STATISTICAL METHODS FOR ESTABLISHING EQUIVALENT SCORES
IN SUCCESSIVE GENERATIONS OF A TESTING PROGRAM

BY

MARJORIE HARUE FUJI PETERSON

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
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CHAPTER 1: INTRODUCTION

1.1 Introduction.

The problem treated in this paper is that of estimating the correspondence between scores on two different forms of a test. The observations available are the scores obtained on one form by one group of examinees, and the scores obtained on the other form by another group of examinees. A common set of items, called an anchor test, is also administered to each group.

This problem arises in many large testing programs where it is desirable, for reasons of insuring that examinees do not have advanced knowledge of the test items, to give different test forms at different administration times. It is desired that different forms of a test measure the same level and type of ability (so that the scores of examinees who took different forms can be used to compare their abilities). Thus, different forms are usually constructed to be as similar as possible in the types and difficulties of the questions. However, some differences between the tests may remain, with the result that a numerical score on one test form will not indicate the same level of ability as the identical numerical score on the other test form. The general problem considered in this paper is that of identifying the scores on the two test forms equivalent to the same level of ability. We shall refer to identifying (estimating) the equivalence between the scores of the two forms as "equating test scores" (although actually it is the levels of ability corresponding to the test scores that are equal, not the scores).

Unfortunately, in the general problem, the group of examinees taking one form of the test does not necessarily have the same abilities as the
group taking the other form. To estimate the correspondence between the scores on the two tests, it is necessary to estimate the difference in the average abilities between the two groups. The anchor test is used for this purpose.

1.2 Statement of the Problem.

1. Abilities of the Two Groups.

Let \( Z_\alpha, \alpha = 1,2,\ldots,2n \), be the (unobserved) level of ability of the \( \alpha \)th examinee.

Let \( Z_\alpha, \alpha = 1,2,\ldots,n \), be independently and identically distributed (i.i.d.) as \( F_1(\cdot) \).

Let \( Z_\alpha, \alpha = n+1,\ldots,2n \) be i.i.d. as \( F_2(\cdot) \).

We shall refer to the set \( \{1,2,\ldots,n\} \) as Group 1 of examinees, \( \{n+1,\ldots,2n\} \) as Group 2 of examinees.

If no assumption is made about whether or not the distributions \( F_1(\cdot) \) and \( F_2(\cdot) \) are equal, then this is sometimes called the "nonrandom groups" (NRG) model. The special case where it is assumed that \( F_1(\cdot) = F_2(\cdot) \) will be called the "random groups" (RG) model.

2. True Scores.

Let the random variables

\[
V^*_\alpha = a + bZ_\alpha
\]

(1.1)

\[
V^*_\alpha = A + BZ_\alpha
\]

\[
V^*_\alpha = \alpha + BZ_\alpha
\]
be the true (i.e. expected) scores (unobserved) on tests $U, V, W$ respectively, for an examinee with level of ability $\beta_\alpha$. For a given test, say $U$, and a given examinee $\alpha$, the true score $U^*_\alpha$ is defined to be the average of the scores that would be observed if examinee $\alpha$ took test $U$ an infinite number of times (independently), and if there is no practice or fatigue effect. The true scores differ from the scores that would be observed because of errors of measurement.

$a, b, A, B, \alpha, \beta$ are unknown constants.

Two tests are said to be congeneric if the correlation coefficient between their true scores is one (that is, there exists a linear relationship between the true scores). Tests which have true scores (1.1) are clearly congeneric. In this paper, consideration will be limited to congeneric tests.

3. **Observed Scores.**

Let the random variables

\[
U_\alpha = U^*_\alpha + e_\alpha
\]

\[
= a + b\beta_\alpha + e_\alpha
\]

\[
V_\alpha = V^*_\alpha + e_\alpha
\]

\[
= A + B\beta_\alpha + e_\alpha
\]

\[
W_\alpha = W^*_\alpha + e_\alpha
\]

\[
= \alpha + \beta\beta_\alpha + e_\alpha
\]

be the scores on tests $U, V, W$ respectively that would be observed if examinee $\alpha$ took tests $U, V, W$. 

3
\( e_\alpha, \bar{e}_\alpha, \varepsilon_\alpha \) are random measurement errors, where

\[
\begin{align*}
e_\alpha &\sim \text{iid } (0, \sigma_{e_1}^2), \ \alpha = 1, \ldots, n \\
e_\alpha &\sim \text{iid } (0, \sigma_{e_2}^2), \ \alpha = n+1, \ldots, 2n \\
\bar{e}_\alpha &\sim \text{iid } (0, \sigma_{\bar{e}_1}^2), \ \alpha = 1, \ldots, n \\
\bar{e}_\alpha &\sim \text{iid } (0, \sigma_{\bar{e}_2}^2), \ \alpha = n+1, \ldots, 2n \\
\varepsilon_\alpha &\sim \text{iid } (0, \sigma_{\varepsilon_1}^2), \ \alpha = 1, \ldots, n \\
\varepsilon_\alpha &\sim \text{iid } (0, \sigma_{\varepsilon_2}^2), \ \alpha = n+1, \ldots, 2n
\end{align*}
\]

\[
\text{Cov}(Z,e) = \text{Cov}(Z,\bar{e}) = \text{Cov}(Z,\varepsilon) = 0
\]

\[
\text{Cov}(e,\bar{e}) = \text{Cov}(e,\varepsilon) = \text{Cov}(\bar{e},\varepsilon) = 0 .
\]

We shall now define the concept of reliability of a test. The reliability of a test is the correlation between the observed scores on two administrations of the same test (on the same group of examinees), where the errors of measurement are assumed to be independent, the variances of the errors of measurement are assumed equal, and it is assumed that there is no practice effect. For example, let \( R_U \) denote the reliability of test \( U \). Then

\[
R_U = \text{corr}(U^* + e, U^* + e') = \frac{\sigma_{U^*}^2}{\sigma_U^2} = \frac{b^2 \sigma^2}{b^2 \sigma^2 + \sigma_{e'}^2} .
\]

Note that the reliability of a test measures the proportion of observed score variation that can be attributed to true score variation. Note
also that the reliability of a test depends not only on the test, but also on the population from which the examinees taking that test are (randomly) selected.

We shall mention later the special case of equally reliable tests. To say that two tests, say U and V, are equally reliable, we must of course specify the two populations from which the examinees taking each test are selected.

If we do not make any assumptions about the reliabilities of the tests, we will call the tests unequally reliable.

4. Observations.

The observations are

\[ U_\alpha, \quad \alpha = 1,2,\ldots,n \]
\[ V_\alpha, \quad \alpha = n+1,\ldots,2n \]
\[ W_\alpha, \quad \alpha = 1,2,\ldots,n,n+1,\ldots,2n \]

This means that Group 1 \([(1,2,\ldots,n)]\) takes tests U and W, and Group 2 \([(n+1,n+2,\ldots,2n)]\) takes tests V and W.

5. Problems.

Problem 1: Estimate the correspondence between the U scores and V scores (called the excluded anchor test problem).

From (1.1), the linear relationship between \( V_\alpha^* \) and \( U_\alpha^* \) is

\[
(1.2) \quad V_\alpha^* = \frac{b}{a} (U_\alpha^* - a) + A .
\]

The problem of equating is then to estimate \( \frac{b}{a}, \; a, \; A \).
Problem 2: Estimate the correspondence between the \( U^W \) scores and the \( V^W \) scores (called the included anchor test problem).

Define

\[
X_\alpha = U_\alpha + W_\alpha \\
Y_\alpha = V_\alpha + W_\alpha .
\]

Then, from (1.1), the linear relationship between \( X_\alpha^* \) and \( Y_\alpha^* \) is

\[
Y_\alpha^* = \frac{B + B^*}{b + B} \left( X_\alpha^* - a - \alpha \right) + A + \alpha .
\]

The problem of equating in the included anchor test problem is then to estimate \( \frac{B + B^*}{b + B} \), \( a + \alpha \), and \( A + \alpha \).

1.3 The General Method.

Recall that the problem of equating in the excluded anchor test problem was shown in section 1.2 to be equivalent to estimating \( \frac{B}{b} \), \( a \), and \( A \) for a specified group of examinees. It will be convenient to express the parameters \( \frac{B}{b} \), \( a \), and \( A \) in terms of parameters \( \mu_{U^*} \), \( \mu_{V^*} \), \( \sigma_{U^*} \), \( \sigma_{V^*} \) defined by:

\[
\begin{align*}
\mu_{U^*} & = \delta(U^*) = a + b_1Z \\
\mu_{V^*} & = \delta(V^*) = A + B_1Z \\
\sigma_{U^*}^2 & = \delta(U^* - \mu_{U^*})^2 = b_2^2 \sigma_Z^2 \\
\sigma_{V^*}^2 & = \delta(V^* - \mu_{V^*})^2 = B_2^2 \sigma_Z^2 .
\end{align*}
\]

where all the expectations above are taken over the same group of examinees.
The linear relationship (1.2) may thus be re-expressed in terms of 
$\mu_{U^*}$, $\mu_{V^*}$, $\sigma_{U^*}$, $\sigma_{V^*}$ as

(1.5) \[ V^*_\alpha = \frac{\sigma_{V^*}}{\sigma_{U^*}} (U^*_\alpha - \mu_{U^*}) + \mu_{V^*}, \]

and the problem of equating is then one of estimating $\mu_{U^*}$, $\mu_{V^*}$, $\sigma_{U^*}$, $\sigma_{V^*}$ for a given group of examinees. Note here that since $\sigma_{U^*}$ and $\sigma_{V^*}$ enter only as the ratio $\frac{\sigma_{V^*}}{\sigma_{U^*}}$, it is only necessary to estimate three parameters, $\mu_{U^*}$, $\mu_{V^*}$, and $\frac{\sigma_{V^*}}{\sigma_{U^*}}$.

Note also that although the parameters defined in (1.4) depend on the group over which the expectations are taken, the relationship (1.5) does not.

In the special case where it is assumed that the tests $U$ and $V$ are equally reliable (i.e., $R_U = \frac{\sigma_{U^*}}{\sigma_U} = R_V = \frac{\sigma_{V^*}}{\sigma_V}$), note that

$$\frac{\sigma_{V^*}}{\sigma_{U^*}} = \frac{\sigma_V}{\sigma_U}.$$ 

Therefore, since it is always true that

$$\mu_{U^*} = E(U^*) = E(U^* + e) = \mu_U$$

$$\mu_{V^*} = E(V^*) = E(V^* + E) = \mu_V,$$

for the equally reliable case, we can use estimates of the observed score parameters $\mu_U$, $\mu_V$, $\sigma_V/\sigma_U$ as estimates of the true score parameters $\mu_{U^*}$, $\mu_{V^*}$, $\sigma_{V^*}/\sigma_{U^*}$ to estimate (1.5), the relationship between $U^*$ and $V^*$. This procedure, used in the equally reliable case, is thus sometimes called standardized observed score equating.
For the case of unequally reliable tests, estimates of the observed score parameters may be used to obtain estimates of the true score parameters through the following relationships:

\[
\begin{align*}
\mu_{U^*} &= \mu_U \\
\mu_{V^*} &= \mu_V \\
\sigma_{U^*} &= \sigma_U \sqrt{R_U} \\
\sigma_{V^*} &= \sigma_V \sqrt{R_V}.
\end{align*}
\]

Thus, with estimates for \( \mu_U, \mu_V, \sigma_U, \sigma_V, \sqrt{R_U}, \sqrt{R_V} \), we can estimate the true score parameters. This procedure, used in the unequally reliable case, is sometimes called standardized true score equating.

Once we have estimated the relationship between \( U^* \) and \( V^* \), it will be used in practice in the following way: if \( U_\alpha \) is observed for examinee \( \alpha \), we will estimate his corresponding \( V^*_\alpha \) score as

\[
\tilde{V}_\alpha^*(U_\alpha) = \left( \frac{\sigma_{V^*}}{\sigma_{U^*}} \right) (U_\alpha - \tilde{\mu}_{U^*}) + \tilde{\mu}_{V^*}.
\]

If we want a predicted observed score \( \tilde{V}_\alpha \), we will use the same estimated relationship. Through this relationship, we will obtain correspondences between the observed scores \( U_\alpha \) and \( V_\alpha \) for every \( \alpha \).

It should be noted here that the method described above treats \( U^* \) and \( V^* \) symmetrically. If we had instead expressed \( U^* \) in terms of \( V^* \), and thus had obtained the estimated relationship

\[
\tilde{U}_\alpha^*(V_\alpha^*) = \left( \frac{\sigma_{U^*}}{\sigma_{V^*}} \right) (V_\alpha^* - \tilde{\mu}_{V^*}) + \tilde{\mu}_{U^*},
\]
the resulting correspondences between \( U_\alpha \) and \( V_\alpha \), for every \( \alpha \), would have been the same.

The equating problem in the included anchor test problem may be handled similarly, as follows.

The linear relationship (1.3) may be re-expressed in terms of \( \mu_X^*, \sigma_X^*, \sigma_Y^* \) as

\[
Y_\alpha^* = \frac{\sigma_Y^*}{\sigma_X^*} (X_\alpha^* - \mu_X^*) + \mu_Y^*,
\]

and the problem of equating is one of estimating \( \mu_X^*, \mu_Y^*, \frac{\sigma_Y^*}{\sigma_X^*} \) for a given group of examinees. The resulting estimated relationship between \( X_\alpha^* \) and \( Y_\alpha^* \) will be used to obtain correspondences between the observed scores \( X_\alpha \) and \( Y_\alpha \) for every \( \alpha \).

1.4 Outline of Paper.

In chapter 2 we review some estimators developed for certain special cases of our problem.

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<td>2. Random groups, unequally reliable tests (RG-UR)</td>
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<td>3. Nonrandom groups, equally reliable tests (NRG-ER)</td>
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<td>4. Nonrandom groups, unequally reliable tests (NRG-UR)</td>
<td>Levine</td>
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In chapter 3, we propose for consideration several different estimators of the unknown parameters for the general equating scores problem. The method of moments is used to obtain the estimators. The estimators are shown to be consistent in the general case, which includes all four special cases treated in chapter 2.

The contribution of the estimators proposed in chapter 3 to the equating scores problem can be summarized as follows. The estimators proposed in chapter 3 for the excluded anchor test problem are a small contribution: they are similar to Levine's estimator for the NRG-UR case; in fact, one is identical to Levine's estimator. On the other hand, the estimators proposed in chapter 3 for the included anchor test problem are a larger contribution. We shall see in chapter 2 that for the included anchor test problem, the estimators proposed by Levine are incorrect, because he uses a relationship among reliabilities and correlations that is incorrect. The Lord estimator is appropriate only when the assumption of random groups holds. In the included anchor test problem, then, there exist no trustworthy estimators (among those reviewed in chapter 2). In chapter 3, some estimators are obtained, which are shown to perform well in the general case.

In chapter 4, we find the asymptotic biases of the estimators of other authors reviewed in chapter 2. It is shown that, although some estimated scores are consistent under the special assumptions for which they were obtained, all of the estimators are not consistent for the general equating problem considered in this paper. The expressions for the asymptotic biases are given, along with some numerical examples to indicate magnitudes of the asymptotic biases.
Chapter 5 examines the asymptotic variances of the consistent estimators proposed in chapter 3. It is shown that none of the proposed estimators is uniformly best for all distributions, $F_1(\cdot)$ and $F_2(\cdot)$, of ability, and for all distributions of the errors $e, E,$ and $e$. One estimate is recommended that is best for some distributions, and is never the worst.
CHAPTER 2: A REVIEW OF ESTIMATORS FOR THE
SPECIAL CASES

2.1 Introduction.

This chapter will review estimators, for both the excluded anchor
test problem and the included anchor test problem, that have been proposed
for the following special cases of the equating problem:

Case 1: Random groups, equally reliable tests (RG-ER)

Case 2: Random groups, unequally reliable tests (RG-UR)

Case 3: Nonrandom groups, equally reliable tests (NRG-ER)

Case 4: Nonrandom groups, unequally reliable tests (NRG-UR)

It was shown in section 1.2 that the excluded anchor test problem of
equating reduces to that of estimating \( \mu_{U^*}, \sigma_{V^*}/\sigma_{U^*} \) for a given
group of examinees, and thus obtaining an estimated relationship between
\( U^*_\alpha \) and \( V^*_\alpha \):

\[
\tilde{V}^*_\alpha (U^*_\alpha) = \frac{\tilde{\sigma}_{V^*}}{\tilde{\sigma}_{U^*}}(U^*_\alpha - \tilde{\mu}_{U^*}) + \tilde{\mu}_{V^*}.
\] (2.1)

In the included anchor test problem, the problem of equating reduces
to that of estimating \( \mu_{X^*}, \sigma_{Y^*}/\sigma_{X^*} \) for a given group of examinees,
and thus obtaining an estimated relationship between \( X^*_\alpha \) and \( Y^*_\alpha \):

\[
\tilde{Y}^*_\alpha (X^*_\alpha) = \frac{\tilde{\sigma}_{Y^*}}{\tilde{\sigma}_{X^*}}(X^*_\alpha - \tilde{\mu}_{X^*}) + \tilde{\mu}_{Y^*}.
\] (2.2)

It is shown in appendix E that the models discussed below are indeed,
as claimed, special cases of the general model outlined in section 1.2.
2.2 Random Groups, Equally Reliable Tests (RG-ER).

We will review two approaches to this problem, one proposed by Tucker (Gulliksen [3]) and another by Lord [7]. Although the approaches are different, they lead to the same solution.

Recall from section 1.3 that estimators of the observed score parameters \( \mu_U, \mu_V, \sigma_V \sigma_U \) suffice as estimators of the true score parameters \( \mu_{U*}, \mu_{V*}, \sigma_{V*} \sigma_{U*} \), since in this special case, the tests are equally reliable.

We review the Tucker solution first. Tucker chooses to obtain estimators of \( \mu_{U_T}, \mu_{V_T}, \sigma_{U_T}^2, \sigma_{V_T}^2 \), where the subscript \( T \) denotes parameters for the total of groups 1 and 2. Since the groups 1 and 2 do not take both tests \( U \) and \( V \), the parameters \( \mu_{U_T}, \mu_{V_T}, \sigma_{U_T}^2, \sigma_{V_T}^2 \) cannot, unfortunately, be estimated directly. They can, however, be estimated indirectly.

First consider \( \mu_{U_T} \). Assume that the \( U \) and \( W \) scores have a relationship of the form

\[
U_\alpha = c + dW_\alpha + f_\alpha.
\]

(2.3)

We can then write

\[
\overline{U}_1 = c + d\overline{W}_1 + \overline{f}_1,
\]

\[
\overline{U}_T = c + d\overline{W}_T + \overline{f}_T,
\]

where
\[
\bar{U}_1 = \frac{1}{n} \sum_{\alpha=1}^{n} U_\alpha \quad \text{(observable)},
\]
\[
\bar{U}_T = \frac{1}{2n} \sum_{\alpha=1}^{2n} U_\alpha \quad \text{(unobservable)},
\]
\[
\bar{W}_1 = \frac{1}{n} \sum_{\alpha=1}^{n} W_\alpha \quad \text{(observable)},
\]
and
\[
\bar{W}_T = \frac{1}{2n} \sum_{\alpha=1}^{2n} W_\alpha \quad \text{(observable)}.
\]

Hence, \( \bar{U}_T \) (an unobservable) can be expressed as
\[
\bar{U}_T = \bar{U}_1 + \bar{d}(\bar{W}_T - \bar{W}_1) + \bar{r}_T - \bar{r}_1.
\]

Thus, \( \mu_{U_T} \) can be estimated by
\[
\hat{\mu}_{U_T} = \bar{U}_T = \bar{U}_1 + \hat{d}(\bar{W}_T - \bar{W}_1),
\]
where
\[
\hat{d} = \frac{\sum_{\alpha=1}^{n} (U_\alpha - \bar{U}_1)(W_\alpha - \bar{W}_1)}{\sum_{\alpha=1}^{n} (W_\alpha - \bar{W}_1)^2}
\]
is the usual (regression) estimator of the slope parameter \( d \).

Now consider \( \mu_{V_T} \). Assume that the \( V \) and \( W \) scores have a relationship of the form
\[
V_\alpha = c' + d'W_\alpha + f'_\alpha.
\]

Find an estimator of \( \mu_{V_T} \) by the same method used above for \( \mu_{U_T} \).
\[ \hat{\mu}_{VT} = \bar{V}_T = \bar{V}_2 + \tilde{d}'(\bar{W}_T - \bar{W}_2) , \]

where

\[ \tilde{d}' = \frac{s_{VW}}{s_{W2}^2} = \frac{\sum_{\alpha=n+1}^{2n} (V_\alpha - \bar{V}_2)(W_\alpha - \bar{W}_2)}{\sum_{\alpha=n+1}^{2n} (W_\alpha - \bar{W}_2)^2} \]

is the usual (regression) estimator of the slope parameter \( d' \).

Now consider \( \sigma_{U_T}^2 \). Observe from (2.3) that

\[ \sigma_{U_1}^2 = d^2 \sigma_{W1}^2 + \sigma_{f_1}^2 + 2d\sigma_{WF_1} \]

\[ \sigma_{U_T}^2 = d^2 \sigma_{W_T}^2 + \sigma_{f_T}^2 + 2d\sigma_{WF_T} , \]

where

\( \sigma_{WF_1} = \text{cov}(W,f) \) for group 1,

\( \sigma_{WF_T} = \text{cov}(W,f) \) for group T.

Tucker makes the assumptions that

\[ \sigma_{f_1}^2 = \sigma_{f_T}^2 \]

and

\[ \sigma_{WF_1} = \sigma_{WF_T} , \]

Hence, \( \sigma_{U_T}^2 = \sigma_{U_1}^2 + d^2(\sigma_{W_T}^2 - \sigma_{W_1}^2) \), and \( \sigma_{U_T}^2 \) can then be estimated by

\[ \hat{\sigma}_{U_T}^2 = s_{U_1}^2 + \tilde{d}^2(s_{W_T}^2 - s_{W_1}^2) , \]

where

\[ s_{U_1}^2 = \frac{1}{n} \sum_{\alpha=1}^{n} (U_\alpha - \bar{U}_1)^2 , \]
\[
\begin{align*}
    s_{W_T}^2 &= \frac{1}{2n} \sum_{\alpha=1}^{2n} (W_\alpha - \bar{W}_T)^2, \\
    s_{W_1}^2 &= \frac{1}{n} \sum_{\alpha=1}^{n} (W_\alpha - \bar{W}_1)^2.
\end{align*}
\]

Similarly, estimate \( \sigma_{V_T}^2 \) by
\[
\tilde{\sigma}_{V_T}^2 = s_{V_2}^2 + \tilde{d}_1^2 (s_{W_T}^2 - s_{W_2}^2),
\]
where
\[
\begin{align*}
    s_{V_2}^2 &= \frac{1}{n} \sum_{\alpha=n+1}^{2n} (V_\alpha - \bar{V}_2)^2, \\
    s_{W_2}^2 &= \frac{1}{n} \sum_{\alpha=n+1}^{2n} (W_\alpha - \bar{W}_2)^2.
\end{align*}
\]

(For the included anchor test problem, make the following changes in the above to obtain the estimators: substitute X for U, and Y for V.)

Thus, substituting the estimators \( \tilde{\mu}_{U_T}, \tilde{\mu}_{V_T}, \tilde{\sigma}_{U_T}, \tilde{\sigma}_{V_T} \) for the true score parameters in (2.1), the foregoing RG-ER solution would lead to an estimated relationship between \( U_\alpha^* \) and \( V_\alpha^* \):

\[
\begin{align*}
    \tilde{V}_\alpha^*(U_\alpha^*) &= \bar{V}_2 + \frac{1}{2} \left[ b_{VW} + b_{UW} \left( \frac{s_{V_2}^2 + b_{VW}^2 (s_{W_T}^2 - s_{W_2}^2)}{s_{U_1}^2 + b_{UW}^2 (s_{W_T}^2 - s_{W_1}^2)} \right) \right] (\bar{V}_1 - \bar{V}_2) \\
    &+ \frac{1}{2} \left[ \frac{s_{V_2}^2 + b_{VW}^2 (s_{W_T}^2 - s_{W_2}^2)}{s_{U_1}^2 + b_{UW}^2 (s_{W_T}^2 - s_{W_1}^2)} \right] (U_\alpha^* - \bar{U}_1),
\end{align*}
\]

where
\[ b_{UW} \equiv \frac{s_{UW}}{s_{W_1}^2}, \]
\[ b_{VW} \equiv \frac{s_{VW}}{s_{W_2}^2}. \]

We now review the Lord solution. Lord [7] assumes that the scores \( U \) and \( W \), and the scores \( V \) and \( W \), have bivariate normal distributions. Maximum likelihood estimators of \( \mu_U \), \( \mu_V \), \( \sigma_U \), \( \sigma_V \) are obtained for the population from which the groups taking tests \( U \) and \( V \) are randomly selected.

The maximum likelihood estimators are:

\[ \tilde{\mu}_U = \overline{U}_1 + \frac{1}{2} b_{UW} (\overline{W}_2 - \overline{W}_1) \]
\[ \tilde{\mu}_V = \overline{V}_2 + \frac{1}{2} b_{VW} (\overline{W}_1 - \overline{W}_2) \]
\[ \tilde{\sigma}_U^2 = s_{U_1}^2 + b_{UW}^2 (s_{W_1}^2 - s_{W_1}^2) \]
\[ \tilde{\sigma}_V^2 = s_{V_2}^2 + b_{VW}^2 (s_{W_2}^2 - s_{W_2}^2) . \]

Substituting these estimators for the true score parameters in (2.1), the Lord solution gives the estimated relationship between \( U^*_\alpha \) and \( V^*_\alpha \):

\[ V^*_\alpha (U^*_\alpha) = \overline{V}_2 + \frac{1}{2} b_{VW} + b_{UW} \left( \frac{s_{V_2}^2 + b_{VW} (s_{W_2}^2 - s_{W_2}^2)}{s_{V_2}^2 + b_{UW} (s_{W_1}^2 - s_{W_1}^2)} \right) \left( \overline{W}_1 - \overline{W}_2 \right) \]
\[ + \left[ \frac{s_{V_2}^2 + b_{VW} (s_{W_2}^2 - s_{W_2}^2)}{s_{U_1}^2 + b_{UW} (s_{W_1}^2 - s_{W_1}^2)} \right]^{\frac{1}{2}} (U^*_\alpha - \overline{U}_1) . \]
The solution in the included anchor test problem is the same, with \( X \) substituted for \( U \), and \( Y \) for \( V \). Note that the estimated relationships between \( U^* \) and \( V^* \) obtained by Tucker and Lord are identical. Hereafter, in this paper, this common solution will be referred to as the RG-ER solution, or the Lord solution.

2.3 Random Groups, Unequally Reliable Tests (RG-UR).

Levine [6] extends the Lord solution to the RG-UR case. The maximum likelihood estimators for \( \mu_{U*}, \mu_{V*}, \sigma_{U*}, \sigma_{V*}, \rho_{UW}, \rho_{VW} \) obtained in the bivariate normal framework are used to obtain maximum likelihood estimators of the true score parameters \( \mu_{U*}, \mu_{V*}, \sigma_{U*}, \sigma_{V*} \), as shown below. Again, estimators are obtained for the population from which the two groups are selected.

Recall that the following relationships between the observed score parameters and the true score parameters hold:

\[
\begin{align*}
\mu_{U*} &= \mu_U \\
\mu_{V*} &= \mu_V \\
\sigma_{U*} &= \sigma_U \sqrt{R_U} \\
\sigma_{V*} &= \sigma_V \sqrt{R_V} 
\end{align*}
\]

(2.4)

Levine seeks to express the reliabilities in terms of parameters which occur naturally in the bivariate normal framework. Estimators of the reliabilities are obtained in terms of estimators of those parameters. Then, the estimators of the reliabilities, together with the maximum
likelihood estimators of $\mu_U$, $\mu_V$, $\sigma_U$, $\sigma_V$ obtained for the previous case, can be used to obtain estimators of the true score parameters in (2.4).

First consider the excluded anchor test problem. Since the tests are congeneric, we have

$$\frac{\rho_{UW}}{\sqrt{R_U} \sqrt{R_W}} = 1$$

(2.5)

$$\frac{\rho_{VW}}{\sqrt{R_V} \sqrt{R_W}} = 1,$$

where $\rho_{UW}$ is the correlation between $U$ and $W$, and $\rho_{VW}$ is the correlation between $V$ and $W$.

Therefore, substituting $\frac{\rho_{UW}}{\sqrt{R_U}}$ for $\sqrt{R_U}$ and $\frac{\rho_{VW}}{\sqrt{R_W}}$ for $\sqrt{R_W}$ in (2.4), we have

$$\frac{\sigma_{V^*}}{\sigma_{U^*}} = \frac{\rho_{VW}}{\rho_{UW}} \frac{\sigma_V}{\sigma_U}.$$

Thus, an estimator of $\frac{\sigma_{V^*}}{\sigma_{U^*}}$ is

$$\tilde{\frac{\sigma_{V^*}}{\sigma_{U^*}}} = \frac{\rho_{VW}}{\rho_{UW}} \frac{\sigma_V}{\sigma_U} = \frac{b_{VW}}{b_{UW}}.$$

Therefore, the estimated relationship between $U^*_\alpha$ and $V^*_\alpha$ in the RG-UR case is:

$$\hat{\nu}_{\alpha} = \left(\frac{U^*_\alpha}{V^*_\alpha}\right) = \overline{V}_2 + b_{VW}(\overline{W}_1 - \overline{W}_2) + \frac{b_{VW}}{b_{UW}} (U^*_\alpha - \overline{U}_1).$$

The included anchor test problem is slightly more difficult to handle. Recall here that we are equating $X^* (= U^* + W^*)$ and $Y^* (= V^* + W^*)$. 

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Again Levine attempts to express $R_X$ in terms of $R_W$ and $\rho_{XW}$, and $R_Y$ in terms of $R_W$ and $\rho_{YW}$.

He uses a result of Angoff [1],

$$R_X = \frac{R_W}{\rho_{XW}^2}$$

(2.6)

$$R_Y = \frac{R_W}{\rho_{YW}^2}.$$

We note that this is incorrect, since if the relationship holds then, since $\rho_{XW}^2 = \rho_{WX}^2$, it must be true that $\frac{R_W}{R_X} = \frac{R_X}{R_W}$, implying that $R_X = \pm R_W$, and then for equation (2.6) to hold, $\rho_{XW}^2 = 1$ would be necessary, and contrary to fact.

To get the correct relationship, note that since $X$ and $W$ are congeneric, we have

$$\frac{\text{cov}^2(X^*,W^*)}{\text{var}(X^*)\text{var}(W^*)} = 1.$$  

Dividing numerator and denominator by $\text{var}(X)\text{var}(W)$, we obtain

$$\frac{\text{cov}^2(X^*,W^*)}{\text{var}(X)\text{var}(W)} \cdot \frac{1}{R_X R_W} = 1,$$

or

$$\frac{\text{cov}^2(X^*,W^*)}{\text{cov}^2(X,W)} \cdot \rho_{XW}^2 \frac{1}{R_X R_W} = 1.$$  

This leads directly to the relationship

$$\rho_{XW}^2 = R_X R_W \frac{\text{cov}^2(X,W)}{\text{cov}^2(X^*,W^*)}.$$
Although we have shown that Levine uses an incorrect relationship in obtaining his estimators, we nevertheless record his method and estimated relationship here for purposes of comparison.

The maximum likelihood estimators of $\mu_{X*}$ and $\mu_{Y*}$ were found in the previous section. They are:

$$
\tilde{\mu}_{X*} = \bar{X}_1 + \frac{1}{2} b_{XW_1} (\bar{W}_2 - \bar{W}_1)
$$

$$
\tilde{\mu}_{Y*} = \bar{Y}_2 + \frac{1}{2} b_{YW_2} (\bar{W}_1 - \bar{W}_2).
$$

Now obtain an estimator of $\sigma_{Y*}/\sigma_{X*}$. This can be expressed in terms of reliabilities and observed score parameters as follows:

$$
\frac{\sigma_{Y*}}{\sigma_{X*}} = \frac{\sigma_Y}{\sigma_X} \sqrt{\frac{R_Y}{R_X}}.
$$

From the previous section, maximum likelihood estimators of $\sigma^2_X$ and $\sigma^2_Y$ are:

$$
\tilde{\sigma}^2_X = s^2_{X_1} + b^2_{XW_1} (s^2_{W_1} - s^2_{W_T})
$$

$$
\tilde{\sigma}^2_Y = s^2_{Y_2} + b^2_{YW_2} (s^2_{W_2} - s^2_{W_T}).
$$

From relation (2.5), the ratio $\sigma_{Y*}/\sigma_{X*}$ is equal to

$$
\frac{\sigma_{Y*}}{\sigma_{X*}} = \frac{\sigma_Y^{\beta_{XW}}}{\sigma_X^{\beta_{YW}}}.\]

Thus, estimate $\sigma_{Y*}/\sigma_{X*}$ by

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\[
\frac{\bar{\sigma}_{Y^*}}{\bar{\sigma}_{X^*}} = \frac{b_{XW_1}}{b_{YW_2}} \cdot \left[ \frac{s_{Y_2}^2 + b_{YW_2}^2 (s_{W_T}^2 - s_{W_2}^2)}{s_{X_1}^2 + b_{XW_1}^2 (s_{W_T}^2 - s_{W_1}^2)} \right].
\]

These maximum likelihood estimators of the unknown parameters are substituted into (2.2).

Thus, the estimated relationship between \(X^*_\alpha\) and \(Y^*_\alpha\) in the RG-UR case is:

\[
\bar{Y}^*_\alpha(X^*_\alpha) = \bar{Y}_2 + \frac{1}{2} b_{YW} \left( 1 + \frac{b_{XW} (s_{Y_2}^2 + b_{YW}^2 (s_{W_T}^2 - s_{W_2}^2))}{b_{YW} (s_{X_1}^2 + b_{XW}^2 (s_{W_T}^2 - s_{W_1}^2))} \right) (\bar{X}_1 - \bar{X}_1).
\]

2.4 Nonrandom Groups, Equally Reliable Tests (NRG-ER).

Levine [6] obtains estimators for the observed score parameters for group T, the total of groups 1 and 2. Recall from section 1.3 that in order to obtain estimators of \(\mu_{U_T}, \mu_{V_T}, \sigma_{V_T}/\sigma_{U_T}\), it suffices to find estimators of \(\mu_{U_T}, \mu_{V_T}, \sigma_{V_T}/\sigma_{U_T}\), since the tests are equally reliable.

Levine makes the following assumptions:

(1) \[
\frac{\mu_{U_T}^*}{\sigma_{U_T}^*} - \frac{\mu_{U_1}^*}{\sigma_{U_1}^*} = \frac{\mu_{U_T}^*}{\sigma_{U_T}^*} - \frac{\mu_{U_1}^*}{\sigma_{U_1}^*},
\]

\[
\frac{\mu_{V_T}^*}{\sigma_{V_T}^*} - \frac{\mu_{V_2}^*}{\sigma_{V_2}^*} = \frac{\mu_{V_T}^*}{\sigma_{V_T}^*} - \frac{\mu_{V_2}^*}{\sigma_{V_2}^*}.
\]
\[ \frac{\sigma_{\text{U*}}}{\sigma_{\text{W*}}} = \frac{\sigma_{\text{U*}}}{\sigma_{\text{W*}}} \]

\[ \frac{\sigma_{\text{V*}}}{\sigma_{\text{W*}}} = \frac{\sigma_{\text{V*}}}{\sigma_{\text{W*}}} \]

\[ \frac{\sigma_{\text{V*}}}{\sigma_{\text{W*}}} = \frac{\sigma_{\text{V*}}}{\sigma_{\text{W*}}} \]

(3) \[ \sigma_{\text{U}}^2 (1 - R_{\text{U}}) = \sigma_{\text{U}}^2 (1 - R_{\text{U}}) \]

\[ \sigma_{\text{V}}^2 (1 - R_{\text{V}}) = \sigma_{\text{V}}^2 (1 - R_{\text{V}}) \]

\[ \sigma_{\text{W}}^2 (1 - R_{\text{W}}) = \sigma_{\text{W}}^2 (1 - R_{\text{W}}) \]

\[ \sigma_{\text{W}}^2 (1 - R_{\text{W}}) = \sigma_{\text{W}}^2 (1 - R_{\text{W}}) \]

First obtain an estimator of \( \mu_{\text{U*}} \). From assumptions (1) and (2), \( \mu_{\text{U*}} \) can be expressed as

\[ \mu_{\text{U*}} = \mu_{\text{U*}} + \frac{\sigma_{\text{U*}}}{\sigma_{\text{W*}}} (\mu_{\text{W*}} - \mu_{\text{W*}}) \]

Hence an estimator of \( \mu_{\text{U*}} \) is

(2.8) \[ \tilde{\mu}_{\text{U*}} = \tilde{\mu}_{\text{U*}} + \frac{\sigma_{\text{U*}}}{\sigma_{\text{W*}}} (\tilde{\mu}_{\text{W*}} - \tilde{\mu}_{\text{W*}}) \), where

\[ \left( \frac{\sigma_{\text{U*}}}{\sigma_{\text{W*}}} \right) = \left( \frac{\sigma_{\text{U*}}}{\sigma_{\text{W*}}} \right) \]
Similarly, an estimator of $\mu_{VT}$ is

\begin{equation}
\tilde{\mu}_{VT} = \bar{V}_2 + \left( \frac{\sigma_{VT}}{\sigma_{V_2}} \right) (\bar{W}_T - \bar{W}_2), \quad \text{where}
\end{equation}

\begin{align*}
\left( \begin{array}{c}
\tilde{\sigma}_{VT}^2 \\
\tilde{\sigma}_{V_2}^2
\end{array} \right) &= \left( \begin{array}{c}
\sigma_{VT} \sqrt{R_{VT}} \\
\sigma_{V_2} \sqrt{R_{V_2}}
\end{array} \right).
\end{align*}

Now obtain an estimator for $\frac{\sigma_{VT}}{\sigma_{U_T}}$. From assumptions (2) and (3),

\begin{align*}
\sigma_{U_T}^2 &= \sigma_{U_1}^2 + \frac{\sigma_{U_1}^2}{\sigma_{W_1}^2} \left( \frac{1}{\sigma_{W_1}^2} - \sigma_{V_1}^2 \right), \\
\sigma_{VT}^2 &= \sigma_{V_2}^2 + \frac{\sigma_{V_2}^2}{\sigma_{W_2}^2} \left( \frac{1}{\sigma_{W_2}^2} - \sigma_{W_1}^2 \right).
\end{align*}

Hence, estimate $\sigma_{U_T}^2$ by

\begin{equation}
\tilde{\sigma}_{U_T}^2 = s_{U_1}^2 + \frac{\sigma_{U_1}^2}{\sigma_{W_1}^2} \left( s_{W_T}^2 - s_{W_1}^2 \right).
\end{equation}

Similarly, estimate $\sigma_{VT}^2$ by

\begin{equation}
\tilde{\sigma}_{VT}^2 = s_{V_2}^2 + \frac{\sigma_{V_2}^2}{\sigma_{W_2}^2} \left( s_{W_T}^2 - s_{W_2}^2 \right).
\end{equation}
It remains, then, to find estimators for

\[
\frac{\sigma^2_{U_1 R_{U_1}}}{\sigma^2_{W_1 W_1}} \quad \text{and} \quad \frac{\sigma^2_{V_2 R_{V_2}}}{\sigma^2_{W_2 W_2}}.
\]

(The above discussion is in terms of excluded anchor test scores. For the anchor test problem, substitute \(X\) for \(U\), and \(Y\) for \(V\).)

At this point, the excluded anchor test problem is treated separately from the included anchor test problem.

We treat the excluded anchor test problem first. Recall, from (2.5), that \(R_{U_1}\) and \(R_{V_2}\) can be expressed as

\[
R_{U_1} = \frac{\rho_{UW}}{R_{W_1}}
\]

(2.5)

\[
R_{V_2} = \frac{\rho_{VW}}{R_{W_2}}.
\]

Then, \(\frac{\sigma^2_{U_1 R_{U_1}}}{\sigma^2_{W_1 W_1}}\) can be rewritten as

\[
\frac{\sigma^2_{U_1 R_{U_1}}}{\sigma^2_{W_1 W_1}} = \frac{\sigma^2_{U_1 W_1}}{\sigma^2_{W_1 W_1}} \cdot \frac{\rho_{UW}^2}{R_{W_1}^2}.
\]

Hence an estimator of \(\frac{\sigma^2_{U_1 R_{U_1}}}{\sigma^2_{W_1 W_1}}\) is
\begin{equation}
\left( \frac{\sigma^2_{U_1}}{\sigma^2_{W_1}} \right) = \frac{b_{UW_1}}{R^2_{W_1}}.
\end{equation}

Similarly, an estimator of \( \frac{\sigma^2_{V_2}}{\sigma^2_{W_2}} \) is

\begin{equation}
\left( \frac{\sigma^2_{V_2}}{\sigma^2_{W_2}} \right) = \frac{b_{VW_2}}{R^2_{W_2}}.
\end{equation}

Using (2.12) and (2.13) in (2.8), (2.9), (2.10), (2.11), obtain estimators of \( \mu_{U_T}, \mu_{V_T}, \sigma_{V_T}/\sigma_{U_T} \):

\begin{align*}
\tilde{\mu}_{U_T} &= \overline{U}_1 + \frac{b_{UW_1}}{R^2_{W_1}} (\overline{W}_T - \overline{W}_1) \\
\tilde{\mu}_{V_T} &= \overline{V}_2 + \frac{b_{VW_2}}{R^2_{W_2}} (\overline{W}_T - \overline{W}_2) \\
\left( \frac{\tilde{\sigma}_{V_T}}{\tilde{\sigma}_{U_T}} \right) &= \left[ \frac{s^2_{V_2} + \frac{b^2_{VW_2}}{R^2_{W_2}} (s^2_{W_T} - s^2_{W_2})}{s^2_{U_1} \frac{b^2_{UW_1}}{R^2_{W_1}} (s^2_{W_T} - s^2_{W_1})} \right]^{\frac{1}{2}}.
\end{align*}

Therefore, by substituting these estimators for the true score parameters in (2.1), the estimated relationship between \( \hat{U}_\alpha \) and \( \hat{V}_\alpha \) in

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the NRG-ER case is

\[
\tilde{V}_\alpha (\gamma) = \bar{V}_2 + \frac{1}{2} \left( \frac{b_{W_2}}{R_{W_2}} + \frac{b_{W_1}}{R_{W_1}} \right) \left( \frac{s_{V_2}^2 + \frac{b_{W_2}^2}{R_{W_2}^2} (s_{W_2}^2 - s_{W_1}^2)}{s_{U_1}^2 + \frac{b_{W_1}^2}{R_{W_1}^2} (s_{W_1}^2 - s_{W_2}^2)} \right)^{1/2} (\bar{W}_1 - \bar{W}_2).
\]

We now look at the included anchor test problem. The relationship (2.6) was again used:

\[
R_x = \frac{R_W}{\rho_{XW}}
\]

(2.6)

\[
R_y = \frac{R_W}{\rho_{YW}}.
\]

Although (2.6) was shown to be incorrect, we nevertheless record below the (incorrect) solution obtained by its use.

Estimators are sought for \( \mu_{X_T}, \mu_{Y_T}, \sigma_{X_T}, \sigma_{Y_T} \).

From (2.8), an estimator of \( \mu_{X_T} \) is
\[ \tilde{\mu}_{X_T} = \bar{X}_1 + \left( \frac{\sigma_{X_1} \sqrt{R_{X_1}}}{\sigma_{W_1} \sqrt{R_{W_1}}} \right) (\bar{W}_T - \bar{W}_1). \]

Now, from (2.6), an estimator of \( \frac{\sigma_{X_1} \sqrt{R_{X_1}}}{\sigma_{W_1} \sqrt{R_{W_1}}} \) is

\[
\left( \frac{\sigma_{X_1} \sqrt{R_{X_1}}}{\sigma_{W_1} \sqrt{R_{W_1}}} \right) = \left( \frac{\sigma_{X_1} \sqrt{R_{W_1}}}{\sigma_{W_1} \sqrt{R_{X_1}}} \right) = \left( \frac{\sigma_{X_1} \sigma_{W_1}}{\sigma_{W_1} \sigma_{X_1}} \right) = \frac{1}{\beta_{W^X_1}}.
\]

Therefore, an estimator of \( \tilde{\mu}_{X_T} \) is

\[ (2.14) \quad \tilde{\mu}_{X_T} = \bar{X}_1 + \frac{1}{\beta_{W^X_1}} (\bar{W}_T - \bar{W}_1). \]

Similarly, an estimator of \( \tilde{\mu}_{Y_T} \) is

\[ (2.15) \quad \tilde{\mu}_{Y_T} = \bar{Y}_2 + \frac{1}{\beta_{W^Y_2}} (\bar{W}_T - \bar{W}_2). \]

We now obtain an estimator of \( \sigma_{Y_T}/\sigma_{X_T} \). Again, using (2.6) in (2.10) and (2.11), estimators of \( \sigma_{X_T} \) and \( \sigma_{Y_T} \) are:

\[
\sigma_{X_T}^2 = s_{X_1}^2 + \frac{1}{\beta_{W^X_1}} (s_{W_T}^2 - s_{W_1}^2)
\]

\[
\sigma_{Y_T}^2 = s_{Y_2}^2 + \frac{1}{\beta_{W^Y_2}} (s_{W_T}^2 - s_{W_2}^2).
\]
Hence, an estimator of \( \frac{\sigma_{Y_T}}{\sigma_{X_T}} \) is:

\[
\left( \frac{\sigma_{Y_T}}{\sigma_{X_T}} \right) = \left( \frac{s_{Y_2}^2 + \frac{1}{2} (s_{W_T}^2 - s_{W_2}^2)}{b_{Wy_2}} \right)^{\frac{1}{2}} \cdot \left( \frac{s_{X_1}^2 + \frac{1}{2} (s_{W_T}^2 - s_{W_1}^2)}{b_{WX_1}} \right)^{\frac{1}{2}}.
\]

(2.16)

Therefore, substituting the estimators (2.14), (2.15), (2.16) for the true score parameters in (2.2), the estimated relationship between \( X_\alpha^* \) and \( Y_\alpha^* \) in the NRG-ER case is

\[
\tilde{Y}_\alpha^*(X_\alpha^*) = \bar{Y}_2 + \frac{1}{2} \left[ \frac{s_{Y_2}^2 + \frac{1}{2} (s_{W_T}^2 - s_{W_2}^2)}{b_{Wy_2}} \left( \frac{s_{X_1}^2 + \frac{1}{2} (s_{W_T}^2 - s_{W_1}^2)}{b_{WX_1}} \right)^{\frac{1}{2}} \right] (\bar{W}_1 - \bar{W}_2)
\]

\[
+ \left( \frac{s_{Y_2}^2 + \frac{1}{2} (s_{W_T}^2 - s_{W_2}^2)}{b_{Wy_2}} \right)^{\frac{1}{2}} \cdot \left( \frac{s_{X_1}^2 + \frac{1}{2} (s_{W_T}^2 - s_{W_1}^2)}{b_{WX_1}} \right)^{\frac{1}{2}} (X_\alpha^* - \bar{X}_1).
\]

2.5 Nonrandom Groups, Unequally Reliable Tests (NRG-UR).

This special case was also treated by Levine [6]. Estimators are sought for \( \mu_{UT_T}, \nu_{VT_T}, \sigma_{UT_T}, \sigma_{VT_T} \) in the excluded anchor test problem, and for \( \mu_{MT_T}, \nu_{MT_T}, \sigma_{MT_T}, \sigma_{MT_T} \) in the included anchor test problem.
We treat the excluded anchor test problem first. Estimators of $\mu_{T}^{*}$ and $\mu_{T}^{*}$ were found in the previous section:

\begin{equation}
\tilde{\mu}_{T}^{*} = \bar{U}_{1} + \frac{b_{W_{1}}}{\tilde{R}_{W_{1}}} (\bar{W}_{T} - \bar{W}_{1}) \tag{2.17}
\end{equation}

\begin{equation}
\tilde{\mu}_{T}^{*} = \bar{V}_{2} + \frac{b_{W_{2}}}{\tilde{R}_{W_{2}}} (\bar{W}_{T} - \bar{W}_{2}) \tag{2.18}
\end{equation}

Now obtain an estimator of $\frac{\sigma_{T}^{*}}{\sigma_{U_{1}^{*}}}$. From assumption (2) in section 2.4, dividing the second equation by the first will yield the relationship:

\begin{equation}
\frac{\sigma_{T}^{*}}{\sigma_{U_{1}^{*}}} = \frac{\sigma_{T}^{*}}{\sigma_{V_{2}^{*}}} \cdot \frac{\sigma_{W_{1}^{*}}}{\sigma_{W_{2}^{*}}} .
\end{equation}

This can be rewritten in terms of observed score parameters and reliabilities as

\begin{equation}
\frac{\sigma_{T}^{*}}{\sigma_{U_{1}^{*}}} = \frac{\sigma_{V_{2}^{*}} \sqrt{R_{V_{2}}} \cdot \sigma_{W_{1}^{*}} \sqrt{R_{W_{1}}}}{\sigma_{U_{1}^{*}} \sqrt{R_{U_{1}}} \cdot \sigma_{W_{2}^{*}} \sqrt{R_{W_{2}}}} .
\tag{2.19}
\end{equation}

Using the relationship (2.5), this becomes

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\[
\frac{\sigma_{V^*}}{\sigma_{U^*}} = \frac{\sigma_V^2 \cdot \rho_{VW_2}}{\sqrt{R_{W_2}}} \cdot \frac{\sigma_{W_1} \sqrt{R_{W_1}}}{\sigma_{W_2} \sqrt{R_{W_2}}}
\]

\[
\frac{R_{W_1}}{R_{W_2}} \frac{\sigma_{W_2}}{\sigma_{W_2}} \cdot \frac{\sigma_{W_1}^2}{\sigma_{W_1}}.
\]

Therefore, an estimator of \( \frac{\sigma_{V^*}}{\sigma_{U^*}} \) is

\[
\begin{pmatrix}
\hat{\sigma}_{V^*} \\
\sigma_{U^*}
\end{pmatrix}
= \begin{pmatrix}
b_{VW_2} & \tilde{R}_{W_1} \\
b_{W_2} & \tilde{R}_{W_2}
\end{pmatrix}
\]

(2.20)

Substituting the estimators in (2.17), (2.18), (2.20) into (2.1), the estimated relationship between \( U^*_\alpha \) and \( V^*_\alpha \) in the NRG-UR case is

\[
\tilde{V}^*_\alpha(U^*_\alpha) = \tilde{V}_2 + \frac{b_{VW_2}}{\tilde{R}_{W_2}} (\tilde{W}_1 - \tilde{W}_2) + \frac{b_{VW_2}}{b_{W_1}} \frac{\tilde{R}_{W_1}}{R_{W_2}} (U^*_\alpha - \bar{U}_1).
\]

We now treat the included anchor test problem. Estimators of \( \mu_{x^*_T} \) and \( \mu_{y^*_T} \) were found in the previous section in (2.14) and (2.15):

\[
\tilde{U}_{x^*_T} = \bar{X}_1 + \frac{1}{b_{W_1}} (\bar{W}_T - \bar{W}_1)
\]

(2.21)
(2.22) \[ \hat{\mu}_{y*} = \bar{y}_2 + \frac{1}{p_{wy_2}} (\bar{w}_1 - \bar{w}_2). \]

Now obtain an estimator of \( \frac{\sigma_{y*}}{\sigma_{x*}} \). From (2.19), we can express \( \frac{\sigma_{y*}}{\sigma_{x*}} \) as

\[
\frac{\sigma_{y*}}{\sigma_{x*}} = \frac{\sigma_y \sqrt{R_{y2}}}{\sigma_{x*}} \frac{\sigma_w \sqrt{R_{w1}}}{\sigma_{w*}}.
\]

Using the relationship (2.6), this becomes

\[
\frac{\sigma_{y*}}{\sigma_{x*}} = \frac{\sigma_y \sqrt{R_{y2}}}{\sigma_{x*}} \frac{\sigma_w \sqrt{R_{w1}}}{\sigma_{w*}} = \frac{\sigma_y^2}{\sigma_{wy_2}} \frac{\sigma_w^2}{\sigma_{wx_1}}.
\]

Therefore, an estimator of \( \frac{\sigma_{y*}}{\sigma_{x*}} \) is

\[
\begin{pmatrix} \hat{\sigma}_{y*} \cr \hat{\sigma}_{x*} \end{pmatrix} = \begin{pmatrix} a_{wy_1} \cr b_{wy_2} \end{pmatrix}.
\]

Substituting (2.21), (2.22), (2.23) for the true score parameters in (2.2), the estimated relationship between \( \hat{x}_c^* \) and \( \hat{y}_c^* \) in the NRG-UR case is
\[
\tilde{y}_u(x_\alpha^*) = \overline{y}_2 + \frac{1}{b_{WY_2}} \left( \overline{w}_1 - \overline{w}_2 \right) + \frac{b_{WX_2}}{b_{WY_2}} (x_\alpha^* - \overline{x}_1). 
\]

A disturbing feature of the estimators proposed in sections 2.4 and 2.5 is that, as \( |R_Y - R_U| \to 0 \), i.e., the reliabilities approach each other, the estimated relationship for the unequally reliable case does not reduce to the estimated relationship for the equally reliable case.

Recall also that the estimators for the included anchor test problem in sections 2.3, 2.4, and 2.5 are incorrect because of the use of relationship (2.6). In chapter 3, consistent estimators will be proposed for the included anchor test problem.

This concludes the review of estimators for the special cases. The asymptotic behavior of these estimators, for both the special cases under which they were derived and the general case, is discussed in chapter 4.
CHAPTER 3: SOME PROPOSED ESTIMATORS

3.1 Introduction.

In this chapter, we consider the general (i.e. no assumption that the tests are equally reliable or that the groups are randomly selected from the same population) equating scores problem. We propose some estimators of the parameters $\sigma_{Y^*}/\sigma_{U^*}, \mu_{Y^*}, \mu_{Y^*}$ for the excluded anchor test problem, and of the parameters $\sigma_{Y^*}/\sigma_{X^*}, \mu_{X^*}, \mu_{X^*}$ for the included anchor test problem. Recall from section 1.3 that these parameters define the relationship,

$$V^*_{\alpha} = \mu_{V^*} + \left( \frac{\sigma_{V^*}}{\sigma_{U^*}} \right) (U^*_{\alpha} - \mu_{U^*}),$$

between the scores $U^*$ and $V^*$ (in the excluded anchor test problem), and the relationship,

$$Y^*_{\alpha} = \mu_{Y^*} + \left( \frac{\sigma_{Y^*}}{\sigma_{X^*}} \right) (X^*_{\alpha} - \mu_{X^*}),$$

between the scores $X^*$ and $Y^*$ (in the included anchor test problem).

Recall also, from section 1.3, that these parameters are defined as expectations over some (arbitrary) specified population of examinees. The population that is chosen in this chapter is the population of examinees from which Group 1 comes.

Hence, we shall propose some estimators of $\mu_{Y^*1}, \mu_{Y^*1}, \sigma_{Y^*}/\sigma_{U^*1} (\equiv \frac{b}{b})$ (for the excluded anchor test problem) and of $\mu_{X^*1}, \mu_{X^*1}, \sigma_{Y^*}/\sigma_{X^*1} (\equiv \frac{B+b}{b+B})$ (for the included anchor test problem). The subscript 1 denotes that
the parameters are defined as expectations over the population from which
Group 1 is drawn.

Most of the following discussion will be in terms of the excluded
anchor test problem. The appropriate modifications of the procedure for
the included anchor test problem will then be indicated.

The method used to obtain estimators will be the method of moments.

3.2 Proposed Estimators for the Excluded Anchor Test Problem.

First, consider the parameters (means) \( \mu_{1*} \) and \( \mu_{2*} \). The parameter
\( \mu_{1*} \) can be estimated directly from the data, since Group 1 takes test U.
However, the parameter \( \mu_{2*} \) cannot, since Group 1 does not take test V.
Therefore, re-express \( \mu_{2*} \) in terms of parameters whose corresponding
sample values are observed.

From (1.1) we have

\[
\mu_{2*} = A + B \mu_{2*} \\
= A + B \mu_{2*} + B(\mu_{2} - \mu_{2}) \\
= \mu_{2*} + \frac{B}{\beta} (\mu_{2*} - \mu_{2*}) \\
\]

So, in place of the parameter \( \mu_{2*} \), we now have the four parameters,
\( \mu_{2}, \mu_{1*}, \mu_{2*}, \) and \( \frac{B}{\beta} \). We have, in fact, "reduced" the problem of
finding estimators for \( \mu_{1*}, \mu_{2*}, \) \( \frac{B}{\beta} \) to that of finding estimators for
\( \mu_{1*}, \mu_{2*}, \mu_{1*}, \mu_{2*}, \frac{B}{\beta} \) and \( \frac{B}{\beta} \).

Estimators of the means are readily available, using the means of
the observed scores.
(1) $\tilde{\mu}_1^* = \bar{U}_1 = \frac{1}{n} \sum_{\alpha=1}^{n} U_\alpha$

(2) $\tilde{\mu}_2^* = \bar{V}_2 = \frac{1}{n} \sum_{\alpha=1}^{2n} V_\alpha$

(3) $\tilde{\mu}_1^* = \bar{W}_1 = \frac{1}{n} \sum_{\alpha=1}^{n} W_\alpha$

(4) $\tilde{\mu}_2^* = \bar{W}_2 = \frac{1}{n} \sum_{\alpha=n+1}^{2n} W_\alpha$

These are, in fact, the first four equations in the method of moments (where we set the sample moments equal to their respective population moments).

The estimated relationship between $U^*$ and $V^*$ is thus:

$$\tilde{V}_2^* = \bar{V}_2 + \left( \frac{3}{b} \right) (\bar{W}_1 - \bar{W}_2) + \left( \frac{2}{b} \right) (U_1^* - \bar{U}_1)$$

It remains, then, to find estimators of $\frac{B}{B}$ and $\frac{b}{b}$. We will hereafter denote the estimator of $\frac{B}{B}$ by $C_W$ (since it is the coefficient of the term involving $W$ scores), and the estimator of $\frac{b}{b}$ by $C_U$.

Define the following sample variances and covariances:

$$s_{U_1}^2 = \frac{1}{n-1} \sum_{\alpha=1}^{n} (U_\alpha - \bar{U}_1)^2$$

$$s_{V_2}^2 = \frac{1}{n-1} \sum_{\alpha=n+1}^{2n} (V_\alpha - \bar{V}_2)^2$$

$$s_{W_1}^2 = \frac{1}{n-1} \sum_{\alpha=1}^{n} (W_\alpha - \bar{W}_1)^2$$

(3.1)
\[ s_{w_2}^2 = \frac{1}{n-1} \sum_{\alpha=n+1}^{2n} (\bar{w}_{\alpha}-\bar{w}_2)^2 \]

\[ s_{uw_1} = \frac{1}{n-1} \sum_{\alpha=1}^{n} (\bar{u}_\alpha-\bar{u}_1)(\bar{w}_\alpha-\bar{w}_1) \]

\[ s_{vw_2} = \frac{1}{n-1} \sum_{\alpha=n+1}^{2n} (\bar{v}_\alpha-\bar{v}_2)(\bar{w}_\alpha-\bar{w}_2). \]

Note that the expectations of these quantities are as follows:

\[ \mathbb{E}(s_{u_1}^2) = b^2 + \sigma_{Z_1}^2 \]

\[ \mathbb{E}(s_{v_2}^2) = b^2 + \sigma_{Z_2}^2 \]

\[ \mathbb{E}(s_{w_1}^2) = b^2 + \sigma_{Z_1}^2 + \sigma_{\varepsilon_1}^2 \]

\[ \mathbb{E}(s_{w_2}^2) = b^2 + \sigma_{Z_2}^2 + \sigma_{\varepsilon_2}^2 \]

\[ \mathbb{E}(s_{uw_1}) = b \beta \sigma_{Z_1}^2 \]

\[ \mathbb{E}(s_{vw_2}) = b \beta \sigma_{Z_2}^2. \]

Recall that we are trying to find estimators of \( \frac{b}{\beta} \) and \( \frac{b}{\beta} \). Take the following ratios:

\[ \frac{\mathbb{E}(s_{u_1}^2)}{\mathbb{E}(s_{uw_1})} = \frac{b^2 + \sigma_{Z_1}^2}{b \beta \sigma_{Z_1}^2} = \frac{b^2}{b} \frac{1}{\beta} \frac{1}{R_{u_1}}. \]

\[ \frac{\mathbb{E}(s_{v_2}^2)}{\mathbb{E}(s_{vw_2})} = \frac{b^2 + \sigma_{Z_2}^2}{b \beta \sigma_{Z_2}^2} = \frac{b^2}{b} \frac{1}{\beta} \frac{1}{R_{v_2}}. \]

(3.3)
\[
(c) \quad \frac{\varepsilon(s_{W_1}^2)}{\varepsilon(s_{UW_1})} = \frac{\beta^2 \sigma_{Z_1}^2 + \sigma_{\varepsilon_1}^2}{b \beta \sigma_{Z_1}^2} = \frac{\beta}{b} \frac{1}{R_{W_1}}
\]

\[
(d) \quad \frac{\varepsilon(s_{W_2}^2)}{\varepsilon(s_{VW_2})} = \frac{\beta^2 \sigma_{Z_2}^2 + \sigma_{\varepsilon_2}^2}{b \beta \sigma_{Z_2}^2} = \frac{\beta}{b} \frac{1}{R_{W_2}}
\]

where \( R_{U_1}, R_{V_2}, R_{W_1}, \) and \( R_{W_2} \) are the reliabilities defined by

\[
R_{U_1} = \frac{b \sigma_{Z_1}^2}{b \sigma_{Z_1}^2 + \sigma_{\varepsilon_1}^2}
\]

\[
R_{V_2} = \frac{B \sigma_{Z_2}^2}{B \sigma_{Z_2}^2 + \sigma_{\varepsilon_2}^2}
\]

\[
R_{W_1} = \frac{\beta \sigma_{Z_1}^2}{\beta \sigma_{Z_1}^2 + \sigma_{\varepsilon_1}^2}
\]

\[
R_{W_2} = \frac{\beta \sigma_{Z_2}^2}{\beta \sigma_{Z_2}^2 + \sigma_{\varepsilon_2}^2}
\]

Thus, we can express \( \frac{B}{\beta} \) as

\[
(3.4) \quad \frac{B}{\beta} = \frac{\varepsilon(s_{V_2}^2)}{\varepsilon(s_{VW_2})} \quad \text{from (b)}
\]

and

\[
\frac{B}{\beta} = \frac{\varepsilon(s_{W_2}^2)}{\varepsilon(s_{W_2}^2)} \quad \frac{1}{R_{W_2}} \quad \text{from (d)}
\]
Similarly, we can express \( \frac{B}{b} \) as

\[
\frac{B}{b} = \frac{\varepsilon(s_{V2}^2)}{\varepsilon(s_{U1}^2)} \cdot \frac{\varepsilon(s_{UW1}^2)}{\varepsilon(s_{W2}^2)} \cdot \frac{R_{V2}}{R_{U1}} \quad \text{from (a) and (b)},
\]

\[
\frac{B}{b} = \frac{\varepsilon(s_{W1}^2)}{\varepsilon(s_{U1}^2)} \cdot \frac{\varepsilon(s_{VW2}^2)}{\varepsilon(s_{W2}^2)} \cdot \frac{R_{W1}}{R_{W2}} \quad \text{from (c) and (d)},
\]

(3.5)

\[
\frac{B}{b} = \frac{\varepsilon(s_{V2}^2)}{\varepsilon(s_{W2}^2)} \cdot \frac{\varepsilon(s_{W1}^2)}{\varepsilon(s_{U1}^2)} \cdot \frac{R_{V2}R_{W1}}{R_{U1}R_{W2}} \quad \text{from (b) and (c)},
\]

and

\[
\frac{B}{b} = \frac{\varepsilon(s_{UW1}^2)}{\varepsilon(s_{U1}^2)} \cdot \frac{\varepsilon(s_{VW2}^2)}{\varepsilon(s_{W2}^2)} \cdot \frac{1}{R_{U1}R_{W2}} \quad \text{from (a) and (d)}.
\]

Note that estimators of the quantities on the right hand sides of (3.4) and (3.5) can be obtained through the method of moments equations where we set the sample moments in (3.1) equal to their expectations in (3.2). The equations are:

(5) \( \varepsilon(s_{U1}^2) = s_{U1}^2 \)

(6) \( \varepsilon(s_{V2}^2) = s_{V2}^2 \)

(7) \( \varepsilon(s_{W1}^2) = s_{W1}^2 \)

(8) \( \varepsilon(s_{W2}^2) = s_{W2}^2 \)

(9) \( \varepsilon(s_{UW1}^2) = s_{UW1} \)

(10) \( \varepsilon(s_{VW2}^2) = s_{VW2}^2 \)
We will thus obtain the following candidate estimators:

For \( \frac{B}{b} \):

\[
C_{W_a} = \frac{\tilde{R}_V}{\tilde{R}_W} \frac{s_{V_2}^2}{s_{V_2} s_{W_2}}
\]

\[
C_{W_b} = \frac{1}{\tilde{R}_W} \frac{s_{V_2} s_{W_2}}{s_{V_2} s_{W_2}^2}
\]

\[
C_{W_c} = \left[ \frac{\tilde{R}_V}{\tilde{R}_W} \frac{s_{V_2}^2}{s_{W_2}^2} \right]^{1/2}
\]

The geometric mean of \( C_{W_a} \) and \( C_{W_b} \).

For \( \frac{B}{b} \):

\[
C_{U_a} = \frac{\tilde{R}_V}{\tilde{R}_U} \frac{s_{V_2}^2}{s_{V_2} s_{W_2}} \cdot \frac{s_{U_1}}{s_{U_1}}
\]

\[
C_{U_b} = \frac{1}{\tilde{R}_W} \frac{s_{W_1}^2}{s_{U_1} s_{W_2}} \cdot \frac{s_{V_2} s_{W_2}}{s_{V_2} s_{W_2}}
\]

\[
C_{U_c} = \frac{\tilde{R}_V}{\tilde{R}_U} \frac{s_{V_2}^2}{s_{V_2} s_{W_1}} \cdot \frac{s_{W_1}}{s_{W_1}^2}
\]

\[
C_{U_d} = \frac{1}{\tilde{R}_U} \frac{s_{U_1}^2}{s_{U_1} s_{W_2}} \cdot \frac{s_{V_2} s_{W_2}}{s_{V_2} s_{W_2}}
\]

\[
C_{U_e} = \left[ \frac{\tilde{R}_V}{\tilde{R}_U} \frac{s_{V_2}^2}{s_{W_1}^2} \cdot \frac{s_{U_1}}{s_{U_1}^2} \right]^{1/2}
\]

The geometric mean of \( C_{U_a} \) and \( C_{U_b} \); or of \( C_{U_c} \) and \( C_{U_d} \); or of \( C_{U_a}, C_{U_b}, C_{U_c} \), \( C_{U_d} \), \( C_{U_e} \).
Estimates of the reliabilities may be obtained through existing methods, such as split-half reliabilities or Kuder-Richardson-20.

Note that Levine's estimators $C_{W_1}$ and $C_{U_1}$ for the NRG-UR case in the excluded anchor test problem are identical to the estimators $C_{W_b}$ and $C_{U_b}$ proposed above.

3.3 Proposed Estimators for the Included Anchor Test Problem.

We will now obtain estimators of the parameters $\mu_{X^*1}, \mu_{Y^*1}, \frac{\sigma_{Y^*1}}{\sigma_{X^*1}}$ ($= \frac{B+\beta}{b+\beta}$) in the included anchor test problem, following the same procedure described in section 3.2.

The parameter $\mu_{X^*1}$ can be estimated directly from the data. Re-express $\mu_{Y^*1}$ as

$$\mu_{Y^*1} = \mu_{Y^*_2} + \frac{B+\beta}{b+\beta} (\mu_{W^*_1} - \mu_{W^*_2}).$$

The means will be estimated by the respective sample means:

$$\tilde{\mu}_{X^*1} = \bar{X}_1$$
$$\tilde{\mu}_{Y^*1} = \bar{Y}_1$$
$$\tilde{\mu}_{W^*_1} = \bar{W}_1$$
$$\tilde{\mu}_{W^*_2} = \bar{W}_2.$$

The estimates relationship between $X^*$ and $Y^*$ is thus

$$\tilde{Y}^*_a = \bar{Y}_2 + \frac{B+\beta}{b+\beta} (\bar{W}_1 - \bar{W}_2) + \frac{B+\beta}{b+\beta} (\bar{U}_a - \bar{U}_1).$$
It remains, then, to find estimators of \( \frac{B + \beta}{\beta} \) and \( \frac{b + \beta}{b + \beta} \). The ratios for the included anchor test problem corresponding to (3.3) are:

(a) \[
\frac{\bar{e}(s^2_{X_1})}{\bar{e}(s^2_{XW_1})} = \frac{(b + \beta)^2 \sigma^2_{Z_1} + \sigma^2_{\varepsilon_1} + \sigma^2}{\beta (b + \beta) \sigma^2_{Z_1}} = \frac{b + \beta}{\beta} \frac{1}{R_{X_1}}
\]

(b) \[
\frac{\bar{e}(s^2_{Y_2})}{\bar{e}(s^2_{YW_2})} = \frac{(B + \beta)^2 \sigma^2_{Z_2} + \sigma^2_{\varepsilon_2} + \sigma^2}{\beta (B + \beta) \sigma^2_{Z_2}} = \frac{B + \beta}{\beta} \frac{1}{R_{Y_2}}
\]

(c) \[
\frac{\bar{e}(s^2_{W_1})}{\bar{e}(s^2_{XW_1})} = \frac{\beta^2 \sigma^2_{Z_1} + \sigma^2}{\beta (b + \beta) \sigma^2_{Z_1}} = \frac{\beta}{b + \beta} \frac{1}{R_{W_1}}
\]

(d) \[
\frac{\bar{e}(s^2_{W_2})}{\bar{e}(s^2_{YW_2})} = \frac{\beta^2 \sigma^2_{Z_2} + \sigma^2}{\beta (B + \beta) \sigma^2_{Z_2}} = \frac{\beta}{B + \beta} \frac{1}{R_{W_2}}
\]

From the ratios in (3.6), and substituting sample moments for their expected values, we obtain the following candidate estimators:

For \( \frac{B + \beta}{\beta} \):

\[
C^i_{Wa} = \tilde{R}_{Y_2} \frac{s^2_{Y_2}}{s^2_{YW_2}}
\]

\[
C^i_{Wb} = \frac{1}{\tilde{R}_{W_2}} \frac{s^2_{YW_2}}{s^2_{W_2}}
\]

\[
C^i_{Wc} = \left[ \frac{\tilde{R}_{Y_2} s^2_{Y_2}}{\tilde{R}_{W_2} s^2_{W_2}} \right]^{\frac{1}{2}}, \text{ the geometric mean of } C^i_{Wa} \text{ and } C^i_{Wb}
\]

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For \( \frac{B+\beta}{\beta+\beta} \):

\[
C_{X_a} = \frac{s_{X_2}^2}{s_{X_1}} \frac{s_{Y_2}^2}{s_{YW_2}} \frac{s_{XW_1}}{s_{X_1}^2}
\]

\[
C_{X_b} = \frac{s_{W_1}}{s_{X_2}} \frac{s_{YW_2}}{s_{W_2}^2} \frac{s_{W_1}^2}{s_{XW_1}}
\]

\[
C_{X_c} = \frac{s_{Y_2}^2}{s_{YW_2}} \frac{s_{W_1}^2}{s_{XW_1}}
\]

\[
C_{X_d} = \frac{1}{s_{W_2}^2} \frac{s_{YW_2}}{s_{W_1}^2} \frac{s_{XW_1}}{s_{X_1}^2}
\]

\[
C_{X_e} = \left[ \frac{s_{Y_2}^2}{s_{X_1}^2} \frac{s_{W_1}^2}{s_{X_2}^2} \frac{s_{W_1}^2}{s_{X_1}^2} \frac{s_{XW_1}}{s_{X_1}^2} \right]^{\frac{1}{2}}
\]

The geometric mean of \( C_{X_a} \), \( C_{X_b} \), or \( C_{X_c} \), and \( C_{X_d} \) or \( C_{X_e} \), \( C_{X_4} \), \( C_{X_5} \), \( C_{X_6} \), \( C_{X_7} \), \( C_{X_8} \), and \( C_{X_9} \); or of \( C_{X_a} \), \( C_{X_b} \), \( C_{X_c} \), \( C_{X_d} \), \( C_{X_e} \), \( C_{X_4} \), \( C_{X_5} \), \( C_{X_6} \), \( C_{X_7} \), \( C_{X_8} \), \( C_{X_9} \), and \( C_{X_10} \).

3.4 Properties of the Proposed Estimators.

The estimators proposed in this chapter are necessarily consistent by the manner in which they were constructed. Each estimator is a continuous function of sample moments, which are consistent estimators of the corresponding population moments. Thus the proposed estimators are themselves consistent.

Since all of the proposed estimators are consistent, a choice among them must be made on some other basis. In chapter 5, we compare the asymptotic variances of the proposed estimators.
CHAPTER 4: THE ASYMPTOTIC BIASES OF ESTIMATORS
FOR THE SPECIAL CASES

4.1 Introduction.

In this chapter, we will obtain the probability limits of each of the estimators reviewed in chapter 2, under various special cases. We thereby determine when the estimators are consistent. There are four special cases under consideration, as outlined in section 2.1.

The estimated relationship between $U_\alpha^*$ and $V_\alpha^*$ for the excluded anchor test problem is

$$V_\alpha^*(U_\alpha^*) = V_2 + \left( \frac{\sigma_{V^*}}{\sigma_{W^*}} \right) (\overline{W}_1 - \overline{W}_2) + \left( \frac{\sigma_{V^*}}{\sigma_{U^*}} \right) (U_\alpha^* - \overline{U}_1).$$

Let $C_W$ denote $\left( \frac{\sigma_{V^*}}{\sigma_{W^*}} \right)$, the estimator of the coefficient of $\overline{W}_1 - \overline{W}_2$, and $C_U$ denote $\left( \frac{\sigma_{V^*}}{\sigma_{U^*}} \right)$, the estimator of the coefficient of $U_\alpha^* - \overline{U}_1$.

The special cases and estimators proposed by others for these cases are given in the following table:

<table>
<thead>
<tr>
<th>Equally Reliable Tests</th>
<th>Random Groups</th>
<th>Nonrandom Groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>Special Case 1</td>
<td>Lord estimators</td>
<td>Special Case 3</td>
</tr>
<tr>
<td></td>
<td>$C_{W_1}$</td>
<td>Levine estimators</td>
</tr>
<tr>
<td></td>
<td>$C_{U_1}$</td>
<td>$C_{W_3}$</td>
</tr>
<tr>
<td>Unequally Reliable Tests</td>
<td>Special Case 2</td>
<td>Special Case 4</td>
</tr>
<tr>
<td></td>
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<td></td>
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</tr>
<tr>
<td></td>
<td>$C_{U_2}$</td>
<td>$C_{U_4}$</td>
</tr>
</tbody>
</table>
where

I. \( c_{W_1} = \frac{1}{2} \left[ b_{WW_2} + b_{UW_1} \right] \left\{ \frac{s_{W_2}^2 + b_{WW_2}^2}{2} \left( \frac{s_{W_1}^2}{2} - \frac{s_{W_2}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right) \right\}^{\frac{1}{2}} \)

II. \( c_{W_2} = b_{WW_2} \)

III. \( c_{W_3} = \frac{1}{2} \left[ b_{WW_2} + b_{UW_1} \right] \left\{ \frac{s_{W_2}^2 + b_{WW_2}^2}{2} \left( \frac{s_{W_1}^2}{2} - \frac{s_{W_2}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right) \right\}^{\frac{1}{2}} \)

\( b_{WW_2}, b_{UW_1} \)
\[ C_{U_3} = \left[ \frac{s_{W_2}^2 + \frac{b_{W_2}^2}{R_{W_2}} \left( \frac{s_{W_1}^2}{2} - \frac{s_{W_2}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right)}{s_{U_1}^2 + \frac{b_{U_1}^2}{R_{W_1}} \left( \frac{s_{W_2}^2}{2} - \frac{s_{W_1}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right)} \right]^{1/2} \]

IV. \[
C_{W_3} = \frac{b_{W_2}}{R_{W_2}}
\]

\[
C_{U_4} = \frac{b_{W_2}}{b_{U_1}} \frac{\bar{R}_{W_1}}{R_{W_2}}
\]

The estimated relationship between \( X^*_\alpha \) and \( Y^*_\alpha \) for the included anchor test case is

\[ (4.2) \quad \hat{Y}^*_\alpha (X^*_\alpha) = \bar{Y}_2 + C_{W} \left( \frac{\bar{W}_1 - \bar{W}_2}{2} \right) + C_{X} (X^*_\alpha - \bar{X}_1), \]

where \( C_{W} \) is an estimator for \( \sigma_{Y*/W}^2 / \sigma_{W*}^2 \), and \( C_{X} \) for \( \sigma_{Y*/X}^2 / \sigma_{X*}^2 \).

The special cases and estimators are

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<td>Special Case 4</td>
</tr>
<tr>
<td></td>
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<td>( C_{W_4} )</td>
</tr>
<tr>
<td></td>
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<td>( C_{X_4} )</td>
</tr>
</tbody>
</table>
where

\[ C_{W_1} = \frac{1}{2} b_{YW_2} + b_{XW_1} \left( \frac{s_{W_1}^2}{2} - \frac{s_{W_2}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right) \]

\[ C_{X_1} = \left[ \frac{s_{W_1}^2 + b_{YW_2}^2 \left( \frac{s_{W_1}^2}{2} - \frac{s_{W_2}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right)}{s_{X_1}^2 + b_{XW_1}^2 \left( \frac{s_{W_1}^2}{2} - \frac{s_{W_2}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right)} \right]^{1/2} \]

\[ C_{W_2} = \frac{1}{2} b_{YW_2} \left[ 1 + \left\{ \frac{b_{XW_1}}{s_{Y_2} + b_{YW_2}^2 \left( \frac{s_{W_1}^2}{2} - \frac{s_{W_2}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right)} \right\} \right] \]

\[ C_{X_2} = \left[ \frac{s_{X_1}^2 + b_{XW_1}^2 \left( \frac{s_{W_1}^2}{2} - \frac{s_{W_2}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right)}{s_{X_1}^2 + b_{XW_1}^2 \left( \frac{s_{W_1}^2}{2} - \frac{s_{W_2}^2}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2 \right)} \right]^{1/2} \]
III. \[ C'_{W3} = \frac{1}{2} \left( \frac{\frac{s^2_{W1}}{2} - \frac{s^2_{W2}}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2}{s^2_{X1} + \frac{1}{2} \frac{s^2_{W1}}{2} - \frac{1}{2} \frac{s^2_{W2}}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2} \right)^{\frac{1}{2}} \]

\[ C'_{X3} = \left( \frac{s^2_{Y2} + \frac{1}{2} \frac{s^2_{W1}}{2} - \frac{s^2_{W2}}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2}{s^2_{X1} + \frac{1}{2} \frac{s^2_{W1}}{2} - \frac{1}{2} \frac{s^2_{W2}}{2} + \frac{1}{4} (\bar{W}_1 - \bar{W}_2)^2} \right)^{\frac{1}{2}} \]

IV. \[ C'_{W4} = \frac{1}{b_{WY2}} \]

\[ C'_{X4} = \frac{b_{WX1}}{b_{WY2}} \]

The probability limits of each of the estimators will now be investigated in turn, under the four different sets of assumptions above. The assumption of congeneric tests will underlie all calculations.

Because all of the estimators are continuous functions of the sample moments, their probability limits are easily obtained by substituting population moments for the respective sample moments.

In section 4.2, the probability limits of the estimators in the excluded anchor test problem are obtained and discussed. It is found that some of the proposed estimators are not consistent, even in those specific cases for which the estimators are obtained. In section 4.4,
the size of the asymptotic biases in the inconsistently estimated scores are examined numerically for some special cases.

Sections 4.3 and 4.5 give corresponding results for the included anchor test problem.

4.2 Probability Limits: Excluded Anchor Test Problem.

For this case, the assumption of random groups means that $\mu_{W_1} = \mu_{W_2}$, $\sigma^2_{W_1} = \sigma^2_{W_2}$, and $R_{W_1} = R_{W_2}$. The assumption of equal reliability means that $R_{\pi_1} = R_{\pi_2}$. Recall from (2.5) that the assumption of congeneric tests implies that

$$R_{\pi_1} R_{\pi_2} = \rho_{\pi W_1}^2$$

(2.5)

$$R_{\pi_2} R_{\pi_2} = \rho_{\pi W_2}^2$$

It is expected that the estimators would perform best (with respect to consistency) in those situations for which they were obtained. Therefore, the probability limit of each estimator is first obtained under the appropriate set of assumptions or model for which it was derived. It is also important to assess the behavior of each estimator when the special assumptions do not hold. The most extreme situation is when neither the assumption of equal reliability nor that of random groups holds (special case 4). This is the next situation for which the probability limits are obtained. The probability limits in the remaining two situations are then calculated.
I. Random groups, equally reliable tests.

Under RG-ER, the limits are

\[
\lim_{n \to \infty} C_{W_1} = \frac{\sigma_{V*}}{\sigma_{U*}} \cdot R_{W_2}
\]

\[
\lim_{n \to \infty} C_{U_1} = \frac{\sigma_{V*}}{\sigma_{U*}}
\]

It is seen that even under the assumptions of equal reliability and random groups, the situation for which the estimators were obtained, \(C_{W_1}\) is not a consistent estimate, because \(\sigma_{V*}/\sigma_{W*}\). \(C_{W_1}\) underestimates \(\sigma_{V*}/\sigma_{W*}\) by a factor \(R_{W_2}\). This would result in underestimating the score \(V\) (4.1) if \((\overline{W}_1 - \overline{W}_2) > 0\), and overestimating it if \((\overline{W}_1 - \overline{W}_2) < 0\). Since under the RG assumption, \(\overline{|W_1 - W_2|} \to 0\), the asymptotic bias from the inconsistency of \(C_{W_1}\) also vanishes, since \(C_{W_1} \overline{|W_1 - W_2|} \to 0\).

Under the extreme case of NRG-UR the probability limits are

\[
\lim_{n \to \infty} C_{W_1} = \frac{1}{2} \left[ \sqrt{R_{V_2} \frac{\sigma_{V_2}}{\sigma_{W_2}}} + \sqrt{R_{U_1} \frac{\sigma_{U_1}}{\sigma_{W_1}}} \right]
\]

\[
\left\{ \frac{\sigma_{V_2}^2 + R_{V_2} \frac{\sigma_{V_2}}{\sigma_{W_2}} \left( \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right)}{\sigma_{U_1}^2 + R_{U_1} \frac{\sigma_{U_1}}{\sigma_{W_1}} \left( \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right)} \right\}^{1/2}
\]
\[
\frac{\sigma_{v*}}{\sigma_{w*}} = \frac{R_{w_2}}{2} \left\{ 1 + \frac{\sigma_{W_2}^2}{\frac{R_{v_2} R_{w_2}}{2}} - \frac{\sigma_{W_1}^2}{\frac{R_{u_1} R_{w_1}}{2}} + \frac{1}{4} (\mu_{w_1} - \mu_{w_2})^2 \right\}^{\frac{1}{2}}
\]

\[
\lim_{n \to \infty} C_{U_1} = \frac{\sigma_{v_1}^2 + \sigma_{w_1}^2}{\frac{R_{v_1} R_{w_1}}{2} \sigma_{w_2}^2} \left( \frac{\sigma_{W_1}^2}{\frac{R_{w_1}}{2}} - \frac{\sigma_{W_2}^2}{\frac{R_{w_2}}{2}} + \frac{1}{4} (\mu_{w_1} - \mu_{w_2})^2 \right) \left( \frac{\sigma_{U_1}^2 + \sigma_{U_1}^2}{\frac{R_{w_1}}{2} \sigma_{w_1}^2} \right)^{\frac{1}{2}}
\]

\[
= \frac{\sigma_{v*}}{\sigma_{w*}} \left[ \frac{\sigma_{W_1}^2}{\frac{R_{v_2} R_{w_2}}{2}} + \frac{1}{4} (\mu_{w_1} - \mu_{w_2})^2 \right]^{\frac{1}{2}}
\]

The asymptotic biases here can lead to either underestimates or overestimates of the V-score, depending on the relative values of the parameters.

If either the assumption of random groups or that of equally reliable tests does not hold, then again neither \( C_{W_1} \) nor \( C_{U_1} \) is a consistent estimate of the respective parameters.
Under RG-UR, the limits are

\[
\lim_{n \to \infty} C_{W_1} = \frac{\sigma_{W^*}}{\sigma_{W^*}} \frac{R_{W_2}}{2} \left[ 1 + \left( \frac{R_{U_1}}{R_{V_2}} \right)^{\frac{1}{2}} \right]
\]

\[
\lim_{n \to \infty} C_{U_1} = \frac{\sigma_{V^*}}{\sigma_{U^*}} \left[ \frac{R_{U_1}}{R_{V_2}} \right]^{\frac{1}{2}}
\]

Under NRG-ER, the limits are

\[
\lim_{n \to \infty} C_{W_1} = \frac{\sigma_{W^*}}{\sigma_{W^*}} \frac{R_{W_2}}{2} \left[ 1 + \frac{\sigma^2_{W_2}}{R_{V_2} R_{W_2}} + \frac{\sigma^2_{W_1}}{2} - \frac{\sigma^2_{W_2}}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right]^{\frac{1}{2}}
\]

\[
\lim_{n \to \infty} C_{U_1} = \frac{\sigma_{U^*}}{\sigma_{U^*}} \left[ \frac{R_{U_1}}{R_{V_2} R_{W_2}} + \frac{R_{W_2}}{R_{W_1}} \left( \frac{\sigma^2_{W_1}}{2} - \frac{\sigma^2_{W_2}}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right) \right]^{\frac{1}{2}}
\]

II. Random groups, unequally reliable tests.

These estimators were obtained under special case 2, random groups and unequally reliable tests. For that situation, the limits are
\[ \lim_{n \to \infty} C_{W_2} = \frac{\sigma_{Y^*}}{\sigma_{W^*}} R_{W_2} \]

\[ \lim_{n \to \infty} C_{U_2} = \frac{\sigma_{Y^*}}{\sigma_{U^*}} . \]

It is seen that \( C_{U_2} \) is a consistent estimator of \( \frac{\sigma_{Y^*}}{\sigma_{U^*}} \) but \( C_{W_2} \) underestimates \( \frac{\sigma_{Y^*}}{\sigma_{W^*}} \) by a factor \( R_{W_2} \).

In none of the situations do the estimators lead to consistently estimated V-scores, except when \( \mu_{W_1} - \mu_{W_2} = 0 \).

Under NRG-ER or NRG-UR, the limits are

\[ \lim_{n \to \infty} C_{W_2} = \frac{\sigma_{Y^*}}{\sigma_{W^*}} R_{W_2} \]

\[ \lim_{n \to \infty} C_{U_2} = \frac{\sigma_{Y^*}}{\sigma_{U^*}} \frac{R_{W_2}}{R_{W_1}} . \]

Under RG-ER, the limits are as for under RG-UR.

III. Nonrandom groups, equally reliable tests.

These estimators were obtained under special case 3, nonrandom groups and equally reliable tests. For that situation, the limits are
\[
\text{plim } C_{W_3} = \frac{\sigma_{V*}}{\sigma_{W*}} \frac{1}{2} \left[ 1 + \left\{ \frac{R_{W_2} \sigma_{W_2}^2}{R_{V_2}} + \frac{2}{2} - \frac{2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right\} \right]^{\frac{1}{2}}.
\]

\[
\text{plim } C_{U_3} = \frac{\sigma_{V*}}{\sigma_{U*}} \left[ \frac{R_{W_2} \sigma_{W_2}^2}{R_{V_2}} + \frac{2}{2} - \frac{2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right]^{\frac{1}{2}}.
\]

It is clear that these estimators are inconsistent, except for very special values of the parameters which appear in the brackets.

Under the extreme case of NRG-UR, the limits are

\[
\text{plim } C_{W_3} = \frac{\sigma_{V*}}{\sigma_{W*}} \frac{1}{2} \left[ 1 + \left\{ \frac{R_{W_2} \sigma_{W_2}^2}{R_{V_2}} + \frac{2}{2} - \frac{2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right\} \right]^{\frac{1}{2}}.
\]

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\[
\begin{align*}
\text{plim}_{n \to \infty} C_{U_3} &= \frac{\sigma_{Y*}}{\sigma_{U*}} \left[ \frac{R_{W_2} \sigma_{W_2}^2 + \sigma_{W_1}^2 - \frac{\sigma_{W_1}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2}{R_{V_2}^2} \right] \\
&= \frac{R_{W_1} \sigma_{W_1}^2 + \sigma_{W_1}^2 - \frac{\sigma_{W_1}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2}{R_{U_1}^2} \\
&\quad \times \left[ \frac{R_{U_1}}{R_{V_2}} \right]^{\frac{1}{2}}
\end{align*}
\]

Under RG-ER, the estimators are consistent.

Under RG-UR, the limits are

\[
\text{plim}_{n \to \infty} C_{W_3} = \frac{\sigma_{Y*}}{\sigma_{W*}} \left( 1 + \left[ \frac{R_{U_1}}{R_{V_2}} \right]^{\frac{1}{2}} \right)
\]

\[
\text{plim}_{n \to \infty} C_{U_3} = \frac{\sigma_{Y*}}{\sigma_{U*}} \left[ \frac{R_{U_1}}{R_{V_2}} \right]^{\frac{1}{2}}
\]

These are overestimates if \( R_{U_1} > R_{V_2} \), and underestimates if \( R_{U_1} < R_{V_2} \).

IV. Nonrandom Groups, Unequally Reliable Tests.

The estimators are consistent under all of the special cases. Note that \( C_{W_4} \) is identical to \( C_{W_b} \) and \( C_{U_4} \) to \( C_{U_b} \), and \( C_{W_b} \) and \( C_{U_b} \) were noted in chapter 3 to be consistent.

The effects of these inconsistently estimated coefficients on the estimated V-score were examined numerically for a few special cases. The results are presented and discussed in section 4.4.
4.3 Probability Limits: Included Anchor Test Problem

We will now investigate the consistency of the estimators in the case of the included anchor test.

Again the assumption of random groups means that \( \sigma^2_{w_1} = \sigma^2_{w_2} \), \( \mu_{w_1} = \mu_{w_2} \), and \( R_{w_1} = R_{w_2} \). The assumption of equal reliability in this case means that \( R_{x_1} = R_{y_2} \). There is no simple relationship among the reliabilities and correlation coefficients (as there was in the excluded anchor test case), although it is still assumed that \( X, Y \) and \( W \) are congeneric tests. The relationship among the reliabilities and correlations is, from (2.6),

\[
\rho^2_{xw_1} = \frac{\sigma^2_{x_1} \sigma^2_{w_1}}{\text{cov}^2(x_{x_1}^{*},w^{*})} \frac{\text{cov}^2(x_{x_1}^{*},w^{*})}{\text{cov}^2(x+w,w)}
\]

\[
= \frac{R_{x_1} R_{w_1}}{\left( \frac{\beta (b+\beta) \sigma^2_{z}}{\beta (b+\beta) \sigma^2_{z} + \sigma^2_{e}} \right)^2}
\]

\[
= \frac{R_{x_1} R_{w_1}}{\left( \frac{R_{w_1} (b+\beta)}{b R_{w_1} + \beta} \right)^2}
\]

Let \( \theta_1 = \frac{R_{w_1} (b+\beta)}{b R_{w_1} + \beta} \) and \( \theta_2 = \frac{R_{w_2} (b+\beta)}{b R_{w_2} + \beta} \). Note that for \( 0 < R_{w_1} < 1 \), \( \theta_1 \) is always less than 1. Then the relationships are

\[
\rho^2_{xw_1} = \frac{R_{x_1} R_{w_1} \theta^2_1}{\text{cov}^2(x_{x_1}^{*},w^{*})} \text{ and }
\]

\[
\rho^2_{yw_2} = \frac{R_{y_2} R_{w_2} \theta^2_2}{\text{cov}^2(y_{y_2}^{*},w^{*})}
\]

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The probability limits of the estimators are found below.

I. Random Groups, Equally Reliable Tests.

The estimators were obtained under special case 1. The limits in this case are

\[
\lim_{n \to \infty} C_{W_1}^* = \frac{\sigma_{Y*}}{\sigma_{W*}} \frac{R_{W_2} \theta_2}{2} \left( 1 + \frac{\theta_1}{\theta_2} \right)
\]

\[
\lim_{n \to \infty} C_{X_1} = \frac{\sigma_{Y*}}{\sigma_{X*}}.
\]

In this case, \( C_{X_1} \) is a consistent estimator of \( \frac{\sigma_{Y*}}{\sigma_{X*}} \), but \( C_{W_1}^* \) is not a consistent estimator of \( \frac{\sigma_{Y*}}{\sigma_{W*}} \).

The estimators are not consistent under any of the other sets of assumptions.

Under NRG-UR, the limits are

\[
\lim_{n \to \infty} C_{W_1}^* = \frac{\sigma_{Y*}}{\sigma_{W*}} \frac{R_{W_2} \theta_2}{2}
\cdot \left[ 1 + \left\{ \left( \frac{\sigma_{W_2}^2}{2} + \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right) \right\}^{\frac{1}{2}} \right]
\]
\[
\lim_{n \to \infty} C_{\chi_1} = \frac{\sigma_{Y*}^2}{\sigma_{\chi_1}^*} \left[ \frac{R_{W_2} \theta_2^2}{2} \left( \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right) \right]^{1/2}.
\]

Under RG-UR, the limits are

\[
\lim_{n \to \infty} C_{\chi_1} = \frac{\sigma_{Y*}^2}{\sigma_{\chi_1}^*} \left( 1 + \frac{\theta_1}{\theta_2} \right) \frac{\sqrt{R_{X_1}}}{\sqrt{R_{Y_2}}}.
\]

\[
\lim_{n \to \infty} C_{\chi_1} = \frac{\sigma_{Y*}^2}{\sigma_{\chi_1}^*} \frac{\sqrt{R_{X_1}}}{\sqrt{R_{Y_2}}}.
\]

Under NRG-ER, the limits are

\[
\lim_{n \to \infty} C_{\chi_1} = \frac{\sigma_{Y*}^2}{\sigma_{\chi_1}^*} \left[ 1 + \frac{\left( \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right)}{\left( \frac{R_{Y_2} R_{W_2} \theta_2^2}{2} + \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right)} \right]^{1/2}.
\]
\[
\lim_{n \to \infty} C_{X_1} = \frac{\sigma_{Y*}}{\sigma_{X*}} \left[ \begin{array}{c}
\frac{R_{W_2}}{R_{Y_2}} + R_{W_2}^2 \sigma_{W_2}^2 \left( \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right) \\
\frac{R_{W_1}}{R_{Y_2}} + R_{W_1}^2 \sigma_{W_1}^2 \left( \frac{\sigma_{W_2}^2}{2} - \frac{\sigma_{W_1}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right)
\end{array} \right]^{1/2}.
\]

II. Random Groups, Unequally Reliable Tests.

The estimators were obtained under special case 2, random groups and unequal reliability. In this case the limits are

\[
\lim_{n \to \infty} C_{W_2}^* = \frac{\sigma_{Y*}}{\sigma_{W*}} \frac{R_{W_2}}{2} \left( \frac{\sqrt{R_{X_1}}}{R_{Y_2}} \frac{\theta_1}{\theta_2} \right) \left( 1 + \frac{\sqrt{R_{X_1}}}{R_{Y_2}} \frac{\theta_1}{\theta_2} \right)
\]

\[
\lim_{n \to \infty} C_{X_2} = \frac{\sigma_{Y*}}{\sigma_{X*}} \frac{R_{X_1}}{R_{Y_2}} \frac{\theta_1}{\theta_2}.
\]

These are clearly not consistent estimators.

Under NRG-UR, the limits are
\[
\lim_{n \to \infty} C_{W_2} = \frac{\sigma_{Y*}}{\sigma_{W*}} \frac{R_{W_2} \theta_2}{\theta_2} \left[ 1 + \left( \frac{R_{X_1} R_{W_1}}{R_{X_2} R_{W_2}} \right)^{\frac{1}{2}} \frac{\theta_1}{\theta_2} \frac{\sigma_{W_1}}{\sigma_{W_2}} \frac{\sigma_{Y_2}}{\sigma_{Y_1}} \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right]
\]

\[
\lim_{n \to \infty} C_{X_2} = \frac{\sigma_{Y*}}{\sigma_{X*}} \frac{R_{W_2} \theta_2}{\theta_2} \left[ \frac{\sigma_{W_2}^2}{R_{W_2} R_{X_2} \theta_2} + \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right]
\]

Under RG-ER, the limits reduce to

\[
\lim_{n \to \infty} C_{W_2} = \frac{\sigma_{Y*}}{\sigma_{W*}} \frac{R_{W_2} \theta_2}{\theta_2} \left( 1 + \frac{\theta_1}{\theta_2} \right)
\]

\[
\lim_{n \to \infty} C_{X_2} = \frac{\sigma_{Y*}}{\sigma_{X*}} \frac{\theta_1}{\theta_2}
\]

Under NRG-ER, the limits are
\[
\lim_{n \to \infty} C_{W_2}^* = \frac{\sigma_{W_2}^*}{\sigma_{W_2}^*} \frac{R_{W_2} \theta_2}{2} \left[ 1 + \frac{\sqrt{R_{W_1} \theta_1}}{\sqrt{R_{W_2} \theta_2}} \frac{\sigma_{W_1}}{\sigma_{W_2}} \right] \left\{ \frac{\sigma_{W_2}^2 + R_{W_2} \theta_2}{2} \left( \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right) \right\} \left( \frac{\sigma_{W_2}^2 + R_{W_2} \theta_2}{2} \left( \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2 \right) \right) \right.
\]

\[
\lim_{n \to \infty} C_{X_2} = \frac{\sigma_{X_1}}{\sigma_{X_2}^*} \frac{R_{W_2} \theta_2}{2} \left[ 1 + \frac{R_{W_1} R_{Y_2} \theta_2}{2} \right] \left\{ \frac{\sigma_{W_2}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2}{\sigma_{W_1}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2} \right\} \left[ \frac{R_{W_1} R_{Y_2} \theta_2}{2} \right] \left\{ \frac{\sigma_{W_2}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2}{\sigma_{W_1}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2} \right\} \right. \right.
\]

III. Nonrandom Groups, Equally Reliable Tests.

The estimators were obtained under special case 3, nonrandom groups and equal reliability. The limits in this case are

\[
\lim_{n \to \infty} C_{W_3}^* = \frac{\sigma_{W_2}^*}{\sigma_{W_2}^*} \frac{1}{2} \frac{R_Y \theta_2}{2} \left[ 1 + \left\{ \frac{R_{W_2} R_{Y_2} \theta_2^2 \sigma_{W_2}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2}{\sigma_{W_2}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2} \right\} \left[ \frac{R_{W_2} R_{Y_2} \theta_2^2 \sigma_{W_2}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2}{\sigma_{W_2}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2} \right] \right\} \right.
\]

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\[
\lim_{n \to \infty} C_{x_j} = \frac{\sigma_{Y*}}{\sigma_{X*}} \left[ \frac{R_{Y_2} R_{W_2} \sigma_{W_2}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2}{R_{Y_2} R_{W_1} \sigma_{W_1}^2 + \frac{\sigma_{W_2}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2} \right]^{\frac{1}{2}}
\]

Even in this case for which the estimators were obtained, they are not consistent.

Under NRG-UR, the limits are

\[
\lim_{n \to \infty} C'_{W_j} = \frac{\sigma_{Y*}}{\sigma_{W*}} \left\{ \frac{1}{2} \frac{1}{R_{Y_2}} \right\} \left[ \frac{R_{Y_2} R_{W_2} \sigma_{W_2}^2 + \frac{\sigma_{W_1}^2}{2} - \frac{\sigma_{W_2}^2}{2} + \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2}{R_{X_1} R_{W_1} \sigma_{W_1}^2 + \frac{\sigma_{W_2}^2}{2} - \frac{1}{4} (\mu_{W_1} - \mu_{W_2})^2} \right]^{\frac{1}{2}}
\]

Under RG-ER, the limits are

\[
\lim_{n \to \infty} C'_{x_j} = \frac{\sigma_{Y*}}{\sigma_{W*}} \frac{1}{R_{X_1}} \frac{1}{R_{Y_2}}
\]

62
\[ \text{plim } C_{X_{\infty}} = \frac{\sigma_Y^*}{\sigma_X^*}. \]

Under RG-UR, the limits are

\[ \text{plim } C_{W_{\infty}} = \frac{\sigma_Y^*}{\sigma_{W^*}} \left( \frac{1}{2} \right) \left( \frac{1}{R_Y \theta_2} \frac{1}{R_Y \theta_2} \right)^{1/2} \left( 1 + \left( \frac{R_Y \theta_2}{R_X \theta_2} \right)^{1/2} \right) \]

\[ \text{plim } C_{X_{\infty}} = \frac{\sigma_Y^*}{\sigma_X^*} \left( \frac{R_X \theta_1}{R_Y \theta_2} \right)^{1/2}. \]

IV. Nonrandom Groups, Unequally Reliable Tests.

The estimators were obtained under special case 4, nonrandom groups and unequal reliability. Under that special case or under RG-UR, the limits are

\[ \text{plim } C_{W_{\infty}} = \frac{\sigma_Y^*}{\sigma_{W^*}} \frac{1}{R_Y \theta_2} \]

\[ \text{plim } C_{X_{\infty}} = \frac{\sigma_Y^*}{\sigma_X^*} \frac{R_X \theta_1}{R_Y \theta_2}. \]

Under either RG-ER or NRG-ER, the limits are

\[ \text{plim } C_{W_{\infty}} = \frac{\sigma_Y^*}{\sigma_{W^*}} \frac{1}{R_Y \theta_2}. \]
\[
\text{plim } C_n X_{\ell_1} = \frac{\sigma_{Y^*}}{\sigma_{X^*}} \frac{\theta_1}{\theta_2}.
\]

It is seen that under none of these sets of assumptions are the
estimators consistent.

We have seen that even under the special assumptions for which
they were designed, most of these estimates do not consistently esti-
mate \( \sigma_{Y^*}/\sigma_{W^*} \) or \( \sigma_{Y^*}/\sigma_{X^*} \). The effects of this on the estimated score \( \tilde{Y}_{\alpha} \) are examined numerically for a few special cases and are discussed
in section 4.5.

4.4 Numerical Examples in Excluded Anchor Test Problem.

It was seen in section 4.2 that most of the proposed estimators
are asymptotically biased, even under those specific assumptions for
which the estimators were obtained. In order to assess how seriously
these asymptotically biased coefficients affect the estimated scores,
some numerical examples are worked out.

The general form of the estimated relationship between \( U_{\alpha}^* \) and
\( Y_{\alpha}^* \) is

\[
\tilde{Y}_{\alpha}^*(U_{\alpha}^*) = \tilde{V}_2 + \left( \frac{\tilde{\sigma}_{Y^*}}{\tilde{\sigma}_{W^*}} \right) (\tilde{W}_1 - \tilde{W}_2) + \left( \frac{\tilde{\sigma}_{Y^*}}{\tilde{\sigma}_{U^*}} \right) (U_{\alpha}^* - \tilde{U}_1).
\]

The probability limit of each estimator of \( \frac{\sigma_{Y^*}}{\sigma_{W^*}} \) can be expressed
as \( \frac{\sigma_{Y^*}}{\sigma_{W^*}} \cdot f_{W_j} \), where \( \frac{\sigma_{Y^*}}{\sigma_{W^*}} (f_{W_j} - 1) \) is the asymptotic bias of the \( j \)th
estimator. Likewise, the probability limit of each estimator of \( \frac{\sigma_{Y^*}}{\sigma_{U^*}} \)
can be expressed as \( \frac{\sigma_{Y^*}}{\sigma_{U^*}} \cdot f_{U_j} \).
To simplify the problem somewhat, we chose to study cases where 
\( \sigma_{U^*} = \sigma_{V^*} \) and \( \sigma_{W^*} = c \sigma_{V^*} \), with \( c \) an assigned constant between 0 and 1.

Thus, \( \frac{\sigma_{V^*}}{\sigma_{W^*}} = \frac{1}{c} \) and \( \frac{\sigma_{V^*}}{\sigma_{U^*}} = \frac{1}{1} \). The probability limit of the \( j \)th estimated score is then

\[
\bar{y}_j(\alpha^*) = \mu_{\bar{y}_2} + \frac{1}{c} f_{W_j}(\mu_{\bar{w}_1} - \mu_{\bar{w}_2}) + f_{U_j}(\alpha^* - \mu_{\bar{u}_1}).
\]

Since \( \mu_{\bar{y}_2} \) is present in each estimated score, it was subtracted from both sides and we calculated

\[
\bar{y}_j(\alpha^*) - \mu_{\bar{y}_2} = \frac{1}{c} f_{W_j}(\mu_{\bar{w}_1} - \mu_{\bar{w}_2}) + f_{U_j}(\alpha^* - \mu_{\bar{u}_1}).
\]

The numerical values which were used were:

\( \mu_{\bar{w}_1} - \mu_{\bar{w}_2} \): -100, -50, -25, 0, 25, 50, 100
\( U_0^* - \mu_{\bar{u}_1} \): -150, -75, 0, 75, 150
Variance of \( W_1 \): 10000, 22500
\( c \): .40, .50, .60
Reliabilities: .75, .85, .95
(Instead of considering all $3^4 = 81$ ways of assigning the reliabilities, a Graeco-Latin square was picked to ease the computing.

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(The cells marked with asterisks are represented in Table 4.1 and Table 4.3.)

The asymptotic bias in the estimated $V$-score can be expressed for the $j$-th estimate as

$$b_j = \frac{1}{c} (f_{W_j} \cdot 1)(\mu_{W_1} - \mu_{W_2}) + (f_{U_j} \cdot 1)(\mu_{U} - \mu_{U_1})$$

Note that $f_{W_j} = 1$ and $f_{U_j} = 1$ imply that the estimators are consistent.

The biases below are for the special case where $c = 0.40$ and $\sigma_{W_1}^2 = 10000$.

Table 4.1 summarizes the asymptotic biases in a few cases. The asymptotic bias, $b_j$, of Levine's estimator (proposed for the nonrandom groups, unequally reliable case) is always 0 and hence is omitted from Table 4.1.
Table 4.2 summarizes the values of $f^j_{W_i}$ and $f^j_{U_i}$. Again, the values $f^j_{W_i}$ and $f^j_{U_i}$ are always 1 for Levine's estimator (proposed for NRG-UR case), and hence are omitted from Table 4.2.

We first look at $b_1$, the asymptotic bias in the estimator for the RG-ER case. Recall that this estimator was obtained under the assumptions of equally reliable tests and random groups.

$f^j_{U_1}$ is consistently estimated when $R^j_{W_1} = R^j_{W_2}$ and $R^j_{U} = R^j_{V}$. However, in this situation, $f^j_{W_1} = 0$, so that $f^j_{W_1}$ is always underestimated. This will lead to overestimating $V$ when $\mu_{W_1} - \mu_{W_2} < 0$, and underestimating $V$ when $\mu_{W_1} - \mu_{W_2} > 0$. $\tilde{V}_1$ is consistently estimated only in the very special case where $\mu_{W_1} - \mu_{W_2} = 0$, $R^j_{W_1} = R^j_{W_2}$, and $R^j_{U} = R^j_{V}$. In the special case where $R^j_{W_1} = R^j_{W_2} = .95$, and $R^j_{U} = R^j_{V} = .75$, the $\tilde{V}_1$ score is overestimated by 62 points when $\mu_{W_1} - \mu_{W_2} = -100$ and underestimated by 62 points when $\mu_{W_1} - \mu_{W_2} = +100$ (regardless of the value of $U^k_{\alpha} - \mu_{U_1}$). The $\tilde{V}_1$ score is overestimated by 31 points when $\mu_{W_1} - \mu_{W_2} = -50$, and underestimated by the same amount when $\mu_{W_1} - \mu_{W_2} = 50$.

Both $f^j_{U_1}$ and $f^j_{W_1}$ are underestimated whenever $R^j_{W_2} < R^j_{W_1}$. This would lead to overestimating a $V$ score if both $\mu_{W_1} - \mu_{W_2} < 0$ and $U^k_{\alpha} - \mu_{U_1} < 0$. The $V$ score would be underestimated if both $\mu_{W_1} - \mu_{W_2} > 0$ and $U^k_{\alpha} - \mu_{U_1} > 0$. Even if $\mu_{W_1} - \mu_{W_2} = 0$ (which means that the two groups are of equal ability) and $R^j_{U} = R^j_{V} = .75$ or .85 (the equally reliable case), $\tilde{V}_1$ is overestimated by as much as 41 points if $U^k_{\alpha} - \mu_{U_1} = -150$ (i.e., for an examinee whose ability is below the group average) and underestimated by 41 points if $U^k_{\alpha} - \mu_{U_1} = +150$ (i.e., for an examinee who is above the group average).
Table 4.1

Asymptotic biases in the excluded anchor test problem

\[ b_j = \frac{1}{c} (f_{W_j} - 1)(\mu_{W_1} - \mu_{W_2}) + (f_{U_j} - 1)(\mu_{U_1}^* - \mu_{U_2}^*) \]

Let \( c = .40 \), \( \sigma_{W_1}^2 = 10,000 \)

\[ b_1 = \text{asymptotic bias in (Lord's) estimator (proposed for the random groups, equally reliable case)} \]
\[ b_2 = \text{asymptotic bias in (Levine's) estimator (proposed for the random groups, unequally reliable case)} \]
\[ b_3 = \text{asymptotic bias in (Levine's) estimator (proposed for the nonrandom groups, equally reliable case)} \]

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68
Values of $f_{W_j}$ and $f_{U_j}$, where

$$b_j = \frac{1}{c} \left( f_{W_j} - 1 \right) \left( \mu_{W_1} - \mu_{W_2} \right) + \left( f_{U_j} - 1 \right) \left( \mu_{U_1} - \mu_{U_2} \right).$$

$c = .40, \quad \sigma_{W_1}^2 = 10000, \quad \mu_{W_1} - \mu_{W_2} = -100$

$b_1 = \text{asymptotic bias in (Lord's) estimator (proposed for the random groups, equally reliable case)}$

$b_2 = \text{asymptotic bias in (Levine's) estimator (proposed for the random groups, unequally reliable case)}$

$b_3 = \text{asymptotic bias in (Levine's) estimator (proposed for the nonrandom groups, equally reliable case)}$

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Of the special cases that were considered, the largest asymptotic bias occurred when \( R_{w_1} = .95, R_{w_2} = .75, \) and \( R_U = R_V = .85. \) In this case, \( \hat{V}_1 \) overestimates \( V \) by 107 points in the case where

\[
\mu_{w_1} - \mu_{w_2} = -100 \quad \text{and} \quad U^*_{\alpha} - U_{\alpha} = -150,
\]

and underestimates it by 107 points when \( \mu_{w_1} - \mu_{w_2} = +100 \) and \( U^*_{\alpha} - U_{\alpha} = +150. \)

If \( f_{u_1} \) is overestimated whenever \( R_{w_1} = R_{w_2} = .85 \) or \(.95, \) or when \( R_{w_1} < R_{w_2}. \) In these cases, \( f_{w_1} \) is still underestimated, except when

\[
R_{w_1} = R_{w_2} = .95. \]

In this special case \( f_{w_1} \) is almost consistent (ranges from .999 to 1.010). The result is that a V-score is underestimated by as much as 19 points for \( U^*_{\alpha} - U_{\alpha} < 0, \) and overestimated for \( U^*_{\alpha} - U_{\alpha} > 0. \)

When \( R_{w_1} = .75 \) and \( R_{w_2} = .95, \) there is serious underestimation of the V-score when \( U^*_{\alpha} - U_{\alpha} < 0 \) (by as much as 56 points) and serious overestimations when \( U^*_{\alpha} - U_{\alpha} > 0 \) (again by as much as 56 points).

It is seen, then, that \( \hat{V}_1 \) is consistent only in the very special case where \( R_U = R_V, \) \( R_{w_1} = R_{w_2}, \) and \( \mu_{w_1} - \mu_{w_2} = 0. \) If any of these assumptions does not hold, then serious biases could result.

We now look at the asymptotic bias \( b_2, \) which behaves very much like \( b_1. \) \( f_{w_2} = R_{w_2}, \) so \( f_{w_2} \) is always underestimated. \( f_{u_2} = \frac{R_{w_2}}{R_{w_1}}, \) so \( f_{u_2} \) is overestimated when \( R_{w_2} > R_{w_1}, \) underestimated when \( R_{w_2} < R_{w_1}, \) and consistently estimated when \( R_{w_1} = R_{w_2}. \)

\( \hat{V}_2 \) was obtained under the assumptions of random groups and unequally reliable tests. For this special case \( (R_{w_1} = R_{w_2}, \mu_{w_1} - \mu_{w_2} = 0, \) \( R_U \neq R_V), \) \( \hat{V}_2 \) consistently estimates \( V. \) If, however, either \( R_{w_1} \neq R_{w_2} \)
or $\mu_{W_1} - \mu_{W_2} \neq 0$, then the estimated score contains asymptotic biases. For example, if $R_{W_1} = .75$ and $R_{W_2} = .95$, then even if $\mu_{W_1} - \mu_{W_2} = 0$, $\tilde{V}_2$ underestimates $V$ by 40 points.

Among the special cases that were considered, the most serious overestimate occurred when $R_{W_1} = .95$, $R_{W_2} = .75$ and $\mu_{W_1} - \mu_{W_2} = -100$ (by 94 points). The most serious underestimate occurred when $R_{W_1} = .95$, $R_{W_2} = .75$ and $\mu_{W_1} - \mu_{W_2} = +100$ (again by 94 points).

It is seen, then, that $\tilde{V}_2$ is a consistent estimator for the special case for which it was obtained. Any departure from these assumptions would result in an inconsistently estimated score.

We look now at $b_3$. $V$ is consistently estimated by $\tilde{V}_3$ when $R_{W_1} = R_{W_2}$ and $R_U = R_V$; or when $R_{W_1} = R_U$ and $R_{W_2} = R_V$.

$f_{W_3}$ and $f_{W_3}$ are both underestimated when $R_{W_1} > R_{W_2}$ and $R_U < R_V$. This would tend to result in overestimation of $V$ when $U_{U_1}^{*} - \mu_{U_1} < 0$ (by as much as 38 points), and underestimation of $V$ when $U_{U_1}^{*} - \mu_{U_1} > 0$ (again by as much as 38 points). When $U_{U_1}^{*} - \mu_{U_1} = 0$, the scores are overestimated when $\mu_{W_1} - \mu_{W_2} < 0$, and underestimated when $\mu_{W_1} - \mu_{W_2} > 0$.

$f_{U_3}$ and $f_{W_3}$ are both overestimated when $R_{W_1} < R_{W_2}$, or when $R_{W_1} = R_{W_2}$ and $R_U > R_V$. In these cases, the scores tend to be underestimated when $U_{U_1}^{*} - \mu_{U_1} < 0$ (by as much as 30 points), and overestimated when $U_{U_1}^{*} - \mu_{U_1} > 0$ (again by as much as 30 points).

Recall that $\tilde{V}_3$ was obtained for the case where the assumption of random groups does not hold. However, it appears that the estimation of $V$ is seriously affected by this very case.
In summary, \( \tilde{V}_1 \) and \( \tilde{V}_2 \) are appropriate under their respective assumptions. Their use when the assumptions do not hold would lead to inconsistently estimated scores. \( \tilde{V}_3 \) is not even correct for the special case for which it was obtained. It is affected by the departure from the assumption of random groups.

4.5 Numerical Examples in Included Anchor Test Problem.

As in the excluded anchor test problem, most of the proposed estimators in the included anchor test problem were found to be asymptotically biased. Some numerical examples were worked out to indicate the possible importance of these asymptotic biases.

The general form of the estimated relationship between \( X_\alpha^* \) and \( Y_\alpha^* \) is

\[
\hat{Y}_\alpha^* (X_\alpha^*) = \hat{V}_2 + \left( \frac{\hat{\sigma}_{Y^*}}{\hat{\sigma}_{W^*}} \right) (\hat{W}_1 - \hat{W}_2) + \left( \frac{\hat{\sigma}_{Y^*}}{\hat{\sigma}_{X^*}} \right) (X_\alpha^* - \bar{X}_1).
\]

The probability limit of each estimator of \( \frac{\sigma_{Y^*}}{\sigma_{W^*}} \) can be expressed as
\[
\frac{\sigma_{Y^*}}{\sigma_{W^*}} \cdot f_{W_j} \quad \text{where} \quad \frac{\sigma_{Y^*}}{\sigma_{W^*}} (f_{W_j} - 1) \quad \text{is the asymptotic bias of the } j^{th} \text{ estimator.}
\]

Similarly, the probability limit of each estimator of \( \frac{\sigma_{Y^*}}{\sigma_{X^*}} \) can be expressed as
\[
\frac{\sigma_{Y^*}}{\sigma_{X^*}} \cdot f_{X_j}.
\]

To simplify the problem, we again chose to study cases where
\[
\sigma_{U^*} = \sigma_{Y^*} \quad \text{and} \quad \sigma_{W^*} = c \sigma_{Y^*}, \quad \text{with} \quad c \quad \text{an assigned constant between 0 and 1. Thus} \quad \frac{\sigma_{Y^*}}{\sigma_{W^*}} = \frac{1+c}{c} \quad \text{and} \quad \frac{\sigma_{Y^*}}{\sigma_{X^*}} = 1.
\]

The probability limit of the \( j^{th} \) estimated score is then
\[ Y^*_\alpha(x^*_\alpha) = \mu_{Y_2} + \frac{1+c}{c} \cdot f_{W_j}^{'}(\mu_{W_{1}} - \mu_{W_{2}}) + f_{X_j}^{'}(x^*_\alpha - \mu_{X_1}) \cdot \]

was subtracted from both sides and the following was calculated for the proposed estimators:

\[ Y^*_\alpha(x^*_\alpha) - \mu_{Y_2} = \frac{1+c}{c} \cdot f_{W_j}^{'}(\mu_{W_{1}} - \mu_{W_{2}}) + f_{X_j}^{'}(x^*_\alpha - \mu_{X_1}) \cdot \]

The numerical values which were used were:

\[ \mu_{W_{1}} - \mu_{W_{2}}: \quad -100, \quad -50, \quad -25, \quad 0, \quad 25, \quad 50, \quad 100 \]

\[ X_{\alpha}^* - \mu_{X_1}: \quad -150, \quad -75, \quad 0, \quad 75, \quad 150 \]

variance of \( W_{1} \): 10000

\( c \): .40, .50, .60

Reliabilities: .75, .85, .95

The same Graeco-Latin square that was used to assign reliabilities in the excluded anchor test case was used here. To calculate the reliabilities \( R_X \) and \( R_Y \), the following were used (there are appropriate in the special case where \( b = B, \beta = cB \)).

\[ R_X = \frac{(1+c)^2}{\left(\frac{1}{R_W} + \frac{c^2}{R_{W_{1}}} + 2c\right)} \]

\[ R_Y = \frac{(1+c)^2}{\left(\frac{1}{R_Y} + \frac{c^2}{R_{W_{2}}} + 2c\right)} \]

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The asymptotic bias in the estimated Y-score can be expressed for the \( j \)th estimate as

\[
b'_j = \frac{1+c}{c} \left( f'_{W_j} - 1 \right) (\mu_{W_1} - \mu_{W_2}) + \left( f'_{X_j} - 1 \right) (X^* - \mu_{X_1}).
\]

When the estimates are consistent, \( f'_{W_j} = 1 \) and \( f'_{X_j} = 1 \).

The biases in the discussion are for the special case where \( c = .40 \), \( \sigma^2_{W_1} = 10,000 \).

Table 4.3 summarizes the biases in a few cases.

Table 4.4 summarizes the values of \( f'_{W_j} \) and \( f'_{X_j} \).

The biases were investigated numerically for \( \bar{Y}_2, \bar{Y}_3, \) and \( \bar{Y}_4 \).

\( \bar{Y}_1 \) has the same functional form as \( \bar{Y}_1 \), and the biases in \( \bar{Y}_1 \) would tend to behave like the biases in \( \bar{Y}_1 \). Therefore it is not included in either Table 4.3 or Table 4.4.

We first look at \( b'_2 \). When \( R_{W_1} < R_{W_2} \), the scores \( Y \) are underestimated for \( X^* - \mu_{X_1} < 0 \), and overestimated for \( X^* - \mu_{X_1} > 0 \) (by as much as 112 points).

When \( R_{W_1} > R_{W_2} \), \( Y \) is overestimated when \( \mu_{W_1} - \mu_{W_2} < 0 \), and underestimated when \( \mu_{W_1} - \mu_{W_2} > 0 \). The most serious asymptotic bias occurred when \( R_{W_1} = .95 \) and \( R_{W_2} = .75 \). Then, when \( \mu_{W_1} - \mu_{W_2} = -100 \), and \( X^* - \mu_{X_1} = -150 \), \( Y \) is overestimated by 197 points.

\( \bar{Y}_2 \) was obtained under the assumptions of random groups and unequally reliable tests. However, it is seen that even in this situation, biases will occur. Look at \( R_{W_1} = R_{W_2} = .95 \) and \( \mu_{W_1} - \mu_{W_2} = 0 \), the random groups case. \( Y \) is underestimated by 21 points. When \( R_{W_1} = R_{W_2} = .85 \) and \( \mu_{W_1} - \mu_{W_2} = 0 \), then \( Y \) is underestimated by 9 points.
### Table 4.3

Asymptotic biases in the included anchor test problem

\[
b_j' = \frac{1+c}{c} (f^1_{Wj} - 1) (\mu_{W1} - \mu_{W2}) + (f^1_{Xj} - 1) (\mu_X - \mu_X)
\]

Let \( c = 0.40 \), \( \sigma^2_{W1} = 10,000 \)

\( b_2' \) = asymptotic bias in (Levine's) estimator (proposed for random groups, unequally reliable case)

\( b_3' \) = asymptotic bias in (Levine's) estimator (proposed for nonrandom groups, equally reliable case)

\( b_4' \) = asymptotic bias in (Levine's) estimator (proposed for nonrandom groups, unequally reliable case)

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Table 4.4

Values of $f'_{Wj}$ and $f'_{Xj}$, where

$$b'_{j} = \frac{1+c}{c} (f'_{Wj} - 1)(\mu_{W1} - \mu_{W2}) + (f'_{Xj} - 1)(X^*_X - X^*_L) .$$

$c = .40$, $\sigma^2_{W1} = 10000$, $\mu_{W1} - \mu_{W2} = -100$.

$b'_2$ = asymptotic bias in (Levine's) estimator (proposed for random groups, unequally reliable case)

$b'_3$ = asymptotic bias in (Levine's) estimator (proposed for nonrandom groups, equally reliable case)

$b'_4$ = asymptotic bias in (Levine's) estimator (proposed for nonrandom groups, unequally reliable case)

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We now look at $b'_3$. The most important factor affecting the bias in $\bar{Y}_3$ is the relative sizes of $R_X$ and $R_Y$.

When $R_X > R_Y$, then $Y$ is underestimated when $\mu_{W_1} - \mu_{W_2} < 0$, and overestimated when $\mu_{W_1} - \mu_{W_2} > 0$. The largest bias of 134 points occurred when $R_X = .844$ and $R_Y = .835$.

When $R_X < R_Y$, then $f'_{W_3}$ is overestimated and $f'_{X_3}$ is underestimated. Thus, the direction of the bias depends on the relative sizes of $(\mu_{W_1} - \mu_{W_2})$ and $(X^*_a - \mu_{X_1})$. For example, when $R_{W_1} = .85$, $R_{W_2} = .95$, $R_X = .905$, and $R_Y = .970$, we have $f'_{W_3} = 1.045$ and $f'_{X_3} = .899$. So the bias is positive whenever

$$\frac{1}{c}(1.045 - 1)(\mu_{W_1} - \mu_{W_2}) + (.899 - 1)(X^*_a - \mu_{X_1}) > 0,$$

or whenever $\mu_{W_1} - \mu_{W_2} > .898(X^*_a - \mu_{X_1})$.

$\bar{Y}_3$ was obtained under the assumptions of nonrandom groups and equally reliable tests. We look at one such case and find that biases will occur whenever the group means are different. When $R_X = R_Y = .949$, then $Y$ is underestimated by 54 points when $\mu_{W_1} - \mu_{W_2} = -100$.

Finally, we look at $b'_4$. Since $f'_{W_4} = \frac{1}{R_Y^2}$, it is always overestimated for the values we considered ($R_Y$ is < 1, and $\theta_2 < 1$ whenever $c < 1$).

$f'_{X_4}$ is overestimated when $R_X > R_Y$. This results in underestimation of $Y$ when $\mu_{W_1} - \mu_{W_2} < 0$, and overestimation when $\mu_{W_1} - \mu_{W_2} > 0$. Some serious biases occur when $R_{W_1} = .85$, $R_{W_2} = .75$, $R_X = .844$, and $R_Y = .835$. Then $Y$ is underestimated by 117 points when $\mu_{W_1} - \mu_{W_2} = -100$. 

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and $X_{\alpha}^* - \mu_{X_1} = -150$ (or overestimated by 117 points when $\mu_{W_1} - \mu_{W_2} = 100$
and $X_{\alpha}^* - \mu_{X_1} = 150$).

$f_{X_4}$ is underestimated when $R_X < R_Y$. The Y-scores are then always
underestimated when $\mu_{W_1} - \mu_{W_2} = -100$, and overestimated when $\mu_{W_1} - \mu_{W_2} = 100$
(by as much as 62 points). When $\mu_{W_1} - \mu_{W_2} = -50$, $\tilde{Y}_4$ tends to under-
estimate Y in most of the cases that were considered. When $\mu_{W_1} - \mu_{W_2} = 50$, Y tends to be overestimated.

When $\mu_{W_1} - \mu_{W_2} = 0$, Y is overestimated for $X_{\alpha}^* - \mu_{X_1} < 0$ and under-
estimated for $X_{\alpha}^* - \mu_{X_1} > 0$ (by as much as 23 points).

$\tilde{Y}_4$ was obtained under the assumptions of nonrandom groups and unequally reliable tests. But it is seen that serious biases can result
even when these assumptions hold.

The numerical results show that serious errors in estimating can
occur when using these estimators if the assumptions of their respective
cases do not hold. It should be noted here that some of the extreme
asymptotic biases were obtained when $\mu_{W_1} - \mu_{W_2} = \pm 100$, which may be an
unrealistic situation. However, asymptotic biases do appear even when
$\mu_{W_1} - \mu_{W_2} = \pm 25$, a difference in group means that is realistic. Hence,
for the excluded anchor test problem it is important to establish the
validity of the assumptions before using these procedures to estimate
the relationship between $U^*$ and $V^*$. Similarly, for the included
anchor test problem in the RG-ER case, it is important to establish the
validity of the assumptions before using these procedures to estimate
the relationship between $X^*$ and $Y^*$.

For the included anchor test problem in the other special cases, we
have seen in this chapter that the estimators proposed in chapter 2 are
not consistent even under the assumptions for which they were proposed.
CHAPTER 5: ASYMPTOTIC VARIANCES OF THE CONSISTENT ESTIMATORS

In this chapter, the asymptotic variances of the estimators, in the excluded anchor test problem, proposed in chapter 3 are obtained. It was noted there that all of the proposed estimators are consistent. In order to choose among them, we will compare their asymptotic variances.

We review here the consistent estimators proposed in chapter 3.

For $\frac{B}{b}$:

\[
C_{W_a} = \tilde{R}_V \frac{s^2}{s_{VW}}
\]

\[
C_{W_b} = \frac{1}{\tilde{R}_W} \frac{s_{VW}}{s^2_{W_2}}
\]

\[
C_{W_c} = \left[ \frac{\tilde{R}_V}{\tilde{R}_W} \frac{s^2}{s^2_{W_2}} \right]^{\frac{1}{2}}
\]

For $\frac{B}{b}$:

\[
C_{U_a} = \tilde{R}_V \frac{2}{\tilde{R}_U} \frac{s^2}{s_U} \frac{s_{VW}}{s^2_{VW}}
\]

\[
C_{U_b} = \frac{\tilde{R}_{W_1}}{\tilde{R}_W} \frac{s^2_{W_1}}{s^2_{W_2}} \frac{s_{VW}}{s_{UW}}
\]

\[
C_{U_c} = \tilde{R}_V \tilde{R}_{W_1} \frac{s^2_{VW}}{s_{VW}} \frac{s^2_{W_1}}{s_{UW}}
\]

\[
C_{U_d} = \frac{1}{\tilde{R}_U} \frac{1}{\tilde{R}_W} \frac{s_{UW} s_{VW}}{s_U s^2_{W_2}}
\]

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\[
C_{Ue} = \left[ \begin{array}{ccc}
\tilde{R}_V & \tilde{R}_W & s_V^2 \\
\tilde{R}_U & \tilde{R}_W & s_U^2 \\
& & \end{array} \right]^{1/2}
\]

The asymptotic variances are calculated according to the method described in Kendall and Stuart [4]. An example is worked out in appendix B.

Table 5.1 summarizes the forms of the asymptotic variances of the estimators, an example of which is worked out in appendix B.

Table 5.2 summarizes the necessary variances and covariances of the sample moments involved in obtaining the variances of the estimators. Examples are worked out in appendices C1 to C4.

Table 5.3 summarizes the coefficients of variation which appear in the forms in table 5.1. Examples are worked out in appendices D1 to D4.

The comparisons of the asymptotic variances follow table 5.3.
Table 5.1
Asymptotic Variances of Estimators

To arrive at a more tractable form of the problem, we take the special case where \( R_U, R_V, R_W, \) and \( R_{W_2} \) are known constants.
Let \( \phi = \frac{B}{b}, \psi = \frac{B}{b} \).

\[
\text{var } C_{W_a} = \text{var } \left[ \frac{s^2_{y}}{s_{W}} \right] = \psi^2 \left[ \frac{\text{var } s^2_{y} + \text{var } s_{W}}{(Es^2_{y})^2} - 2 \frac{\text{cov}(s^2_{y}, s_{W})}{(Es^2_{y})(Es_{W})} \right]
\]

\[
\text{var } C_{W_b} = \text{var } \left[ R_{W} \frac{s_{W}}{s_{W_2}} \right] = \phi^2 \left[ \frac{\text{var } s^2_{W} + \text{var } s^2_{W_2}}{(Es^2_{W})^2} - 2 \frac{\text{cov}(s^2_{W}, s_{W_2})}{(Es^2_{W})(Es_{W_2})} \right]
\]

\[
\text{var } C_{W_c} = \text{var } \left[ \frac{R_{W}}{R_{W_2}} \frac{s^2_{W}}{s^2_{W_2}} \right] = \phi^2 \left[ \frac{\text{var } s^2_{W} + \text{var } s^2_{W_2}}{(Es^2_{W})^2} - 2 \frac{\text{cov}(s^2_{W}, s^2_{W_2})}{(Es^2_{W})(Es_{W_2})} \right]
\]

\[
\text{var } C_{U_a} = \text{var } \left[ \frac{R_{V}}{R_{U}} \frac{s^2_{y}}{s_{W}} \frac{s_{UW}}{s^2_{U}} \right] = \psi^2 \left[ \frac{\text{var } s^2_{y} + \text{var } s^2_{U} + \text{var } s_{WU}}{(Es^2_{y})^2} + \frac{\text{var } s_{UW}}{(Es^2_{UW})^2} - 2 \frac{\text{cov}(s^2_{y}, s_{WU})}{(Es^2_{y})(Es_{WU})} \right]
\]

\[
\text{var } C_{U_b} = \text{var } \left[ \frac{R_{W_1}}{R_{W_2}} \frac{s^2_{W_1}}{s_{W_1}} \frac{s_{UW}}{s^2_{W_2}} \right] = \psi^2 \left[ \frac{\text{var } s^2_{W_1} + \text{var } s^2_{W_2} + \text{var } s_{UW}}{(Es^2_{W_1})^2} + \frac{\text{var } s_{UW}}{(Es^2_{UW})^2} - 2 \frac{\text{cov}(s^2_{W_1}, s_{UW})}{(Es^2_{W_1})(Es_{UW})} \right]
\]

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Table 5.1 (Cont.)

\[
\begin{align*}
\text{var } C_{U_c} &= \text{var} \left[ R_{VW_1} R_{VW} \frac{s_V^2}{s_{VW}} \frac{1}{s_{UW}} \right] = \frac{1}{2} \left[ \frac{\text{var } s_U^2}{(\text{Es}_U^2)^2} + \frac{\text{var } s_W_1^2}{(\text{Es}_W_1^2)^2} + \frac{\text{var } s_{VW}^2}{(\text{Es}_{VW}^2)^2} + \frac{\text{var } s_{UW}^2}{(\text{Es}_{UW}^2)^2} \\
&\quad - \frac{\text{cov}(s_V^2, s_{VW}^2)}{(\text{Es}_V^2)(\text{Es}_{VW}^2)} - 2 \frac{\text{cov}(s_{W_1}^2, s_{UW}^2)}{(\text{Es}_{W_1}^2)(\text{Es}_{UW}^2)} \right] \\
\text{var } C_{U_d} &= \text{var} \left[ \frac{1}{R_{UW_2}} R_{UW} \frac{s_{UW}}{s_U} \frac{1}{s_{W_2}} \right] = \frac{1}{2} \left[ \frac{\text{var } s_{UW}^2}{(\text{Es}_{UW}^2)^2} + \frac{\text{var } s_{VW}^2}{(\text{Es}_{VW}^2)^2} + \frac{\text{var } s_U^2}{(\text{Es}_U^2)^2} + \frac{\text{var } s_{W_2}^2}{(\text{Es}_{W_2}^2)^2} \\
&\quad - \frac{\text{cov}(s_{UW}^2, s_{UW}^2)}{(\text{Es}_{UW}^2)(\text{Es}_U^2)} - 2 \frac{\text{cov}(s_{VW}^2, s_{W_2}^2)}{(\text{Es}_{VW}^2)(\text{Es}_{W_2}^2)} \right] \\
\text{var } C_{U_e} &= \text{var} \left[ \frac{R_{VW_1}}{R_{UW_2}} \frac{s_V^2}{s_U} \frac{s_{W_1}^2}{s_{W_2}} \right] = \frac{1}{2} \left[ \frac{\text{var } s_V^2}{(\text{Es}_V^2)^2} + \frac{\text{var } s_U^2}{(\text{Es}_U^2)^2} + \frac{\text{var } s_{W_1}^2}{(\text{Es}_{W_1}^2)^2} + \frac{\text{var } s_{W_2}^2}{(\text{Es}_{W_2}^2)^2} \\
&\quad - \frac{\text{cov}(s_V^2, s_{W_2}^2)}{(\text{Es}_V^2)(\text{Es}_{W_2}^2)} - 2 \frac{\text{cov}(s_U^2, s_{W_1}^2)}{(\text{Es}_U^2)(\text{Es}_{W_1}^2)} \right]
\end{align*}
\]
Table 5.2

Variances and Covariances of Sample Moments

To simplify the calculations, all computations are accurate to order \( n \).
Let \( \mu_i^* = \mu_i - \delta_i \).

\[
\text{var } s_{U}^2 = \frac{1}{n} \left[ \beta \mu_4^*(Z_1) + \mu_4^*(\varepsilon) + 4b^2 \sigma^2 Z_1 \sigma^2 \varepsilon \right]
\]

\[
\text{var } s_{V}^2 = \frac{1}{n} \left[ B^4 \mu_4^*(Z_2) + \mu_4^*(\varepsilon) + 4B^2 \sigma^2 Z_2 \sigma^2 \varepsilon \right]
\]

\[
\text{var } s_{W_1}^2 = \frac{1}{n} \left[ \beta \mu_4^*(Z_1) + \mu_4^*(\varepsilon) + 4B^2 \sigma^2 Z_1 \sigma^2 \varepsilon \right]
\]

\[
\text{var } s_{W_2}^2 = \frac{1}{n} \left[ B^4 \sigma^2 Z_2 + \mu_4^*(\varepsilon) + 4B^2 \sigma^2 Z_2 \sigma^2 \varepsilon \right] - B^2 \sigma^2 Z_2 \]

\[
\text{var } s_{UW}^2 = \frac{1}{n} \left[ \beta \mu_4^*(Z_1) + \mu_4^*(\varepsilon) + 4b^2 \sigma^2 Z_1 \sigma^2 \varepsilon \right] - b^2 \sigma^2 Z_1 \]

\[
\text{var } s_{VW}^2 = \frac{1}{n} \left[ B^2 \sigma^2 Z_2 + \mu_4^*(\varepsilon) + 4B^2 \sigma^2 Z_2 \sigma^2 \varepsilon \right] - B^2 \sigma^2 Z_2 \]

\[
\text{cov}(s_{U}^2, s_{W_1}^2) = \frac{1}{n} \left[ \beta \mu_4^*(Z_1) \right]
\]

\[
\text{cov}(s_{U}^2, s_{W_2}^2) = \frac{1}{n} \left[ \beta \mu_4^*(Z_1) \right]
\]

\[
\text{cov}(s_{U}^2, s_{UW}^2) = \frac{1}{n} \left[ \beta \mu_4^*(Z_1) + 2b \sigma^2 Z_1 \sigma^2 \varepsilon \right]
\]

\[
\text{cov}(s_{V}^2, s_{VW}^2) = \frac{1}{n} \left[ B^2 \mu_4^*(Z_2) + 2B \sigma^2 Z_2 \sigma^2 \varepsilon \right]
\]

\[
\text{cov}(s_{W_1}^2, s_{UW}^2) = \frac{1}{n} \left[ \beta \mu_4^*(Z_1) + 2b \sigma^2 Z_1 \sigma^2 \varepsilon \right]
\]

\[
\text{cov}(s_{V}^2, s_{VW}^2) = \frac{1}{n} \left[ \beta \mu_4^*(Z_1) + 2b \sigma^2 Z_1 \sigma^2 \varepsilon \right]
\]

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Table 5.3

Coefficients of Variation

Let $\gamma^*_2 = \mu_4/\sigma^4$

$$\frac{\text{var } s_U^2}{(\text{Es } s_U^2)^2} = \frac{1}{n} \left[ \gamma^*_2 (Z_1)_U^2 + \gamma^*_2 (e)(1-R_U)^2 + 4R_U(1-R_U) \right]$$

$$\frac{\text{var } s_V^2}{(\text{Es } s_V^2)^2} = \frac{1}{n} \left[ \gamma^*_2 (Z_2)_V^2 + \gamma^*_2 (e)(1-R_V)^2 + 4R_V(1-R_V) \right]$$

$$\frac{\text{var } s_{W_1}^2}{(\text{Es } s_{W_1}^2)^2} = \frac{1}{n} \left[ \gamma^*_2 (Z_1)_{W_1}^2 + \gamma^*_2 (e)(1-R_{W_1})^2 + 4R_{W_1}(1-R_{W_1}) \right]$$

$$\frac{\text{var } s_{W_2}^2}{(\text{Es } s_{W_2}^2)^2} = \frac{1}{n} \left[ \gamma^*_2 (Z_2)_{W_2}^2 + \gamma^*_2 (e)(1-R_{W_2})^2 + 4R_{W_2}(1-R_{W_2}) \right]$$

$$\frac{\text{var } s_{UW}^2}{(\text{Es } s_{UW}^2)^2} = \frac{1}{n} \left[ \gamma^*_2 (Z_1) + \frac{1}{R_U} \frac{1}{R_{W_1}} - 1 \right]$$

$$\frac{\text{var } s_{VW}^2}{(\text{Es } s_{VW}^2)^2} = \frac{1}{n} \left[ \gamma^*_2 (Z_2) + \frac{1}{R_V} \frac{1}{R_{W_2}} - 1 \right]$$

$$\frac{\text{cov}(s_{UW}, s_U^2)}{(\text{Es } s_{UW})(\text{Es } s_U^2)} = \frac{1}{n} \left[ \gamma^*_2 (Z_1)_U + 2(1-R_U) \right]$$

$$\frac{\text{cov}(s_{VW}, s_V^2)}{(\text{Es } s_{VW})(\text{Es } s_V^2)} = \frac{1}{n} \left[ \gamma^*_2 (Z_2)_V + 2(1-R_V) \right]$$

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\[
\frac{\text{cov}(s_{UW}^2, s_{W_1}^2)}{(E_{s_{UW}})(E_{s_{W_1}})} = \frac{1}{n} \left[ \gamma_2^*(Z_1)R_{W_1} + 2(1-R_{W_1}) \right]
\]

\[
\frac{\text{cov}(s_{UW}^2, s_{W_2}^2)}{(E_{s_{UW}})(E_{s_{W_2}})} = \frac{1}{n} \left[ \gamma_2^*(Z_2)R_{W_2} + 2(1-R_{W_2}) \right]
\]

\[
\frac{\text{cov}(s_{U}^2, s_{W_1}^2)}{(E_{s_{U}})(E_{s_{W_1}})} = \frac{1}{n} \gamma_2^*(Z_1)R_{UW_1}
\]

\[
\frac{\text{cov}(s_{V}^2, s_{W_2}^2)}{(E_{s_{V}})(E_{s_{W_2}})} = \frac{1}{n} \gamma_2^*(Z_2)R_{VW_2}
\]
Substituting the results of Table 5.3 into the forms of the variances in Table 5.1, the asymptotic variances of the estimators of the coefficient \( \frac{B}{P} \) are obtained. They are listed below and then compared.

\[
\text{var } C_{W_a} = \frac{\sigma^2}{n} \left[ \gamma^*_2(Z_2)R^2_V + \gamma^*_2(E)(1-R_V)^2 + 4R_V(1-R_V) \right]
\]
\[
+ \gamma^*_2(Z_2) + \frac{1}{R_V} \frac{1}{R_{W_2}} - 1 - 2\gamma^*_2(Z_2)R_V(1-R_V) \right].
\]

Gathering up like terms, we have

\[
= \frac{\sigma^2}{n} \left[ \gamma^*_2(Z_2)(R^2_V + 2R_V) + \gamma^*_2(E)(1-R_V)^2 \right]
\]
\[
+ 4(1-R_V)(R_V - 1) + \frac{1}{R_V} \frac{1}{R_{W_2}} - 1 \right]
\]
\[
= \frac{\sigma^2}{n} \left[ (1-R_V)^2(\gamma^*_2(Z_2) + \gamma^*_2(E) - 4) + \frac{1}{R_V} \frac{1}{R_{W_2}} - 1 \right].
\]

The variances of \( C_{W_b} \) and \( C_{W_c} \) are similarly obtained. In summary, we have for the estimators of \( \frac{B}{P} \), the following variances:

\[
\text{var } C_{W_a} = \frac{\sigma^2}{n} \left[ (1-R_V)^2(\gamma^*_2(Z_2) + \gamma^*_2(E) - 4) + \frac{1}{R_V} \frac{1}{R_{W_2}} - 1 \right]
\]
\[
\text{var } C_{W_b} = \frac{\sigma^2}{n} \left[ (1-R_{W_2})^2(\gamma^*_2(Z_2) + \gamma^*_2(E) - 4) + \frac{1}{R_V} \frac{1}{R_{W_2}} - 1 \right]
\]
\[
\text{var } C_{W_c} = \frac{\sigma^2}{4n} \left[ \gamma^*_2(Z_2)(R_V - R_{W_2})^2 + \gamma^*_2(E)(1-R_V)^2 \right]
\]
\[
+ \gamma^*_2(E)(1-R_{W_2})^2 + 4R_V(1-R_V) + 4R_{W_2}(1-R_{W_2}) \right].
\]

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In the comparisons among the asymptotic variances of the estimators of \( \frac{B}{\beta} \), it is seen that \( \text{var} \ C_{\text{wa}} \) is always smaller than at least one other variance (i.e., of either \( C_{\text{wb}} \) or \( C_{\text{wc}} \)) and in some special cases, \( \text{var} \ C_{\text{wc}} \) is the minimum of the three variances.

In the following discussion, write \( R_{V} = 1-\delta_{V} \), \( R_{W2} = 1-\delta_{W2} \), and \( \gamma_{2}^{*} = \gamma_{2}^{*} \gamma_{2}^{*} \), \( \frac{1}{1-\delta_{V}} = 1 + \delta_{V} + \delta_{V}^{2} + \ldots \). (Note that \( \delta_{V} \) and \( \delta_{W2} \) are always positive.) Then, letting \( A = \delta_{V} + \delta_{W2}^{2} + (\delta_{V} + \delta_{W2})^{2} - \delta_{V} \delta_{W2} \), we can write the variances as

\[
\text{var} \ C_{\text{wa}} = \frac{\sigma^{2}}{n} \left[ A + \delta_{V}^{2} (\gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*}) + \text{terms of higher order in} \ \delta_{V} \ \text{and} \ \delta_{W2} \right]
\]

\[
\text{var} \ C_{\text{wb}} = \frac{\sigma^{2}}{n} \left[ A + \delta_{W2}^{2} (\gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*}) + \text{terms of higher order in} \ \delta_{V} \ \text{and} \ \delta_{W2} \right]
\]

\[
\text{var} \ C_{\text{wc}} = \frac{\sigma^{2}}{n} \left[ A - (\delta_{V} + \delta_{W2})^{2} + \frac{\delta_{V}^{2}}{4} (\gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*}) + \frac{\delta_{W2}^{2}}{4} (\gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*}) - \frac{1}{2} \delta_{V} \delta_{W2} \gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*} \gamma_{2}^{*} \right].
\]

The results of some comparisons of the asymptotic variances of the estimators of \( \frac{B}{\beta} \) are summarized in Propositions 1, 2, and 3. In the comparisons, we will make note of the fact that \( \mu_{4} \geq \sigma_{4}^{*} \), and hence that \( \frac{\mu_{4}}{\sigma_{4}^{*}} - 3 \geq -2 \).

**Proposition 1:** If \( Z, \varepsilon, E \) are distributed as independent normal random variables, then
\[ \text{var } C_{w_c} = \min(\text{var } C_{w_a}, \text{var } C_{w_b}, \text{var } C_{w_c}). \]

**Proof.**

If \( Z, \varepsilon, E \) are normal, then \( \gamma_2(Z) = \gamma_2(\varepsilon) = \gamma_2(E) = 0 \). Then

\[ \text{var } C_{w_a} = \text{var } C_{w_b} = \frac{\sigma^2}{n} [A + \text{positive terms}], \quad \text{and} \]

\[ \text{var } C_{w_c} = \frac{\sigma^2}{n} [A - (\varepsilon_v + \varepsilon_{w_2})^2]. \]

Hence \( \text{var } C_{w_c} = \min(\text{var } C_{w_a}, \text{var } C_{w_b}, \text{var } C_{w_c}). \)

**Proposition 2:** If \( \varepsilon_v = \varepsilon_{w_2} \) (i.e., \( R_v = R_{w_2} \)), and \( \gamma_2(\varepsilon) = \gamma_2(E) \), then

\[ \text{var } C_{w_c} = \min(\text{var } C_{w_a}, \text{var } C_{w_b}, \text{var } C_{w_c}). \]

**Proof.**

\[ \text{var } C_{w_a} = \text{var } C_{w_b} = \frac{\sigma^2}{n} [2\varepsilon_v^2 + 3\varepsilon_v^2 + \varepsilon_v^2(\gamma_2(Z) + \gamma_2(E))] \]

\[ + \text{positive terms} \]

\[ \text{var } C_{w_c} = \frac{\sigma^2}{n} [2\varepsilon_v^2 - \varepsilon_v^2 + \frac{\varepsilon_v^2}{2} \gamma_2(E)]. \]

We have, then,

\[ \text{var } C_{w_a} - \text{var } C_{w_c} = \frac{\sigma^2}{n} [\varepsilon_v^2 (4 + \gamma_2(Z) + \frac{1}{2} \gamma_2(E)) + \text{positive terms}] \geq 0, \text{ since } \gamma_2(Z) \geq -2, \text{ and } \gamma_2(E) \geq -2. \]
Hence, \( \text{var } C_c \leq \min(\text{var } C_a, \text{var } C_b, \text{var } C_c) \).

**Proposition 3:** \( \text{var } C_c \leq \max (\text{var } C_a, \text{var } C_b) \) (i.e., there is always an estimator worse than \( C_c \)).

**Proof.** We will show that \( \text{var } C_c \leq \frac{1}{2} (\text{var } C_a + \text{var } C_b) \).

\[
\frac{1}{2} (\text{var } C_a + \text{var } C_b) - \text{var } C_c
= \frac{\sigma^2}{n} \left[ A + \frac{\hat{s}_V^2}{2} (\gamma_2(Z) + \gamma_2(E)) + \frac{\hat{s}_{W_2}^2}{2} (\gamma_2(Z) + \gamma_2(E)) \right]
+ \text{positive terms} - A - (\hat{s}_V + \hat{s}_{W_2})^2
\]
\[-\frac{\hat{s}_V^2}{4} (\gamma_2(Z) + \gamma_2(E)) - \frac{\hat{s}_{W_2}^2}{4} (\gamma_2(Z) + \gamma_2(E))
+ \frac{1}{2} \hat{s}_V \hat{s}_{W_2} \gamma_2(Z)\]
\[
= \frac{\sigma^2}{n} \left[ \hat{s}_V^2 \left( 1 + \frac{\gamma_2(Z)}{4} \right) + \frac{\gamma_2(E)}{4} \right] + \frac{\hat{s}_{W_2}^2}{2} \left( 1 + \frac{\gamma_2(Z)}{4} + \frac{\gamma_2(E)}{4} \right)
+ \hat{s}_V \hat{s}_{W_2} \left( 2 + \frac{1}{2} \gamma_2(Z) \right) + \text{positive terms}\]
\[\geq 0 \text{ since each term is } \geq 0 \text{ by}\]
\[\gamma_2(Z) \geq -2 , \gamma_2(E) \geq -2 , \text{ and } \gamma_2(e) \geq -2 .\]

Hence \( \text{var } C_c \leq \max(\text{var } C_a, \text{var } C_b) \).
For the special case where the reliabilities of the test are assumed to be known, it has been shown that $C_w^c$ has smaller variance (to order $n$) than either $C_w^a$ or $C_w^b$ in some instances. It has also been shown that $\text{var } C_w^c$ is never the largest among the three variances. Hence, without further information, it is recommended that the geometric mean estimator be used to estimate $\frac{B}{b}$.

We now look at the asymptotic variances of the consistent estimators of $\frac{B}{b}$. Substituting the results of Table 5.3 into the forms of the variances in Table 5.1, the variances of the estimators of $\frac{B}{b}$ are obtained. They are listed below and then discussed.

$$
\text{var } C_w^a = \frac{\psi^2}{n} \left[ \gamma_2^*(z_2) R_v^2 + \gamma_2^*(E) (1-R_v)^2 + 4R_v (1-R_v) + \gamma_2^*(z_1) R_U^2 \\
+ \gamma_2^*(e) (1-R_U)^2 + 4R_U (1-R_U) + \gamma_2^*(z_2) + \frac{1}{R_v} \frac{1}{R_{w_2}} - 1 + \gamma_2^*(z_1) \\
+ \frac{1}{R_U} \frac{1}{R_{w_1}} - 1 - 2\gamma_2^*(z_2) R_v - 4(1-R_v) - 2\gamma_2^*(z_1) R_U - 4(1-R_U) \right].
$$

Combining terms in $\gamma_2^*(z_1)$ and $\gamma_2^*(z_2)$, we have

$$
= \frac{\psi^2}{n} \left[ \gamma_2^*(z_2) (R_v^2 + 1 - 2R_v) + \gamma_2^*(z_1) (R_U^2 + 1 - 2R_U) + \gamma_2^*(E) (1-R_v)^2 \\
+ 4R_v (1-R_v) - 4(1-R_v) + \gamma_2^*(e) (1-R_U)^2 + 4R_U (1-R_U) - 4(1-R_U) \\
+ \frac{1}{R_v} \frac{1}{R_{w_2}} + \frac{1}{R_U} \frac{1}{R_{w_1}} - 2 \right].
$$
\[
= \frac{\psi^2}{n} \left[ (1-R_V)^2 (\gamma_2^*(Z_2) + \gamma_2^*(E)^{-4}) + (1-R_U)^2 (\gamma_2^*(Z_1) + \gamma_2^*(E)^{-4}) \right]
+ \frac{1}{R_V} \frac{1}{R_W_2} + \frac{1}{R_U} \frac{1}{R_W_1} - 2 \right].
\]

The variances of the other estimators of $\frac{B}{b}$ are obtained in similar fashion. They are summarized below.

\[
\text{var } C_{U_a} = \frac{\psi^2}{n} \left[ (1-R_V)^2 (\gamma_2^*(Z_2) + \gamma_2^*(E)^{-4}) + (1-R_U)^2 (\gamma_2^*(Z_1) + \gamma_2^*(E)^{-4}) \right]
+ \frac{1}{R_V} \frac{1}{R_W_2} + \frac{1}{R_U} \frac{1}{R_W_1} - 2 \right]
\]

\[
\text{var } C_{U_b} = \frac{\psi^2}{n} \left[ (1-R_W_1)^2 (\gamma_2^*(Z_2) + \gamma_2^*(E)^{-4}) + (1-R_W_2)^2 (\gamma_2^*(Z_1) + \gamma_2^*(E)^{-4}) \right]
+ \frac{1}{R_V} \frac{1}{R_W_2} + \frac{1}{R_U} \frac{1}{R_W_1} - 2 \right]
\]

\[
\text{var } C_{U_c} = \frac{\psi^2}{n} \left[ (1-R_W_1)^2 (\gamma_2^*(Z_2) + \gamma_2^*(E)^{-4}) + (1-R_V)^2 (\gamma_2^*(Z_2) + \gamma_2^*(E)^{-4}) \right]
+ \frac{1}{R_V} \frac{1}{R_W_2} + \frac{1}{R_U} \frac{1}{R_W_1} - 2 \right]
\]

\[
\text{var } C_{U_d} = \frac{\psi^2}{n} \left[ (1-R_U)^2 (\gamma_2^*(Z_1) + \gamma_2^*(E)^{-4}) + (1-R_W_2)^2 (\gamma_2^*(Z_2) + \gamma_2^*(E)^{-4}) \right]
+ \frac{1}{R_U} \frac{1}{R_W_1} + \frac{1}{R_V} \frac{1}{R_W_2} - 2 \right]
\]
$$\text{var } c_{U_e} = \frac{\psi^2}{4n} \left[ \gamma_2^*(Z_1) (R_U - R_{W_1})^2 + \gamma_2^*(Z_2) (R_V - R_{W_2})^2 + (\gamma_2^*(\varepsilon) - \bar{\varepsilon}) (1-R_U)^2 \\
+ 4(1-R_U) + (\gamma_2^*(E) - \bar{\gamma}) (1-R_V)^2 + 4(1-R_V) + (\gamma_2^*(\varepsilon) - \bar{\varepsilon})(1-R_{W_1})^2 \\
+ 4(1-R_{W_1}) + (\gamma_2^*(\varepsilon) - \bar{\varepsilon})(1-R_{W_2})^2 + 4(1-R_{W_2}) \right].$$

The discussion of the variances of the estimators of $\frac{B}{\beta}$ will follow the lines of the discussion of the variances of the estimators of $\frac{B}{\beta}$.

It will be shown that $\text{var } c_{U_e}$ is always smaller than at least one other variance, and in some special cases, $\text{var } c_{U_e}$ is the minimum of the five variances.

Let $R_U = 1-\delta_U$, $R_V = 1-\delta_V$, $R_{W_1} = 1-\delta_{W_1}$, and $R_{W_2} = 1-\delta_{W_2}$. (Note again that the $\delta$'s are always positive.) Write $\frac{1}{1-\delta} = 1 + \delta + \delta^2 + \cdots$, $\gamma^* = \gamma + 2$.

Let $B = \delta_U + \delta_V + \delta_{W_1} + \delta_{W_2} + (\delta_U + \delta_{W_1})^2 + (\delta_V + \delta_{W_2})^2 + (\delta_U + \delta_{W_1})^2 - \delta_U \delta_{W_1} - \delta_V \delta_{W_2}$.

Then

$$\text{var } c_{U_a} = \frac{\psi^2}{n} [B + \delta_V^2 (\gamma_2(Z_2) + \gamma_2(E)) + \delta_U^2 (\gamma_2(Z_1) + \gamma_2(\varepsilon)) + \text{ terms}]$$

$$\text{var } c_{U_b} = \frac{\psi^2}{n} [B + \delta_{W_1}^2 (\gamma_2(Z_1) + \gamma_2(\varepsilon_1)) + \delta_{W_2}^2 (\gamma_2(Z_2) + \gamma_2(\varepsilon_2)) + \text{ terms}]$$

$$\text{var } c_{U_c} = \frac{\psi^2}{n} [B + \delta_{W_1}^2 (\gamma_2(Z_1) + \gamma_2(\varepsilon_1)) + \delta_V^2 (\gamma_2(Z_2) + \gamma_2(E)) + \text{ terms}]$$

$$\text{var } c_{U_d} = \frac{\psi^2}{n} [B + \delta_U^2 (\gamma_2(Z_1) + \gamma_2(\varepsilon)) + \delta_{W_2}^2 (\gamma_2(Z_2) + \gamma_2(\varepsilon_2)) + \text{ terms}]$$
\[
\text{var } C_{U_e} = \frac{\psi^2}{n} \left[ B - (\delta^2 Y + \delta^2 W_2)^2 - (\delta U + \delta W_1)^2 \right] + \frac{\delta^2 V}{4} (\gamma_2(Z_1) + \gamma_2(e_1)) \\
+ \frac{\delta^2 U}{4} (\gamma_2(Z_2 + \gamma_2(e_2)) + \frac{\delta^2 V}{4} (\gamma_2(Z_2) + \gamma_2(e_2)) \\
+ \frac{\delta^2 W}{4} (\gamma_2(Z_2) + \gamma_2(e_2)) - \frac{1}{2} \delta U \delta W_1 \gamma_2(Z_1) - \frac{1}{2} \delta V \delta W_2 \gamma_2(Z_2).
\]

The results of some comparisons of the asymptotic variances of the estimators of $\frac{B}{b}$ are summarized in Propositions 4, 5, and 6.

**Proposition 4:** If $Z_1, Z_2, e, e_1, e_2$ are distributed as independent normal random variables, then

\[
\text{var } C_{U_e} = \min\left(\text{var } C_{U_a}, \text{var } C_{U_b}, \text{var } C_{U_c}, \text{var } C_{U_d}, \text{var } C_{U_e}\right).
\]

**Proof.**

\[
\gamma_2(Z_1) = \gamma_2(Z_2) = \gamma_2(e) = \gamma_2(e_1) = \gamma_2(e_2) = 0.
\]

Then

\[
\text{var } C_{U_a} = \text{var } C_{U_b} = \text{var } C_{U_c} = \text{var } C_{U_d} = \frac{\psi^2}{n} B + \text{positive terms}.
\]

\[
\text{var } C_{U_e} = \frac{\psi^2}{n} \left[ B - (\delta^2 Y + \delta^2 W_2)^2 - (\delta U + \delta W_1)^2 \right].
\]

We have, then,

\[
\text{var } C_{U_a} - \text{var } C_{U_e} = \frac{\psi^2}{n} \left[ \text{positive terms} + (\delta^2 Y + \delta^2 W_2)^2 + (\delta U + \delta W_1)^2 \right] \geq 0.
\]

Hence $C_{U_e}$ has the minimum asymptotic variance. ||

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Proposition 5: If \( \delta_u = \delta_v = \delta_w_1 = \delta_w_2 = \delta, \gamma_2(z_1) = \gamma_2(z_2), \) and \( \gamma_2(e) = \gamma_2(E) = \gamma_2(\varepsilon_1) = \gamma_2(\varepsilon_2), \) then

\[
\text{var } C_{U_e} = \min(\text{var } C_{U_a}, \text{var } C_{U_b}, \text{var } C_{U_c}, \text{var } C_{U_d}, \text{var } C_{U_e}).
\]

Proof. \[
\text{var } C_{U_a} = \text{var } C_{U_b} = \text{var } C_{U_c} = \text{var } C_{U_d} = \\
\frac{\psi^2}{n} [4\delta^2 + 6\delta^2 + 2\delta^2 \gamma_2(z) + 2\delta^2 \gamma_2(e)]
\]

\[
\text{var } C_{U_e} = \frac{\psi^2}{n} [4\delta^2 - 2\delta^2 + \delta^2 \gamma_2(e)].
\]

We have, then,

\[
\text{var } C_{U_a} - \text{var } C_{U_e} = \frac{\psi^2}{n} [\delta^2 (4 + 2\gamma_2(z) + \gamma_2(e))] \\
\geq 0 \text{ since } \gamma_2(z) \geq -2, \gamma_2(e) \geq -2.
\]

Hence \( C_{U_e} \) has the minimum asymptotic variance. ||

Proposition 6: In general, \( \text{var } C_{U_e} \leq \max(\text{var } C_{U_a}, \text{var } C_{U_b}, \text{var } C_{U_c}, \text{var } C_{U_d}). \)

Proof. We will show \( \text{var } C_{U_e} \leq \frac{1}{4} \left( \text{var } C_{U_a} + \text{var } C_{U_b} + \text{var } C_{U_c} + \text{var } C_{U_d} \right) \)

\[
\frac{1}{4} \left( \text{var } C_{U_a} + \text{var } C_{U_b} + \text{var } C_{U_c} + \text{var } C_{U_d} \right) - \text{var } C_{U_e}
\]

\[
= \frac{\psi^2}{n} \left[ B + \frac{\delta_v^2}{4} (\gamma_2(z_2) + \gamma_2(E)) + \frac{\delta_w_1^2}{4} (\gamma_2(z_1) + \gamma_2(e)) + \frac{\delta_w_2^2}{4} (\gamma_2(z_1) + \gamma_2(e_1)) + \frac{\delta_v^2}{4} (\gamma_2(z_2) + \gamma_2(e_2)) - B \right]
\]

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\begin{align*}
+ (\delta_V^2 + \delta_W^2) + (\delta_U^2 + \delta_W^2) + \frac{1}{2} \delta_U \delta_W \gamma_2(Z_1) + \frac{1}{2} \delta_U \delta_W \gamma_2(Z_2) \\
+ \text{positive terms}]
= \frac{\psi^2}{n} \left[ \delta_V^2 \left( 1 + \frac{\gamma_2(Z_2)}{4} + \frac{\gamma_2(Z_1)}{4} + \frac{\gamma_2(e)}{4} \right) + \delta_U^2 \left( 1 + \frac{\gamma_2(Z_1)}{4} + \frac{\gamma_2(Z_2)}{4} + \frac{\gamma_2(e)}{4} \right) \\
+ \delta_W^2 \left( 1 + \frac{\gamma_2(Z_1)}{4} + \frac{\gamma_2(Z_2)}{4} + \frac{\gamma_2(e)}{4} \right) + \delta_W^2 \left( 1 + \frac{\gamma_2(Z_1)}{4} + \frac{\gamma_2(Z_2)}{4} + \frac{\gamma_2(e)}{4} \right) \\
+ \delta_V \delta_W \left( 1 + \frac{\gamma_2(Z_2)}{2} + \frac{\gamma_2(Z_1)}{2} \right) + \delta_U \delta_W \left( 1 + \frac{\gamma_2(Z_1)}{2} + \frac{\gamma_2(Z_2)}{2} \right) \\
+ \text{positive terms} \right]
\geq 0, \text{ since, by } \gamma_2(\cdot) \geq -2, \text{ each term is positive.}
\end{align*}

Hence, \( \text{var } C_{U_e} \) is never the largest of the asymptotic variances. 

The conclusion, then, is that in some special cases, \( C_{U_e} \) performs better (has smaller asymptotic variance) than any of the other estimators of \( \frac{B}{b} \). In general, it can be said that \( C_{U_e} \) has smaller asymptotic variance than at least one other estimator. So, without further information, it is recommended that the geometric mean estimator be used in estimating \( \frac{B}{b} \).
Appendix A: Notation

$U, V$  test forms and corresponding scores, excluded anchor test problem

$W$  anchor test and corresponding score

$X, Y$  test forms and corresponding scores, included anchor test problem ($X = U+W, Y = V+W$)

$R_{X_1}$  reliability of test $X$ for population $1$

$U^*$  true score corresponding to observed score $U$

$b_{VV} = \frac{s_{VV}^2}{s_{W_2}^2}$

$\mu_4(Z) = E(Z-\mu_Z)^4$

$\mu_4(Z) = \mu_{11}(Z) - \frac{\mu_4(Z)}{\sigma_Z}$

$\gamma_2^*(Z) = \frac{\mu_4^*(Z)}{\sigma_Z^4} = \frac{\mu_4(Z)}{\sigma_Z^4} - 1$

$\gamma_2(Z) = \gamma_2^*(Z) - 2 = \frac{\mu_4(Z)}{\sigma_Z^4} - 3$

$\phi = \frac{\beta}{\bar{b}}$

$\psi = \frac{B}{b}$

$\theta_1 = \frac{R_{W_1}(b+\beta)}{bR_{W_1} + \beta}$

$\theta_2 = \frac{R_{W_2}(B+\beta)}{BR_{W_2} + \beta}$
Appendix B: Standard Error of $C_{W_a}$

According to the method described in Kendall and Stuart ([4], p. 232)), the standard error of a function of random variables, $f(X_1, X_2, \ldots, X_n)$, is approximately (up to order $n$) equal to

$$\text{var} \ f(X_1, X_2, \ldots, X_n) \approx \sum_{i,j=1}^{k} \left[ \frac{\partial f}{\partial X_i} \cdot \frac{\partial f}{\partial X_j} \right] \text{cov}(X_i, X_j),$$

where the square brackets indicate that the derivatives within the brackets are evaluated at the means of the random variables involved.

Using the above, we find the variance of $C_{W_a}$, where, for simplicity, it has been assumed that the reliabilities are known constants.

Let

$$f(s_v^2, s_{vw}) = \frac{s_v^2}{s_{vw}^2} = f.$$ 

Then

$$\text{var} \ C_{W_a} = \text{var} \left[ R_v \frac{s_v^2}{s_{vw}^2} \right]$$

$$= R_v^2 \left\{ \left[ \frac{\partial f}{\partial s_v^2} \right] \text{var} s_v^2 + \left[ \frac{\partial f}{\partial s_{vw}^2} \right] \text{var} s_{vw} + 2 \left[ \frac{\partial f}{\partial s_v^2} \frac{\partial f}{\partial s_{vw}^2} \right] \text{cov}(s_v^2, s_{vw}) \right\}$$

(1)

$$= R_v^2 \left\{ \frac{1}{(Es_{vw})^2} \text{var} s_v^2 + \frac{(Es_{vw})^2}{(Es_{vw})^4} \text{var} s_{vw} - 2 \frac{Es_v^2}{(Es_{vw})^3} \text{cov}(s_v^2, s_{vw}) \right\}.$$ 

Factoring out $\left( \frac{Es_v^2}{Es_{vw}} \right)^2$, (1) becomes

$$R_v^2 \left( \frac{Es_v^2}{Es_{vw}} \right)^2 \left\{ \frac{\text{var} s_v^2}{(Es_v^2)^2} + \frac{\text{var} s_{vw}}{(Es_{vw})^2} - \frac{2\text{cov}(s_v^2, s_{vw})}{(Es_v^2)(Es_{vw})} \right\}.$$
Now,

\[ \frac{R_v^2}{\left( \frac{B^2 \sigma_Z^2}{B^2 \sigma_Z^2 + \sigma_E^2} \right)^2} \]

and

\[ \left( \frac{E_{sv}^2}{E_{svW}} \right)^2 = \left( \frac{B^2 \sigma_Z^2 + \sigma_E^2}{B^2 \sigma_Z^2} \right)^2, \]

and thus

\[ R_v^2 \left( \frac{E_{sv}^2}{E_{svW}} \right)^2 = \left( \frac{B}{\beta} \right)^2. \]

Hence,

\[
\text{var } C_{W_a} = \left( \frac{B}{\beta} \right)^2 \left[ \frac{\text{var } s_v^2}{(E_{sv})^2} + \frac{\text{var } s_{svW}^2}{(E_{svW})^2} - \frac{2 \text{cov}(s_v^2, s_{svW})}{(E_{sv})^2 (E_{svW})^2} \right].
\]
Appendix Cl: Variance of \( s_U^2 \)

The variance of \( s_U^2 \) is approximately (to order \( n^{-1} \)) equal to

\[
\text{var } s_U^2 \approx \frac{1}{n^2} \left\{ \text{E} \left[ \sum (U - \mu_U)^2 \right]^2 - \left[ \text{E}(U - \mu_U)^2 \right]^2 \right\}.
\]

Let

\[
(Cl.1) = \text{E}(\sum (U - \mu_U)^2)^2
\]

\[
(Cl.2) = \text{E}(U - \mu_U)^2
\]

Then,

\[
\text{var } s_U^2 \approx \frac{1}{n^2} \left( (Cl.1) - (Cl.2)^2 \right).
\]

We first find \( (Cl.1) \).

\[
\text{E}(\sum (U - \mu_U)^2)^2
\]

\[
= \text{E}(\Sigma b^2 (Z - \mu_Z)^2 + 2bZ (Z - \mu_Z) e + e^2)^2
\]

\[
= \text{E}(\Sigma b^4 (Z - \mu_Z)^4 + \sum_{i \neq j} b^4 (Z_i - \mu_Z)^2 (Z_j - \mu_Z)^2 + 4b^2 (Z - \mu_Z)^2 e^2
\]

\[
+ 2 \Sigma e^4 + \sum_{i \neq j} e_i^2 e_j^2 + 2(\Sigma b^2 (Z - \mu_Z)^2)(\Sigma e^2) + \text{terms}
\]

with expectation 0]

\[
(Cl.1) = n[b^4 \mu_4(Z) + (n-1)b^2 \sigma_Z^4 + 4b^2 \sigma_Z^2 \sigma_e^2 + \mu_e(e) + (n-1)\sigma_e^4
\]

\[
+ 2nb^2 \sigma_Z^2 \sigma_e^2].
\]

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Now find (Cl.2).

\[ E[\Sigma (U_{ij} - \mu_U)^2] \]

\[ = E[\Sigma b(Z_{ij} - \mu_Z)^2 + 2\Sigma b(Z_{ij} - \mu_Z)e + \Sigma e^2] \]

(Cl.2) \[ = n[b^2 \sigma_{Z}^2 + \sigma_e^2] \cdot \]

Then,

\[ \text{var}\; s_U \cong \frac{1}{n^2} \left[ (\text{Cl.1}) - (\text{Cl.2})^2 \right] \]

\[ = \frac{1}{n} \left[ b^2 \mu_4^*(Z) + b^2 \mu_4^*(e) + 4b^2 \sigma_{Z}^2 \sigma_e^2 \right] , \]

where \[ \mu_4^*(Z) = \mu_4(Z) - \sigma_Z^4 . \]
Appendix C2: Variance of $s_{UW}$

The variance of $s_{UW}$ is approximately (to order $n^{-1}$) equal to

$$\text{var } s_{UW} \approx \frac{1}{n^2} \left\{ \text{E}\left[ \Sigma (U - \mu_{U})(W - \mu_{W}) \right]^2 - \left[ \text{E}(\Sigma (U - \mu_{U})(W - \mu_{W})) \right]^2 \right\}.$$

Let

(C2.1) $= \text{E}[\Sigma (U - \mu_{U})(W - \mu_{W})]^2$

(C2.2) $= \text{E}[\Sigma (U - \mu_{U})(W - \mu_{W})].$

Then,

$$\text{var } s_{UW} \approx \frac{1}{n^2} [(C2.1) - (C2.2)^2].$$

We first find (C2.1).

$$\text{E}[\Sigma (U - \mu_{U})(W - \mu_{W})]^2 = \text{E}[\Sigma b(Z - \mu_Z)^2 + \Sigma b(Z - \mu_Z)\varepsilon + \Sigma b(Z - \mu_Z)e + \Sigma \varepsilon]^2$$

$$= \text{E}[\Sigma b^2(Z - \mu_Z)^2 + \sum_{i \neq j} b^2 (Z_i - \mu_Z)^2 (Z_j - \mu_Z)^2 + \Sigma b^2(Z - \mu_Z)^2 \varepsilon^2$$

$$+ \Sigma b^2(Z - \mu_Z)^2 \varepsilon^2 + \Sigma \varepsilon \varepsilon + \text{terms with expectation 0} ]$$

(C2.1) $= n[b^2 \mu_{\mu}(Z) + (n-1)b^2 \sigma_{Z}^2 + b^2 \sigma_{Z}^2 \varepsilon + \mu_{\mu} \sigma_{Z}^2 e + \sigma_e^2 \varepsilon].$

Now find (C2.2)
\[ E[\Sigma(U-\mu_U)(W-\mu_W)] \]

\[ = E[\Sigma b(Z-\mu_Z)^2 + \Sigma b(Z-\mu_Z)e + \Sigma b(Z-\mu_Z)e + \Sigma e e] \]

(C2.2) \[ = nb\beta \sigma_Z^2. \]

Then,

\[ \text{var} s_{UW} = \frac{1}{n^2} \left\{ (C2.1) - (C2.2)^2 \right\} \]

\[ = \frac{1}{n} \left[ b^2 \beta^2 \mu_i^*(Z) + (\beta^2 \sigma_Z^2 + \sigma_e^2)(b^2 \sigma_Z^2 + \sigma_e^2) - b^2 \beta^2 \sigma_Z^4 \right], \]

where \( \mu_i^*(Z) = \mu_i(Z) - \frac{1}{b} \)

\( \sigma_Z. \)
Appendix C3: Covariance of $s_U^2$ and $s_W^2$

The covariance of $s_U^2$ and $s_W^2$ is approximately (to order $n^{-1}$)
equal to

$$\text{cov}(s_U^2, s_W^2) \approx \frac{1}{n^2} \left\{ E[\Sigma(U-\mu_U)^2 \cdot \Sigma(W-\mu_W)^2] - E[\Sigma(U-\mu_U)^2] \cdot E[\Sigma(W-\mu_W)^2] \right\}.$$

Let

(C3.1) = $E[\Sigma(U-\mu_U)^2 \cdot \Sigma(W-\mu_W)^2]$

(C3.2) = $E[\Sigma(U-\mu_U)^2]\$

(C3.3) = $E[\Sigma(W-\mu_W)^2]\$

Then,

$$\text{cov}(s_U^2, s_W^2) \approx \frac{1}{n^2} \{(C3.1) - (C3.2)(C3.3)\}.$$

We first find (C3.1).

$$E[\Sigma(U-\mu_U)^2 \cdot \Sigma(W-\mu_W)^2]$$

$$= E[\{\Sigma b(Z-\mu_Z)^2 + \Sigma e^2 + 2\Sigma b(Z-\mu_Z)e\} \cdot$$

$$\{\Sigma b(Z-\mu_Z)^2 + \Sigma e^2 + 2\Sigma b(Z-\mu_Z)e\}]$$

$$= E[\Sigma b^2(Z-\mu_Z)^4 + \Sigma \beta^2(Z_1-\mu_Z)^2(Z_j-\mu_Z)^2 +$$

$$\Sigma e^2 \cdot \Sigma e^2 + \Sigma b^2(Z-\mu_Z)^2 \cdot \Sigma e^2 + \Sigma e^2 + \text{terms with expectation 0}].$$
(C3.1) = n[b^2 \beta^2 \mu_h(Z) + (n-1)b^2 \beta^2 \sigma^4_Z + n\sigma_e^2 \sigma_e^2 + n b^2 \sigma_Z^2 \sigma_e^2
+ n b^2 \sigma_Z^2 \sigma_e^2].

Now find (C3.2).

\[ E[\Sigma(U-\mu_U)^2] \]
\[ = E[2b^2(\mu_Z-\mu_Z)^2 + 2b\Sigma(\mu_Z-\mu_Z)e + \Sigma e^2] \]
\[ = n(b^2 \sigma_Z^2 + \sigma_e^2). \]

Now find (C3.3).

\[ E[\Sigma(W-\mu_W)^2] \]
\[ = E[2b^2(\mu_Z-\mu_Z)^2 + 2b\Sigma(\mu_Z-\mu_Z)e + \Sigma e^2] \]
\[ = n(b^2 \sigma_Z^2 + \sigma_e^2). \]

Then,

\[ \text{cov}(s_{U}^2, s_{W}^2) \approx \frac{1}{n^2} [(C3.1) - (C3.2)(C3.3)] \]
\[ = \frac{1}{n} [b^2 \beta^2 \mu_h'(Z)], \]

where \( \mu_h'(Z) = \mu_h(Z) - \sigma^4_Z. \)
Appendix C4: Covariance of $s_U^2$ and $s_{UW}$

The covariance of $s_U^2$ and $s_{UW}$ is approximately (to order $n^{-1}$) equal to

$$
\text{cov}(s_U^2, s_{UW}) \sim \frac{1}{n^2} \left[ E[(\Sigma (U-\mu_U)^2) (\Sigma (U-\mu_U) (W-\mu_W))] \\
- E[(\Sigma (U-\mu_U)^2)] E[(\Sigma (U-\mu_U) (W-\mu_W))] \right].
$$

Let

$$(C4.1) = E[(\Sigma (U-\mu_U)^2) (\Sigma (U-\mu_U) (W-\mu_W))]$$

$$(C4.2) = E[(\Sigma (U-\mu_U)^2)]$$

$$(C4.3) = E[(\Sigma (U-\mu_W) (W-\mu_W))] .$$

Then

$$
\text{cov}(s_U^2, s_{UW}) \sim \frac{1}{n^2} \left( (C4.1) - (C4.2)(C4.3) \right).
$$

First find $(C4.1)$.

$$
E[(\Sigma (U-\mu_U)^2) (\Sigma (U-\mu_U) (W-\mu_W))]
$$

$$
= E[(\Sigma b^2 (Z-\mu_Z)^2 + 2b\Sigma (Z-\mu_Z)e + \Sigma e^2] .
$$

$$
[\Sigma b^2 (Z-\mu_Z)^2 + b\Sigma (Z-\mu_Z)e + \Sigma (Z-\mu_Z)^2]e + \Sigma e\Sigma]
$$

$$
= E[\Sigma b^2 (Z-\mu_Z)^4 + \sum_{i \neq j} b^2 \beta (Z_i-\mu_Z)^2 (Z_j-\mu_Z)^2
+ 2b\beta\Sigma (Z-\mu_Z)^2 e^2 + \Sigma \beta (Z-\mu_Z)^2 \cdot \Sigma e^2
+ \text{terms with expectation 0}]
$$
\[ = n(b^3 \beta \mu_4(Z) + (n-1) b^3 \beta \sigma_Z^4 + 2b\beta \sigma_Z^2 \sigma_e^2 + nb\beta \sigma_Z^2 \sigma_e^2) . \]

Now find (C4.2).

\[ E[\Sigma(U-\mu_U)^2] = E[\Sigma b^2 (Z-\mu_Z)^2 + 2b\Sigma (Z-\mu_Z)e + \Sigma e^2] \]

\[ = n[b^2 \sigma_Z^2 + \sigma_e^2] . \]

Now find (C4.3).

\[ E[(U-\mu_U)(W-\mu_W)] = E[\Sigma b \beta (Z-\mu_Z)^2 + \Sigma b (Z-\mu_Z)e + \Sigma \beta (Z-\mu_Z)e + \Sigma e^2] \]

\[ = nb\beta \sigma_Z^2 . \]

Then,

\[ \text{cov}(s_U^2, s_W) \approx \frac{1}{n} \left( (C4.1) - (C4.2)(C4.3) \right) \]

\[ = \frac{1}{n} \left[ b^3 \beta \mu_4^*(Z) + 2b\beta \sigma_Z^2 \sigma_e^2 \right] , \]

where \( \mu_4^*(Z) = \mu_4(Z) - \frac{1}{n} \sigma_Z^4 . \)
Appendix D1: Coefficient of Variation: \[ \frac{\text{var } s_U^2}{(\psi s_U^2)^2} \]

From appendix C1, we have

\[ (\text{Dl.1}) \quad \frac{\text{var } s_U^2}{(\psi s_U^2)^2} = \frac{1}{n} \left[ \frac{b \mu_i^*(Z) + \mu_i^*(e) + 4b^2 \sigma_Z^2 \sigma_e^2}{b^2 \sigma_Z^2 + \sigma_e^2} \right]. \]

Let \( \gamma_2^*(\cdot) = \frac{\mu_i^*(\cdot)}{\sigma^*}. \)

Dividing numerator and denominator in (Dl.1) by \( \frac{b}{b} \sigma_Z^2 \), we have

\[ (\text{Dl.2}) \quad \frac{1}{n} \left[ \gamma_2^*(Z) + \gamma_2^*(e) \left( \frac{\sigma_e^2}{b^2 \sigma_Z^2} \right)^2 + \frac{4 \sigma_e^2}{b^2 \sigma_Z^2} \right]. \]

where \( R_U = \frac{b^2 \sigma_Z^2}{b \sigma_Z^2 + \sigma_e^2} \).

Now \( \frac{1-R_U}{R_U} = \frac{\sigma_e^2}{b^2 \sigma_Z^2} \), so (Dl.2) becomes

\[ \frac{1}{n} \left[ \gamma_2^*(Z) + \gamma_2^*(e) \left( \frac{1-R_U}{R_U} \right)^2 + \frac{\frac{1-R_U}{R_U}}{R_U} \right]. \]

\[ = \frac{1}{n} \left[ \gamma_2^*(Z) R_U^2 + \gamma_2^*(e) (1-R_U)^2 + 4 R_U (1-R_U) \right]. \]
Appendix D2: Coefficient of Variation: $\frac{\text{var } s_{\text{UW}}}{(s_{\text{UW}})^2}$

From appendix C2, we have

\begin{equation}
\text{(D2.1)} \quad \frac{\text{var } s_{\text{UW}}}{(s_{\text{UW}})^2} = \frac{1}{n} \left[ \frac{b^2 \beta^2 \mu_4^*(Z) + (b^2 \sigma_e^2 + u^2) (\beta^2 \sigma_Z^2 + u^2) - b^2 \beta^2 \sigma_Z^2 h^4}{b^2 \beta^2 \sigma_Z^2 h^4} \right].
\end{equation}

Let $\gamma_2^*(\cdot) = \frac{\mu_4^*(\cdot)}{\sigma^4}$.

Dividing numerator and denominator in (D2.1) by $b^2 \beta^2 \sigma_Z^2 h^4$, we have

\begin{equation}
\text{(D2.2)} \quad \frac{1}{n} \left[ \gamma_2^*(Z) + \frac{1}{R_U} \frac{1}{R_W} - 1 \right].
\end{equation}
Appendix D3: Coefficient of Variation: \[ \frac{\text{cov}(s_U^2, s_W^2)}{(s_U^2)(s_W^2)} \]

From appendix C3, we have

\[ (D3.1) \quad \frac{\text{cov}(s_U^2, s_W^2)}{(s_U^2)(s_W^2)} = \frac{1}{n} \left[ \frac{b^2 \beta^2 \mu_h'(Z)}{(b^2 \sigma_Z^2 + \sigma_e^2)(\beta^2 \sigma_Z^2 + \sigma_e^2)} \right]. \]

Let \( \gamma_2^*(\cdot) \equiv \frac{\mu_h'(\cdot)}{\sigma}. \)

Dividing numerator and denominator in (D3.1) by \( b^2 \beta^2 \sigma_Z^4 \), (D3.1) becomes

\[ \frac{1}{n} \left[ \frac{\gamma_2^*(Z)}{\frac{1}{R_U} \frac{1}{R_W}} \right] \]

\[ = \frac{1}{n} \left[ R_U R_W \gamma_2^*(Z) \right]. \]
Appendix D4: Coefficient of Variation \( \frac{\text{cov}(s_{\text{UW}}, s^2_{U})}{(\varepsilon s_{\text{UW}})(\varepsilon s^2_{U})} \)

From appendix C4, we have

\[
(D4.1) \quad \frac{\text{cov}(s_{\text{UW}}, s^2_{U})}{(\varepsilon s_{\text{UW}})(\varepsilon s^2_{U})} = \frac{1}{n} \left[ \frac{b^3 \beta \mu^*_U(Z) + 2b^2 b \sigma^2_e}{(b^2 \sigma^2_Z + \sigma^2_e)(b^2 \sigma^2_Z)} \right].
\]

Let \( \gamma^*_2(\cdot) = \frac{\mu^*_U(\cdot)}{\sigma^*} \).

Dividing numerator and denominator in (D4.1) by \( b^3 \beta \sigma^*_Z \), (D4.1) becomes

\[
(D4.2) \quad \frac{1}{n} \left[ \frac{\gamma^*_2(Z) + 2}{\frac{\sigma^2_e}{b^2 \sigma^*_Z}} \right],
\]

where \( R_U = \frac{b^2 \sigma^2_Z}{b^2 \sigma^2_Z + \sigma^2_e} \).

Now \( \frac{1 - R_U}{R_U} = \frac{\sigma^2_e}{b^2 \sigma^*_Z} \), so (D4.2) becomes

\[
\frac{1}{n} \left[ \frac{\gamma^*_2(Z) + 2}{\frac{1 - R_U}{R_U}} \right]
\]

\[
= \frac{1}{n} \left[ \gamma^*_2(Z) R_U + 2(1 - R_U) \right].
\]
Appendix E: Models Used in Special Cases.

It should first of all be noted that all of the models in this paper assumed, either explicitly or implicitly, that there exists a linear relationship between the true scores of the tests to be equated. It is only for such a situation that it is meaningful to equate standardized scores.

Without loss of generality, then, it can be assumed that there exists a random variable (ability) such that each true score can be written as a linear function of that random variable.

So it remains to verify that the specific assumptions of the special cases in chapter 2 are indeed special cases of the general model described in section 1.2.

We first look at the Tucker assumptions.

They are:

1. \[ U_\alpha = c + d\hat{W}_\alpha + f_\alpha \] for groups 1 and T

   \[ V_\alpha = c' + d'\hat{W}_\alpha + f'_\alpha \] for groups 2 and T

2. \[ \sigma^2_{f_1} = \sigma^2_{f_T} \]

   \[ \sigma^2_{f_2} = \sigma^2_{f'_T} \]

3. \[ \text{cov}_1(W,f) = \text{cov}_T(W,f) \]

   \[ \text{cov}_2(W,f') = \text{cov}_T(W,f') \].
Assumption 1 means that the same linear relationship exists between $U_\alpha$ and $W_\alpha$ for groups 1 and T, and between $V_\alpha$ and $W_\alpha$ for groups 2 and T. In terms of the model of this paper,

$$c = a - \frac{b}{\beta} \alpha ; \quad c' = A - \frac{B}{\beta} \alpha$$

$$d = \frac{b}{\beta} ; \quad d' = \frac{B}{\beta}$$

$$f_\alpha = e_\alpha + \frac{b}{\beta} e_\alpha ; \quad f'_\alpha = E_\alpha - \frac{B}{\beta} e_\alpha.$$

Assumption 2 is equivalent to $\sigma^2_{e_1} = \sigma^2_{e_T}$, $\sigma^2_{E_2} = \sigma^2_{E_T}$, $\sigma^2_e = \sigma^2 = \sigma^2_T$. This is a special case of our model, since in general, it is not assumed that the error variances are the same for different groups.

Assumption 3 also is equivalent to $\sigma^2_{e_1} = \sigma^2_{e_2} = \sigma^2_e$.

We have thus shown that the Tucker model can be expressed as a special case of the general model.

We now treat Lord's model. Recall that Lord's formulation of the scores is

$$\begin{pmatrix} U \\ W \end{pmatrix} \sim \eta \begin{pmatrix} \mu_U \\ \mu_W \end{pmatrix} , \quad \begin{pmatrix} \sigma^2_U \\ \sigma^2_W \\ \rho_{UW} \sigma^2_U \sigma^2_W \end{pmatrix}$$

$$\begin{pmatrix} V \\ W \end{pmatrix} \sim \eta \begin{pmatrix} \mu_V \\ \mu_W \end{pmatrix} , \quad \begin{pmatrix} \sigma^2_V \\ \sigma^2_W \\ \rho_{VW} \sigma^2_V \sigma^2_W \end{pmatrix} .$$

As a special case in our model, take $Z \sim \eta(\mu_Z, \sigma^2_Z)$ for both populations (random groups assumption), $e \sim \eta(0, \sigma^2_e)$, $E \sim \eta(0, \sigma^2_E)$ and $\varepsilon \sim \eta(0, \sigma^2_\varepsilon)$. 

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Then we have

\[
\begin{pmatrix}
U \\
W
\end{pmatrix}
\sim 
\mathcal{N}
\begin{pmatrix}
\alpha + b\mu_Z \\
\alpha + b\mu_Z
\end{pmatrix},
\begin{pmatrix}
b^2 \sigma_Z^2 + \sigma_e^2 & b\beta \sigma_Z^2 \\
b\beta \sigma_Z^2 & \beta^2 \sigma_Z^2 + \sigma_e^2
\end{pmatrix}
\]

\[
\begin{pmatrix}
V \\
W
\end{pmatrix}
\sim 
\mathcal{N}
\begin{pmatrix}
\alpha + B\mu_Z \\
\alpha + b\mu_Z
\end{pmatrix},
\begin{pmatrix}
B^2 \sigma_Z^2 + \sigma_E^2 & B\beta \sigma_Z^2 \\
B\beta \sigma_Z^2 & \beta^2 \sigma_Z^2 + \sigma_e^2
\end{pmatrix}
\]

To find the special case in our model that corresponds to Lord's model, make the following identifications:

\[
\mu_U = a + b\mu_Z
\]
\[
\mu_V = A + B\mu_Z
\]
\[
\mu_W = \alpha + b\mu_Z
\]
\[
\sigma_U^2 = b^2 \sigma_Z^2 + \sigma_e^2
\]
\[
\sigma_V^2 = B^2 \sigma_Z^2 + \sigma_E^2
\]
\[
\sigma_W^2 = \beta^2 \sigma_Z^2 + \sigma_e^2
\]
\[
\rho_{VW} \sigma_V \sigma_W = B\beta \sigma_Z^2
\]

This will yield a special case in our model, though it is not determined uniquely.

We now look at Levine's assumptions. They are
(1) \[
\mu_{U_1}^{*} - \frac{\sigma_{U_1}^{*}}{2\sigma_{W_1}^{*}} \mu_{U_1}^{*} = \mu_{U_T}^{*} - \frac{\sigma_{U_T}^{*}}{2\sigma_{W_T}^{*}} \mu_{W_T}^{*}
\]

\[
\mu_{V_2}^{*} - \frac{\sigma_{V_2}^{*}}{2\sigma_{W_2}^{*}} \mu_{W_2}^{*} = \mu_{V_T}^{*} - \frac{\sigma_{V_T}^{*}}{2\sigma_{W_T}^{*}} \mu_{W_T}^{*}
\]

(2) \[
\frac{\sigma_{U_1}^{*}}{\sigma_{W_1}^{*}} = \frac{\sigma_{U_T}^{*}}{\sigma_{W_T}^{*}}
\]

\[
\frac{\sigma_{V_2}^{*}}{\sigma_{W_2}^{*}} = \frac{\sigma_{V_T}^{*}}{\sigma_{W_T}^{*}}
\]

(3) \[
\sigma_{U_1}^{2} (1-R_{U_1}) = \sigma_{U_T}^{2} (1-R_{U_T})
\]

\[
\sigma_{V_2}^{2} (1-R_{V_2}) = \sigma_{V_T}^{2} (1-R_{V_T})
\]

\[
\sigma_{W_1}^{2} (1-R_{W_1}) = \sigma_{W_T}^{2} (1-R_{W_T})
\]

\[
\sigma_{W_2}^{2} (1-R_{W_2}) = \sigma_{W_T}^{2} (1-R_{W_T}).
\]

Both assumptions (1) and (2) hold true in our model, since (1) translates to
\[ b_{\mu Z_1} - \frac{b_\sigma Z_1}{\beta \sigma Z_1} \beta_{\mu Z_1} = b_{\mu Z_T} - \frac{b_\sigma Z_T}{\beta \sigma Z_T} \beta_{\mu Z_T} \]

\[ B_{\mu Z_2} - \frac{B_\sigma Z_2}{\beta \sigma Z_2} \beta_{\mu Z_2} = B_{\mu Z_T} - \frac{B_\sigma Z_T}{\beta \sigma Z_T} \beta_{\mu Z_T} , \]

and (2) translates to

\[ \frac{b_\sigma Z_1}{\beta \sigma Z_1} = \frac{b_\sigma Z_T}{\beta \sigma Z_T} \]

\[ \frac{B_\sigma Z_2}{\beta \sigma Z_2} = \frac{B_\sigma Z_T}{\beta \sigma Z_T} . \]

It is thus only necessary to check assumption (3). It is equivalent to

\[ \sigma^2_{e_1} = \sigma^2_{e_T} \]

\[ \sigma^2_{\hat{e}_T} = \sigma^2_{\hat{e}_T} \]

\[ \sigma^2_{\hat{e}_1} = \sigma^2_{\hat{e}_2} = \sigma^2_{\hat{e}_T} . \]

Thus the Levine model corresponds to the special case in our model that assumes (E.1).
REFERENCES


