Finally, \( \text{Var} \left[ \sqrt{2n} \left( U_N^* - E[U_N^*] \right) \right] = \frac{2}{n^3} \text{Var}[n^2 U_n^*] \); and this yields the expression in the statement of the theorem.

Verifying that \( A(\Delta), B\left(\frac{1}{\Delta}\right), \ldots \) are equal to the stated quantities is straightforward. Only \( Y(\Delta) \) presents any problem. Its derivation is given in Lemma 2. q.e.d.

**Lemma 2:**

\[ Y(\Delta) = \int_0^\infty \Phi(\omega) \frac{1}{\Delta} \phi(\omega) \omega d\omega = \frac{1}{4} \left(1 + \frac{2}{n} \sin^{-1}\left(\frac{\Delta}{\sqrt{1+\Delta^2}}\right)\right). \]

**Proof:**

\[ Y(\Delta) = \int_0^\infty \Phi(\omega) \frac{1}{\Delta} \phi(\omega) \omega d\omega \]

\[ = \int_0^\infty \left[ \int_0^\omega \phi(t) dt \right] \frac{1}{\Delta} \phi(\omega) d\omega \]
\[
\begin{align*}
&= \frac{1}{2\pi\Delta} \int_0^\infty \left[ \int_{-\infty}^\infty e^{-\frac{1}{2}t^2} dt \right] e^{-\frac{1}{2}(\Delta)^2} \, d\omega \\
&= \frac{1}{2\pi\Delta} \int_0^\infty \int_{-\infty}^0 e^{-\frac{1}{2}[t^2 + 2t\omega + (1 + \frac{1}{\Delta^2})\omega^2]} \, dt \, d\omega \\
&= \frac{1}{\sqrt{1 + \frac{1}{\Delta^2}}} \int_0^\infty \int_{-\infty}^0 e^{-\frac{1}{2}[t^2 - 2t\Delta/\sqrt{1 + \Delta^2} + \Delta^2]} \, dt \, d\omega \\
&= \int_{-\infty}^0 \int_{-\infty}^0 dN \left( \left( \frac{0}{0} \right), \left( \frac{1}{\rho} \frac{\rho}{1} \right) \right)
\end{align*}
\]

where \( \rho = \Delta/\sqrt{1 + \Delta^2} \). Thus

\[
X(\Delta) = \frac{1}{4} \left[ 1 + \frac{2}{\pi} \sin^{-1} \left( \frac{\Delta}{\sqrt{1 + \Delta^2}} \right) \right].
\]
q.e.d.

Theorem 5: Let \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \) be random samples from \( N(\mu, \sigma^2) \) and \( N(\nu, \tau^2) \) distributions respectively. Let

\[
U_n = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n X(|X_i - \mu|, |Y_j - \nu|)
\]

Then \( \sqrt{2n \left( U_n - E[U_n] \right)} / \sqrt{\text{Var} \left( \sqrt{2n \left( U_n - E[U_n] \right)} \right)} \) is asymptotically \( N(0,1) \).
Moreover

\[ E(U_N) = C(\Delta) \]

and

\[ \text{Var} \left[ \sqrt{2n} (U_N - C(\Delta)) \right] = 2[(A(\Delta) + B(C(\Delta))) - \frac{1}{n}(A(\Delta) + B(C(\Delta))) - C(\Delta)] \]

where \( A(\Delta) \), \( B(\Delta) \) and \( C(\Delta) \) are given in the statement of Theorem 4.

Proof: Now \( \text{Var}[n^2 U_N] = n^2 \epsilon(\Delta) + n^2 (n-1) \delta(\Delta) + \gamma(\Delta) + \sum_{i=1}^{n} \epsilon^2(\Delta) - n \epsilon^2(\Delta) \)

where \( \epsilon(\Delta) = P(|X_1 - \mu| < |Y_1 - \nu|) \)

\[ \delta(\Delta) = P(|X_1 - \mu| < |Y_1 - \nu| \text{ and } |X_2 - \mu| < |Y_1 - \nu|) \]

\[ \gamma(\Delta) = P(|X_1 - \mu| < |Y_1 - \nu| \text{ and } |X_1 - \mu| < |Y_2 - \nu|) \]

Also \( E(U_N) = P(|X_1 - \mu| < |Y_1 - \nu|) = C(\Delta) \).

Now \( U_N \) is asymptotically normal since it is in fact just the ordinary Mann-Whitney statistic. We now verify the expression for \( \text{Var} \left[ \sqrt{2n} (U_N - C(\Delta)) \right] \).

From the above (or from Ex. 21, page 253 in Lehmann [12]) we deduce that

\[ \text{Var}[n^2 U_N] = n^2 \int_0^{\infty} (2\Phi(\omega)-1) \frac{2}{\Delta} \phi(\omega/\Delta) d\omega \]

\[ + (n^3 - n) \left\{ \int_0^{\infty} [2\Phi(\omega)-1]^2 \frac{2}{\Delta} \phi(\omega/\Delta) d\omega \right\} \]

\[ + \int_0^{\infty} 4(1-\Phi(\omega))^2 2\Phi(\omega) d\omega \]

\[ + (n^4 - 2n^3 + n^2) \left[ \int_0^{\infty} (2\Phi(\omega)-1) \frac{2}{\Delta} \phi(\omega/\Delta) d\omega \right]^2 \]

69
\[ - n \int_0^\infty (2\Phi(\omega)-1) \frac{\omega}{\Delta} \varphi(\frac{\omega}{\Delta}) d\omega \]

Thus

\[ \text{Var}\left( \sqrt{2n} (U_N - C(\Delta)) \right) = \frac{2}{n^3} \text{Var} \left[ n^2 U_N \right] \]

\[ = 2[(A(\Delta)+B(\frac{1}{\Delta})-C^2(\Delta)) - \frac{1}{n} (A(\Delta)+B(\frac{1}{\Delta})-C^2(\Delta)-C(\Delta))] \]

q.e.d.
VI. REFERENCES


Key to Tables 4, 5, 6 and 7

In Tables 4-7 the single entries ending with a series of 9's describe a particular rank ordering. Thus 00011011 9999999999999 refers to the rank ordering WWWWWWWW. Directly below the rank ordering description appear two rows of entries. The first row gives the probability of the particular rank ordering for the value of $\Delta$ associated with that column; the second row gives the sum of the probabilities of the rank orderings enumerated above for the values of $\Delta$ associated with that column. Thus the circled entries in Table 4 tell us that when $\Delta = \sqrt{2}$ the rank ordering WWWWWWWW has probability .055081 and the union of the rank orderings WWWWWWWW and WWWWWWWW and WWWWWWWW has probability .262079. In Table 4 $m = n = 20$, $\Psi(\cdot)$ is a normal c.d.f., and $W_k$ is in fact $V_5$. In Table 5 $m = n = 21$, $\Psi(\cdot)$ is a normal c.d.f., and $W_k$ is in fact $V_7$. In Table 6 $m = n = 20$, $\Psi(\cdot)$ is a uniform c.d.f., and $W_k$ is in fact $R_5$. In Table 7 $m = n = 21$, $\Psi(\cdot)$ is a uniform c.d.f., and $W_k$ is in fact $R_7$. 

74
TABLE 4

Exact Probabilities of Rank Sets Comprising the 10% Critical Region of the

$Y_5$-test when $m = n = 20$ and $\Psi(\cdot)$ is a Normal c.d.f.

| SUBGROUPS OF X'S AND 4 SUBGROUPS OF Y'S WHEN SAMPLING FROM THE NORMAL DISTRIBUTION |
|---------------------------------|---------------------------------|---------------------------------|
| SIZE OF SUBGROUPS IS 5         | 4                               | 5                               |
| 0001111999999999999999         | 0.001485143                      | 0.001485143                     |
| 0.001485143 0.001485143         | 0.001485143                      | 0.001485143                     |
| 0.001485143 0.001485143         | 0.001485143                      | 0.001485143                     |
| 0.001485143 0.001485143         | 0.001485143                      | 0.001485143                     |
| 0.001485143 0.001485143         | 0.001485143                      | 0.001485143                     |

$\Delta = 1 \quad \Delta = 1.2 \quad \Delta = \sqrt{2} \quad \Delta = 1.7 \quad \Delta = 2 \quad \Delta = \sqrt{6} \quad \Delta = 3 \quad \Delta = 4$

TABLE 5

Exact Probabilities of rank sets comprising the 10% Critical Region of the

$Y_5$-test when $m = n = 21$ and $\Psi(\cdot)$ is a Normal c.d.f.

| SUBGROUPS OF X'S AND 3 SUBGROUPS OF Y'S WHEN SAMPLING FROM THE NORMAL DISTRIBUTION |
|---------------------------------|---------------------------------|---------------------------------|
| SIZE OF SUBGROUPS IS 6         | 3                               | 5                               |
| 0011111999999999999999         | 0.001485143                      | 0.001485143                     |
| 0.001485143 0.001485143         | 0.001485143                      | 0.001485143                     |
| 0.001485143 0.001485143         | 0.001485143                      | 0.001485143                     |
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| 0.001485143 0.001485143         | 0.001485143                      | 0.001485143                     |

$\Delta = 1 \quad \Delta = 1.2 \quad \Delta = \sqrt{2} \quad \Delta = 1.7 \quad \Delta = 2 \quad \Delta = \sqrt{6} \quad \Delta = 3 \quad \Delta = 4$
### Table 6

Exact Probabilities of Rank Sets Comprising the 10% Critical Region of the \( R_5 \)-test when \( m = n = 20 \) and \( \psi(\cdot) \) is a Uniform c.d.f.

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<th>4 Subgroups of ( X ), ( 5 ) and 4 Subgroups of ( Y ), ( 5 ) Giving Ranges of Size 5 from the Uniform Distribution</th>
<th>DEL = 1.0</th>
<th>DEL = 1.2</th>
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### Table 7

Exact Probabilities of Rank Sets Comprising the 10% Critical Region of the \( R_7 \)-test when \( m = n = 21 \) and \( \psi(\cdot) \) is a Uniform c.d.f.

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# Table 8

Mean, Variance, Error Terms and Power Associated with a Test Statistic of Sukhatme

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</table>
\[ \Psi(\cdot) \text{ is a normal c.d.f.} \]

- Classical F-test \( m=n=20 \)
- Box-Anderson \( m=n=20 \)
- Moses \( V_7 \)-test \( k=7, m=n=21 \)
- Moses \( V_6 \)-test \( k=5, m=n=20 \)

Power Functions of Various Tests when \( \Psi(\cdot) \)

is a Normal c.d.f.

Figure 4a
\[ \Psi(\cdot) \text{ IS A NORMAL c.d.f.} \]

CLASSICAL F-TEST \hspace{1cm} m = n = 20

BOX-ANDERSON \hspace{1cm} m = n = 20

LEVENE Z-TEST \hspace{1cm} m = n = 20

BOX (OLDS-BRANSON-ODEH)(k=6) \hspace{1cm} m = n = 24

\[ \Delta^2 \]

Power Functions of Various Tests when \( \psi(\cdot) \) is a Normal c.d.f.

\text{Figure 4b}
\[ \Psi(\cdot) \text{ is a NORMAL c.d.f.} \]

- CLASSICAL F-TEST \( m=n=20 \)
- BOX-ANDERSON \( m=n=20 \)
- SUKHATME (KNOWN MEDIAN) \( m=n=20 \)
- SUKHATME (UNKNOWN MEDIAN) \( m=n=20 \)

Power Functions of Various Tests when \( \Psi(\cdot) \) is a Normal c.d.f.

Figure 4c
\( \Psi(\cdot) \) IS A NORMAL c.d.f.

Power Functions of Various Tests when \( \Psi(\cdot) \)
is a Normal c.d.f.

Figure 4d
\[ \psi(\cdot) \text{ is a uniform c.d.f.} \]

- Levene Z-Test \( m = n = 20 \)
- Box-Anderson \( m = n = 20 \)
- Moses \( V_7 \)-Test \( k = 7 \) \( m = n = 21 \)
- Moses \( V_5 \)-Test \( k = 5 \) \( m = n = 20 \)

Power Functions of Various Tests when \( \psi(\cdot) \) is a Uniform c.d.f.

Figure 5
Power Functions of Various Tests when $\Psi(\cdot)$

is a Double Exponential c.d.f.

Figure 6