JOINT DISTRIBUTION OF THE ROOTS OF CERTAIN MATRICES IN MULTIVARIATE ANALYSIS UNDER THE RANK ONE ALTERNATIVE WITH POWER STUDIES OF THE LARGEST ROOT TEST

BY

NIRA HERRMANN

TECHNICAL REPORT NO. 41
MAY 1976

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
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INTRODUCTION

Several statistics have been proposed for the testing of the general linear hypothesis in multivariate analysis. These test statistics are based on the roots of the determinantal equations $|H-\rho E| = 0$ and $|H-\lambda(E+H)| = 0$, where $H(p \times p)$ is the sum of products matrix for the null hypothesis and $E(p \times p)$ is the sum of products matrix for error. For example, in the one way analysis of variance, for testing the equality of means in $k$ populations, we have

$E = \sum_{i=1}^{k} \sum_{j=1}^{N_i} (Y_{ij} - \bar{Y}_i)(Y_{ij} - \bar{Y}_i)'$

and

$H = \sum_{i=1}^{k} N_i (\bar{Y}_i - \bar{Y})(\bar{Y}_i - \bar{Y})'$

where $Y_{ij}(p \times 1)$ is the $j^{th}$ observation on $p$ variates from the $i^{th}$ population; $\bar{Y}_i(p \times 1)$ is the mean of the $i^{th}$ population; $\bar{Y}(p \times 1)$ is the overall mean and $N_i$ is the number of observations from the $i^{th}$ population ($j = 1, \ldots, N_i; i = 1, \ldots, k$).

Some of the proposed statistics are:

1. $\hat{\lambda}_1$ or $\lambda_1$, the largest root, (Roy 1953):

2. $\sum_{i=1}^{p} \hat{\lambda}_1 = \text{tr } HE^{-1}$, the Lawley-Hotelling trace, (Lawley 1938; Hotelling 1947);

3. $\sum_{i=1}^{p} \hat{\lambda}_i = \text{tr } (H+E)^{-1}$, the Pillai trace, (Pillai 1955);

4. $\prod_{i=1}^{p} (1+\hat{\lambda}_i)^{-1} = \frac{|E|}{|E+H|}$, the likelihood ratio test, (Wilks 1932).
It should be noted that there is a one-to-one transformation relating the \( \{\hat{\lambda}_i\} \) and the \( \{\lambda_i\} \) which is given by

\[
\lambda_i = \frac{\hat{\lambda}_i}{1 + \hat{\lambda}_i}, \quad i = 1, \ldots, p.
\]

A natural question arising from the availability of several test statistics for the same hypothesis is: which is the best? To assess the performance of the test statistics, their behavior under the null and alternative hypotheses can be compared. Since the roots are invariant, tests based on the roots are invariant tests and admissibility has been shown for these tests by Kiefer and Schwartz (1965). The joint distribution of the roots \( \{\lambda_i\} \) in the central case was found independently by Fisher (1939), Girshick (1939), Hsu (1939) and Roy (1939) and is stated in Theorems 1.1\* and 1.2\*. Central distributions for some of the other statistics have also been found (e.g., Nanda (1948a), (1951); Pillai (1956a), (1964a), (1964b), (1965); Heck (1960); Bagai (1962b); Ito (1962); Olkin and Rubin (1964); Pillai and Jayachandran (1967); Pillai and Young (1971); Krishnaiah and Chang (1971a), (1971b), (1972); Krishnaiah and Waiker (1971a); Krishnaiah, Schuurmann, and Waiker (1973)); and tables of percentage points have been published (e.g., Nanda (1951); Pillai (1956a), (1964a), (1965); Ito (1962); Pillai and Jayachandran (1967); Pillai and Young (1971); Krishnaiah, Schuurmann and Waiker (1973); Harris (1975)).

The non-central distributions of the roots and statistics based on these roots are considerably more complex to derive. A variety of
approaches have been tried and are summarized in Kshirsagar (1972), 
Mathai (1973) and Crowther and Young (1974). Among the approaches 
used for both central and non-central distributions are asymptotic 
expansions (T. W. Anderson (1948), (1958); Ito (1956), (1960); G. A. 
Anderson (1965), (1970); Chang (1970); Bingham (1972)) and expansions 
using zonal polynomials (James (1960), (1964); Constantine (1963); 
Tumura (1965); Khatri (1967); Pillai and Jayachandran (1967); Pillai 
and Al-Ani (1970); Krishnaiah and Chang (1971a), (1971b); Krishnaiah 
and Waikar (1971a), (1971b). Additional results have been obtained by 
restricting the non-null case to be linear or planar (Anderson and 
Girshick (1944); Sitgreaves (1952); Kshirsagar (1961), (1972); Bagai 
(1962a); Khatri and Pillai (1965); A. K. Gupta (1971)). 

Studies have also been made to compare the power of these statistics. 
Monotonicity of the power functions has been shown (Roy and Mikhail (1961); 
Anderson, Das Gupta, and Mudholker (1964); Eaton and Perlman (1974)) and 
bounds on the power function investigated (Roy (1957)). Monte Carlo 
studies have been done to compare the various tests and zonal poly-
nomial and other expansions have been computed for several cases (Ito 
(1962); Dempster and Schatzoff (1965); Gnanadesikan et al (1965); 
Pillai and Jayachandran (1967); Lee (1971); Olson (1974)). However, 
the zonal polynomial expansions converge very slowly in some cases 
and the polynomials are currently available only through degree twelve. 
Power studies based on these expansions have used zonal polynomials 
through degree six (see Pillai's work).
In this study we investigate the joint distribution of the roots \( \{\lambda_i\} \) and \( \{\varphi_i\} \) and the behavior of the largest root under the rank one alternative (the non-null linear case). We use the techniques of Sitgreaves (1952), Bowker (1961) and Tumura (1965). In Chapter 1 we present the canonical form for the problem as developed by S. N. Roy (1957). This canonical form is used to derive the joint densities of the roots \( \{\lambda_i\} \) and also of the \( \{\varphi_i\} \) in the non-null (linear) case. The distribution of the largest root is found in Chapter 2 using the results of the first chapter. Chapter 3 contains the derivations of the limiting joint distributions of the roots and the limiting distribution for the largest root as the sample size goes to infinity.

Finally, in Chapter 4, we present tables of the power function for the largest root test and power comparisons of the largest root test with the other tests listed in (1)-(4). The appendix contains copies of the programs used to obtain the tables in Chapter 4.

**Notation**

Unless otherwise specified, we use the following notation. Capital letters denote matrices or vectors; e.g., \( A (p \times q) \) is a rectangular matrix with \( p \) rows and \( q \) columns. The determinant of a matrix \( A (p \times p) \) is denoted by \( |A| \) and the trace by \( \text{tr} \ A \). The elements of \( A (p \times q) \) are denoted by \( a_{ij} \), \( i = 1, \ldots, p; \ j = 1, \ldots, q \). The notation \( D_\gamma (p \times p) \) is used for a diagonal matrix which has the elements \( \gamma_1, \ldots, \gamma_p \) on the diagonal and zeros elsewhere; \( T (p \times p) \) denotes a lower triangular matrix and \( I_p \) denotes the \( (p \times p) \) identity matrix.
The capital letter \( J \) is used as a generic and in each case refers to the Jacobian of the transformation being discussed. The \( p \)-variate normal distribution is denoted by \( \mathcal{N}_p(\mu, \Sigma) \), where \( \mu(p \times 1) \) is the mean vector and \( \Sigma(p \times p) \) is the covariance matrix. The central Wishart distribution is denoted by \( \mathcal{W}_p(n, \Sigma) \), where \( \Sigma \) is a \( (p \times p) \) matrix and \( n \) is the degrees of freedom. The symbol "\( \sim \)" means "distributed as"; e.g., \( A(p \times p) \sim \mathcal{W}_p(n, \Sigma) \) means that the elements of the matrix \( A \) have a joint central Wishart distribution. The symbol "\( \text{d}A \)" is a shorthand for \( \prod_{i,j} \text{d}a_{ij} \), where \( A \) is a matrix with the elements \( \{a_{ij}\} \). Note that for a symmetric matrix, \( \text{d}A \) represents each element only once: \( \text{d}A = \prod_{i \leq j} \text{d}a_{ij} \).

The remaining notation is defined as it is needed.
1. THE JOINT DISTRIBUTION OF THE CHARACTERISTIC ROOTS UNDER THE RANK ONE ALTERNATIVE

1.1 Introduction

In this chapter we derive the non-null (linear) distribution of the roots of the equation $|H - \phi E| = 0$, with $H(p \times p)$ and $E(p \times p)$, using the canonical form for the problem developed by S. N. Roy (1957). The non-null (linear) alternative denotes the case of only one non-zero root of the equation $|\eta \eta' - \gamma \gamma'| = 0$, where $\eta(p \times n_1)$ is the matrix of mean differences and $\gamma(p \times p)$ is the covariance matrix, and corresponds to the situation of all the populations having equal means except for one in the one way analysis of variance. To obtain the canonical form, we begin with

$$H = Y_1'Y_1' \quad \text{and} \quad E = Y_2'Y_2'$$

where $Y_1(p \times n_1)$ has expectation $EY_1 = \eta$, the columns of $Y_1$ are distributed independently as $N_p(\eta_{i}, \frac{\gamma}{\lambda})$, $i = 1, \ldots, n_1$ and $\eta_{i}(p \times 1)$ is the $i^{th}$ column of $\eta$; $Y_2(p \times n_2)$ has independently and identically distributed columns distributed as $N_p(0, \frac{\gamma}{\lambda})$; and $Y_1$ is distributed independently of $Y_2$. Additionally, we have $n_2 \geq p \geq n_1$ or $n_2 \geq n_1 > p$. This structure follows from the usual assumptions of normality in the multivariate analysis of variance (see Anderson (1958), Chapter 8).

Since $\gamma$ is positive definite and $\eta \eta'$ is positive semidefinite, there is a non-singular matrix $L(p \times p)$ such that
\[ L^H L' = I_p \quad \text{and} \quad \det L = D_\gamma = \text{diag}(\gamma_1, \ldots, \gamma_p), \]

where the \( \{\gamma_i, i = 1, \ldots, p\} \) are the solutions of the determinantal equation \( |\eta \eta' - \gamma \eta \gamma'| = 0 \) and \( \gamma_p \leq \cdots \leq \gamma_1 \). Note that at most \( s = \min(p, n) \) of the \( \{\gamma_i, i = 1, \ldots, p\} \) are positive and the rest are zero. In addition, there exists an orthogonal matrix \( R(n \times n) \) such that

\[ \xi = L\eta R = \begin{pmatrix} D_{\sqrt{\gamma_1}} & 0 \\ 0 & 0 \end{pmatrix}, \]

where \( D_{\sqrt{\gamma_i}}(s \times s) \) is a diagonal matrix with \( \{\sqrt{\gamma_i}, i = 1, \ldots, s\} \) along the diagonal.

We transform to \( X_1(p \times n) \) and \( X_2(p \times n) \), where

\[ \begin{array}{l}
X_1 = L Y R \\
X_2 = LX_1, \end{array} \]

to obtain the canonical form for the problem. Then \( \xi X_1 = \xi \) and the columns of \( X_1 \) are distributed independently with normal distributions and covariance matrix \( I_p \) independently of the columns of \( X_2 \), which are distributed identically and independently as \( N_p(0, I_p) \). The null distribution can now be characterized by

\[ H_0: \gamma_1 = \cdots = \gamma_s = 0 \]

and the various alternatives can be characterized by
\[ H_j: \gamma_1, \ldots, \gamma_j > 0 ; \quad \gamma_{j+1} = \ldots = \gamma_s = 0 \quad (j = 1, \ldots, s) \]
since only the first \( s \) roots can be greater than zero. It remains only to notice that the roots of \( |H - \rho E| = |Y_1 Y'_1 - \rho Y_2 Y'_2| = 0 \) are the same as the roots of \( |X_1 X'_1 - \rho X_2 X'_2| = 0 \) since
\[
|Y_1 Y'_1 - \rho Y_2 Y'_2| = |LX_1 RR'X'_1 L' - \rho LX_2 X'_2 L'| = |L||X_1 X'_1 - \rho X_2 X'_2||L'|
\]
and \( |L| \neq 0 \).

So without loss of generality, we can set \( H = X_1 X'_1 \) and \( E = X_2 X'_2 \). The linear alternative is \( H_1: \gamma_1 > 0; \gamma_2 = \ldots = \gamma_s = 0 \) and we shall obtain the joint distribution of the roots \( \{\rho_i, i = 1, \ldots, s\} \) under \( H_1 \) by first obtaining the joint distribution of the roots \( \{\lambda_i, i = 1, \ldots, s\} \), which are the roots of the determinantal equation \( |H - \lambda (H + E)| = 0 \), and then making the transformation \( \lambda_i = \rho_i (1 + \phi_1)^{-1}, \quad i = 1, \ldots, s \).

1.2 Central Case Results

For the sake of comparison, we give the joint densities of the \( \{\rho_i\} \) and \( \{\lambda_i\} \) in the central case. The results were derived independently by Fisher (1959), Girshick (1959), Hsu (1959) and S. N. Roy (1939). The results and derivations can be found in Anderson (1958, Chapter 13) for the case of two Wishart matrices. We state here the two main results using our notation.

Case 1: \( n_1 \leq p \)
Theorem 1.1*:

(i) If \( H(p \times p) \) is distributed as \( \Sigma_{\alpha=1}^{n} \xi_{\alpha} \xi_{\alpha}' \), where the \( \xi_{\alpha}(p \times 1) \) are independent, each with the distribution \( N_{p}(0, \Sigma) \), \( n_{1} \leq p \), and \( E(p \times p) \) is independently distributed according to \( \mathcal{W}_{p}(n_{2}, \Sigma) \), \( n_{2} \geq p \), then the density of the non-zero roots of \( |H-\lambda(H+E)| = 0 \) is given by

\[
(1.1) \quad p_{0}(\lambda_{1}, \ldots, \lambda_{n_{1}}) = C_{1} \cdot \prod_{i=1}^{n_{1}} \left\{ \lambda_{1}^{\frac{1}{2}(p-n_{1}-1)} (1-\lambda_{1})^{\frac{1}{2}(n_{2}-p-1)} \right\} \cdot \prod_{i<j} (\lambda_{i}-\lambda_{j})
\]

for \( 0 \leq \lambda_{1} \leq \cdots \leq \lambda_{2} \leq \lambda_{n_{1}} \leq 1 \), where

\[
C_{1} = \pi^{\frac{1}{2}n_{1}} \cdot \prod_{i=1}^{n_{1}} \Gamma\left(\frac{1}{2}(n_{1}+n_{2}+p-1)\right)
\]

\[
(1.2) \quad \cdot \left\{ \prod_{i=1}^{n_{1}} \Gamma\left(\frac{1}{2}(n_{1}+n_{2}+p-1)\right) \Gamma\left(\frac{1}{2}(n_{1}+n_{2}+p-1)\right) \Gamma\left(\frac{1}{2}(n_{1}+n_{2}+p-1)\right) \right\}^{-1}.
\]

(ii) Under the same assumptions as in (i), the density of the non-zero roots of \( |H-\phi E| = 0 \) is given by

\[
p_{0}(\phi_{1}, \ldots, \phi_{n_{1}}) = C_{1} \cdot \prod_{i=1}^{n_{1}} \left\{ \phi_{1}^{\frac{1}{2}(p-n_{1}-1)} (1+\phi_{1})^{\frac{1}{2}(n_{1}+n_{2})} \right\} \cdot \prod_{i<j} (\phi_{i}-\phi_{j})
\]

for \( 0 \leq \phi_{1} \leq \cdots \leq \phi_{2} \leq \phi_{n_{1}} \leq \infty \), where \( C_{1} \) is given in (1.2).

Case 2: \( n_{1} > p \)

The results for \( n_{1} > p \) can be obtained from Theorem 1.1* by
replacing $n_1$ by $p$, $p$ by $n_1$ and $n_2$ by $n_1 + n_2 - p$ in the expressions for the densities to get

**Theorem 1.2**: 

(i) If $H$ and $E$ are distributed independently according to $w_p(n_1, \frac{p}{n_1})$ and $w_p(n_2, \frac{p}{n_2})$, respectively, $(n_1 \geq p, n_2 \geq p)$, the joint density of the roots of $|H - \lambda(H+E)| = 0$ is given by

$$p_0(\lambda_1, \ldots, \lambda_p) = C_2 \cdot \prod_{i=1}^{p} \left\{ \frac{1}{\lambda_i^{\frac{3}{2}(n_1-p-1)}} \left(1-\lambda_i^{-\frac{3}{2}(n_2-p-1)} \right) \right\} \cdot \prod_{i<j} \left(\lambda_i - \lambda_j \right)$$

for $0 \leq \lambda_1 \leq \ldots \leq \lambda_p \leq 1$, where

$$C_2 = \frac{\pi^{3p}}{\left( \prod_{i=1}^{p} \Gamma(\frac{1}{2}(n_1+n_2+1-i)) \right)^{\frac{1}{2}} \cdot \left( \prod_{i=1}^{p} \Gamma(\frac{1}{2}(n_2+1-i)) \Gamma(\frac{1}{2}(n_1+1-i)) \Gamma(\frac{1}{2}(p+1)) \right)^{-1}}$$

(1.3)

(ii) Under the same assumptions as in (i), the density of the non-zero roots of $|H - \rho E| = 0$ is given by

$$p_0(\rho_1, \ldots, \rho_p) = C_2 \cdot \prod_{i=1}^{p} \left\{ \rho_i^{\frac{1}{2}(n_1-p-1)} \left(1+\rho_i^{-\frac{1}{2}(n_1+n_2)} \right) \right\} \cdot \prod_{i<j} \left(\rho_i - \rho_j \right)$$

for $0 \leq \rho_{p_1} \leq \ldots \leq \rho_1 < \infty$, where $C_2$ is given in (1.3).
1.3 Linear Case Results

Before presenting the derivations of the joint distributions of the roots in the non-null (linear) case, we give the analogues of Theorems 1.1* and 1.2*:

Case 1: \( n_1 \leq p \)

Theorem 1.1:

(i) Let \( Y_1(p \times n_1) \) and \( Y_2(p \times n_2) \) be distributed independently such that the columns of \( Y_1 \) are distributed independently as \( N_p(\xi_i, \Phi) \), \( i = 1, \ldots, n_1 \), where \( \xi = (\xi_1, \ldots, \xi_{n_1}) = \xi Y_1 \) is of rank 1; and the columns of \( Y_2 \) are distributed independently, each as \( N_p(0, \Phi) \). Let \( \gamma_1 \) be the nonzero solution of \( |\xi^{\prime}_i - \gamma_1^2| = 0 \). Then, if \( n_2 \geq p \geq n_1 \), the density of the non-zero roots of \( |Y_1Y_1' - \lambda (Y_1Y_1' + Y_2Y_2')| = 0 = |H - \lambda (H + E)| \), where \( H = Y_1Y_1' \) and \( E = Y_2Y_2' \), is given by

\[
P_{\gamma_1}(\lambda_1, \ldots, \lambda_{n_1}) = e^{-\frac{1}{2} \gamma_1} \cdot C_1 \cdot \prod_{i=1}^{n_1} \left\{ \lambda_i^\frac{1}{2} (p-n_1-1) \left(1 - \lambda_i\right)^{\frac{1}{2}} \frac{(n_2-p-1)}{\pi} \right\} \cdot \prod_{i<j} (\lambda_i - \lambda_j) \]

\[
\cdot \left(1 + \sum_{k_0=1}^k \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+2k_0)\right) \Gamma\left(\frac{1}{2}p\right) \Gamma\left(\frac{1}{2}n_1\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2)\right) \Gamma\left(\frac{1}{2}p+k_0\right) \Gamma\left(\frac{1}{2}n_1+k_0\right)} \cdot \left(\frac{\gamma_1}{2}\right)^{k_0} \right) .
\]

\[
\sum_{k_1=0}^k \sum_{k_2=0}^{k_1} \sum_{k_{n_1-1}=0}^{k_{n_1-2}} \left(\prod_{i=1}^{n_1-1} \frac{\Gamma(k_{i-1}-k_{i-1}+\frac{1}{2})}{\Gamma(k_{i-1}+1)} \cdot \frac{\Gamma(k_{n_1-1}-\frac{1}{2})}{\Gamma(k_{n_1-1}+\frac{1}{2})} \cdot \frac{\Gamma(k_{n_1-1}+\frac{1}{2})}{\Gamma(k_{n_1-1}+1)} \cdot \frac{1}{\pi}\right) \cdot \left(\prod_{i=1}^{n_1-1} \lambda_i \right) \right) \cdot \left(\prod_{i=1}^{n_1} \lambda_i \right) \}
\]

for \( 0 \leq \lambda_{n_1} \leq \cdots \leq \lambda_1 \leq 1 \) and \( C_1 \) as in (1.2).
(ii) Under the same assumptions, the density of the non-zero roots of 

\[ |H - \lambda (H + E)| = 0 \text{ for } n_2 \geq p \geq n_1 \]

is given by

\[
P_{\gamma_1}(\phi_1, \ldots, \phi_{n_1}) = e^{-\frac{1}{2} \gamma_1} \cdot C_1 \cdot \prod_{i=1}^{n_1} \left\{ \phi_i^{\frac{1}{2}(p-n_1-1)} \left( 1 + \phi_i \right)^{-\frac{1}{2}(n_1+n_2)} \right\} \cdot \prod_{i<j} (\phi_i - \phi_j)
\]

\[ \cdot \left\{ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(\frac{1}{2}(n_1+n_2)+k_0)\Gamma(\frac{1}{2}p)\Gamma(\frac{1}{2}n_2)}{\Gamma(\frac{1}{2}(n_1+n_2))\Gamma(\frac{1}{2}p+k_0)\Gamma(\frac{1}{2}n_2+k_0)} \cdot \left( \frac{\gamma_1}{2} \right)^{k_0} \right\} \]

\[ \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{n_1-1}=0}^{k_{n_1-2}} \frac{\Gamma(k_{i-1}-k_i+\frac{1}{2})}{\Gamma(k_{i-1}-k_i+1)^2} \right\} \cdot \frac{\Gamma(k_{n_1-1}+\frac{1}{2})}{\Gamma(k_{n_1-1}+1)^2} \]

\[
\begin{bmatrix}
\left( \frac{n_1-1}{i=1} \left( \frac{\phi_i^{k_i-1}k_i}{i+\phi} \right) \right)
\left( \frac{n_1-1}{i=1} \left( \frac{\phi_i^{k_i-1}k_i}{i+\phi} \right) \right)
\end{bmatrix}
\]

for \( 0 \leq \phi_1 \leq \cdots \leq \phi_{n_1} < \infty \) and \( C_1 \) as in (1.2).

Case 2: \( n_1 > p \)

As in the central case, the results for \( n_1 > p \) can be obtained from Theorem 1.1 by replacing \( n_1 \) by \( p \), \( p \) by \( n_1 \) and \( n_2 \) by \( n_1 + n_2 - p \) to get

Theorem 1.2:

(i) For \( H \) and \( E \) as in Theorem 1.1 and \( n_2 \geq n_1 > p \), the density of the non-zero roots of \( |H - \lambda (H + E)| = 0 \) is given by
\[ p_{\gamma_1}(\lambda_1, \ldots, \lambda_p) = e^{-\frac{1}{2} \gamma_1} \cdot C_2 \cdot \prod_{i=1}^p \left\{ \frac{1}{2}(n_1-p-1) \right\} \cdot \prod_{i<j} (\lambda_i \cdot \lambda_j) \]

\[ \cdot \left\{ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_1+n_2)+k_0\right)\Gamma\left(\frac{1}{2}p\right)\Gamma\left(\frac{1}{2}n_1\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2)\right)\Gamma\left(\frac{1}{2}p+k_0\right)\Gamma\left(\frac{1}{2}n_1+k_0\right)} \cdot (\frac{\gamma_1}{2})^{k_0} \right\} \]

\[ \sum_{k_1=0}^{k_0} \cdots \sum_{k_{p-1}=0}^{k_{p-2}} \left( \frac{p-1}{p} \cdot \frac{\Gamma(k_{i-1}+\frac{1}{2})}{\Gamma(k_{i-1}+\frac{1}{2})} \cdot \frac{\Gamma(k_{p-1}+\frac{1}{2})}{\Gamma(k_{p-1}+\frac{1}{2})} \cdot \left( \frac{\lambda_1}{\lambda_{p-1}} \right) \cdot \frac{\gamma_{1-p}}{\gamma_{1-p}} \cdot \prod_{i=1}^{p-1} (\phi_{i-1} \cdot \phi_i) \right\} \]

for \( 0 \leq \lambda_p \leq \cdots \leq \lambda_1 \leq 1 \) and \( C_2 \) as in (1.3).

(ii) For \( H \) and \( E \) as in Theorem 1.1 and \( n_2 \geq n_1 > p \), the density of the non-zero roots of \( |H-\phi E| = 0 \) is given by

\[ p_{\gamma_1}(\phi_1, \ldots, \phi_p) = e^{-\frac{1}{2} \gamma_1} \cdot C_2 \cdot \prod_{i=1}^p \left\{ \frac{1}{2}(n_1-p-1) \right\} \cdot \prod_{i<j} (\phi_i \cdot \phi_j) \]

\[ \cdot \left\{ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_1+n_2)+k_0\right)\Gamma\left(\frac{1}{2}p\right)\Gamma\left(\frac{1}{2}n_1\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2)\right)\Gamma\left(\frac{1}{2}p+k_0\right)\Gamma\left(\frac{1}{2}n_1+k_0\right)} \cdot (\frac{\gamma_1}{2})^{k_0} \right\} \]

\[ \sum_{k_1=0}^{k_0} \cdots \sum_{k_{p-1}=0}^{k_{p-2}} \left( \frac{p-1}{p} \cdot \frac{\Gamma(k_{i-1}+\frac{1}{2})}{\Gamma(k_{i-1}+\frac{1}{2})} \cdot \frac{\Gamma(k_{p-1}+\frac{1}{2})}{\Gamma(k_{p-1}+\frac{1}{2})} \cdot \left( \frac{\phi_1}{\phi_{p-1}} \right) \cdot \prod_{i=1}^{p-1} (\phi_{i-1} \cdot \phi_i) \right\} \]

for \( 0 \leq \phi_p \leq \cdots \leq \phi_1 < \infty \) and \( C_2 \) as in (1.3).
It should be noted that the non-central density becomes the central
density when $\gamma_1 = 0$. The non-central density can be expressed in
slightly different forms for computing ease (see Chapter 4). The
derivation of the results in Theorems 1.1 and 1.2 shall be in three
cases: $n_1 < p$, $n_1 > p$ and $n_1 = p$ and is based on the canonical form
developed above (Section 1.1).

1.4 Linear Case Derivation - Case 1: $n_1 < p$

The derivation of the distribution of the $\{\phi_i, i = 1, \ldots, n_1\}$ involves several steps:

Step 1: Derivation of the joint density of the elements of the
$(n_1 \times n_1)$ matrix $\tilde{H} = X_1'X_1$, which has a non-null (linear)
Wishart distribution.

Step 2: Derivation of the joint density of the elements of the
$(n_1 \times n_1)$ matrix $\tilde{X} = (\tilde{X} + \tilde{H})^{-\frac{1}{2}} \tilde{H}(\tilde{X} + \tilde{H})^{-\frac{1}{2}}$ whose characteristic
roots are $\{\lambda_i, i = 1, \ldots, n_1\}$.

Step 3: Use of the Tumura (1965) representation of orthogonal
matrices to obtain the joint density of $\{\lambda_i, i = 1, \ldots, n_1\}$ -
Theorem 1.1 (i).

Step 4: Transformation from $\{\lambda_i, i = 1, \ldots, n_1\}$ to
$\{\phi_i, i = 1, \ldots, n_1\}$ - Theorem 1.1 (ii).

Step 1:

We begin by noting that the $\{\phi_i, i = 1, \ldots, n_1\}$ are the
characteristic roots of the $(n_1 \times n_1)$ matrix $\Phi = X_1'E^{-1}X_1$ and that
$E \sim W_p(n_2, I_p)$ since $E = X_2X_2'$. There exists an orthogonal $(p \times p)$ matrix $G$ such that

$$X_1'G' = \begin{pmatrix} \tilde{H}^{\frac{1}{2}} & 0 \\ n_1X_1 & n_1X(p-n_1) \end{pmatrix},$$

where $\tilde{H}^{\frac{1}{2}}$ is the lower triangular matrix square root of $\tilde{H} = X_1'X_1$ with positive elements along the diagonal. This gives

$$\tilde{\Phi} = X_1'G'E\tilde{G}' \tilde{G}X_1$$

$$= (\tilde{H}^{\frac{1}{2}} 0)E^{x-1} \begin{pmatrix} \tilde{H}^{\frac{1}{2}}' \\ 0 \end{pmatrix}$$

$$= \tilde{H}^{\frac{1}{2}}E^{x-1} \tilde{H}^{\frac{1}{2}}'.$$

where $E^{x-1}(n_1X_1)$ is the upper left hand corner of $E^{x-1}$.

Now $G$ is a random orthogonal matrix independent of $E$ so that $E^* = GE_1G' \sim W_p(n_2, I_p)$. Furthermore, if we set $\tilde{A} = (E^{x-1})^{-1}$, we have that $\tilde{A} \sim W_{n_1}(n_2-p+n_1, I_{n_1})$ (see, for example, Bowker (1961)). Hence we seek the distribution of the characteristic roots of the matrix $\tilde{\Phi} = \tilde{H}^{\frac{1}{2}}A^{-1}\tilde{H}^{\frac{1}{2}}$, where $\tilde{H}$ and $\tilde{A}$ are independently distributed. We have the distribution of $\tilde{A}$ and need to obtain the distribution of $\tilde{H}$.

Recall that $\tilde{H} = X_1'X_1$ and that
\[ \mathbf{x}_{\mathbf{1}}^{*} = \begin{bmatrix} \gamma_{1}^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1n_{1}} \\ x_{21} & \cdots & x_{2n_{1}} \\ \vdots & \vdots & \vdots \\ x_{pl} & \cdots & x_{pn_{1}} \end{bmatrix} = \begin{bmatrix} \tilde{x}_{1} \\ x_{1}^{*} \end{bmatrix}, \]

where \( \tilde{x}_{1} = (x_{11}, \ldots, x_{1n_{1}}) \) is the first row of \( X_{1} \) and \( x_{1}^{*} (\mathbf{p-1} \times n_{1}) \) is given by

\[ X_{1}^{*} = \begin{bmatrix} x_{21} & \cdots & x_{2n_{1}} \\ \vdots & \vdots & \vdots \\ x_{pl} & \cdots & x_{pn_{1}} \end{bmatrix}. \]

Then \( \mathcal{E}X_{1}^{*} = 0 \) and

\[ \tilde{H} = X_{1}^{*}X_{1} = \tilde{x}_{1}^{\prime}\tilde{x}_{1} + X_{1}^{*}X_{1} = \tilde{x}_{1}^{\prime}\tilde{x}_{1} + \tilde{H}^{*}, \]

where \( \tilde{H}^{*} = X_{1}^{*}X_{1}^{*} \) is distributed as \( W_{n_{1}} (p-1, I_{n_{1}}) \) independently of

\[ \tilde{x}_{1}^{\prime} \sim N_{n_{1}} \begin{bmatrix} \gamma_{1}^{\frac{1}{2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, I_{n_{1}}. \]

The joint distribution of \( \tilde{H}^{*} \) and \( \tilde{x}_{1} \) is

\[ p(\tilde{H}^{*}, \tilde{x}_{1}) = 2^{\frac{1}{2} n_{1} - 2} \pi \left( \frac{1}{2} \right)^{\frac{n_{1}}{2}} \Gamma\left( \frac{1}{2} (p-1) \right)^{-1} \]

\[ \cdot e^{- \frac{1}{2} \left( \frac{1}{p-1} n_{1} \right)^{2} - \frac{1}{2} x_{11}^{2}} e^{- \frac{1}{2} x_{22}^{2}} - \cdots - \frac{1}{2} x_{n_{1}n_{1}}^{2}. \]
Let \( \tilde{H} = \tilde{H}^* + \tilde{x}_1^2 \). Then \( \text{tr} \tilde{H} = \text{tr} \tilde{H}^* + \tilde{x}_{11}^2 + \tilde{x}_{12}^2 + \cdots + \tilde{x}_{1n_1}^2 \) and

\[
p(\tilde{H}, \tilde{x}_1) = C_3 \cdot e^{-\frac{1}{2} \text{tr} \tilde{H} + \frac{1}{2} \tilde{x}_{11}^2} \frac{1}{n_1^{\frac{1}{2} (p-n_1-2)}} |\tilde{H} - \tilde{x}_1^2|^{-\frac{1}{2} (p-n_1-2)},
\]

where

\[
C_3 = \left( 2^{\frac{1}{2} \pi n_1 (n_1-1)/4 + \frac{1}{2} n_1 \prod_{i=1}^{n_1} \Gamma\left( \frac{1}{2} (p-i) \right) \right)^{-1}.
\]

Since \( \tilde{H} \) is symmetric and positive definite with probability one, there exists a non-singular, lower triangular matrix \( T(n_1 x_{11}) \), with \( t_{11} = \tilde{H}_{11}^* \), such that \( \tilde{H} = TT' \). If we make the transformation \( \tilde{U} = T^{-1} \tilde{x}' \), then the Jacobian is \( J = |T| = |\tilde{H}|^{\frac{1}{2}} \), \( x_{11} = \tilde{H}_{11}^* \tilde{u}_1 \), where \( \tilde{U}' = (\tilde{u}_1, \ldots, \tilde{u}_{n_1}) \), and

\[
p(\tilde{H}, \tilde{U}) = C_3 \cdot |\tilde{H}|^{\frac{1}{2} (p-n_1-1)} \cdot |I-\tilde{U}U'|^{\frac{1}{2} (p-n_1-2)}
\]
\[
\cdot e^{-\frac{1}{2} \text{tr} \tilde{H} + \frac{1}{2} \tilde{x}_{11}^2} \tilde{u}_1 \gamma_{11}^{-\frac{1}{2}} \gamma_1
\]
\[
= C_3 \cdot |\tilde{H}|^{\frac{1}{2} (p-n_1-1)} \cdot (1-\tilde{u}_1^2 - \cdots - \tilde{u}_{n_1}^2)^{\frac{1}{2} (p-n_1-2)}
\]
\[
\cdot e^{-\frac{1}{2} \text{tr} \tilde{H} + \frac{1}{2} \tilde{x}_{11}^2} \tilde{u}_1 \gamma_{11}^{-\frac{1}{2}} \gamma_1.
\]

We have used the fact that \( |I-\tilde{U}U'| = |I-\tilde{U} \tilde{U}| \) to obtain the last expression. To get the marginal distribution of \( \tilde{H} \), we must integrate out the elements of \( \tilde{U} \) over the space where \( I-\tilde{U}U' \) is positive definite. (This follows from the fact that \( \tilde{H}^* \) is positive definite
with probability one). We proceed by making the transformation
\[
v_1 = \tilde{u}_2 / (1 - \tilde{u}_1^2)^{1/2}
\]
\[
v_2 = \tilde{u}_3 / (1 - \tilde{u}_1^2)^{1/2}(1 - v_1^2)^{1/2}
\]
\[\vdots\]
\[
v_{n_1-1} = \tilde{u}_{n_1} / ((1 - \tilde{u}_1^2)^{1/2} \prod_{i=1}^{n_1-2} (1 - v_i^2)^{1/2})
\]
where, by convention, \(\prod_{i=1}^{n_1-2} (1 - v_i^2)^{1/2} = 1\) if \(n_1 = 1\) or \(2\) and the Jacobian is
\[
J = (1 - \tilde{u}_1^2)^{n_1-2} \prod_{i=1}^{n_1-2} (1 - v_i^2)^{n_1-1-i}.
\]
This gives
\[
p(\tilde{H}, \tilde{u}_1, v_1, \ldots, v_{n_1-1}) = C_3 \cdot |\tilde{H}|^{1/2(p-n_1-1)} \cdot (1 - \tilde{u}_1^2)^{1/2(p-3)}
\]
\[(1.4)\]
\[
\cdot \prod_{i=1}^{n_1-1} (1 - v_i^2)^{1/2(p-3-i)} \cdot e^{-\frac{1}{2} tr \tilde{H} + \frac{1}{2} \tilde{u}_1^2 + \frac{1}{2} \tilde{u}_1^2 \gamma_1^2}.
\]
where \(\tilde{u}_1\) and \(v_i, i = 1, \ldots, n_1-1\) range over the interval \((-1, 1)\).

To integrate out \(\tilde{u}_1\), we expand \(e^{\frac{1}{2} \tilde{u}_1^2 \gamma_1^2 + \frac{1}{2} \tilde{u}_1^2 \gamma_1^2}\) in a power series and, after interchanging the order of summation and integration, integrate term by term to obtain...
\[
\int_{-1}^{1} (1-\tilde{u}_1^2)^{\frac{1}{2}}(p-3) e^{\frac{1}{2}k_0 \tilde{u}_1 \gamma_1} d\tilde{u}_1 \\
= \sum_{k_0=0}^{\infty} \frac{\frac{1}{2}k_0}{k_0} \int_{-1}^{1} (1-\tilde{u}_1^2)^{\frac{1}{2}}(p-3) k_0 \tilde{u}_1 d\tilde{u}_1 \\
= \sum_{k_0=0}^{\infty} \frac{\frac{1}{2}k_0}{\Gamma(2k_0+1)} \int_{0}^{\pi/2} (1-\tilde{u}_1^2)^{\frac{1}{2}}(p-3) 2k_0 \tilde{u}_1 d\tilde{u}_1 \\
= \sum_{k_0=0}^{\infty} \frac{\frac{1}{2}k_0}{\Gamma(2k_0+1)} \int_{0}^{\pi/2} \sin^{p-2} \theta \cos \theta d\theta \\
= \sum_{k_0=0}^{\infty} \frac{\Gamma(\frac{1}{2}(p-1)) \Gamma(k_0+\frac{1}{2}) k_0 k_0}{\Gamma(k_0+\frac{1}{2}p) \Gamma(k_0+1)} \frac{k_0 k_0}{\tilde{u}_1 \gamma_1} \\
= \sum_{k_0=0}^{\infty} \frac{\sqrt{\pi} \Gamma(\frac{1}{2}(p-1))}{\Gamma(k_0+\frac{1}{2}p) \Gamma(k_0+1)^2} \frac{k_0 k_0}{\tilde{u}_1 \gamma_1},
\]

where we have used the duplication formula to simplify:

\[
\Gamma(k_0+\frac{1}{2}) = \frac{\Gamma(2k_0+1) \sqrt{\pi}}{2k_0}.
\]

Integrating the \( v_i, i = 1, \ldots, n_1-1 \) yields

\[
\prod_{i=1}^{n_1-1} \int_{-1}^{1} (1-v_i^2)^{\frac{1}{2}}(p-3-i) dv_i = \frac{1}{2}(n_1-1) \frac{\Gamma(\frac{1}{2}(p-n_1))}{\Gamma(\frac{1}{2}(p-1))}.
\]
Substituting these results in (1.4) yields

\[
\begin{bmatrix}
\frac{1}{\gamma_1^2} & 0 & \cdots & 0 \\
0 & \gamma_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \gamma_1
\end{bmatrix}
\]

with columns which are distributed independently as \( N_p(\xi_i, I_p) \), where \( \xi_i \) is the \( i \)th column of \( \xi \) and \( n_1 < p \). Let \( \tilde{H}(n_1 x_{n_1}) \) be defined by \( \tilde{H} = X_1^T X_1 \). Then the elements of \( \tilde{H} \) have the joint density

\[
p_{\gamma_1}(\tilde{H}) = \left\{ 2^{\frac{1}{2} n_1} \pi^{\frac{1}{2} p n_1} \prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(p-i)\right) \right\}^{-1} \cdot
\]

\[
e^{-\frac{1}{2} \gamma_1} \cdot |\tilde{H}|^{\frac{1}{2}(p-n_1-1)} \cdot e^{-\frac{1}{2} \text{tr} \tilde{H}} \cdot \sum_{k_0=0}^{\infty} \frac{1}{\Gamma\left(\frac{1}{2} p + k_0\right) \Gamma(k_0 + 1) 2^{k_0} h_{11}^{k_0} \gamma_1}.
\]

**Step 2:**

Recall that \( \tilde{\gamma} = \gamma_1^{\frac{1}{2}} [\gamma_2^{-1/2}]' \), where \( \tilde{\gamma} \) and \( \gamma_2 \) are independently distributed, so we can write directly

\[
p_{\gamma_1}(\tilde{A}, \tilde{H}) = p(\tilde{A}) \cdot p_{\gamma_1}(\tilde{H})
\]

\[
= \left\{ 2^{\frac{1}{2} (n_1 + n_2)} \pi^{\frac{1}{2} p (n_1 + n_2)} \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(n_1 + n_2 - p + i - 1)\right) \right\}^{-1} \cdot
\]

\[
|\tilde{A}|^{\frac{1}{2}(n_2 - p - 1)} \cdot |\tilde{H}|^{\frac{1}{2}(p - n_1 - 1)} \cdot e^{-\frac{1}{2} \text{tr} (\tilde{A} + \tilde{H})} \cdot e^{-\frac{1}{2} \gamma_1}.
\]

(1.5) \[
\sum_{k_0=0}^{\infty} \frac{1}{\Gamma\left(\frac{1}{2} p + k_0\right) \Gamma(k_0 + 1) 2^{k_0} h_{11}^{k_0} \gamma_1}.
\]
We will use this density to obtain the joint density of the elements of the \((n_1 \times n_1)\) matrix

\[
\tilde{\Lambda} = (\tilde{A} + \tilde{H})^{-\frac{1}{2}} \tilde{H} (\tilde{A} + \tilde{H})^{-\frac{1}{2}},
\]

where \((\tilde{A} + \tilde{H})^{-\frac{1}{2}}\) is the lower triangular square root of \(\tilde{A} + \tilde{H}\) with positive diagonal elements. The characteristic roots of \(\tilde{\Lambda}\) are the \(\{\lambda_i, i = 1, \ldots, n_1\}\).

Let \(\tilde{B} = \tilde{A} + \tilde{H}\). The Jacobian of this transformation is \(J = 1\) and from (1.5) we get

\[
p_{\gamma_1}(\tilde{B}, \tilde{H}) = C_4 \cdot \frac{1}{|\tilde{B} - \tilde{H}|^{\frac{1}{2}(n_2 - p - 1)}} \cdot \frac{1}{|\tilde{H}|^{\frac{1}{2}(p - n_1 - 1)}} \cdot e^{-\frac{1}{2} \text{tr} \tilde{B}} \cdot e^{-\frac{1}{2} \gamma_1} \cdot \sum_{k_0 = 0}^{\infty} \frac{1}{\Gamma(\frac{1}{2} p + k_0) \Gamma(k_0 + 1) 2^{k_0} n_{11}} \frac{k_0}{\gamma_1},
\]

where

\[
C_4 = \frac{1}{2} \frac{(n_1 + n_2) n_1 n_1 (n_1 - 1)/2}{\prod_{i=1}^{n_1} \Gamma(\frac{1}{2} (n_1 + n_2 - p + 1 - i))} \frac{n_1 - 1}{\Gamma(\frac{1}{2} (p - 1))}\]

Since \(\tilde{B}\) is symmetric and positive definite with probability one, there is a lower triangular, non-singular \((n_1 \times n_1)\) matrix square root \(T\) of \(\tilde{B}\) with \(t_{11} = b_{11}^{\frac{1}{2}}\). The transformation \(\tilde{H} = T \tilde{B} T'\) has Jacobian \(J = |T|^{\frac{1}{2}(n_1 + 1)}\) (see Deemer and Olkin (1951)) and \(\tilde{h}_{11} = \tilde{b}_{11}^{\frac{1}{2}} \tilde{h}_{11}\), where \(\tilde{\Lambda} = (\tilde{\lambda}_{i,j})\), so we get
\[ p_{\gamma_1}(\tilde{B}, \lambda) = c_4 \cdot |\tilde{B}|^{\frac{1}{2}(n_2-1)} \cdot |I-\lambda|^{\frac{1}{2}(n_2-p-1)} \cdot |\lambda|^{\frac{1}{2}(p-n_1-1)} \]

\[ \times e^{-\frac{1}{2} \text{tr} \tilde{B}} \cdot e^{-\frac{1}{2} \lambda_1} \cdot \sum_{k_0=0}^{\infty} \frac{l}{\Gamma\left(\frac{1}{2}p+k_0\right)\Gamma(k_0+1)^2} \]

To get the marginal distribution of \( \lambda \), we must integrate out the elements of the matrix \( \tilde{B} \). Sitgreaves (1952) gives the following formula (Equation (1.4)) for integrating over all the elements of a symmetric, positive definite matrix \( \tilde{B}^{*}(kxk) \) except for the upper left hand corner element \( b_{11}^{*} \):

\[
\int \cdots \int |\tilde{B}|^{\frac{1}{2}(n-k+1)} e^{-\text{tr} \tilde{B}^{*}} \; db_{12}^{*} \cdots db_{pp}^{*}
\]

\( \tilde{B}_{(1)} \) pos. def.

\[-\infty \leq b_{11}^{*} / b_{11}^{*} \leq \infty \]

\[ = 2^{\frac{1}{2}(k-1)(n+2)} \frac{k(k-1)}{4} \frac{1}{b_{11}^{*}} e^{-\frac{1}{2} b_{11}^{*} b_{11}} \frac{k-1}{i=1} \Gamma\left(\frac{1}{2}(n+2-i)\right), \]

for \( i, j = 2, \ldots, k \) where \( \tilde{B}_{(1)}^{*} = (b_{i,j}^{*} b_{i,j}^{*} / b_{11}^{*}) \).

Setting \( n = n_1 + n_2 - 2 \), \( k = n_1 \) and integrating over all the elements of \( \tilde{B} \) except \( b_{11} \) using this formula yields
\[ p_{\gamma_1}(\mathbf{\tilde{\Sigma}}_{\text{ll}}, \mathbf{\tilde{X}}) = \]
\[
\frac{1}{2}(n_1+n_2) \pi \frac{n_1}{4} \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(n_1+n_2-p+1-i)\right) \prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(p-i)\right)^{-1} \cdot \]
\[
\prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(n_2+i+1)\right) \cdot e^{-\frac{1}{2} \gamma_{11} \cdot |I-\mathbf{\tilde{X}}|} \cdot \left|\mathbf{\tilde{X}}\right|^\frac{1}{2}(n_2-p-1) \cdot \left|\mathbf{\tilde{X}}\right|^\frac{1}{2}(p-n_1-1) \cdot \]
\[
\sum_{k_0=0}^{\infty} \frac{1}{2 \pi} \frac{k_0 \cdot k_0 \cdot \gamma_{11} \cdot \left|\mathbf{\tilde{X}}\right|}{\Gamma\left(\frac{1}{2}p+k_0\right) \Gamma\left(\frac{1}{2}k_0+1\right)} \cdot e^{-\frac{1}{2} \gamma_{11}} \cdot \left|\mathbf{\tilde{X}}\right| \cdot \left|\mathbf{\tilde{X}}\right|^\frac{1}{2}(n_2-p-1) \cdot \left|\mathbf{\tilde{X}}\right|^\frac{1}{2}(p-n_1-1) \cdot \]

Finally we integrate \( \mathbf{\tilde{\Sigma}}_{\text{ll}} \) over the range \((0, \infty)\) to get the density of \( \mathbf{\tilde{X}} \).

**Lemma 1.2:** Let \( \mathbf{\tilde{H}}(n_1 x_{n_1}) \) be defined as in Lemma 1.1 and let \( \mathbf{\tilde{X}}(n_1 x_{n_1}) \) be distributed as \( W_{n_1} (n_2-p+n_1, I_{n_1}) \). Then the distribution of \( \mathbf{\tilde{X}} = (\mathbf{\tilde{A}}+\mathbf{\tilde{H}})^{\frac{1}{2}} \), where \( (\mathbf{\tilde{A}}+\mathbf{\tilde{H}})^{\frac{1}{2}} \), is the lower triangular matrix square root of \( \mathbf{\tilde{A}}+\mathbf{\tilde{H}} \) with positive diagonal elements, is given by

\[
p_{\gamma_1}(\mathbf{\tilde{X}}) = \prod_{i=1}^{n_1} \frac{n_1(n_1-1)/4 \cdot \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(n_1+n_2-p+1-i)\right) \prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(p-i)\right)}{\prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(n_2+i+1)\right) \cdot e^{-\frac{1}{2} \gamma_{11} \cdot |I-\mathbf{\tilde{X}}|} \cdot \left|\mathbf{\tilde{X}}\right|^\frac{1}{2}(n_2-p-1) \cdot \left|\mathbf{\tilde{X}}\right|^\frac{1}{2}(p-n_1-1)} \cdot \]

\[
\sum_{k_0=0}^{\infty} \frac{1}{2 \pi} \frac{k_0 \cdot k_0 \cdot \gamma_{11} \cdot \left|\mathbf{\tilde{X}}\right|}{\Gamma\left(\frac{1}{2}p+k_0\right) \Gamma\left(\frac{1}{2}k_0+1\right)} \cdot e^{-\frac{1}{2} \gamma_{11}} \cdot \left|\mathbf{\tilde{X}}\right| \cdot \left|\mathbf{\tilde{X}}\right|^\frac{1}{2}(n_2-p-1) \cdot \left|\mathbf{\tilde{X}}\right|^\frac{1}{2}(p-n_1-1) \cdot \]

where the elements of \( \mathbf{\tilde{X}} \) range over the space where \( \mathbf{\tilde{X}} \) and \( I-\mathbf{\tilde{X}} \)
are positive definite (this follows from the derivation, beginning with the fact that the matrix \( \tilde{A} \) is positive definite).

[Note: See also work of Kshirsagar (1961) and Khatri and Pillai (1965)].

Step 3:

By construction, we have that \( \tilde{A} \) is a symmetric matrix and therefore there is an orthogonal matrix \( L(n_1 \times n_1) \) such that

\[
\tilde{A} = LD_{\lambda}L',
\]

where \( D_{\lambda} = \text{diag}(\lambda_1, \ldots, \lambda_{n_1}) \) with \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_{n_1} \leq 1 \).

An \( (n_1 \times n_1) \) orthogonal matrix corresponds to a rotation in \( n_1 \)-dimensional space and can therefore be represented (see, for example, Tumura (1965)) in terms of the individual angles of rotation as

\[
(1.7) \quad L = L_1(\theta_1) \cdot L_2(\theta_2) \cdots L_{n_1-1}(\theta_{n_1-1}^{n_1-1}),
\]

where

\[
L_v(\theta_v) = R_{n_1-1}(\theta_v, n_1-1) \cdot R_{n_1-2}(\theta_v, n_1-2) \cdots R_v(\theta_v, v),
\]

and

\[
R_v(\theta) = \begin{bmatrix}
I_{v-1} & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta \\
0 & 0 & I_{n_1-v-1}
\end{bmatrix}.
\]
Note that $R_{\nu}^{-1}(\theta)$ is a single rotation matrix and that every rotation in $n_1$-dimensional space consists of $\frac{1}{2}n_1(n_1-1)$ single rotations. Hence the $(n_1 \times n_1)$ orthogonal matrix $L^{(n_1)}(\theta_{ij})$ is the product of $\frac{1}{2}n_1(n_1-1)$ matrices, each representing an independent single rotation.

To make this transformation we need the results of

Lemma 1.3: (Theorem 2.1, Tumira (1965)).

The Jacobian of the transformation $\hat{\lambda} = LD_{\lambda}L'$ is

$$J = \text{det} \left( \sum_{i=1}^{n_1-2} \sum_{j=1}^{n_1-2} \sin^{n_1-j-1} \theta_{ij} \right),$$

where $\hat{\lambda}$ is an $(n_1 \times n_1)$ symmetric matrix and $L = L^{(n_1)}(\theta_{ij})$ is an orthogonal matrix defined by (1.7).

We require additionally that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}$ so the $\theta$'s will all range over the interval $0 \leq \theta_{ij} \leq \pi$, $i = 1, \ldots, n_1-1$; $j = i, \ldots, n_1-1$.

Since $|I-\hat{\lambda}| = |I-D_{\lambda}|$ and $|\hat{\lambda}| = |D_{\lambda}|$, we only need the explicit representation of $\hat{\lambda}_{11}$ under this transformation for Equation (1.6). Hence we need to know the first row, $\Delta$, of $L$, since $\hat{\lambda}_{11} = \Delta D_{\lambda} \Delta'$. To obtain $\Delta$ we note that

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\( L_{\nu}^{-1}(\theta_{\nu,j}) = R_{n_{1}-1}(\nu, n_{1}-1) \cdots R_{\nu}^{-1}(\theta_{\nu,0}) \)

\[
\begin{bmatrix}
1
& 0
& 0
& \vdots \\
0
& \cos \theta_{\nu,0}
& -\sin \theta_{\nu,0}
& 0 \\
\sin \theta_{\nu,0} \cos \theta_{\nu,1}
& \cos \theta_{\nu,0} \cos \theta_{\nu,1}
& \vdots \\
\vdots 
& \vdots 
& \vdots \\
\left( \frac{n_{1}-2}{j=0} \sin \theta_{\nu,j} \cos \theta_{\nu,j} \right)
& \cos \theta_{\nu,0} \left( \frac{n_{1}-2}{j=0} \sin \theta_{\nu,j} \cos \theta_{\nu,j} \right)
& \vdots \\
\left( \frac{n_{1}-2}{j=0} \sin \theta_{\nu,j} \cos \theta_{\nu,n_{1}-1} \right)
& \cos \theta_{\nu,0} \left( \frac{n_{1}-2}{j=0} \sin \theta_{\nu,j} \cos \theta_{\nu,n_{1}-1} \right)
& \cos \theta_{\nu,n_{1}-1}
\end{bmatrix}
\]

The only terms needed for \( \Delta \) are the terms \( \cos \theta_{\nu,0} \) and \(-\sin \theta_{\nu,0}\) from each \( L_{\nu}^{-1} \). From (1.7) we have

\( L = L_{\nu}^{-1}(\theta_{ij}) = L_{1}^{-1}(\theta_{ij})L_{2}^{-1}(\theta_{2j}) \cdots L_{n_{1}-1}(\theta_{n_{1}-1j,n_{1}-1}) \)

\[
\begin{bmatrix}
\cos \theta_{11}
& -\sin \theta_{11}
& 0
& \cdots \\
\vdots
& \vdots
& \vdots
& \ddots \\
\vdots
& \vdots
& \vdots
& \ddots \\
0
& \cos \theta_{22}
& -\sin \theta_{22}
& 0
& \cdots \\
\vdots
& \vdots
& \vdots
& \ddots
\end{bmatrix}
\]

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\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cos \theta_{33} & -\sin \theta_{33} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
I_{n_1-3} & 0 & 0 & 0 \\
0 & \cos \theta_{n_1-2,n_1-2} & -\sin \theta_{n_1-2,n_1-2} & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
I_{n_1-2} & 0 \\
0 & \cos \theta_{n_1-1,n_1-1} & -\sin \theta_{n_1-1,n_1-1} \\
\sin \theta_{n_1-1,n_1-1} & \cos \theta_{n_1-1,n_1-1} \\
\end{bmatrix}
\]

Carrying out the required multiplications yields

\[\Delta = (\cos \theta_{11}, -\sin \theta_{11}, \cos \theta_{22}, \sin \theta_{22}, \sin \theta_{22} \cos \theta_{33}, \cdots, \]

\[(-1)^{n_1-2} \prod_{i=1}^{n_1-1} \sin \theta_{i1} \cos \theta_{n_1-1,n_1-1}, (-1)^{n_1+1} \prod_{i=1}^{n_1-1} \sin \theta_{i1}).\]

Therefore,
\[ z_{ll} = \Delta D \Delta' = \]

\[ \lambda_1 \cos^2 \theta_{11} + \lambda_2 \sin^2 \theta_{11} \cos^2 \theta_{22} + \cdots + \lambda_{n-1} \left( \prod_{i=1}^{n-2} \sin^2 \theta_{ii} \right) \cos^2 \theta_{n-1,n-1} \]

\[ + \lambda_n \prod_{i=1}^{n-1} \sin^2 \theta_{ii}. \]

We are now able to make the transformation \( \tilde{z} = \Delta D \Delta' \) in (1.6) to get

\[ p_{y_l}(L^{(n_l)},D_{\lambda}) = \]

\[ \prod_{i=1}^{n_l(n_l-1)/4} \frac{n_l}{\Gamma \left( \frac{1}{2} (n_1+n_2-p+1) \right)} \prod_{i=1}^{n_l} \left( \frac{1}{\Gamma \left( \frac{1}{2} p \right)} \right)^{-1} \cdot \]

\[ \prod_{i=1}^{n_l} \Gamma \left( \frac{1}{2} (n_2+1) \right) \cdot e^{-\frac{1}{2} \gamma_1} \cdot \prod_{i=1}^{n_l} \left\{ \lambda_i \left( 1 - \lambda_i \right)^{\frac{1}{2} (n_2-p-1)} \right\} \cdot \]

\[ \prod_{i<j} \left( \lambda_i - \lambda_j \right) \cdot \prod_{i=1}^{n_l-2} \prod_{j=1}^{n_l-1} \sin \theta_{ij}. \]

\[ \sum_{k_0=0}^{(\frac{1}{2} n_2 + k_0)^+} \frac{\Gamma \left( \frac{1}{2} (n_2-k_0) \right)}{\Gamma \left( \frac{1}{2} (n_2+k_0) \right) \Gamma \left( k_0 + 1 \right)} \cdot \left( \frac{1}{2} \right) \cdot \left( \lambda_1 \cos^2 \theta_{11} + \lambda_2 \sin^2 \theta_{11} \cos^2 \theta_{22} + \cdots + \lambda_{n_l-1} \left( \prod_{i=1}^{n_l-2} \sin^2 \theta_{ii} \right) \cos^2 \theta_{n_l-1,n_l-1} + \lambda_n \prod_{i=1}^{n_l-1} \sin^2 \theta_{ii} \right)^{k_0} \]

We need only integrate over the \( \theta_{ij} \) to get the density of the roots \( \{ \lambda_i, i = 1, \ldots, n_l \} \) in the linear case. We shall first integrate out all those \( \theta_{ij} \) for which \( i = j \). Note that
\[ \begin{align*}
&= \sin \theta_{i j} \left( \prod_{i=1}^{n_1-2} \prod_{j=1}^{n_1-2} \sin \theta_{i j} \right) \left( \prod_{i=1}^{n_1-3} \prod_{j=1}^{n_1-3} \sin \theta_{i j} \right) \cdot \sin \theta_{11} \\
&= \sin \theta_{i j} \cdot \sin \theta_{11} \cdot \sin \theta_{11} \cdot \sin \theta_{11} \\
\end{align*} \]

Now we use the binomial expansion to get that

\[ \left( \lambda_1 \cos^2 \theta_{11} + \lambda_2 \sin^2 \theta_{11} \cos^2 \theta_{22} + \cdots + \lambda_{n_1} \sin^2 \theta_{11} \cdots \sin^2 \theta_{n_1-1,n_1-1} \right)^{k_0} \]

\[ \begin{align*}
&= \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{n_1-1}=0}^{k_{n_1-2}} (k_0) (k_1) \cdots (k_{n_1-2})(k_{n_1-1})(k_{n_1-1}) \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_{n_1}^{k_{n_1-1}} \\
&= \sum_{i=1}^{n_1-1} \cos \frac{2(k_{i-1} - k_i)}{2} \theta_{ii} \cdot \sin \frac{2k_i}{2} \theta_{ii} \\
&= \sum_{i=1}^{n_1-1} \cos \frac{2(k_{i-1} - k_i)}{2} \theta_{ii} \cdot \sin \frac{2k_i}{2} \theta_{ii} \\
\end{align*} \]

Combining all the terms with \( \theta_{ii}, i = 1, \ldots, n_1-1 \), in (1.9) and (1.10) gives

\[ \begin{align*}
&= \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{n_1-1}=0}^{k_{n_1-2}} (k_0) (k_1) \cdots (k_{n_1-2})(k_{n_1-1})(k_{n_1-1}) \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_{n_1}^{k_{n_1-1}} \\
&= \sum_{i=1}^{n_1-2} \cos \frac{2(k_{i-1} - k_i)}{2} \theta_{ii} \cdot \sin \frac{2k_i}{2} \theta_{ii} \\
&= \sum_{i=1}^{n_1-2} \cos \frac{2(k_{i-1} - k_i)}{2} \theta_{ii} \cdot \sin \frac{2k_i}{2} \theta_{ii} \\
&= \sum_{i=1}^{n_1-2} \cos \frac{2(k_{i-1} - k_i)}{2} \theta_{ii} \cdot \sin \frac{2k_i}{2} \theta_{ii} \\
\end{align*} \]
We integrate over \( \theta_{i,i} \) in the range \((0, \pi)\) using the formula

\[
\int_0^\pi \sin^m \theta \cos^n \theta \, d\theta = \frac{\Gamma\left(\frac{1}{2}(m+1)\right) \Gamma\left(\frac{1}{2}(2n+1)\right)}{\Gamma\left(\frac{1}{2}(m+2n+1)\right)} \quad \text{(see Sitgreaves 1952)}
\]

to get

\[
\sum_{k_0}^{k_1} \cdots \sum_{k_{n_1-1}=0}^{n_1-2} \frac{k_{n_1-2} \ldots k_0 \ldots k_{n_1-2} \ldots \left(\frac{1}{2}\right)^{n_1-1}}{\Gamma\left(\frac{1}{2}(k_{i-1}-k_i) + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}(n_{i-1} - \frac{1}{2})\right)} \cdot \frac{1}{\Gamma\left(\frac{1}{2} n_1 + k_0\right)} .
\]

We still must integrate the \( \theta_{i,j} \) for \( j = i+1, \ldots, n_1-1 \); \( i = 1, \ldots, n_1-2 \). From (1.9) we have the first term of the product, namely

\[
\sum_{i=1}^{n_1-3} \sum_{j=i+1}^{n_1-2} \sin^{n_1-j-1} \theta_{i,j} .
\]

Let

\[
\theta_{i,j} = \int_0^\pi \sin^{n_1-j-1} \theta_{i,j} \, d\theta_{i,j} = \frac{\Gamma\left(\frac{1}{2}(n_{i}-j)\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}(n_1-j+1)\right)} .
\]

Then

\[
\sum_{j=i+1}^{n_1-2} \theta_{i,j} = \frac{\pi}{\frac{1}{2}(n_1-i-2)} = \pi \frac{1}{\Gamma\left(\frac{1}{2}(n_1-i)\right)} .
\]
and
\[ 
\prod_{i=1}^{n_1-3} \prod_{j=i+1}^{n_1-2} \theta_{ij} = \frac{\frac{n_1^2}{2} - \frac{5}{2} n_1 + 3 + \frac{1}{2}}{\prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(n_1-1)\right)}.
\]

The only \( \theta_{ij} \)'s still not integrated have indices \( i = 1, \ldots, n_1-2, \)
\[ j = n_1-1 \] and \( \prod_{i=1}^{n_1-2} \int_0^\pi d\theta_i n_1-1 = \pi^{n_1-2} \). So the total term from integrating over \( \theta_{ij}, i \neq j \), is

\[ \frac{n_1^2}{2} - \frac{5}{2} n_1 + 3 + \frac{1}{2} + \frac{n_1-2}{4} = \frac{n_1^2}{2} \pi^{n_1-2} \frac{\prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(n_1-1)\right)}{\prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(n_1-1)\right)}.
\]

Substituting these results in (1.8) yields the distribution of the roots \( \{\lambda_1\} \):

\[ p_{\gamma_1}(\lambda_1, \ldots, \lambda_{n_1}) = \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(n_1+n_2-p+1-1)\right) \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(p-i)\right)\Gamma\left(\frac{1}{2}(n_1-i)\right)^{-1}.
\]

\[ e^{-\frac{1}{2} \gamma_1 \prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(n_1+1)i\right) \prod_{i=1}^{n_1} \left\{\lambda_i \prod_{i=1}^{\frac{1}{2}(p-n_1-1)} \prod_{i=1}^{\frac{1}{2}(2p-n_1-1)} (\lambda_i - \lambda_j) \right\} \right}\] (1.11)

\[ \sum_{k_0=0}^{k_0} \frac{\Gamma\left(\frac{1}{2}(n_1+n_2)+k_0\right)}{\Gamma\left(\frac{1}{2}p+k_0\right)\Gamma\left(\frac{1}{2}n_1+k_0\right)} \cdot \gamma_1^{k_0} \cdot \frac{1}{2}.
\]

\[ \sum_{k_1=0}^{k_0} \sum_{k_1=0}^{k_0} \cdots \sum_{k_1=0}^{k_0} \cdots \sum_{k_1=0}^{k_0} \left\{\prod_{i=1}^{n_1-1} \Gamma\left(k_i\right)\right\} \left(\prod_{i=1}^{n_1-1} \Gamma\left(k_i-\frac{1}{2}\right)\right)^{-1} \Gamma\left(k_i-\frac{1}{2}\right) \Gamma\left(k_{n_1-1}+\frac{1}{2}\right) \cdot \gamma_1^{k_0} \cdot \frac{1}{2}.
\]

\[ \sum_{k_1=0}^{k_0} \cdots \sum_{k_1=0}^{k_0} \cdots \sum_{k_1=0}^{k_0} \left\{\prod_{i=1}^{n_1-1} \Gamma\left(k_i-k_i\right)\right\} \left(\prod_{i=1}^{n_1-1} \lambda_i^2 \right)^{\frac{k_1-1-k_1}{2}} \lambda_{n_1}^{k_1-1}.
\]

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To get the density into the same form as in Theorem 1.1 (i) we must rearrange some of the terms. If we let \( k_0 = 0 \) in the infinite sum, we get the "zero term", which is

\[
\frac{\frac{1}{2n_1} \Gamma\left(\frac{1}{2}(n_1+2n_2)\right)}{\Gamma\left(\frac{1}{2}p\right) \Gamma\left(\frac{1}{2}n_1\right)}.
\]

The constant term in (1.11) is

\[
C_5 = \frac{n_{-1}}{\prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(n_2+i)\right)} \frac{n_{-1}}{\prod_{i=1}^{n_{n_2-p+1}} \Gamma\left(\frac{1}{2}(p-i)\right)} \frac{n_{-1}}{\prod_{i=1}^{n_{1-l}} \Gamma\left(\frac{1}{2}(n_1-i)\right)}
\]

Combining the zero term with \( C_5 \) gives

\[
C_1 = \frac{\frac{1}{2}n_1}{\prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(n_1+n_2+p+1)\right)} \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+1-l)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2-p+1-l)\right)\Gamma\left(\frac{1}{2}(n_1-l)\right)\Gamma\left(\frac{1}{2}(p-l)\right)}
\]

which agrees with the definition of \( C_1 \) in (1.2). Now let

\[
S^* = C_5 \cdot \sum_{k_0=1}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+k_0)\right)}{\Gamma\left(\frac{1}{2}k_0\right)\Gamma\left(\frac{1}{2}n_1+k_0\right)\Gamma\left(\frac{1}{2}k_0+1\right)} \cdot \frac{\gamma_{-1}^{k_0}}{n_1-1} \frac{\ldots}{\sum_{k_1=0}^{n_1-2} \ldots \sum_{k_{n_1-1}=0}^{n_1-2}} \frac{\Gamma\left(k_{n_1-1}+\frac{1}{2}\right)}{\left(\prod_{i=1}^{n_1-1} \lambda_i\right)^{k_{n_1-1}}}.
\]

The terms in \( S^* \) can be rearranged by noting that
\[
\begin{align*}
\binom{k_0}{k_1} & \binom{k_1}{k_2} \cdots \binom{k_{n_1-2}}{k_{n_1-1}} = \frac{\Gamma(k_0+1)}{\prod_{i=1}^{n_1-1} \Gamma(k_{i-1}+1)} \\
\end{align*}
\]

and by combining the terms of \( C_5 \) with the gamma functions inside the infinite sum to get

\[
S^* = C_1 \cdot \sum_{k_0=1}^{\infty} \frac{\Gamma\left(\tfrac{1}{2}(n_1+n_2)+k_0\right) \Gamma\left(\tfrac{1}{2}n_1\right)}{\Gamma\left(\tfrac{1}{2}(n_1+n_2)+k_0\right) \Gamma\left(\tfrac{1}{2}n_1+k_0\right)} \cdot \gamma_{\frac{1}{2}}^{k_0} \cdot \gamma_{\frac{n_1-1}{2}}^{k_{n_1-1}} \cdot \gamma_{\frac{k_1-1}{2}}^{k_{1-1}}
\]

\[
\sum_{k_1=0}^{k_0} \sum_{k_{n_1-1}=0}^{k_{n_1-2}} \left( \prod_{i=1}^{n_1-1} \frac{\Gamma(k_{i-1}+\frac{1}{2})}{\Gamma(k_{i-1}-k_{i}+\frac{1}{2})(n_i^2)} \right) \frac{\Gamma(k_{n_1-1}+\frac{1}{2})}{\Gamma(k_{n_1-1}+1)(n_1^2)}
\]

\[
= C_1 \cdot S ,
\]

where \( S \) is all the terms in the infinite sum.

Substituting into (1.11) yields

\[
p_{\gamma_1}(\lambda_1, \ldots, \lambda_{n_1}) = e^{-\frac{1}{2} \gamma_1} \cdot C_1 \cdot \prod_{i=1}^{n_1} \lambda_1^{\frac{1}{2}(p-n_1-1)} (1-\lambda_1)^{\frac{1}{2}(n_2-p-1)} \prod_{i<j} (\lambda_i - \lambda_j)
\]

\[
\cdot \{1 + S\}
\]

which agrees with the result in Theorem 1.1 (i).
Step 4

At this point it is straightforward to get the joint density of the \( \{\rho_i, i = 1, \ldots, n_1\} \) from the joint density of the \( \{\lambda_i, i = 1, \ldots, n_1\} \): as mentioned earlier, the roots of \( |H-\lambda(H+E)| = 0 \) are related to the roots of \( |H-\rho E| = 0 \) by the one-to-one transformation

\[
\lambda_i = \frac{\rho_i}{1 + \rho_i}, \quad i = 1, \ldots, n_1.
\]

The Jacobian is \( J = \prod_{i=1}^{n_1} (1+\rho_i)^{-2} \) and we note that \( \rho_i > \rho_j \) whenever \( \lambda_i > \lambda_j \) and, for \( 0 \leq \lambda_i \leq 1 \), we have \( 0 \leq \rho_i \leq \infty \) for \( i = 1, \ldots, n_1 \). Applying this transformation to (1.12) yields the result in Theorem 1.1 (ii). ||

1.5 Linear Case Derivation - Case 2: \( n_1 > p \)

As in Case 1 (Section 1.4), we note that the \( \{\rho_i, i = 1, \ldots, p\} \) are the characteristics roots of the \((p \times p)\) matrix \( \Phi = HE^{-1} = X_1X_1' E^{-1} \), where \( E \sim W_p(n_2, I_p) \). Let

\[
X_1 = \begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1n_1} \\
\vdots & \ddots & \ddots & \vdots \\
x_{p1} & x_{p2} & \cdots & x_{pn_1}
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_1^*
\end{bmatrix},
\]

where \( x_1' = (x_{11}, \ldots, x_{p1}) \) is the first row of \( X_1' \) and
\[ X_1^* = \begin{bmatrix}
  x_{12} & \cdots & x_{1n_1} \\
  \vdots & \ddots & \vdots \\
  x_{p2} & \cdots & x_{pn_1}
\end{bmatrix} \]

so that \( X_1^* \) is a \((p \times n_1 - 1)\) matrix with \( E X_1^* = 0 \). Then we can proceed as in Case 1 by setting

\[ H = X_1^* X_1' = x_1 x_1' + X_1^* X_1^* = x_1 x_1' + H^* , \]

where \( H^* = X_1^* X_1'^* \sim \mathcal{W}(n_1 - 1, I_p) \) independently of \( x_1 \sim \mathcal{N}_p \left( \begin{bmatrix} \gamma_{11}^1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, I_p \right) \).

The distribution of \( H \) can be obtained from the distribution of \( \tilde{H} \) by interchanging \( p \) and \( n_1 \) in the derivation of Step 1 to get

\[ \text{Lemma 1.4}: \text{Let } X_1(p \times n_1) \text{ have expectation } E X_1 = \begin{bmatrix}
  \gamma_{11}^1 & 0 & \cdots & 0 \\
  0 & \vdots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & 0 \\
  0 & \cdots & 0 & 0
\end{bmatrix} = \xi
\]

with independently distributed columns, each distributed as \( \mathcal{N}_p (\xi_1, I_p) \), where \( \xi_1 \) is the \( i \)th column of \( \xi \), and \( n_1 > p \). Let \( H(p \times p) \) be defined by \( H = X_1 X_1' \). Then the elements of \( H \) have the joint density
\[ p_{\gamma_1}(H) = \left\{ 2^{\frac{1}{2} p n_1} \prod_{i=1}^{p-1} \frac{\Gamma\left(\frac{1}{2}(n_i-1)\right)}{\Gamma\left(\frac{1}{2}(n_{i+1})\right)} \right\}^{-1} \cdot e^{-\frac{1}{2} \gamma_1 \cdot |H|^{\frac{1}{2}(n_1-p-1)}} \cdot e^{-\frac{1}{2} \text{tr}H} \cdot \sum_{k_0=0}^{\infty} \frac{1}{\Gamma\left(\frac{1}{2}(n_1^{-1}+k_0)\right) \Gamma(k_0+1)2^{k_0}} \cdot h_{11}^{-1} \cdot k_0 k_0. \]

We now set \( A = E \) and recall that \( A \) and \( H \) are independently distributed so we can write the joint density of \( A \) and \( H \) as

\[ p_{\gamma_1}(A,H) = \left\{ 2^{\frac{1}{2} (n_1+n_2)p} \prod_{i=1}^{p} \frac{\Gamma\left(\frac{1}{2}(n_{i+1})\right)}{\Gamma\left(\frac{1}{2}(n_i-1)\right)} \right\}^{-1} \cdot e^{-\frac{1}{2} \text{tr}(A+H)} \cdot e^{-\frac{1}{2} \gamma_1 \cdot |A|^{\frac{1}{2}(n_2-p-1)} \cdot |H|^{\frac{1}{2}(n_1-p-1)}}. \]

(1.13)

If we replace \( p \) by \( n_1 \), \( n_1 \) by \( p \) and \( n_2 \) by \( n_2-p+n_1 \) in (1.13), we get \( p_{\gamma_1}(\tilde{A},\tilde{H}) \) from Step 2, (1.5). Consequently, we can obtain the distributions of the roots \( \{\lambda_i, i = 1, \ldots, p\} \) and \( \{\hat{\lambda}_i, i = 1, \ldots, p\} \) in Theorem 1.2 by similar changes in the parameters \( p, n_1 \) and \( n_2 \) in the results of Theorem 1.1.

1.6 Linear Case Derivation - Case 3: \( n_1 = p \)

In order to use the Wishart distributions for the matrices \( \tilde{H}^* \) and \( \tilde{H}^* \) in Sections 1.4 and 1.5, it is necessary to assume a strict inequality between \( n_1 \) and \( p \); i.e., \( \tilde{H}^* \sim W_{n_1}(p-1, I_{n_1}) \) and the
Wishart density exists only for \( p-1 \geq n_1 \). A somewhat different approach is required if, for example, \( n_1 \leq p \). Instead of deriving the distribution of \( \hat{\Sigma} \), we let \( A = E \sim W_p(n_2, I_p) \) and follow the techniques of Sitgreaves (1952).

The joint density of \( A(p \times p) \) and \( X_1(p \times n_1) \) may be written as

\[
P(A, X_1) = \frac{1}{2(n_1 + n_2)p p^{(p-1)/2} n_1 p} \prod_{i=1}^{p} \Gamma \left( \frac{1}{2}(n_2 + 1 - i) \right)^{-1} \cdot \\
\begin{array}{l}
\frac{1}{2(n_2-p-1)} |A|^{-\frac{1}{2}} \exp \left( \frac{1}{2} tr(A + X_1^T X_1') + x_1' \bar{x}_1 - \frac{1}{2} y_1 \right),
\end{array}
\]

where \( x_1 \) is the upper left hand corner element of \( X_1 \).

Let \( B = A + X_1 X_1' \) (earlier we had \( B = A + H \)) to get \( p_{\gamma_1}(B, X_1) \) and then transform to \( U = T^{-1} X_1 \), where \( U \) is \( (p \times n_1) \) and \( T(p \times p) \) is the lower triangular square root of \( B \) with \( t_{11} = \frac{b_{11}}{2} \) (i.e., \( B = T T' \)). The Jacobian of this transformation is \( |B|^{-\frac{1}{2}n_1} \) and the new density is

\[
P_{\gamma_1}(B, U) = \frac{1}{2(n_1 + n_2)p p^{(p-1)/2} n_1 p} \prod_{i=1}^{p} \Gamma \left( \frac{1}{2}(n_2 + 1 - i) \right)^{-1} \cdot \\
\begin{array}{l}
\frac{1}{2(n_2-p-1)} |B|^{-\frac{1}{2}} |U' U|^{-\frac{1}{2}} \exp \left( \frac{1}{2} trB + \frac{1}{2} b_{11} u_1 u_1' \right),
\end{array}
\]

where we have again used the fact that \( |I - U U'| = |I - U' U| \). Integrating out \( B \) (using (14) of Sitgreaves (1952) and expanding \( \frac{1}{2} b_{11}^2 u_1^2 \) in a Taylor series) leaves
\[
\gamma_1 p(U) = \prod_{i=1}^{n_1} \frac{\Gamma(\frac{1}{2}(n_2+1))}{\Gamma(\frac{1}{2}(n_1+1))} \cdot \prod_{i=1}^{n_1} \frac{\Gamma(\frac{1}{2}(n_1-p+1))}{\pi^{\frac{1}{2}n_1-p}} \cdot \frac{1}{2^{n_1-p-1}}.
\]

\[
|I-U'U|^{\frac{1}{2}(n_2-p)} \cdot e^{-\frac{1}{2} \gamma_1} \cdot \sum_{k_0=0}^{\infty} \frac{\Gamma(\frac{1}{2}(n_1+n_2+k_0))}{\Gamma(k_0+1)} \frac{1}{2^{\frac{k_0}{2}} k_0} \frac{1}{2^{k_0}} \gamma_1 \cdot U_{11}
\]

where \( U(p \times n_1) \) ranges over the region where \( I-U'U \) is positive definite.

The roots \( \{\lambda_i, \ i = 1, \ldots, n_1\} \) are the characteristic roots of the matrix \( \tilde{A} = U'U \) (\( \tilde{A} \) is \( n_1 \times n_1 \)), so we want to obtain the distribution of \( \tilde{A} \) and then proceed as in Case 1 to get the distribution of the roots.

Let \( U = (U_1, \ldots, U_{n_1}) \) where \( U_i (p \times 1) \) is the \( i \)th column of \( U \).

Then

\[
\tilde{A} = U'U = \begin{bmatrix} U_{1}'U_1 & U_{1}'U_2 & \cdots & U_{1}'U_{n_1} \\ \vdots & \vdots \\ U_{n_1}'U_1 & \cdots & U_{n_1}'U_{n_1} \end{bmatrix}
\]

and has \( \frac{1}{2}n_1(n_1+1) \) elements whereas \( U \) has \( pn_1 \) elements; therefore we must integrate out the extra variables to get the density of \( \tilde{A} \).

We begin by transforming the columns of \( U \) to the columns \( V_1 \) of \( V(p \times n_1) \) by letting
\[ V_1 = U_1 \]
\[ V_2 = G_1 U_2 \]
\[ \vdots \]
\[ V_k = G_{k-1} G_{k-2} \cdots G_2 G_1 U_1, \quad k = 2, \ldots, n_1, \]

where \( G_1 \) (\( p \times p \)) is orthogonal and has first row proportional to \( U_1 \) and

\[
G_{k-1} = \begin{bmatrix}
1 & 0 \\
0 & G_{k-1}^*
\end{bmatrix},
\]

where \( G_{k-1}^* \) (\( (p-k+2) \times (p-k+2) \)) is an orthogonal matrix with first row proportional to the last \( p-k+2 \) elements of \( V_{k-1} \). For example,

\[
G_2 = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & & & \\
\vdots & \ddots & \ddots & \ddots \\
0 & & & 1 \\
0 & & & 0
\end{bmatrix}
\]

and the first row of \( G_2^* \) (\( (p-1) \times (p-1) \)) is proportional to \((v_{i2}, v_{i3}, \ldots, v_{ip_2})\), where \( V_2' = (v_{i2}, v_{i2}, \ldots, v_{ip_2}) \). Since all the \( G_i \), \( i = 1, \ldots, n_1-1 \) are orthogonal, the Jacobian of this transformation is 1; \( U'U = V'V \) and \( u_{11} = v_{11} \). Hence,
\[ p_{\gamma_1}(v) = \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(n_2+1)\right) \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(n_2-p+1)\right) \pi^{-\frac{3}{2}n_1p - 1} \cdot \nabla_{V} \left| I - V \right|^{-\frac{1}{2}(n_2-p-1)} \cdot e^{-\frac{1}{2}\gamma_1} \cdot \sum_{k_0=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+k_0)\right)}{\Gamma(k_0+1)} \frac{1}{2}\gamma_1 n_1 v_1 \cdot \frac{k_0}{k_0} \frac{1}{k_0} \frac{k_0}{k_0} \cdot \right. \]

The matrix \( V(p \times n_1) \) can be rewritten in a "triangular" form by changing notation: let \( V^*_1 = (v_{11}, \ldots, v_{p1}) \) for \( i = 1, \ldots, n_1 \). Then

\[
V = \begin{bmatrix}
V_1 & v_{12} & v_{13} & \cdots & v_{1n_1} \\
& V_2 & v_{23} & \cdots & v_{2n_1} \\
& & \ddots & \ddots & \vdots \\
& & & V_{n_1-1,n_1} & v_{n_1-1,n_1} \\
& & & & V_{n_1}
\end{bmatrix}
\]

We are now in a position to integrate out the "extra" variables by transforming each \( V^*_1 \) into a radius and \( p-i \) angles and then integrating over the angles. So for each \( i, i = 1, \ldots, n \), let

\[
\begin{align*}
\ell_{\frac{3}{2}}^{\frac{1}{2}} & \cdot \ell_{\frac{3}{2}}^{\frac{1}{2}} \cdot i_{i,1}^l \cdot \cdots \cdot i_{i,l-1}^l \cdot \cos \theta_{i,1} \\
\ell_{\frac{3}{2}}^{\frac{1}{2}} & \cdot \ell_{\frac{3}{2}}^{\frac{1}{2}} \cdot i_{i,1}^l \cdot \cdots \cdot i_{i,l-1}^l \cdot \sin \theta_{i,1} \cdot \cos \theta_{i,2} \\
\vdots \\
\ell_{\frac{3}{2}}^{\frac{1}{2}} & \cdot \ell_{\frac{3}{2}}^{\frac{1}{2}} \cdot i_{i,1}^l \cdot \cdots \cdot i_{i,l-1}^l \cdot \sin \theta_{i,1} \cdot \sin \theta_{i,2} \cdot \cdots \cdot \sin \theta_{p-i,1} \cdot \cos \theta_{p-i,2} \\
\ell_{\frac{3}{2}}^{\frac{1}{2}} & \cdot \ell_{\frac{3}{2}}^{\frac{1}{2}} \cdot i_{i,1}^l \cdot \cdots \cdot i_{i,l-1}^l \cdot \sin \theta_{i,1} \cdot \sin \theta_{i,2} \cdot \cdots \cdot \sin \theta_{p-i,1} \cdot \sin \theta_{p-i,2}^l
\end{align*}
\]
The Jacobian is \( J = \prod_{i=1}^{n_1} J_i \), where

\[
J_i = \frac{1}{2} \lambda_{ii+1} \ldots, i-1 \sin^{p-i-1} \theta_{ii} \sin^{p-i-2} \theta_{ii+1} \ldots \sin \theta_{p-i-1,i}
\]

and

\( 0 \leq \theta_{jj} \leq \pi \) (\( j = 1, \ldots, p-i-1, i = 1, \ldots, n_1 \)), \( 0 \leq \theta_{p-i,i} \leq 2\pi \) (\( i = 1, \ldots, n_1 \)). (see Sitgreaves (1952)). Under this transformation, \( V_1^* V_1 = \lambda_{ii+1} \ldots, i-1 \) and \( v_{ll} = \frac{1}{2} \lambda_{ll} \cos \theta_{ll} \). In addition, careful attention to the way the \( V_1 \) are constructed in (1.14), namely that the \( G_i \) always have a first row proportional to the appropriate part of the vector \( v_i \), leads to the result that

\[
I - U'U = I - V'V = I - TT'
\]

where

\[
T = \begin{bmatrix}
\frac{1}{2} \lambda_{ll} & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{2} \lambda_{22} & \cdot & \cdot \\
\cdot & \cdot & \frac{1}{2} \lambda_{33} & \cdot \\
\vdots & \vdots & \vdots & \vdots \\
\cdot & \cdot & \cdot & \frac{1}{2} \lambda_{n_1n_1} \\
\end{bmatrix}
\]

is a lower triangular (\( n_1 \times n_1 \)) matrix.

Under this transformation of the \( \{V_1^*\} \), the \( \theta \)'s appear only in the Jacobian and in place of the \( v_{ll} \) in the infinite sum so that
\[ p(\ell_{i1,i}, ..., i-1; \theta_{j1}; v_{ki}; i = 1, ..., n_1; j = 1, ..., p-1; k = 1, ..., i-1) \]

\[ = \prod_{i=1}^{n_1} \Gamma \left( \frac{1}{2}(n_2 + 1) \right) \left\{ \prod_{i=1}^{n_1} \Gamma \left( \frac{1}{2}(n_2 + 1) \right) \right\} \cdot \frac{\frac{1}{2}n_1 p - 1}{e^{-\frac{1}{2} \gamma_1}}. \]

\[ |I - TT|^{\frac{1}{2}(n_2 - p - 1)} \cdot \sum_{k_0=0}^{\infty} \frac{\Gamma \left( \frac{1}{2}(n_1 + n_2 + k_0) \right)}{\Gamma \left( k_0 + 1 \right)} \cdot \frac{\frac{1}{2}k_0 \frac{1}{2}k_0 \frac{1}{2}k_0}{\gamma_1 \ell_{11}^2 \cos \theta_{11}} \]

\[ \cdot \prod_{i=1}^{n_1} \frac{1}{2} (p-i-1) \sin^{p-i-1} \theta_{1i} \sin^{p-i-2} \theta_{2i} \cdots \sin \theta_{p-i-1,i} \]

for \[ 0 \leq \theta_{j1} \leq \pi \quad (j = 1, ..., p-1; i = 1, ..., n_1) \]

and \[ 0 \leq \theta_{p-i,i} \leq 2\pi \quad (i = 1, ..., n_1) \]

We use again the following formula from Sitgreaves (1952):

\[ \int_0^{\pi} \sin^m \theta \cos^n \theta \, d\theta = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{\Gamma \left( \frac{1}{2}(m+1) \right) \Gamma \left( \frac{1}{2}(n+1) \right)}{\Gamma \left( \frac{1}{2}(m+n)+1 \right)}, & \text{if } n \text{ is even} \end{cases} \]

to get the density of the \( \ell_{i1,i}, ..., i-1 \) and \( v_{ki} \):

\[ p_{\gamma_1}(\ell_{i1,i}, ..., i-1; v_{ki}; i = 1, ..., n_1; k = 1, ..., i-1) \]

\[ = \prod_{i=1}^{n_1} \Gamma \left( \frac{1}{2}(n_2 + p + 1) \right) \cdot \prod_{i=1}^{n_1} \Gamma \left( \frac{1}{2}(p - i) \right) \cdot \frac{n_1(n_1 - 1)/2 - 1}{e^{-\frac{1}{2} \gamma_1}}. \]
\[
\prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(n_2+1)\right) \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(p-i-1)\right) \quad |I-TT'|^{-\frac{1}{2}(n_2-p-1)}
\]

\[
\sum_{k_0=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_1+n_2-k_0)\right)}{\Gamma\left(\frac{1}{2}(n+p-k_0)\right)\Gamma(k_0+1)} \left(\frac{\gamma_1\gamma_0}{\pi}\right)^{k_0} t_{i1}^{2k_0}.
\]

We now have the density of \( \frac{1}{2} n_1(n_1+1) \) elements, some of which are highly suggestive of their relationship to the matrix \( \tilde{\Lambda} \): namely, \( T \) is the lower triangular square root of \( \tilde{\Lambda} \). Hence we would like to transform from \( T \) to \( \tilde{\Lambda} \) by making the transformation \( \tilde{\Lambda} = TT' \).

But the diagonal elements of \( T \) are \( t_{ii} = \frac{1}{2} n_{i-1} \), \( i = 1 \) and the density we have is for \( i = 1 \), \( i = 1 \); so we shall first make the transformation

\[
\tilde{t}_{i1} = t_{ii}^2, \quad i = 1, \ldots, n_1,
\]

which has Jacobian \( J = \prod_{i=1}^{n_1} 2t_{ii} = 2 \prod_{i=1}^{n_1} t_{ii} \) to get

\[
P_{\gamma_1}(T) = C_6 \cdot |I-TT'|^{-\frac{1}{2}(n_2-p-1)} \cdot e^{-\frac{1}{2}\gamma_1} \cdot 2 \prod_{i=1}^{n_1} t_{ii}^{p-i},
\]

\[
\sum_{k_0=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_1+n_2-k_0)\right)}{\Gamma\left(\frac{1}{2}(n+p-k_0)\right)\Gamma(k_0+1)} \left(\frac{\gamma_1\gamma_0}{\pi}\right)^{k_0} t_{i1}^{2k_0},
\]

where

\[
C_6 = \frac{\prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(n_2+1)\right) \cdot \prod_{i=1}^{n_1-1} \Gamma\left(\frac{1}{2}(p-i-1)\right) \cdot \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(p)\right) \cdot \pi^{n_1(n_1-1)/4}}{\prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(n_1+n_2-p+1-i)\right) \cdot \prod_{i=1}^{n_1} \Gamma\left(\frac{1}{2}(i)\right) \cdot \pi^{n_1(n_1-1)/4}}.
\]
From Theorem 4.1 and 5B.1 of Deemer and Olkin (1951), we find that the Jacobian of the transformation $\bar{X} = TT'$ is

$$J = (2^{n_1} \prod_{i=1}^{n_1} t_{i1}^{n_1-i+1})^{-1}.$$ We note that $\prod_{i=1}^{n_1} t_{i1}^{n_1-i+1} = |\bar{X}|^{\frac{1}{2}(p-n_1-1)}$

since $T$ is the lower triangular square root of $\bar{X}$, so that

$$p_{\gamma_1}(\bar{X}) = c_6 \cdot e^{-\frac{1}{2} \gamma_1} \cdot |I - \bar{X}|^{\frac{1}{2}(n_2-p-1)} \cdot |\bar{X}|^{\frac{1}{2}(p-n_1-1)}$$

$$\cdot \sum_{k_0=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_1+n_2)+k_0\right)}{\Gamma\left(\frac{1}{2}p+k_0\right)\Gamma\left(k_0+1\right)} \frac{\gamma_1}{k_0} e^{-\frac{1}{2} \gamma_1} k_0,$$

which agrees with the result in Lemma 1.2 but includes the case $n_1 = p$. From this point, the same derivation (Steps 3 and 4) works to obtain the joint density of $\{\lambda_i, i = 1, \ldots, n_1\}$ and of $\{\phi_i, i = 1, \ldots, n_1\}$. As before, this derivation works for $n_1 > p$ if we replace $n_1$ by $p$, $p$ by $n_1$ and $n_2$ by $n_2 - p + n_1$ so that we have shown two methods to obtain the distribution of the roots of $\bar{X}$ in the non-null (linear) case.

1.7 Summary

In this chapter we present the canonical form of the multivariate analysis of variance problem and use this form to derive the non-null (linear) joint distribution of the characteristic roots $\{\lambda_i\}$ and $\{\phi\}$. These roots are the maximal invariants with respect to the group of affine transformations for this problem and the standard statistical tests are functions of one or more of the roots. For this
reason it is of interest to study the behavior of the roots under alternative hypotheses. The non-null linear hypothesis corresponds to having all but one population root equal to zero and having the remaining root take on positive values. The derivation of the density of the roots under this alternative involves transformations and integrations to reduce the number of variables considered and to obtain the joint density of the desired variables. Two methods are used to obtain the non-null (linear) joint densities of the roots and both lead to the same expressions.

For the sake of completeness, Theorems 1.1* and 1.2* contain the central case results and Theorems 1.1 and 1.2 contain the non-null (linear) results. It should be noted that the densities in Theorems 1.1 and 1.2 reduce to the expressions in Theorems 1.1* and 1.2* when the non-centrality parameter, \( \gamma_1 \), is set to zero; and that the null hypothesis corresponds to the situation when all the population roots, including \( \gamma_1 \), are equal to zero.
2. THE DISTRIBUTION OF THE LARGEST ROOT UNDER THE RANK ONE ALTERNATIVE

2.1 Introduction and Notation

In this chapter we derive the non-null (linear) distribution of the largest root, \( \lambda_1 \), of the determinantal equation \( \det (H - \lambda (H+E)) = 0 \) using the joint density of the roots derived in Chapter 1. This distribution is of interest because it is needed to compute the power function of the largest root test for the multivariate analysis of variance under the rank one alternative. The distribution of the largest root of the equation \( \det (H - \rho E) = 0 \) can be obtained from the distribution of \( \lambda_1 \) by noting that

\[
\Pr\{\rho_1 \leq x\} = \Pr\left\{\frac{\lambda_1}{1 - \lambda_1} \leq x\right\} = \Pr\{\lambda_1 \leq \frac{x}{1+x}\}.
\]

The results are in the form of equations containing incomplete beta integrals for which extensive tables exist (see Pearson (1934)) as well as computer programs. In Chapter 4, the results of this chapter are used to evaluate the power of the largest root test and some comparisons with the power of other tests under the rank one alternative are presented. To derive the distribution of the largest root, we use the techniques and notation of Nanda (1948a) and begin by listing some useful results.

(a) Let

\[
\prod_{1 < j} (\lambda_i - \lambda_j) = \{1, 2, \ldots, p\}.
\]
Then the value of the Vandermonde determinant

\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
\lambda_1 & \lambda_2 & \ldots & \lambda_p \\
\lambda_1^2 & \lambda_2^2 & \ldots & \lambda_p^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{p-1} & \lambda_2^{p-1} & \ldots & \lambda_p^{p-1}
\end{vmatrix}
\]

is equal to \( \prod_{i>j} (\lambda_i - \lambda_j) = (-1)^{p \cdot \{1, 2, \ldots, p\}} \). Therefore,

\[
\begin{vmatrix}
1 & 1 & 1 \\
\lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2
\end{vmatrix} = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1) = -\{1, 2, 3\} ;
\]

but this determinant can also be expanded by minors of the first row to give

\[-[\lambda_1 \lambda_2 \{1, 2\} + \lambda_2 \lambda_3 \{2, 3\} + \lambda_3 \lambda_1 \{3, 1\}]\]

where \( \lambda_i - \lambda_j = \{i, j\} \).

Hence

\[(2.1) \quad \{1, 2, 3\} = \lambda_1 \lambda_2 \{1, 2\} + \lambda_2 \lambda_3 \{3, 1\} + \lambda_2 \lambda_3 \{2, 3\} .\]
(b) For convenience in handling some of the integrations, Nanda has developed a shorthand notation and some useful integral identities.

Let \((a, b; m, n) = y^m(1-y)^n \bigg|_a^b = b^m(1-b)^n - a^m(1-a)^n\),

and

\((a, 1, b; m, n) = \int_a^b \! y^m(1-y)^n \, dy\).

**Lemma 2.1**: (Nanda; 1948):

\[(a, 1, b; m+1, n)\]

\[= - \frac{(a, b; m+1, n+1)}{m+n+2} + \frac{m+1}{m+n+2} (a, 1, b; m, n)\].

**Proof**: Integration by parts.

(c) The notation for double integrals follows.

\[(a, 2, 1, b; m, n) = \int_{a < \lambda_2 < \lambda_1 < b} (\lambda_1 \lambda_2)^m(1-\lambda_1)^n(1-\lambda_2)^n \{1, 2\} \, d\lambda_1 \, d\lambda_2\]

and

\[(a, 2, b, 1, c; m, n) = \int_{a < \lambda_2 < b < \lambda_1 < c} (\lambda_1 \lambda_2)^m(1-\lambda_1)^n(1-\lambda_2)^n \{1, 2\} \, d\lambda_1 \, d\lambda_2\].

(d) A slightly different notation for multiple integrals is given by defining

\[m_{a}^{b; m, n} g(y) = \int_{a}^{b} y^m(1-y)^n g(y) \, dy\];

then

\[m_{a}^{b; m, n} (0, y; r, t) = (a, 1, b; m+r, n+t), \quad (r > 0)\]
and
\[ T_a^{b;m,n}(b,1,c;r,t) = (a,1,b;m,n)(b,1,c;r,t). \]

From the notation given above we prove the following lemma:

**Lemma 2.2:**

\[ T_a^{b;m,n}(a,1,y;r,t) = T_a^{b;r,t}(y,1,b;m,n). \]

**Proof:**

\[ T_a^{b;m,n}(a,1,y;r,t) = \int_a^b \left[ \int_a^y z^{r(1-z)}dz \right] y^{m(1-y)}dy \]

\[ = \int_a^b \left[ \int_a^b y^{m(1-y)}dy \right] z^{r(1-z)}dz = T_a^{b;r,t}(y,1,b;m,n). \]

It follows from Lemma 2.2 that

\[ (2.2) \quad T_a^{b;m,n}(a,1,y;m,n) - T_a^{b;m,n}(y,1,b;m,n) = 0. \]

2.2 Joint Density of the Roots

The joint density of the roots is derived in Chapter 1 (Theorems 1.1 and 1.2). With a few changes in notation the expressions for the cases \( n_1 \leq p \) and \( n_1 > p \) can be combined into one expression:

Let

\[ (2.3) \quad s = \min(p,n_1); \quad \mu = |p-n_1| + 1 \quad \text{and} \quad \nu = n_2 - p + 1. \]

Then \( C_1 \) and \( C_2 \) (see equations (1.2) and (1.3)) can be combined as
\begin{equation}
C = \pi^s \frac{s}{\Gamma(\frac{1}{2}(v+\mu+2s-1)) \Gamma(\frac{1}{2}(\mu+s-1)) \Gamma(\frac{1}{2}(\mu+1))}
\end{equation}

and the density of the non-zero roots of \(|H-H(E)| = 0\) can be written as

\begin{equation}
P_{\gamma_1}(\lambda_1, \ldots, \lambda_s) = e^{-\frac{1}{2}\gamma_1} \cdot C \cdot \prod_{i=1}^{s} \{\lambda_1^{\mu-1}(1-\lambda_i)\gamma_{v-1}\} \cdot \prod_{i<j}^{} (\lambda_i - \lambda_j).
\end{equation}

\begin{align}
\left\{1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(\frac{1}{2}(v+\mu+2s-2+2k_0)) \Gamma(\frac{1}{2}s) \Gamma(\frac{1}{2}(\mu+s-1))}{\Gamma(\frac{1}{2}(v+\mu+2s-2)) \Gamma(\frac{1}{2}s+k_0) \Gamma(\frac{1}{2}(\mu+s-1)+k_0)} \right. \\
\left. \sum_{k_1=0}^{k_0} \cdots \sum_{k_{s-1}=0}^{k_{s-2}} \prod_{i=1}^{s-1} \frac{\Gamma(k_{i-1} - k_{i-1} + \frac{1}{2})}{\Gamma(k_{i-1} - k_{i-1} + 1 + \frac{1}{2})} \cdot \frac{\gamma_1^{k_0}}{\gamma_{1}^{k_0}} \right.
\end{align}

for \(0 \leq \lambda_s \leq \cdots \leq \lambda_1 \leq 1\).

A more convenient expression for obtaining the distribution of the largest root comes from equation (1.11) and, with the changes in notation, is given by

\begin{equation}
P_{\gamma_1}(\lambda_1, \ldots, \lambda_s) = e^{-\frac{1}{2}\gamma_1} \cdot C_0 \cdot \prod_{i=1}^{s} \{\lambda_1^{\mu-1}(1-\lambda_i)\gamma_{v-1}\} \cdot \prod_{i<j}^{} (\lambda_i - \lambda_j).
\end{equation}

\begin{equation}
\sum_{k_0=0}^{\infty} \frac{\Gamma(\frac{1}{2}(v+\mu+2s-2+k_0)) \Gamma(\frac{1}{2}s+k_0) \Gamma(\frac{1}{2}(\mu+s-1)+k_0)}{\Gamma(\frac{1}{2}(v+\mu+2s-2)) \Gamma(\frac{1}{2}s+k_0) \Gamma(\frac{1}{2}(\mu+s-1)+k_0)} \cdot \left(\frac{\gamma_1}{2}\right)^{k_0}.
\end{equation}
\[
\begin{align*}
\sum_{k_1=0}^{k_0} \cdots \sum_{k_{s-1}=0}^{k_{s-2}} (\prod_{i=1}^{s-1} \frac{\Gamma(k_{i-1-k_i+\frac{1}{2}})}{\Gamma(k_{i-1-k_i})} \cdot \frac{\Gamma(k_{s-1-k_{s-1}+\frac{1}{2}})}{\Gamma(k_{s-1-k_{s-1}})} \cdot \prod_{i=1}^{s-1} \frac{\Gamma(\lambda_i)}{\Gamma(\lambda_i-1)} \cdot \frac{\Gamma(\lambda_{s-1})}{\Gamma(\lambda_{s-1}-1)}),
\end{align*}
\]

where

\[
(2.7) \quad C_0 = \frac{\prod_{i=1}^{s-1} \Gamma(\frac{1}{2}(\nu+2\mu-2-i))}{\prod_{i=1}^{s-1} \Gamma(\frac{1}{2}(\nu+s-1)) \prod_{i=1}^{s-1} \Gamma(\frac{1}{2}(s-i)) \prod_{i=1}^{s-1} \Gamma(\frac{1}{2}(\mu+s-1-i))}
\]

An additional notational simplification is obtained by letting

\[
(2.8) \quad \ell = \frac{\mu}{2} - \frac{1}{2} = \frac{1}{2}(|p-n_1|-1); \quad n = \frac{\nu}{2} - \frac{1}{2} = \frac{1}{2}(n_2-p-1).
\]

2.3 Central Case Results

The distribution of the largest root in the central case for \( s = 2, 3, 4, 5 \) was obtained by Nanda (1948a). Due to the increasing complexity of the results in the non-null (linear) case, we shall consider only the cases \( s = 2 \) and \( s = 3 \). We present here the statements of the two central case results and note that the proofs are special cases of the non-central results which follow (i.e., for \( \gamma_1 = 0 \)).

Theorem 2.1* (Nanda, 1948a): Under the assumptions of Theorems 1.1*(i) and 1.2*(i), the distribution of the largest root of \( |H-\lambda(H+E)| = 0 \), for \( s = \min(p,n_1) = 2 \), is given by
\[ \Pr_{2,0}(\lambda_1 \leq x) = \frac{c}{x+n+2} \left\{ \begin{array}{l} 2(0,1,x;2l+1,2n+1) \\ - (0,x;\ell+1,n+1) \cdot (0,1,x;\ell,n) \end{array} \right\} \\
= \frac{c}{m+n+2} \left\{ \begin{array}{l} 2 \int_0^x y^{2l+1}(1-y)^{2n+1}dy \\ -x^{\ell+1}(1-x)^{n+1} \int_0^x y^{\ell}(1-y)^{n}dy \end{array} \right\} , \]

where \( c \) is given by (2.4) and \( \ell \) and \( n \) are given in (2.8).

**Theorem 2.2** *(Nanda, 1948a):* Under the assumptions of Theorems 1.1* (i) and 1.2* (i), the distribution of the largest root of 
\[ |H-\lambda H+E| = 0, \text{ for } s = \min(p,n_1) = 3, \] is given by

\[ \Pr_{3,0}(\lambda_1 \leq x) = \frac{c}{x+n+3} \left\{ \begin{array}{l} 2(0,1,x;2l+3,2n+1) \cdot (0,1,x;\ell,n) \\ - 2(0,1,x;2l+2,2n+1) \cdot (0,1,x;\ell+1,n) \\ - (0,x;\ell+2,n+1) \cdot (0,2,1,x;\ell,n) \end{array} \right\} , \]

where \( c \) is given in (2.4) and \( \ell \) and \( n \) are defined in (2.8).

### 2.4 Linear Case Results - \( s = 2 \)

In this section we state the analogue to Theorem 2.1* for the non-null (linear) case and present the derivation of the largest root distribution in the next section. Using the notation of Sections 2.1 and 2.2, with \( s = 2 \), we can write the joint non-null (linear) density of \( (\lambda_1, \lambda_2) \) as
\[ p_{\gamma_1}(\lambda_1, \lambda_2) = e^{-\frac{1}{2} \gamma_1} \cdot C_0 \cdot \frac{\Gamma(\ell+n+3+k_0)}{\Gamma(k_0+1)\Gamma(\ell+\frac{3}{2}+k_0)} \cdot (\frac{1}{2})^k_0 \cdot \]

where \( C_0 \) is given in (2.7), and \( \ell \) and \( n \) are defined in (2.8).

Using this density we obtain

**Theorem 2.1:** Under the assumptions of Theorems 1.1 (i) and 1.2 (i), the distribution of the largest root, \( \lambda_1 \), of \( |H-\lambda(H+E)| = 0 \), for \( s = 2 \), is given by

\[ \Pr_{\gamma_1}(\lambda_1 \leq x) = e^{-\frac{1}{2} \gamma_1} \cdot C_0 \cdot \frac{\Gamma(\ell+n+3+k_0)}{\Gamma(k_0+1)\Gamma(\ell+\frac{3}{2}+k_0)} \cdot (\frac{1}{2})^k_0 \cdot S^*_{k_0,k_1}, \]

for

\[ S^*_{k_0,k_1} = \frac{\Gamma(k_0+\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(k_0+1)\Gamma(1)} \left[ \frac{1}{(\ell+n+k_0+2)} \left( 2 \int_0^x y^{(1-y)2n+1} \, dy \right) \right. \]

\[ + \left. \sum_{i=1}^{k_0} \left( \prod_{j=1}^{i-1} \frac{\ell+k_0+2-j}{\ell+n+k_0+2-j} \right) \frac{1}{(\ell+n+k_0+2)} \left( 2 \int_0^x y^{(1-y)2n+1} \, dy \right) \right] \]

\[ - \frac{\ell+k_0+1}{(1-x)^{n+1}} \int_0^x y^{(1-y)^n} \, dy \]

\[ \begin{align*}
\left[ \frac{1}{(\ell+n+k_0+2)} \left( 2 \int_0^x y^{(1-y)2n+1} \, dy \right) \right] \]

\[ - \frac{\ell+k_0+1}{(1-x)^{n+1}} \int_0^x y^{(1-y)^n} \, dy \right] \]
\[ \begin{align*}
&+ \sum_{k_1=1}^{\left\lceil \frac{k_0}{2} \right\rceil} \left\{ \frac{\Gamma(k_0-k_1+\frac{1}{2})\Gamma(k_1+\frac{1}{2})\Gamma(k_0-k_1+\frac{3}{2})\Gamma(k_1-\frac{3}{2})}{\Gamma(k_0-k_1+1)\Gamma(k_1+1)\Gamma(k_0-k_1+2)\Gamma(k_1)} \right\}.

&\left\{ \frac{1}{x+n+k_0-k_1+2} \left( 2 \int_0^x \frac{y^{\ell+k_0+1}}{(1-y)^{2n+1}} dy \right) \\
&- \frac{x^{\ell+k_0-k_1+1}}{(1-x)^{n+1}} \left( \int_0^x y^{\ell+k_1}(1-y)^{n} dy \right) \right\}

&+ \sum_{i=1}^{k_0-2k_1} \left( \int_{j=1}^{l+k_0-k_1+2-j} \frac{1}{x+n+k_0-k_1+2-j} \left( 2 \int_0^x \frac{y^{\ell+k_0+1-j}}{(1-y)^{2n+1}} dy \right) \\
&- \frac{x^{\ell+k_0-k_1+1-j}}{(1-x)^{n+1}} \left( \int_0^x y^{\ell+k_1}(1-y)^{n} dy \right) \right\},
\end{align*} \]

where \( \left\lceil \frac{k_0}{2} \right\rceil = k \) for \( k \) the largest integer such that \( k \leq \frac{k_0}{2} \); \( \ell \) and \( n \) are defined in (2.8), \( C_0 \) is defined in (2.7) and, by convention, \( \sum_{i=1}^0 = 0 \).

2.5 Linear Case Derivation - \( s = 2 \)

The derivation of the distribution of \( \lambda_1 \) involves reducing the double integral

\[ \int_{0 < \lambda_2 < \lambda_1 < x} p_{\gamma_1}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \]

to single integrals of the incomplete beta type, where \( p_{\gamma_1}(\lambda_1, \lambda_2) \)
is given in (2.9). We begin by defining $S_{k_0,k_1}$ as

$$S_{k_0,k_1} = \sum_{k_1=0}^{k_0} \frac{\Gamma(k_0-k_1+\frac{1}{2})\Gamma(k_1+\frac{1}{2})}{\Gamma(k_0-k_1+1)\Gamma(k_1+1)} (1-\lambda_1)^n(1-\lambda_2)^n \lambda_1^{\ell+k_0-k_1+1} \lambda_2^{\ell+k_1}.$$

By rearranging the terms in the summation, we obtain

$$S_{k_0,k_1} = \sum_{k_1=0}^{k_0} \frac{\Gamma(k_0-k_1+\frac{1}{2})\Gamma(k_1+\frac{1}{2})}{\Gamma(k_0-k_1+1)\Gamma(k_1+1)} (1-\lambda_1)^n(1-\lambda_2)^n \lambda_1^{\ell+k_0-k_1+1} \lambda_2^{\ell+k_1}$$

$$- \sum_{k_1=1}^{k_0+1} \frac{\Gamma(k_0-k_1+\frac{3}{2})\Gamma(k_1-\frac{1}{2})}{\Gamma(k_0-k_1+2)\Gamma(k_1+1)} (1-\lambda_1)^n(1-\lambda_2)^n \lambda_1^{\ell+k_0-k_1+1} \lambda_2^{\ell+k_1}$$

$$= \frac{\Gamma(k_0+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k_0+1)\Gamma(1)} \left\{(1-\lambda_1)^n(1-\lambda_2)^n \lambda_1^{\ell+k_0+1} \lambda_2^{\ell+k_0} \right\}$$

$$- (1-\lambda_1)^n(1-\lambda_2)^n \lambda_1^{\ell+k_0-k_1+1} \lambda_2^{\ell+k_1},$$

where

$$G(k_0,k_1) = \frac{\Gamma(k_0-k_1+\frac{1}{2})\Gamma(k_1+\frac{1}{2})}{\Gamma(k_0-k_1+1)\Gamma(k_1+1)} - \frac{\Gamma(k_0-k_1+\frac{3}{2})\Gamma(k_1-\frac{1}{2})}{\Gamma(k_0-k_1+2)\Gamma(k_1+1)}.$$

As $k_1$ goes from 1 to $k_0$, notice that $\ell+k_0-k_1+1$ and $\ell+k_1$ go through the same values, but in reverse order. Therefore, if we define $[z]$ to be the greatest integer function, i.e., $[z]$ is
the largest integer less than or equal to \( z \), we can write

\[
S_{k_0, k_1} = \frac{\Gamma(k_0 + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k_0 + 1)\Gamma(1)} \{(1-\lambda_1)^n(1-\lambda_2)^n\lambda_1^{l+k_0+1}\lambda_2^{\ell+1}\}
\]

\[
= -(1-\lambda_1)^n(1-\lambda_2)^n\lambda_1^{\ell+k_0+1}\lambda_2^{l+1}\}
\]

(2.11)

\[
\frac{[z]}{2} + \sum_{k_1=1}^{\left[\frac{z}{2}\right]} G(k_0, k_1) \{(1-\lambda_1)^n(1-\lambda_2)^n\lambda_1^{\ell+k_0-k_1+1}\lambda_2^{\ell+k_1+1}\}
\]

\[
- (1-\lambda_1)^n(1-\lambda_2)^n\lambda_1^{\ell+k_1+1}\lambda_2^{l+k_0-k_1+1}\}
\]

This expression holds for even and odd values of \( k_0 \) where by convention we define \( \sum_{k_1=1}^{\left[\frac{z}{2}\right]} = 0 \). To get the cumulative distribution of \( \lambda_1 \), we integrate over \( \lambda_1 \) and \( \lambda_2 \). Define

\[
S^*_{k_0, k_1} = \int_{0-\lambda_1<\lambda_2<x} S_{k_0, k_1} d\lambda_1 d\lambda_2 .
\]

Then using Nanda's notation, we can write

\[
S^*_{k_0, k_1} = \frac{\Gamma(k_0 + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k_0 + 1)\Gamma(1)} \{T_0^{x;\ell,n}(y, l, x; \ell+k_0+1, n)
\]

\[-T_0^{x;\ell,n}(0, l, y; \ell+k_0+1, n)\}
\]

\[
\frac{[z]}{2} + \sum_{k_1=1}^{\left[\frac{z}{2}\right]} G(k_0, k_1) \{T_0^{x;\ell+1,n}(y, l, x; \ell+k_0-k_1+1, n)
\]

\[-T_0^{x;\ell+1,n}(0, l, y; \ell+k_0-k_1+1, n)\}
\]

\[
= \frac{\Gamma(k_0 + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k_0 + 1)\Gamma(1)} \{T_0^{x;\ell,n}(y, l, x; \ell+k_0+1, n)
\]

\[-T_0^{x;\ell,n}(0, l, y; \ell+k_0+1, n)\}
\]

\[
+ \sum_{k_1=1}^{\left[\frac{z}{2}\right]} G(k_0, k_1) \{T_0^{x;\ell+1,n}(y, l, x; \ell+k_0-k_1+1, n)
\]

\[-T_0^{x;\ell+1,n}(0, l, y; \ell+k_0-k_1+1, n)\}
\]

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We now use Lemma 2.1 and Equation (2.2) to obtain that

\[
\begin{align*}
T_0 \times_{\ell+k_1,n} &= (y,1,x,\ell+k_0-k_1+l,n) - (0,1,y,\ell+k_0-k_1+l,n) \\
&= T_0 \frac{1}{\ell+k_0-k_1+n+2} \left[ -(y,x;\ell+k_0-k_1+l,n+1) \\
&\quad + (0,y;\ell+k_0-k_1+l,n+1) \right] \\
&\quad + \frac{k_0-k_1+l}{\ell+k_0-k_1+n+2} \left[ (y,1,x;\ell+k_0-k_1,n) - (0,1,y;\ell+k_0-k_1,n) \right] \\
&= T_0 \frac{1}{\ell+k_0-k_1+n+2} \left[ -(y,x;\ell+k_0-k_1+l,n+1) \\
&\quad + (0,y;\ell+k_0-k_1+l,n+1) \right] \\
&\quad + \sum_{i=1}^{k_0-2k_1} \left( \frac{i}{\ell+k_0-k_1+n+2-j} \right) \frac{1}{\ell+k_0-k_1+n+2} \left[ -(y,x;\ell+k_0-k_1+l-i,n+1) \\
&\quad + (0,y;\ell+k_0-k_1+l-i,n+1) \right].
\end{align*}
\]

Substituting in $S^*_{k_0,k_1}$ and simplifying using the results of Section 2.1 (d) yields

\[
S^*_{k_0,k_1} = \frac{\Gamma(k_0+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k_0+1)\Gamma(1)} \left\{ \frac{1}{\ell+k_0+n+2} \left[ 2(0,1,x;2\ell+k_0+1,2n+1) \\
&\quad - (0,x;\ell+k_0+1,n+1) \cdot (0,1,x;\ell,n) \right] \\
&\quad + \sum_{i=1}^{k_0} \left( \frac{i}{\ell+k_0+n+2-j} \right) \frac{1}{\ell+k_0+n+2} \left[ 2(0,1,x;2\ell+k_0+1-l+i,2n+1) \\
&\quad - (0,x;\ell+k_0+1-l+i,n+1) \cdot (0,1,x;\ell,n) \right] \right\}.
\]

(2.12)
\[ + \sum_{k_1=1}^{[k_0/2]} G(k_0, k_1) \cdot \frac{1}{\ell+k_0-k_1+n+2} [2(0,1,x;\ell+k_0+1,2n+1) - (0,x;\ell+k_0-k_1+1,n+1) \cdot (0,1,x;\ell+k_1,n)]^t, \]

\[ + \sum_{i=1}^{k_0-2k_1} \left( \prod_{j=1}^{i} \frac{\ell+k_0-k_1+2-j}{\ell+k_0-k_1+n+2-j} \right) \frac{1}{\ell+k_0-k_1+n+2} [2(0,1,x;2\ell+k_0+1-i,2n+1) - (0,x;\ell+k_0-k_1+1-i,n+1) \cdot (0,1,x;\ell+k_1,n)]^t, \]

where \( G(k_0, k_1) \) is defined in (2.10).

2.6 Linear Case Results - \( s = 3 \)

In this section we state the analogue of Theorem 2.2* for the non-null (linear) case and, in the next section, present the derivation of the distribution. We begin by writing the joint non-null (linear) density of \((\lambda_1, \lambda_2, \lambda_3)\), using the notation of Sections 2.1 and 2.2 with \( s = 3 \), as

\[ p_{\gamma_1}(\lambda_1, \lambda_2, \lambda_3) = e^{-\gamma_1} \cdot C_0 \cdot \sum_{k_0=0}^{\infty} \frac{\Gamma(\ell+n+k_0)}{\Gamma^3(k_0)\Gamma(\ell+2+k_0)} \cdot \left( \frac{1}{2} \right)^k_0 \cdot \]

\[ \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \frac{\Gamma(k_0-k_1+\frac{1}{2})\Gamma(k_1-k_2+\frac{1}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_0-k_1+1)\Gamma(k_1-k_2+1)\Gamma(k_2+1)} \cdot (1-\lambda_1)^n(1-\lambda_2)^n(1-\lambda_3)^n, \]

\[ \lambda_1 \cdots \lambda_3 \cdot \{1,2,3\}, \]

where \( C_0 \) is defined in (2.7); \( s = \min(p,n_1) = 3 \); and \( \ell \) and \( n \) are
given in (2.8). Using this density we obtain the analogue of Theorem 2.2* for the non-null (linear) case:

**Theorem 2.2:** Under the assumptions of Theorems 1.1 (1) and 1.2 (1), the distribution of the largest root of \(|H-\lambda(H+E)| = 0\), for \(s = 3\), is given by

\[
\Pr_{\gamma_1, \gamma_2}(\lambda_1 \leq x) = e^{-\frac{1}{2}\gamma_1} \cdot c_0 \cdot \sum_{k_0=0}^{\infty} \frac{\Gamma(\ell+n+k_0)}{\Gamma(\frac{3}{2}+k_0)\Gamma(\ell+2+k_0)} \cdot \left(\frac{\gamma_1}{2}\right)^{k_0} \cdot S_{k_0, k_1, k_2}^*,
\]

for

\[
S_{k_0, k_1, k_2}^* = \sum_{k_1=0}^{k_0} \frac{\Gamma(k_0-k_1+1)}{\Gamma(k_0-k_1+1)} \left\{ \sum_{i=0}^{k_1} \frac{\Gamma(k_1+\frac{2}{3})\Gamma(\frac{1}{3})}{\Gamma(k_1+1)\Gamma(1)} \right\}.
\]

\[
\left(\prod_{j=1}^{\ell+1} \frac{\ell+k_1+3-j}{\ell+n+k_1+3-j}\right) \cdot \frac{1}{l+k_2+n+3} \cdot \{2(0,1,x;2\ell+k_0-k_1+2-1,2n+1) \cdot (0,l,x;\ell+1,n)
\]

\[
-(0,x;\ell+k_0-k_1,n) \cdot T_0 \cdot \frac{x}{l+k_0-k_1,n} \cdot ((y,1,x;\ell+1,n)-(0,l,y;\ell+1,n))
\]

\[
+ \sum_{k_2=1}^{k_1} \left\{ \frac{\Gamma(k_1-k_2+1)\Gamma(1)}{\Gamma(k_1-k_2+1)\Gamma(1)} - \frac{\Gamma(k_1-k_2+\frac{3}{2})\Gamma(k_2-\frac{1}{2})}{\Gamma(k_1-k_2+2)\Gamma(k_2)} \right\}.
\]

\[
\sum_{i=0}^{k_1-2k_2} \left(\prod_{j=1}^{\ell+1} \frac{\ell+k_1-k_2+3-j}{\ell+n+k_1-k_2+3-j}\right) \cdot \frac{1}{l+k_1-k_2+n+3} \cdot \{2(0,1,x;2\ell+k_0-k_1+2-1,2n+1) \cdot (0,l,x;\ell+k_2+1,n)
\]

\[
-(0,l,x;\ell+k_0-k_1,n) \cdot T_0 \cdot \frac{x}{l+k_0-k_1,n} \cdot ((y,1,x;\ell+1,n)-(0,l,y;\ell+1,n))
\]

\[
- 2(0,1,x;2\ell+k_0-k_2+2-1,2n+1) \cdot (0,l,x;\ell+k_2+1,n)
\]

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\[(0,x,\ell+k_0-k_1+2-i,n+1) \cdot T_0(x,\ell+k_0-k_1,n) \cdot ((y,1,x,\ell+1,n)-(0,1,y,\ell+1,n))\]

where \([\frac{k_1}{2}] = k\) for \(k\) the largest integer such that \(k \leq \frac{k_1}{2}\); \(\ell\) and \(n\) as in (2.8); \(C_0\) as in (2.7); and, where \(\sum_{i=0}^{\infty} = 0\) and \(\prod_{i=1}^{\infty} = 1\) by convention.

Note: In the expression for \(S_{k_0,k_1,k_2}^x\), there is a double integration, namely

\[x,\ell+k_0-k_1,n \cdot T_0((y,1,x,\ell+k_2+1,n) - (0,1,y,\ell+k_2+1,n))\]

which must be evaluated. When \(\ell+k_2+1 = \ell+k_0-k_1\), this expression is equal to zero by Equation (2.2). When \(\ell+k_2+1 > \ell+k_0-k_1\), we use Lemma 2.1 and then Equation (2.2) to get that

\[x,\ell+k_0-k_1,n \cdot T_0((y,1,x,\ell+k_2+1,n) - (0,1,y,\ell+k_2+1,n))\]

\[= \frac{1}{\ell+k_2+m+2} \left\{ 2(0,1,x,2\ell+k_0-k_1+k_2+1,2n+1) - (0,x,\ell+k_2+1,n+1) \cdot (0,1,x,\ell+k_0-k_1,n) \right.\]

\[+ \sum_{i=1}^{k_2-k_0+k_1} \prod_{j=1}^{\ell+k_2+2-i,j}(2(0,1,x,2\ell+k_0-k_1+k_2+1-i,2n+1) - (0,x,\ell+k_2+1-i,n+1) \cdot (0,1,x,\ell+k_0-k_1,n)) \right\}.

When \(\ell+k_2+1 < \ell+k_0-k_1\), we interchange the order of integration and then
apply Lemma 2.1 and Equation (2.2) to obtain

\[
T_0 (0,1,y;_\ell+k_0 -k_1,n) - (y,1,x;_\ell+k_0 -k_1,n) = - \frac{1}{\ell + k_0 - k_1 + n + 1} \left\{ 2(0,1,x;2\ell + k_0 - k_1 + k_2 + 1,2n + 1) - (0,x;\ell + k_0 - k_1, n + 1) \cdot (0,1,x;\ell + k_2 + 1,n) \right. \\
\left. + \sum_{i=1}^{k_0 - k_1 - k_2 - 2} \left( \prod_{j=1}^{\ell + k_0 - k_1 + 1 - j} (\ell + n + k_0 - k_1 + n + 1 - j) \right) 2(0,1,x;2\ell + k_0 - k_1 + k_2 + 1 - i,2n + 1) - (0,x;\ell + k_0 - k_1 - i, n + 1) \cdot (0,1,x;\ell + k_2 + 1,n) \right\}.
\]

2.7 Linear Case Derivation - \( s = 3 \)

The derivation of the distribution of \( \lambda_1 \) involves reducing the integral

\[
\int_{0 < \lambda_3 < \lambda_2 < \lambda_1 < x} p_{\gamma_1} (\lambda_1, \lambda_2, \lambda_3) d\lambda_1 d\lambda_2 d\lambda_3
\]

to an expression containing just single integrals of the incomplete beta type, where \( p_{\gamma_1} (\lambda_1, \lambda_2, \lambda_3) \) is given in (2.13).

As in the case for \( s = 2 \), it is necessary to rearrange the terms in the summations to simplify the expressions and avoid repetition of certain terms. So we begin by defining
\[ S_{k_0, k_1, k_2} = \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \frac{\Gamma(k_0 - k_1 + \frac{1}{2}) \Gamma(k_1 - k_2 + \frac{1}{2}) \Gamma(k_2 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1) \Gamma(k_1 - k_2 + 1) \Gamma(k_2 + 1)}. \]

\[ (1 - \lambda_1)^n (1 - \lambda_2)^n (1 - \lambda_3)^n \lambda_1^{k_0 - k_1} \lambda_2^{k_1 - k_2} \lambda_3^{k_2} \{1, 2, 3\}. \]

Recall from (2.1) that

\[ \{1, 2, 3\} = \lambda_1 \lambda_2 \{1, 2\} + \lambda_3 \lambda_1 \{2, 1\} + \lambda_2 \lambda_3 \{2, 3\}. \]

Note the symmetries in \( k_0 - k_1 \), \( k_1 - k_2 \) and \( k_2 \) as \( k_1 \) and \( k_2 \) go through their ranges of values in \( S_{k_0, k_1, k_2} \). Combining these results with a change in indices we can rewrite \( S_{k_0, k_1, k_2} \) as

\[ S_{k_0, k_1, k_2} = \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \frac{\Gamma(k_0 - k_1 + \frac{1}{2}) \Gamma(k_1 - k_2 + \frac{1}{2}) \Gamma(k_2 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1) \Gamma(k_1 - k_2 + 1) \Gamma(k_2 + 1)}. \]

\[ (1 - \lambda_1)^n (1 - \lambda_2)^n (1 - \lambda_3)^n \left( \lambda_1^{k_0 - k_1 + 1} \lambda_2^{k_1 - k_2 + 1} \lambda_3^{k_2 + 2k_0 - k_1} \{1, 2\} + \lambda_1^{k_0 - k_1 + 1} \lambda_2^{k_1 - k_2 + 1} \lambda_3^{k_2 + 1} \{2, 3\} \right). \]

\[ = \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \frac{\Gamma(k_0 - k_1 + \frac{1}{2}) \Gamma(k_1 - k_2 + \frac{1}{2}) \Gamma(k_2 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1) \Gamma(k_1 - k_2 + 1) \Gamma(k_2 + 1)}. \]

\[ (1 - \lambda_1)^n (1 - \lambda_2)^n (1 - \lambda_3)^n \lambda_3^{k_0 - k_1} \lambda_1^{k_1 - k_2 + 1} \lambda_2^{k_2 + 1} \{1, 2\}. \]
\[
\begin{align*}
&\sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \frac{\Gamma(k_0-k_1+\frac{1}{2})\Gamma(k_1-k_2+\frac{1}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_0-k_1+1)\Gamma(k_1-k_2+1)\Gamma(k_2+1)} \\
&\left(1-\lambda_1\right)^n(1-\lambda_2)^n(1-\lambda_3)^n \lambda_2^{\ell+k_0-k_1} \lambda_1^{k_2+1} \lambda_1^{k_0-k_1-\frac{1}{2}} \lambda_3^{k_2+1} \quad \{1,3\} \\
&\sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \frac{\Gamma(k_0-k_1+\frac{1}{2})\Gamma(k_1-k_2+\frac{1}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_0-k_1+1)\Gamma(k_1-k_2+1)\Gamma(k_2+1)} \\
&\left(1-\lambda_1\right)^n(1-\lambda_2)^n(1-\lambda_3)^n \lambda_2^{\ell+k_0-k_1} \lambda_1^{k_2+1} \lambda_1^{k_0-k_1-\frac{1}{2}} \lambda_3^{k_2+1} \quad \{2,3\},
\end{align*}
\]

where we use the fact that \{1,3\} = -\{3,1\}. The three summands in (2.14) are formally very similar except that \(0 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \leq 1\). If we now define

\[
S_{k_0, k_1, k_2}^* = \int_{0 < \lambda_3 < \lambda_2 < \lambda_1 < 1} S_{k_0, k_1, k_2} d\lambda_1 d\lambda_2 d\lambda_3,
\]

and change variables, we can write

\[
S_{k_0, k_1, k_2}^* = \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \frac{\Gamma(k_0-k_1+\frac{1}{2})\Gamma(k_1-k_2+\frac{1}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_0-k_1+1)\Gamma(k_1-k_2+1)\Gamma(k_2+1)} \\
\left[ \int_{0 < \lambda_3 < \lambda_2 < \lambda_1 < 1} \left(1-\lambda_1\right)^n(1-\lambda_2)^n(1-\lambda_3)^n \lambda_2^{k_0-k_1} \lambda_1^{k_2+1} \lambda_1^{k_0-k_1-\frac{1}{2}} \lambda_3^{k_2+1} \quad \{1,2\} d\lambda_1 d\lambda_2 d\lambda_3 \right].
\]

(2.15)
where the integrands are identical for the three integrals. To facilitate simplification of \( S^{*}_{k_0, k_1, k_2} \), we define the formal quantities

\[
S^{*}_{k_0, k_1, k_2} = \sum_{k_0=0}^{k_0} \sum_{k_1=0}^{k_1} \sum_{k_2=0}^{k_2} \frac{\Gamma(k_0 - k_1 + \frac{1}{2})\Gamma(k_1 - k_2 + \frac{1}{2})\Gamma(k_2 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1)\Gamma(k_1 - k_2 + 1)\Gamma(k_2 + 1)} .
\]

\[
\int (1 - \lambda_1)^{n(1 - \lambda_2)^{n(1 - \lambda_3)^{n_{\lambda_3}}}} \lambda_1^{\ell + k_0 - k_1} \lambda_1^{\ell + k_1 - k_2} \lambda_2^{\ell + k_2 + 1} [1, 2] d\lambda_1 d\lambda_2 d\lambda_3
\]

\[
= \sum_{k_1}^{k_1} \frac{\Gamma(k_1 - k_2 + \frac{1}{2})\Gamma(k_2 + \frac{1}{2})}{\Gamma(k_1 - k_2 + 1)\Gamma(k_2 + 1)} \int (1 - \lambda_3)^{n_{\lambda_3}} \lambda_3^{\ell + k_0 - k_1} .
\]

\[
\sum_{k_2=0}^{k_2} \frac{\Gamma(k_1 - k_2 + \frac{1}{2})\Gamma(k_2 + \frac{1}{2})}{\Gamma(k_1 - k_2 + 1)\Gamma(k_2 + 1)} . \int (1 - \lambda_1)^{n(1 - \lambda_2)^{n_{\lambda_1}}}(1 - \lambda_2)^{\lambda_1^{\ell + k_1 - k_2} + 1} \lambda_2^{\ell + k_2 + 1} [1, 2] d\lambda_1 d\lambda_2 d\lambda_3
\]

and

\[
S^{*}_{k_1, k_2} = \sum_{k_2=0}^{k_2} \frac{\Gamma(k_1 - k_2 + \frac{1}{2})\Gamma(k_2 + \frac{1}{2})}{\Gamma(k_1 - k_2 + 1)\Gamma(k_2 + 1)} . \int (1 - \lambda_1)^{n(1 - \lambda_2)^{n_{\lambda_1}}}(1 - \lambda_2)^{\lambda_1^{\ell + k_1 - k_2} + 1} \lambda_2^{\ell + k_2 + 1} [1, 2] d\lambda_1 d\lambda_2 d\lambda_3
\]

As in the case for \( S^{*}_{k_0, k_1} \) when \( s = 2 \) (in Equation (2.11)), we rearrange the terms in \( S^{*}_{k_1, k_2} \) to get

\[
S^{*}_{k_1, k_2} = \frac{\Gamma(k_1 + \frac{1}{2})\Gamma(k_2 + \frac{1}{2})}{\Gamma(k_1 + 1)\Gamma(k_2 + 1)} \int (1 - \lambda_1)^{n(1 - \lambda_2)^{n_{\lambda_1}}}(1 - \lambda_2)^{\lambda_1^{\ell + k_1 - k_2} + 1} \lambda_2^{\ell + k_2 + 1} [1, 2] d\lambda_1 d\lambda_2 d\lambda_3
\]

(2.16)

\[
\sum_{k_1=0}^{k_1} \sum_{k_2=0}^{k_2} G(k_1, k_2) \int (1 - \lambda_1)^{n(1 - \lambda_2)^{n_{\lambda_1}}}(1 - \lambda_2)^{\lambda_1^{\ell + k_1 - k_2} + 1} \lambda_2^{\ell + k_2 + 1} [1, 2] d\lambda_1 d\lambda_2 d\lambda_3
\]

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where, again, \( \sum_{k_2=1}^{0} = 0 \) by convention and \([k_2/2]\) is the greatest integer function; and

\[
G(k_1, k_2) = \frac{\Gamma(k_1-k_2+\frac{5}{2})\Gamma(k_2+\frac{3}{2})}{\Gamma(k_1-k_2+1)\Gamma(k_2+1)} - \frac{\Gamma(k_1-k_2+\frac{3}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_1-k_2+2)\Gamma(k_2)}
\]

as in (2.10).

At this point we must proceed more carefully and recall that \( \overline{S}_{k_1, k_2}^* \) is a formal quantity representing the three integrals in (2.15), each having different limits of integration. Substituting from (2.16) into (2.15), we can write

\[
\overline{S}_{k_0, k_1, k_2}^* = \sum_{k_1=0}^{k_0} \frac{\Gamma(k_0-k_1+\frac{5}{2})}{\Gamma(k_0-k_1+1)} \int_{0}^{x} (1-\lambda_3)^n \lambda_3^{l+k_0-k_1-\frac{1}{2}} \frac{\Gamma(k_1+\frac{1}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_1+1)\Gamma(k_2+1)} \left( \int_{\lambda_2 \leq \lambda_1 \leq \lambda_1 \leq x} (1-\lambda_1)^n (1-\lambda_2)^n (\lambda_1^{l+k_1+2} \lambda_2^{l+1} - \lambda_1^{l+1} \lambda_2^{l+k_1+2}) \right) d\lambda_1 d\lambda_2
\]

(2.17)

\[
= \sum_{k_1=0}^{k_1/2} \frac{\Gamma(k_1+\frac{3}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_1+1)\Gamma(k_2+1)} \left( \int_{\lambda_2 \leq \lambda_1 \leq \lambda_1 \leq x} (1-\lambda_1)^n (1-\lambda_2)^n (\lambda_1^{l+k_1-k_2+2} \lambda_2^{l+k_2+1} - \lambda_1^{l+k_2+1} \lambda_2^{l+k_1-k_2+2}) \right) d\lambda_1 d\lambda_2
\]

\[
+ \sum_{k_2=1}^{k_1} G(k_1, k_2) \left( \int_{\lambda_3 \leq \lambda_2 \leq \lambda_1 \leq x} (1-\lambda_3)^n (1-\lambda_2)^n (\lambda_1^{l+k_1-k_2+2} \lambda_2^{l+k_2+1} - \lambda_1^{l+k_2+1} \lambda_2^{l+k_1-k_2+2}) \right) d\lambda_3.
\]

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We first integrate over \( \lambda_1 \) and \( \lambda_2 \), leaving \( \lambda_3 \) fixed.

Therefore, we have two types of integrals to work out:

\[
I_1 = \int_{a < \lambda_2 < b} \int_{a < \lambda_1 < b} (1-\lambda_1)^\eta (1-\lambda_2)^\eta (\lambda_1 - \lambda_2) \lambda_1 \lambda_2 - \lambda_1 \lambda_2 \lambda_1^{-k_2+2} \lambda_1^{-k_2+1} \lambda_1^{-k_2+2} \lambda_1 d\lambda_1 d\lambda_2
\]

and

\[
I_2 = \int_{a < \lambda_2 < c} \int_{a < \lambda_1 < c} (1-\lambda_1)^\eta (1-\lambda_2)^\eta (\lambda_1 - \lambda_2) \lambda_1 \lambda_2 - \lambda_1 \lambda_2 \lambda_1^{-k_2+2} \lambda_1^{-k_2+1} \lambda_1^{-k_2+2} \lambda_1 d\lambda_1 d\lambda_2 .
\]

With a change in variables we can write

\[
I_1 = \int_a^b \int_a^b (1-\lambda_2)^\eta \lambda_2 \lambda_2 - \lambda_2 \lambda_2 \lambda_1^{-k_2+2} \eta \lambda_2 \lambda_2 - \lambda_2 \lambda_2 \lambda_1^{-k_2+2} \lambda_1 d\lambda_1 d\lambda_2
\]

\[
+ \int_a^b \int_a^b (1-\lambda_2)^\eta \lambda_2 \lambda_2 - \lambda_2 \lambda_2 \lambda_1^{-k_2+2} \eta \lambda_2 \lambda_2 - \lambda_2 \lambda_2 \lambda_1^{-k_2+2} \lambda_1 d\lambda_1 d\lambda_2
\]

\[
= \int_a^b \int_a^b \frac{b; \lambda_2+1}{n} \lambda_2 \lambda_2 \lambda_2 - \lambda_2 \lambda_2 \lambda_2 \lambda_2 \lambda_2 d\lambda_1 d\lambda_2
\]

\[
= \int_a^b \frac{b; \lambda_2+1, n}{n} \lambda_2 \lambda_2 \lambda_2 - \lambda_2 \lambda_2 \lambda_2 \lambda_2 \lambda_2 d\lambda_1 d\lambda_2
\]

Now we apply Lemma 2.1 several times and then Equation (2.2) to finally obtain

\[
I_1 = \frac{1}{\lambda_1^{-k_2+2} + n+3} \{ 2(a, b; 2\lambda_1+3, 2n+1)
\]

\[
- ((0, a; \lambda_1-k_2+2, n+1) + (0, b; \lambda_1-k_2+2, n+1)) \cdot (a, b; \lambda_1-k_2+1, n) \}
\]

(2.18)

\[
+ \sum_{i=1}^{k_1-2k_2} (-1)^{i+1} \frac{\lambda_1-k_2+3-i}{\lambda_1-k_2+3-i} \frac{1}{\lambda_1-k_2+3-i} .
\]

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\[ \{2(a,1,b;2\ell+k_1+3-1,2n+1) \]

\[ - ((0,a;\ell+k_1-k_2+2-1,n+1) + (0,b;\ell+k_1-k_2+2-1,n+1)) \cdot (a,1,b;\ell+k_2+1,n) \]  

Similarly,

\[ I_2 = \int_a^b \int_c^d (1-\lambda_2)^{\ell+k_2+1} (1-\lambda_1)^{\ell+k_1-k_2+2} d\lambda_1 d\lambda_2 \]

\[ - \int_a^b \int_c^d (1-\lambda_2)^{\ell+k_2+1} (1-\lambda_1)^{\ell+k_1-k_2+2} d\lambda_1 d\lambda_2 \]

\[ = T_a (b,1,c;\ell+k_1-k_2+2,n) - T_b (a,1,b;\ell+k_1-k_2+2,n) \]

\[ \frac{1}{\ell+k_1-k_2+n+3} \{ T_b (a,b;\ell+k_1-k_2+2,n+1) \]  

\[ - T_a (b,c;\ell+k_1-k_2+2,n+1) \]  

\[ \sum_{i=1}^{k_1-2k_2} \left( \frac{1}{\ell+n+k_1-k_2+2-j} \right) \frac{1}{\ell+k_1-k_2+n+3} \cdot \]

\[ \{ T_b (a,b;\ell+k_1-k_2+2-1,n+1) - T_a (b,c;\ell+k_1-k_2+2-1,n+1) \}, \]

by repeated applications of Lemma 2.1 and lastly of Equation (2.2).

Combining the terms yields
\[ I_2 = \frac{1}{k_l-k_2+n+3} \{- (a, l, b; l+k_2+1, n)(0, c; l+k_1-k_2+2, n+1) \]

\[- (b, l, c; l+k_2+1, n)(0, a; l+k_1-k_2+2, n+1) \]

\[+ (a, l, c; l+k_2+1, n)(0, b; l+k_1-k_2+2, n+1) \} \]

\[ \sum_{i=1}^{k_l-2k_2} \left( \prod_{j=1}^{l+k_2+3-2} \right) \frac{1}{l+k_1-k_2+n+3} \cdot \]

\[\{(a, l, b; l+k_2+1, n)(0, c; l+k_1-k_2+2, n+1) \]

\[- (b, l, c; l+k_2+1, n)(0, a; l+k_1-k_2+2, n+1) \]

\[+ (a, l, c; l+k_2+1, n)(0, b; l+k_1-k_2+2, n+1) \} \]

There are two sets of integrals in (2.17), and in each set the first and third integrals are like \( I_1 \), with \( a = \lambda_3 \), \( b = x \) and \( a = 0 \), \( b = \lambda_3 \), respectively; while the middle integral is like \( I_2 \) with \( a = 0 \), \( b = \lambda_3 \), \( c = x \).

Hence, using (2.18) and (2.19), we see that a typical term of \( S_{X_0, k_1, k_2} \) is

\[ \int (1-\lambda_1)^n(1-\lambda_2)^n(\lambda_1^\lambda_2-\lambda_1\lambda_2^\lambda_2-\lambda_1-k_2+2) d\lambda_1 d\lambda_2 \]

\[1 \leq \lambda_2 < \lambda_1 \leq x \]

\[0 < \lambda_2 < \lambda_3 < \lambda_1 \leq x \]

\[0 < \lambda_2 < \lambda_1 < \lambda_3 \]
\[
\begin{align*}
&= \frac{1}{k_1 - k_2 + n + 3} \left[ 2(\lambda_3, 1, x; 2l + k_2 + 3, 2n + 1) \\
&\quad - (0, \lambda_3; l + k_1 - k_2 + 2, n + 1) + (0, x; l + k_1 - k_2 + 2, n + 1) \cdot (\lambda_3, 1, x; l + k_2 + 1, n) \\
&\quad + (0, 1, \lambda_3; l + k_2 + 1, n) \cdot (0, x; l + k_1 - k_2 + 2, n + 1) \\
&\quad - (0, 1, x; l + k_2 + 1, n) \cdot (0, \lambda_3; l + k_1 - k_2 + 2, n + 1) \\
&\quad + 2(0, 1, \lambda_3; 2l + k_1 + 3, 2n + 1) - (0, \lambda_3; l + k_1 - k_2 + 2, n + 1)(0, 1, \lambda_3; l + k_2 + 1, n) \right] \\
&\quad + \frac{k_1 - 2k_2}{\sum_{i=1}^{1} \left( \frac{1}{i} \cdot \frac{1}{l + k_1 - k_2 + n + 3 - j} \cdot \frac{1}{l + k_1 - k_2 + n + 3} \right)} \left[ 2(\lambda_3, 1, x; 2l + k_1 + 3 - i, 2n + 1) \\
&\quad - (0, \lambda_3; l + k_1 - k_2 + 2 - i, n + 1) + (0, x; l + k_1 - k_2 + 2 - i, n + 1) \cdot (\lambda_3, 1, x; l + k_2 + 1, n) \\
&\quad + (0, 1, \lambda_3; l + k_2 + 1, n) \cdot (0, x; l + k_1 - k_2 + 2 - i, n + 1) \\
&\quad - (0, 1, x; l + k_2 + 1, n) \cdot (0, \lambda_3; l + k_1 - k_2 + 2 - i, n + 1) \\
&\quad + 2(0, 1, \lambda_3; 2l + k_1 + 3 - i, 2n + 1) - (0, \lambda_3; l + k_1 - k_2 + 2 - i, n + 1)(0, 1, \lambda_3; l + k_2 + 1, n) \right] \\
&= \frac{1}{l + k_1 - k_2 + n + 3} \left[ 2(0, 1, x; 2l + k_1 + 3, 2n + 1) \\
&\quad - 2(0, \lambda_3; l + k_1 - k_2 + 2, n + 1)(0, 1, x; l + k_2 + 1, n) \\
&\quad - (0, x; l + k_1 - k_2 + 2, n + 1)((\lambda_3, 1, x; l + k_2 + 1, n) - (0, 1, \lambda_3; l + k_2 + 1, n)) \right]
\end{align*}
\]
\[ k_1 - 2k_2 + \sum_{i=1}^{k_1} \left( \frac{1}{\frac{i}{j}} \frac{\frac{l+k_1-k_2+3-i}{j}}{l+k_1-k_2+3+n-i} \right) \frac{1}{l+k_1-k_2+n+3} \{2(0,1,x;2l+k_1+3-i,2n+1) \]

\[-2(0,0,\lambda_2;\frac{l+k_1-k_2+2}{n}-1,n+1)(0,1,x;\frac{l+k_2-1}{n},n) \]

\[-(0,x;\frac{l+k_1-k_2+2}{n}-1,n+1)((\lambda_2,1,x;\frac{l+k_2-1}{n},n) - (0,\lambda_2;\frac{l+k_2-1}{n},n)) \]

Note that the first set of integrals in (2.17) corresponds to having \( k_2 = 0 \) so that (2.20) holds for both sets of integrals for appropriate values of \( k_2 \) and we can substitute this result in (2.17) to get

\[ S_{k_0}^{k_1,k_2} = \sum_{k_1}^{k_0} \frac{\Gamma(k_0-k_1+\frac{1}{2})}{\Gamma(k_0-k_1)} \left\{ \sum_{i=0}^{k_1} \frac{\Gamma(k_1+\frac{1}{2}+i)}{\Gamma(k_1+1)\Gamma(1)} \right\} \]

\[-2(0,1,x;2l+k_0+2-i,2n+1) \cdot (0,1,x;\frac{l+k_0-k_1}{n}) \]

\[-(0,x;\frac{l+k_1+2-i}{n}+1,n+1) \cdot T_0 \quad \{(\frac{x}{y},1,x;\frac{l+1}{n}) - (0,1,y;\frac{l+1}{n})\} \]

\[ + \sum_{k_2=1}^{k_1} \left\{ \frac{\Gamma(k_1-k_2+\frac{3}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_1-k_2+2)\Gamma(k_2+1)} - \frac{\Gamma(k_1-k_2+\frac{3}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_1-k_2+2)\Gamma(k_2+1)} \right\} \]

\[ \sum_{i=0}^{k_1} \frac{\Gamma(k_1-k_2+3-i)}{\Gamma(k_1-k_2+n+3-i)} \]
\[(2.21) \cdot \frac{1}{\ell+k_1-k_2+n+3} \{2(0,1,x;2\ell+k_1+3-1,2n+1) \cdot (0,1,x;\ell+k_0-k_1,n) \\
- 2(0,1,x;2\ell+k_0-k_2+2-1,2n+1) \cdot (0,1,x;\ell+k_2+1,n) \\
- (0,x;\ell+k_1-k_2+2-1,n+1) \cdot T_0^{x;\ell+k_0-k_1,n} ((y,1,x;\ell+k_2+1,n) \\
- (0,1,y;\ell+k_2+1,n))\}\cdot.
\]

Now

\[\Pr_{\gamma_1}^{x;\lambda_1 \leq x} = e^{-\frac{1}{2} \gamma_1} \cdot c_0 \cdot \sum_{k_0=0}^{\infty} \frac{\Gamma(\frac{1}{2}(\nu+\mu+k)+k_0)}{\Gamma\left(\frac{\nu}{2}+k_0\right)\Gamma\left(\frac{\mu}{2}(\mu+2)-k_0\right)} \cdot \left(\frac{\gamma_1}{2}\right)^k \cdot S_{k_0}^{x;\lambda_1,\lambda_2},\]

where \[S_{k_0}^{x;\lambda_1,\lambda_2}\] is defined in (2.21), can be compared with Theorem 2.2* and is seen to be a generalization of the central case result: the individual terms in the braces are similar to the central case distribution.

2.8 Summary

In this chapter we use the joint non-null (linear) distribution of the roots \(\{\lambda_1\}\), derived in Chapter 1, to obtain the distribution function for the largest root when \(s = 2,3\). This distribution is of interest since the power function of the largest root test for the multivariate analysis of variance is a function of the non-centrality parameter \(\gamma_1\) and is given by \[\Pr_{\gamma_1}^{x;\lambda_1 > x} = 1 - \Pr_{\gamma_1}^{x;\lambda_1 \leq x},\]
where \[\Pr_{\gamma_1}^{x;\lambda_1 \leq x}\] is the distribution function of the largest root.
Hence it is possible to compute the power function of $\lambda_1$, when $s = 2$ or $3$, on a high speed computer using the distributions derived in this chapter. (See Chapter 4 for numerical results).

The chapter includes some results on integrating functions of the beta type and some notation developed by Nanda (1943a). For completeness, we include, in Theorems 2.1* and 2.2*, statements of the central case results and, in Theorems 2.1 and 2.2, the non-null (linear) results. The central case results are a special case of the results in Theorems 2.1 and 2.2 and can be obtained directly by setting $\gamma_1 = 0$. 

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3. THE LIMITING DISTRIBUTIONS OF ALL THE ROOTS AND OF THE LARGEST ROOT

3.1 Introduction

In this chapter we find the limiting joint distribution of the roots $\{\phi_1\}$ of the determinantal equation $|H-\Phi E| = 0$ and the limiting joint distribution of the roots $\{\lambda_1\}$ of the equation $|H-\lambda(H+E)| = 0$ as $n_2 \to \infty$ using the joint densities of the roots derived in Chapter 1. In addition, we find the limiting distribution as $n_2 \to \infty$ of the largest root, $\lambda_1 = \max_{1 \leq i \leq s} \lambda_i$, for $s = 2,3$ using the methods of Chapter 2. Letting $n_2 \to \infty$ corresponds to allowing the sample size to become infinite so the limiting distributions provide a look at the large sample behavior of these statistics. For example, the limiting distributions of the $\{\xi_1 = n\lambda_1\}$ and the $\{\psi_1 = m\phi_1\}$ are the same, where $n$ and $m$ are both functions of $n_2$.

The notation in this chapter is the same as in Chapter 2. A useful expression for simplifying terms in what follows is the asymptotic formula

$$\Gamma(az+b) \sim \sqrt{2\pi} e^{-az} (az)^{az+b-\frac{1}{2}} \quad (|\arg z| < \pi, \; a > 0)$$

which we give here for convenient reference. (See Abramowitz and Stegun (1964) Equation 6.1.39).

3.2 Central Case Results

The limiting joint distribution of the roots $\{\lambda_1\}$ in the central case ($\gamma_1 = 0$) and the limiting distribution of $\lambda_1$ in the central
case for $s = 2, 3$ were obtained by Nanda (1948b). The limiting joint
distribution is given in

**Theorem 3.1* (Nanda, 1948b):** Under the assumptions of Theorems 1.1*
(i) and 1.2* (i), if we let $\xi_i = n\lambda_i$, $i = 1, \ldots, s$, where
$n = \frac{1}{2}(n_2 - p - 1)$, then as $n$ tends to infinity (i.e., $n_2$ tends to
infinity, $p$ fixed), the limiting distribution of the $\{\xi_i\}$ is given
by

$$p^\infty(\xi_1, \ldots, \xi_s) = \pi^{s/2} \left( \prod_{i=1}^{s} \frac{2\ell+1}{2} \Gamma(\frac{4}{2}) \right)^{-1} \cdot \prod_{i=1}^{s} \xi_i^{\ell} \cdot \prod_{1 < j < s}(\xi_j - \xi_j)^{-1} \cdot \sum_{i=1}^{s} \xi_i,$$

where $0 \leq \xi_s \leq \xi_{s-1} \leq \cdots \leq \xi_1$, $\ell = \frac{1}{2}(|p-n_1|-1)$ and $s = \min(p, n_1)$.

The limiting central distribution of $\lambda_1$ when $s = 2$ is given in

**Theorem 3.2* (Nanda, 1948b):** Under the assumptions of Theorems 1.1*
(i) and 1.2* (i), if we let $\xi_1 = n\lambda_1$, where $n = \frac{1}{2}(n_2 - p - 1)$, then as
$n$ tends to infinity (i.e., $n_2 \to \infty$, $p$ fixed), the limiting distribution
of $\xi_1$ is given by

$$Pr_{2,0}^\infty(\xi_1 \leq x) = \lim_{n \to \infty} Pr_{2,0}(n\lambda_1 \leq x)$$

$$= \frac{2^{2\ell+1}}{\Gamma(2\ell+2)} \left\{ 2 \int_0^x u^{2\ell+1} e^{-2u} du - x^{\ell+1} e^{-x} \int_0^x u^\ell e^{-u} du \right\},$$

where $\ell = \frac{1}{2}(|p-n_1|-1)$ and $s = \min(n_1, p) = 2$. 

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Finally, the limiting central distribution of $\lambda_1$ when $s = 3$ is given in

**Theorem 3.3***(Nanda, 1945): Under the assumptions of Theorem 1.1** (i) and 1.2** (i), if we let $t_1 = n\lambda_1$, where $n = \frac{1}{2}(n-1-p-1)$, then as $n$ tends to infinity (i.e., $n_2 \to \infty$, $p$ fixed), the limiting distribution of $t_1$ is given by

$$\Pr_0(t_1 \leq x) = \lim_{n \to \infty} \Pr(n\lambda_1 \leq x)$$

$$= \frac{2^{2\ell+2}}{\Gamma(\ell+1)\Gamma(2\ell+3)} \left[ 2 \int_0^x u^{2\ell+3}e^{-2u}du \int_0^x u^\ell e^{-u}du \right.$$

$$- 2 \int_0^x u^{2\ell+2}e^{-2u}du \int_0^x u^{\ell+1}e^{-u}du$$

$$- x^{\ell+2}e^{-x}(2 \int_0^x u^{\ell+1}e^{-2u}du - x^{\ell+1}e^{-x} \int_0^x u^\ell e^{-u}du) \right] ,$$

where $\ell = \frac{1}{2}(|p-n|-1)$ and $s = \min(n_1,p) = 3$.

3.3 Linear Case Results: Limiting Joint Distribution of the $\{\lambda_i\}$

To get the limiting joint distribution of the roots of $|H-\lambda(H+E)| = 0$ in the non-null (linear) case, we begin with the joint distribution of the $\{\lambda_i\}$ from Equation (2.5) which is in terms of the parameters $\mu$, $\nu$ and $s$:  

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\[ p_{\gamma_1}(\lambda_1, \ldots, \lambda_s) = e^{-\frac{1}{2}y_1} \cdot C \cdot \prod_{i=1}^{s} \left\{ \lambda_i^{\frac{1}{2} \mu - 1} (1 - \lambda_i)^{\frac{1}{2} \nu - 1} \right\} \cdot \prod_{1 < j} (\lambda_1 - \lambda_j). \]

\[
\left\{ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(\frac{1}{2}(\nu + \mu + 2s - 2) + k_0)}{\Gamma(\frac{1}{2}(\nu + \mu + 2s - 2)) \Gamma(\frac{1}{2} \mu + s - l) \Gamma(\frac{1}{2} \mu + s - l + k_0)} \right\}
\]

\[
\gamma_1^r \left( \begin{array}{c}
\frac{k_0}{2} \\
\frac{k_0}{2} \\
s - 2 \\
\frac{k_s - 2}{2} \\
\frac{k_s - 2}{2} \\
\frac{k_{s - 1} - 1}{2} \\
\frac{k_{s - 1} - 1}{2} \\
\frac{k_{s - 1} - 1}{2} \\
\frac{k_{s - 1} - 1}{2}
\end{array} \right) \cdot \prod_{i=1}^{s-1} \frac{\Gamma(k_i - 1 - k_i^r + \frac{1}{2})}{\Gamma(k_i - 1 - k_i^r + \frac{1}{2}) \pi^2} \cdot \frac{\Gamma(k_s - 1 - \frac{1}{2})}{\Gamma(k_s - 1 + \frac{1}{2}) \pi^2},
\]

for 0 \leq \lambda_s \leq \cdots \leq \lambda_1 \leq 1, \ s = \min(n_1, p), \ \mu = |p - n_1| + 1, \ 
\nu = n_2 - p + 1 \ \text{and}
\]

\[ C = \pi^{\frac{1}{2} s} \prod_{i=1}^{s} \frac{\Gamma(\frac{1}{2}(\nu + \mu + 2s - 1 - i))}{\Gamma(\frac{1}{2}(\nu + s - 1)) \Gamma(\frac{1}{2}(s + 1 - i)) \Gamma(\frac{1}{2}(\mu + s - 1))} \cdot \]

As in Chapter 2, additional notational simplification is obtained by letting

\[ \ell = \frac{1}{2} \mu - 1 = \frac{1}{2} (|p - n_1| - 1) \]

(3.2) and

\[ n = \frac{1}{2} \nu - 1 = \frac{1}{2} (n_2 - p - 1) \]

to get
\( p_{\gamma_1}(\lambda_1, \ldots, \lambda_s) = e^{-\frac{1}{2} \gamma_1} \cdot C \cdot \prod_{i=1}^{s} \left( \lambda_i \gamma_1 (1 - \frac{\gamma_1}{2}) \right) \cdot \prod_{1 \leq i < j} (\lambda_i - \lambda_j) \cdot \right.

\left\{ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(\ell+n+s+k_0)\Gamma(\frac{3}{2}s)\Gamma(\ell+\frac{3}{2}s+\lambda)}{\Gamma(\ell+n+s)\Gamma(\frac{3}{2}s+k_0)\Gamma(\ell+\frac{3}{2}s+\lambda+k_0)} \right\}.

(3.3)

\[
\gamma_1 = \frac{1}{2} \cdot \frac{k_0}{(2^\frac{k_0}{2})} \cdot \sum_{k_1=0}^{k_0} \cdots \sum_{k_{s-2}=0}^{k_{s-2}} \left( \prod_{i=1}^{s-1} \frac{\Gamma(k_{i-1} - k_i + \frac{1}{2})}{\Gamma(k_{i-1} - k_i + \frac{1}{2})} \right) \cdot \frac{\Gamma(k_{s-2} + \frac{1}{2})}{\Gamma(k_{s-1} + 1)\pi^{\frac{k_{s-2}}{2}}}.
\]

\[
(\frac{1}{1 - \lambda_i - k_i}) \cdot \frac{k_{s-1}}{\lambda_s} \right) \right],
\]

where

(3.4)

\[ C = \frac{1}{2^\frac{k_0}{2}} \cdot \frac{\Gamma(\ell+n+s+\frac{3}{2} - \frac{1}{2})}{\Gamma(n+\frac{s}{2} - \frac{1}{2})\Gamma(\frac{s}{2}(s+1-1))\Gamma(\ell+\frac{s}{2} - \frac{1}{2})} \right).

and \( s = \min(n_1, p) \).

Using this density, we obtain the analogue of Theorem 3.1 for the non-null (linear) case:

**Theorem 3.1:** Under the assumptions of Theorems 1.1 (i) and 1.2 (i), if we let \( \xi_i = n\lambda_i, i = 1, \ldots, s \), then as \( n \to \infty \) (i.e., \( n_2 \to \infty \), \( p \) fixed) the limiting distribution of the \( \{\xi_i\} \) is given by

\[
\lim_{n \to \infty} p_{\gamma_1}(\xi_1, \ldots, \xi_s) = e^{-\frac{1}{2} \gamma_1} \cdot C \cdot \left( \prod_{i=1}^{s} \xi_i \right) \cdot \exp \sum_{i=1}^{s} \xi_i - \prod_{1 \leq i < j} (\xi_i - \xi_j) \right).
\]

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\[
\left\{ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(\frac{1}{2} s)^{\frac{3}{2}}} {\Gamma(\frac{1}{2} s+k_0)^{\frac{3}{2}} \Gamma(\frac{1}{2} s+k_0^{\frac{1}{2}})} \cdot \left( \frac{k_0}{2} \right)^{\frac{3}{2}} \Gamma(\frac{1}{2} s+k_0^{\frac{1}{2}}) \right. \\
\left. \sum_{k_1=0}^{k_0} \ldots \sum_{k_{s-1}=0}^{k_{s-2}} \frac{\Gamma(k_{i}-k_{1}^{\frac{1}{2}})} {\Gamma(k_{i}-k_{1}^{\frac{1}{2}}+1)^{\frac{3}{2}}} \cdot \frac{\Gamma(k_{s-1}^{\frac{1}{2}})} {\Gamma(k_{s-1}^{\frac{1}{2}}+1)^{\frac{3}{2}}} \right) \sum_{k_{s-1}=0}^{k_{s-2}} \ldots \sum_{k_{1}=0}^{k_{0}} \frac{\Gamma(k_{i}-k_{1}^{\frac{1}{2}})} {\Gamma(k_{i}-k_{1}^{\frac{1}{2}}+1)^{\frac{3}{2}}} \cdot \frac{\Gamma(k_{s-1}^{\frac{1}{2}})} {\Gamma(k_{s-1}^{\frac{1}{2}}+1)^{\frac{3}{2}}} \right) \right\},
\]

where \( \ell = \frac{1}{2} (|p-n_1| - 1) \), \( n = \frac{1}{2} (n_2 - p - 1) \), \( s = \min(p, n_1) \) and

\[
C^\infty = \pi^{\frac{3}{2} s} \prod_{i=1}^{s} \Gamma\left(\frac{2i+1}{2}\right)^{-1}
\]

for \( 0 \leq \zeta_s \leq \cdots \leq \zeta_1 \).

**Proof:** We begin by making the transformation

\[
\lambda_i = \frac{\zeta_i}{n}, \quad i = 1, \ldots, s,
\]

which has Jacobian \( J = n^{-s} \), in (3.3) to get

\[
\prod_{i=1}^{s} \frac{\zeta_i}{n} - \frac{\zeta_i}{n} \cdot \left( 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(\ell+n+1+s+k_0)} {\Gamma(\ell+n+1+s+k_0^{\frac{1}{2}}) \Gamma(\ell+n+1+s+k_0^{\frac{1}{2}}) \Gamma(\ell+n+1+s+k_0^{\frac{1}{2}}) \Gamma(\ell+n+1+s+k_0^{\frac{1}{2}})} \cdot \left( \frac{k_0}{2} \right)^{\frac{3}{2}} \Gamma(\ell+n+1+s+k_0^{\frac{1}{2}}) \right)
\]

\[
\left( \frac{\zeta_1}{n} \right)_{k_0} \ldots \left( \frac{\zeta_s}{n} \right)_{k_{s-2}} \left( \frac{\zeta_1}{n} \right)_{k_1} \ldots \left( \frac{\zeta_s}{n} \right)_{k_{s-1}} \right) \right\}.
\]
\[ = e^{\frac{k_0}{2} \gamma_1} \cdot C \cdot n^{-(s+\ell s+\frac{1}{2}s)(s-1)} \cdot \prod_{i=1}^{s} \left( \xi_i (1 - \frac{\xi_i}{n}) \right) \cdot \prod_{1<j} \left( \xi_i - \xi_j \right) \cdot \left[ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(\ell+n+1+s+k_0)\Gamma(\ell+\frac{1}{2}s)}{\Gamma(\ell+n+1+s)\Gamma(\ell+\frac{1}{2}s+k_0)\Gamma(\ell+\frac{1}{2}s+\frac{1}{2}+k_0)} \cdot \right. \\
\left. \prod_{k_0=1}^{\infty} \frac{\Gamma(k_0+1)}{\Gamma(k_0+1)s} \cdot \prod_{k_1=0}^{k_0} \frac{\sum_{k_s=0}^{s-1} \Gamma(k_1+1-k_s)}{\Gamma(k_1+1)k_1^s} \cdot \prod_{k_1=0}^{k_0} \frac{\Gamma(k_1+1)}{\Gamma(k_1+1)s} \cdot \prod_{k_s=0}^{s-1} \frac{\Gamma(k_1+1-k_s)}{\Gamma(k_1+1)k_1^{s-k_s}} \cdot \right) \]

The limiting distribution is obtained by letting \( n \to \infty \) (i.e., \( n_2 \to \infty \)) since \( n = \frac{1}{2}(n_2-p-1) \) and observing that

\[ (1 - \frac{\xi_i}{n}) \to e^{-\xi_i} \]

so that

\[ \prod_{i=1}^{s} \left( 1 - \frac{\xi_i}{n} \right) \to e^{-\sum_{i=1}^{s} \xi_i} \]

In addition, we apply the asymptotic formula in (3.1) to get that

\[ \frac{\Gamma(\ell+n+1+s+k_0)}{\Gamma(\ell+n+s+k_0)} \to 1 \]

and, from the definition of \( C \) in (3.4), that
\[ c \cdot n^{-\left(\frac{1}{2} + \frac{1}{2} + s + \ell s\right)} \longrightarrow \pi^{\frac{1}{2}} \left\{ \prod_{i=1}^{s} \Gamma\left(\frac{1}{2}(s+1-i)\right) \Gamma\left(l+1+i \cdot \frac{s}{s} - \frac{i}{s}\right) \right\}^{-1} \]

\[ = \pi^{\frac{1}{2}} \left\{ \prod_{i=1}^{s} \Gamma\left(\frac{1}{2}(2l+1+i)\right) \right\}^{-1} = C^\infty . \]

Substituting these results into (3.5) yields the theorem.

3.4 Linear Case Result: Limiting Distribution of the Largest Root - $s = 2$

In Theorem 3.1 we have the general form for the limiting joint density of the roots \( \{ \zeta_i = n \lambda_i \}, i = 1, \ldots, s \), in the non-null (linear) case. In this section, we use this density with $s = 2$ to get the limiting distribution of $\zeta_1 = n \lambda_1$. The derivation is similar to the derivation of the distribution of $\lambda_1$ when $s = 2$ (in Section 2.5) and we begin by extending the notation of Chapter 2.

Let

\[ (a, b; m, \infty) = \left. y^m e^{-y} \right|_a^b = b^m e^{-b} - a^m e^{-a} , \]

\[ (a, 1, b; m, \infty) = \int_a^b y^m e^{-y} \, dy , \]

and

\[ T_a^b; m, \infty g(y) = \int_a^b y^m e^{-y} g(y) \, dy . \]

Then

\[ T_a^b; m, \infty (0, y; r, \infty) = \int_a^b y^{m+r} e^{-2y} \, dy \quad (r > 0) \]

and

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\[ t_{a}^{b;m;\infty}(a,1,y;r,\infty) = t_{a}^{b;r;\infty}(y,1,b;m,\infty) \]

and, in particular,

\[ (3.6) \quad t_{a}^{b;m;\infty}(a,1,y;m,\infty) = t_{a}^{b;m;\infty}(y,1,b;m,\infty) = 0. \]

An additional useful result is

**Lemma 3.1:** For \( l > 0 \) and \( m \) and \( r \) integers, with \( m \geq 0, r > 0, \)

\[
T_{0}^{-l+m,\infty}(y,1,x;l+m+r,\infty) = T_{0}^{-l+m,\infty}(0,1,y;l+m+r,\infty)
\]

\[
= 2 \int_{0}^{x} y^{2l+2m+r-i-2y} dy - x^{l+m+r-i} \int_{0}^{x} y^{l+m-i-r} dy
\]

\[
+ \sum_{i=1}^{r-1} \left( \frac{1}{i} \right) (l+m+r-i) \left( 2 \int_{0}^{x} y^{2l+2m+r-i-2y} dy - x^{l+m+r-i} \int_{0}^{x} y^{l+m-i-r} dy \right) .
\]

**Proof:** We have

\[
T_{0}^{-l+m,\infty}(y,1,x;l+m+r,\infty) = T_{0}^{-l+m,\infty}(0,1,y;l+m+r,\infty)
\]

\[
= \int_{0}^{x} y^{l+m-i} \left[ \int_{y}^{x} z^{l+m+r-e-z} dz - \int_{y}^{x} z^{l+m+r-i} dz \right] dy
\]

\[
= \int_{0}^{x} y^{l+m-i} \left[ -z^{l+m+r-e-z} \bigg|_{y}^{x} + z^{l+m+r-i} \bigg|_{y}^{x} \right] dy
\]

\[
+ (l+m+r) \left( \int_{y}^{x} z^{l+m+r-i-1} dz - \int_{0}^{x} z^{l+m+r-1-e-z} dz \right) dy
\]

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\[
= 2 \int_0^x y^{2l+2m+r-2y} dy - x^{l+m+r-1} \int_0^x y^{l+m+r-2y} dy \\
+ \sum_{i=1}^{r-1} \left( \prod_{j=1}^i (l+m+r+1-j) \right) (2 \int_0^x y^{2l+2m+r-2y} dy - x^{l+m+r-1} \int_0^x y^{l+m+r-2y} dy) \\
+ \prod_{j=1}^r (l+m+r+1-j) \left( \int_0^x y^{l+m-2y} \left( \int_0^x z^{l+m-2z} dz \right) dy - \int_0^x z^{l+m-2z} dz dy \right) 
\]

by repeated integration by parts. The result follows by noting that the last summand is zero by (3.6).

We now state and then give a proof of the analogue of Theorem 3.2∗ for the non-null (linear) case when \( s = 2 \).

**Theorem 3.2:** Under the assumptions of Theorems 1.1 (i) and 1.2 (i), if we let \( \xi_1 = n^{\lambda_1} \), then as \( n \to \infty \) (i.e., \( n_2 \to \infty \), \( p \) fixed), the limiting distribution of \( \xi_1 \) for \( s = 2 \) is given by

\[
Pr_{2,\gamma_1} (\xi_1 \leq x) = \lim_{n \to \infty} Pr_{2,\gamma_1} (n^{\lambda_1} \leq x) = \\
\frac{1}{2^{2l+1}} \cdot \frac{2^{2l+1}}{(2l+2)!} \cdot \left\{ 2^{2l+1} \int_0^x u^{2l+1} e^{-2u} du - x^{l+1} \int_0^x u^{l+1} e^{-u} du \right\} \\
+ \sum_{k_0=1}^{\infty} \frac{\Gamma(l + \frac{3}{2})}{\Gamma(l + \frac{3}{2} + k_0) \Gamma(k_0 + 1)} \cdot \left( \frac{1}{2} \right)^{k_0} \cdot \frac{\Gamma(k_0)}{k_0 \gamma_1^{k_0}} \cdot \phi_{k_0,k_1}^{\infty} 
\]
\[ S_{k_0, k_1} = \frac{\Gamma(k_0+1/2)\Gamma(1/2)}{\Gamma(k_0+1)\Gamma(1)} \left\{ 2 \int_0^x u^{2+k_0+1} e^{-2u} du - x^{2+k_0+1} e^{-x} \int_0^x u^{2+k_0+1} e^{-u} du \right\} \]

\[ + \sum_{i=1}^{k_0} \frac{i}{\Gamma(k_0+2-j)(2 \int_0^x u^{2+k_0+1} e^{-2u} du)} \left\{ x^{2+k_0+1-i} e^{-x} \int_0^x u^{l+k_0+1-i} e^{-u} du \right\} \]

\[ + \sum_{k_1=1}^{[k_0/2]} \left\{ \frac{\Gamma(k_0-k_1+1/2)\Gamma(k_1+1/2)}{\Gamma(k_0-k_1+1)\Gamma(k_1+1)} - \frac{\Gamma(k_0-k_1+3/2)\Gamma(k_1-1/2)}{\Gamma(k_0-k_1+2)\Gamma(k_1)} \right\} \cdot \left\{ 2 \int_0^x u^{2+k_0+1} e^{-u} du - x^{2+k_0+1} e^{-x} \int_0^x u^{l+k_0+1} e^{-u} du \right\} \]

\[ + \sum_{i=1}^{k_0-k_1} \frac{i}{\Gamma(k_0-k_1+2-j)(2 \int_0^x u^{2+k_0+1} e^{-2u} du)} \left\{ x^{l+k_0-k_1+1-i} e^{-x} \int_0^x u^{l+k_1+1-i} e^{-u} du \right\} , \]

where \([k_0/2]\) is the largest integer \(k\) such that \(k \leq \frac{k_0}{2}\) and \(k = 0\) by convention.

**Proof:** From Theorem 3.1 we have, for \(0 \leq \xi_2 \leq \xi_1 < \infty\),
\[ P_{\gamma_1}^\infty (\zeta_1, \zeta_2) = e^{-\frac{3}{2} \gamma_1} \cdot \frac{e^{\ell+1}}{\Gamma(2\ell+2)} \cdot \{ \pi \ell \cdot e^{-\zeta_1 - \zeta_2 (\zeta_1 - \zeta_2)} \}
\]

\[(3.7) \quad + \sum_{k_0=1}^{\infty} \frac{\Gamma(\ell + \frac{3}{2})}{\Gamma(\ell + \frac{3}{2} + k_0) \Gamma(k_0 + 1)} \cdot (\frac{\gamma_1}{2})^{k_0} \cdot \frac{\Gamma(k_0 - k_1 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1)} \cdot \frac{\Gamma(k_1 + \frac{1}{2})}{\Gamma(k_1 + 1)} \cdot (\zeta_1 \cdot \zeta_2) \cdot (\zeta_1 - \zeta_2) e^{-\zeta_1 - \zeta_2} \}
\]

The limiting distribution of \( \zeta_1 \) is given by

\[ P_{\zeta_1, \zeta_1}^\infty (\zeta_1 \leq x) = \int \int P_{\gamma_1}^\infty (\zeta_1, \zeta_2) d\zeta_2 d\zeta_1 \]

which involves a double integration. To reduce this double integration to single integrals, we proceed as in Section 2.5 by defining

\[ S_{k_0, k_1}^\infty = \sum_{k_1=0}^{k_0} \frac{\Gamma(k_0 - k_1 + \frac{1}{2}) \Gamma(k_1 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1) \Gamma(k_1 + 1)} \cdot (\zeta_1 \cdot \zeta_2) \cdot (\zeta_1 - \zeta_2) e^{-\zeta_1 - \zeta_2} \]

\[ = \frac{\Gamma(k_0 + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(k_0 + 1) \Gamma(1)} \cdot \left\{ e^{-\zeta_1 - \zeta_2 (\zeta_1 - \zeta_2)} \left( (\zeta_1 \cdot \zeta_2) - (\zeta_1^2 - \zeta_2^2) \right) \right\} \]

\[ + \sum_{k_1=1}^{[\frac{1}{2}]} G(k_0, k_1) \cdot e^{-\zeta_1 - \zeta_2 (\zeta_1 \cdot \zeta_2) - (\zeta_1^2 - \zeta_2^2)} \]

where, we recall from (2.10),

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\[ G(k_0, k_1) = \frac{\Gamma(k_0 - \frac{1}{2}) \Gamma(k_1 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1) \Gamma(k_1 + 1)} - \frac{\Gamma(k_0 - \frac{3}{2}) \Gamma(k_1 - \frac{1}{2})}{\Gamma(k_0 - k_1 + 2) \Gamma(k_1)} \]

We now define

\[ S_{k_0, k_1}^{x, \infty} = \int_{\frac{1}{2} \leq \xi \leq x} S_{x, k_1}^{k_0, \infty} \right] \right] = \frac{\Gamma(k_0 + \frac{1}{2}) \Gamma(k_1 + \frac{1}{2})}{\Gamma(k_0 + 1) \Gamma(k_1)} \{ T_0^{x; \infty} (y, l, x; l + k_0 + 1, \infty) - T_0^{x; \infty} (0, l, y; l + k_0 + 1, \infty) \}

\]

\[ \sum_{k_1=l}^{k_0 \left[ \frac{1}{2} \right]} G(k_0, k_1) \{ T_0^{x; \infty} (y, l, x; l + k_0 - k_1 + 1, \infty) - T_0^{x; \infty} (0, l, y; l + k_0 - k_1 + 1, \infty) \}

and apply Lemma 3.1, once with \( m = 0, r = k_0 + 1 \) and once with \( m = k_1 \), \( r = k_0 - 2k_1 + 1 \), to get

\[ S_{k_0, k_1}^{x, \infty} = \frac{\Gamma(k_0 + \frac{1}{2}) \Gamma(k_1 + \frac{1}{2})}{\Gamma(k_0 + 1) \Gamma(k_1)} \left\{ 2 \int_0^x y \frac{2l + k_0 + 1 - 2y}{e^{-2y} dy} - x e^{-x} \int_0^x y e^{-y} dy \right\}

\[ + \sum_{l=1}^{k_0 \left[ \frac{1}{2} \right]} \frac{1}{j=1} \frac{1}{l + k_0 + 2 - j} \left\{ 2 \int_0^x y \frac{2l + k_0 + 1 - x}{e^{-2y} dy} - x e^{-x} \int_0^x y e^{-y} dy \right\}

\]

\[ + \sum_{k_1=l}^{k_0 \left[ \frac{1}{2} \right]} G(k_0, k_1) \left\{ 2 \int_0^x y e^{-y} dy - x e^{-x} \int_0^x y e^{-y} dy \right\} \]

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\[
\begin{align*}
&\sum_{i=1}^{k_0-2k_1} \left( \prod_{j=1}^{i} (E^{k_0-k_1+2-j})(2 \int_{0}^{x} y^{2k_0+1-i} e^{-2y} dy - x^{k_0-k_1+1-i} e^{-x} \int_{0}^{x} y^{k_1+1} e^{-y} dy) \right).
\end{align*}
\]

The remaining double integration in (3.7) is
\[
\int_{0}^{x} \pi(\xi_1^{l+1} \xi_2^{l} \cdot \xi_2^{l+1}) e^{-\xi_1 \xi_2} d\xi_1 d\xi_2
\]
\[
= \pi(T_0^{\infty} l^{\infty}_{y,1,x;\ell+1} - T_0^{\infty} l^{\infty}_{0,1,y;\ell+1})
\]
\[
= \pi(2 \int_{0}^{x} y^{2\ell+1} e^{-2y} dy - x^{\ell+1} e^{-x} \int_{0}^{x} y^{\ell} e^{-y} dy)
\]

Combining these results and letting \( y = u \) gives the theorem.

3.5 Linear Case Results: Limiting Distribution of the Largest Root - \( s = 3 \)

As in the previous section, we use the limiting joint distribution of the roots \( \{\xi_i = n\lambda_1\} \) \( i = 1, \ldots, s \) from Theorem 3.1 to get the limiting distribution of \( \xi_1 = n\lambda_1 \) when \( s = 3 \). The derivation is similar to the derivation of the distribution of \( \lambda_1 \) for \( s = 3 \) (in Section 2.7) and we use the notation of Chapter 2 and Section 3.4. In addition, we need two more results similar in type to Lemma 3.1 of the previous section.
Lemma 3.2: For \( \ell > 0 \) and \( m \) and \( r \) integers with \( m \geq 0, r > 0 \)

\[
T_{\ell, m, r}^{\infty} (\xi_3, 1, x; \ell, m + r, \infty) - T_{\ell, m, r}^{\infty} (0, 1, \xi_3; \ell, m + r, \infty)
\]

\[
= \xi_3 \ell + m r e^{-\xi_3} \int_0^x y \ell + m r e^{-y} dy - x \ell + m r e^{-x} \int_0^\xi_3 y \ell + m r e^{-y} dy
\]

\[
+ \sum_{i=1}^{r-1} \int_{j=1}^i (\ell + m r + 1 - j) (\ell + m r + 1 - j) \left( \xi_3 \ell + m r + 1 e^{-\xi_3} \right) x \ell + m r + 1 e^{-x} \int_0^\xi_3 y \ell + m r e^{-y} dy
\]

\[
- x \ell + m r - 1 e^{-x} \int_0^\xi_3 y \ell + m r e^{-y} dy
\]

Proof: Integrate by parts and use (3.6) to obtain

\[
T_{\ell, m, r}^{\infty} (\xi_3, 1, x; \ell, m + r, \infty) - T_{\ell, m, r}^{\infty} (0, 1, \xi_3; \ell, m + r, \infty)
\]

\[
= \xi_3 \ell + m r e^{-\xi_3} \int_0^x y \ell + m r e^{-y} dy - x \ell + m r e^{-x} \int_0^\xi_3 y \ell + m r e^{-y} dy
\]

\[
+ \sum_{i=1}^{r-1} \int_{j=1}^i (\ell + m r + 1 - j) (\ell + m r + 1 - j) \left( \xi_3 \ell + m r + 1 e^{-\xi_3} \right) x \ell + m r + 1 e^{-x} \int_0^\xi_3 y \ell + m r e^{-y} dy
\]

\[
- x \ell + m r - 1 e^{-x} \int_0^\xi_3 y \ell + m r e^{-y} dy
\]
Lemma 3.3: For \( l > 0 \) and \( m \) and \( r \) integers with \( m \geq 0, r > 0 \),
\[
T_{x \xi_3}^{\xi_3, l+m,r}(y, l+m, l+m+r, \infty) - T_{x \xi_3}^{\xi_3, l+m, r}(y, l+m, l+m+r, \infty)
\]
\[
= 2 \int_{\xi_3}^{x} y^{l+2m+r-2y} dy - \left( x^{l+m+r-x} + \xi_3^{l+m+r-x} \right) \int_{\xi_3}^{x} y^{l+m-r-y} dy
\]
\[
+ \sum_{i=1}^{r-1} \sum_{j=1}^{i} \left( l+m+r+1-j \right) 2 \int_{\xi_3}^{x} y^{l+2m+r-j-y} dy
\]
\[
- \left( x^{l+m+r-j-x} + \xi_3^{l+m+r-j-x} \right) \int_{\xi_3}^{x} y^{l+m-j} dy .
\]

Proof: Integration by parts as in Lemmas 3.1 and 3.2.

We now state and then give a proof of the analogue of Theorem 3.3* in the non-null (linear) case for \( s = 3 \).

Theorem 3.3: Under the assumptions of Theorems 1.1 (i) and 1.2 (i),
if we let \( \xi_1 = n \lambda_1 \), then as \( n \to \infty \) (\( n_2 \to \infty \), \( p \) fixed), the
limiting distribution of \( \xi_1 \) for \( s = 3 \) is given by
\[
Pr_{X_{\xi_1}, Y_1} (\xi_1 \leq x) = \lim_{n \to \infty} Pr_{X_{\xi_1}, Y_1} (n \lambda_1 \leq x)
\]
\[
= e^{-\frac{1}{2} \gamma_1} \frac{2^{2l+3}}{\Gamma(l+1)\Gamma(2l+3)} \cdot \left( \frac{\pi}{2} \right)^{l} \left( 2 \int_{0}^{x} u^{2l+3} e^{-2u} du \int_{0}^{x} u^{l+1} e^{-u} du \right)
\]
\[
- 2 \int_{0}^{x} u^{2l+2} e^{-2u} du \int_{0}^{x} u^{l+1} e^{-u} du .
\]
\[-x^{\ell+2}e^{-x}(2\int_0^x u^{2\ell+1}e^{-2u}du-x^{\ell+1}e^{-x}\int_0^x u e^{-u}du))\]

\[+\sum_{k_0=1}^{\infty} \frac{\Gamma(\frac{3}{2})\Gamma(\ell+2)}{\Gamma(\frac{\ell}{2}+k_0)\Gamma(\ell+2+k_0)} \cdot \left(\frac{1}{2}\right)^{k_0} \cdot \mathcal{S}_{k_0,k_1,k_2}^{*}\]

for

\[\mathcal{S}_{k_0,k_1,k_2}^{*} = \sum_{k_1=0}^{k_0} \frac{\Gamma(k_0-k_1+\frac{1}{2})}{\Gamma(k_0-k_1+1)} \cdot \left\{ \sum_{i=0}^{k_1} \frac{\Gamma(k_1+\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(k_1+1)\Gamma(1)} \right\} \cdot \frac{1}{(\ell+k_1+3-j)^{\ell+k_1+3-j}} \cdot \{2\int_0^x u^{2\ell+k_1+3-i}e^{-2u}du \int_0^x u^{\ell+k_0-k_1}e^{-u}du\]

\[-2\int_0^x u^{\ell+k_0+2-i}e^{-2u}du \int_0^x u^{\ell+1}e^{-u}du\]

\[-x^{\ell+1}e^{-x}T_0^x ((y,1,x;\ell+1,\infty) - (0,1,y;\ell+1,\infty))\]

\[\frac{1}{2} \left\{ \frac{\Gamma(k_1-k_2+\frac{1}{2})\Gamma(k_2+\frac{1}{2})}{\Gamma(k_1-k_2+1)\Gamma(k_2+1)} - \frac{\Gamma(k_1-k_2+\frac{3}{2})\Gamma(2-k_2)}{\Gamma(k_1-k_2+2)\Gamma(k_2)} \right\} \cdot \sum_{k_2=1}^{k_1-2k_2} \frac{1}{\prod_{j=1}^{i} (\ell+k_1-k_2+3-j)} \cdot \{2\int_0^x u^{2\ell+k_1+3-i}e^{-2u}du \int_0^x u^{\ell+k_0-k_1}e^{-u}du\]

\[-2\int_0^x u^{\ell+k_0-k_2+2-i}e^{-2u}du \int_0^x u^{\ell+k_2+1}e^{-u}du\]

\[-x^{\ell+k_1-k_2+2-i}e^{-x}T_0^x ((y,1,x;\ell+k_2+1,\infty) - (0,1,y;\ell+k_2+1,\infty))\} ,\]

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where \( \lfloor \frac{1}{2} \rfloor \) is the greatest integer \( k \) such that \( k \leq \frac{1}{2} \) and \( \sum_{k=1}^{0} \prod_{i=1}^{0} = 1 \) by convention.

Note: In the expression for \( S_{k_0,k_1,k_2}^* \), there is a double integration, namely \( T_0 \int_{0}^{x;\ell+k_0-k_1,\infty} ((y,1,x;\ell+k_2+1,\infty) - (0,1,y;\ell+k_2+1,\infty)) \), which must be evaluated. When \( \ell+k_2+1 = \ell+k_0-k_1 \), this expression is equal to zero by Equation (3.6). When \( \ell+k_2+1 > \ell+k_0-k_1 \), we use Lemma 3.1 with \( m = k_0-k_1 \) and \( r = k_2-k_0+k_1+1 \) to get that

\[
T_0 \int_{0}^{x;\ell+k_0-k_1,\infty} ((y,1,x;\ell+k_2+1,\infty) - (0,1,y;\ell+k_2+1,\infty))
\]

\[
= 2 \int_{0}^{x} y^{2\ell+k_0-k_1+k_2+1} e^{-2y} dy - x^{\ell+k_2+1} e^{-x} \int_{0}^{x} y^{\ell+k_0-k_1} e^{-y} dy
\]

\[
= \sum_{i=1}^{k_2-k_0+k_1} \left( \frac{1}{i!} (\ell+k_2+2-i) \right) \left( 2 \int_{0}^{x} y^{2\ell+k_0-k_1+k_2+1-i} e^{-2y} dy - x^{\ell+k_2+1-i} e^{-x} \int_{0}^{x} y^{\ell+k_0-k_1} e^{-y} dy \right).
\]

When \( \ell+k_2+1 < \ell+k_0-k_1 \), we interchange the order of integration and then apply Lemma 3.1 with \( m = k_2+1 \) and \( r = k_0-k_1-k_2-1 \) to obtain

\[
T_0 \int_{0}^{x;\ell+k_0-k_1,\infty} ((y,1,x;\ell+k_2+1,\infty) - (0,1,y;\ell+k_2+1,\infty))
\]

\[
= T_0 \int_{0}^{x;\ell+k_2+1,\infty} ((0,1,y;\ell+k_0-k_1,\infty) - (y,1,x;\ell+k_0-k_1,\infty)).
\]
\[
\begin{align*}
&= - \left\{ \int_0^x y^{2k_0+k_1+k_2+1} e^{-2y} dy - \int_0^{x} y^{2k_0-k_1} e^{-2y} dy \right. \\
&\quad + \sum_{i=1}^{k_0-k_1-k_2-2} \left( \frac{1}{i!} \right) (\int_0^x y^{k_0-k_1-1} e^{-2y} dy) (2 \int_0^x y^{2k_0-k_1+k_2+1} e^{-2y} dy) \\
&\quad - \left. \int_0^x y^{\ell+k_0-k_1-1} e^{-2y} dy \right\}.
\end{align*}
\]

**Proof of Theorem 3.3:** From Theorem 3.1 we have, for \(0 \leq \xi_3 \leq \xi_2 \leq \xi_1 < \infty,

\[
F_{\gamma_1}^\infty (\xi_1, \xi_2, \xi_3) = e^{-\gamma_1} \cdot \frac{2^2 \pi^{3/2}}{\Gamma(\ell+1) \Gamma(2\ell+3) \pi^{3/2}} \cdot \left\{ \prod_{k=1}^{\infty} \sum_{k_0=1}^{\infty} \frac{\Gamma(3/2) \Gamma(k_0+k_1+k_2+1)}{\Gamma(k_0-k_1-k_2+1) \Gamma(k_0-k_2+1) \Gamma(k_0-k_1+1)} \cdot \frac{\gamma_1^{k_0}}{2^{k_1+k_2-k_0-k_1-k_2}} \left\{ (\xi_1^{\ell+k_0-k_1-k_2} \cdot e^{-\gamma_1}) \right\} \right\}.
\]

(3.8)

Recall from (2.1) that

\[
\left\{ \xi_1 \xi_2 \right\} = \left\{ \xi_1 \xi_2 \right\} + \left\{ \xi_2 \xi_3 \right\} + \left\{ \xi_3 \xi_1 \right\} + \left\{ \xi_2 \xi_3 \right\} = \xi_1 \xi_2 \left\{ \xi_1 \xi_2 \right\} - \xi_1 \xi_3 \left\{ \xi_1 \xi_3 \right\} + \xi_2 \xi_3 \left\{ \xi_2 \xi_3 \right\},
\]

where \(\{i,j\} = \xi_i - \xi_j\).
The limiting distribution of $\xi_1$ is given by

$$Pr_{\gamma_1, \gamma_1}^{\infty}(\xi_1 \leq x) = \int_{0 \leq \xi_2 \leq \xi_3 \leq \xi_1 < x} \rho_{\gamma_1}(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3.$$

To perform the integrations, we proceed as in Section 2.7, Equations (2.15), (2.16), (2.17), and rearrange the terms to obtain

$$S_{k_0, k_1, k_2}^{\infty} = \sum_{k_1=0}^{k_0} \frac{\Gamma(k_0 - k_1 + \frac{1}{2}) \Gamma(k_1 - k_2 + \frac{1}{2}) \Gamma(k_2 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1) \Gamma(k_1 - k_2 + 1) \Gamma(k_2 + 1)} \cdot$$

$$\left\{ \int_{0 \leq \xi_2 \leq \xi_3 \leq \xi_1 < x} \exp\left(-\left((\xi_2 + \xi_3) + \xi_1\right) + \left(\xi_1 - \xi_2 - \xi_3\right)\right) \cdot \frac{\Gamma(k_0 - k_1 - \frac{1}{2}) \Gamma(k_1 - k_2 - \frac{1}{2}) \Gamma(k_2 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1) \Gamma(k_1 - k_2 + 1) \Gamma(k_2 + 1)} \cdot \int_{0 \leq \xi_2 \leq \xi_1 < x} e^{-\xi_2 - k_0 - k_1} \right\}$$

$$= \sum_{k_1=0}^{k_0} \frac{\Gamma(k_0 - k_1 + \frac{1}{2}) \Gamma(k_1 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1) \Gamma(k_1 + 1)} \int_{0}^{x} e^{-\xi_2 - k_0 - k_1} \cdot$$

$$\left\{ \frac{\Gamma(k_1 + \frac{1}{2}) \Gamma(k_2 + \frac{1}{2})}{\Gamma(k_1 + 1) \Gamma(k_2 + 1)} \left( T_{\xi_3} \left(x; k_1 + 1, \infty\right) \cdot (y, 1, x; k_1 + 2, \infty) - T_{\xi_3} \left(x; k_1 + 1, \infty\right) \cdot (\xi_3, l, y; k_1 + 2, \infty) \right) \right\}$$

$$- T_{\xi_3} \left(x; k_1 + 1, \infty\right) \cdot (\xi_3, l, x; k_1 + 2, \infty) + T_{\xi_3} \left(x; k_1 + 1, \infty\right) \cdot (0, 1, \xi_3; k_1 + 2, \infty)$$

$$+ T_{\xi_3} \left(y, l, \xi_3; k_1 + 2, \infty\right) - T_{\xi_3} \left(0, 1, y; k_1 + 2, \infty\right)$$

(3.9)

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where $G(k_1,k_2)$ is given in (2.10).

We can now apply Lemma 3.3, Lemma 3.2 and, also, Lemma 3.1 (with $x$ replaced by $\xi_3$) to get that a typical term in the summation over $k_2$ (before the final integration over $\xi_3$) is

\[
2 \int_0^x \frac{e^{-2y}}{y^{\frac{k_1-2k_2}{2}}} dy - 2 \xi_3 \int_0^{\xi_3} \frac{e^{-y}}{y^{\frac{k_1-k_2+2}{2}}} dy \left( \frac{2^{k_1-k_2+2-1}}{(k_1-k_2+3)!} \right) \left( \int_0^y \frac{e^{-y}}{y^{\frac{k_1-k_2+2}{2}}} dy \right)
\]
\[ s_{k_0, k_1, k_2} = \sum_{k_1=0}^{k_0} \frac{\Gamma(k_0 - k_1 + \frac{1}{2})}{\Gamma(k_0 - k_1 + 1)} \left\{ \sum_{i=0}^{k_1} \frac{\Gamma(k_1 + \frac{1}{2})\Gamma(i)}{\Gamma(k_1 + 1)\Gamma(i+1)} \right\}.
\]

Note that in the first set of integrals in (3.9) the terms correspond to \( k_2 = 0 \) so that (3.10) holds for both sets of integrals for appropriate values of \( k_2 \). Substituting these results into (3.9) yields

\[\begin{align*}
&\sum_{i=0}^{k_1} \frac{\Gamma(k_1 + \frac{1}{2})\Gamma(i)}{\Gamma(k_1 + 1)\Gamma(i+1)} \times \\
&\left( \prod_{j=1}^{i} (l+k_1+3-j) \right) \left( 2 \int_0^1 y^{2l+k_1+3-i} e^{-2y} dy \int_0^{\xi_3} y^{l+k_0-k_1} e^{-\xi_3} d\xi_3 \right) \\
&- 2 \int_0^{\xi_3} y^{l+k_0-k_1-2} e^{-\xi_3} d\xi_3 \int_0^1 y^{l+1} e^{-y} dy \\
&- x^{l+k_1+2-i} e^{-x} T_0 \left( y, l, l+1, \infty \right) - \left( y, l, l+1, \infty \right) \\
&\frac{k_1}{2} \quad \frac{k_1-2k_2}{2} \quad \frac{k_1-2k_2}{2} \quad \frac{k_1-2k_2}{2} \\
&+ \sum_{k_2=1}^{k_1-2k_2} G(k_1, k_2) \sum_{i=0}^{k_1} \left( \prod_{j=1}^{i} (l+k_1-k_2+3-j) \right) \\
&\left( 2 \int_0^1 y^{2l+k_1+3-i} e^{-2y} dy \int_0^{\xi_3} y^{l+k_0-k_1} e^{-\xi_3} d\xi_3 \right) \\
&- 2 \int_0^{\xi_3} y^{l+k_0-k_1+2-2} e^{-\xi_3} d\xi_3 \int_0^1 y^{l+k_2+1} e^{-y} dy \\
&- x^{l+k_0-k_1+2-i} e^{-x} T_0 \left( y, l, l+k_2+1, \infty \right) - \left( y, l, l+k_2+1, \infty \right) \right\}.
\]
where \( \sum_{k=1}^{0} = 0 \) and \( \prod_{i=1}^{0} = 1 \) by convention and \( G(k_1, k_2) \) is defined in (2.10).

The remaining integration in (3.8) is

\[
\int_{0 \leq \xi_3 \leq \xi_2 \leq \xi_1 \leq x} \xi_1 \xi_2 \xi_3 e^{-(\xi_1 + \xi_2 + \xi_3)} \{1, 2, 3\} d\xi_1 d\xi_2 d\xi_3
\]

\[
= 2 \int_{0}^{x} y e^{\xi_3} e^{-y} dy \int_{0}^{x} \xi_3 e^{-\xi_3} d\xi_3 - 2 \int_{0}^{x} \xi_3 e^{2\xi_2 - e^{-2\xi_3}} d\xi_3 \int_{0}^{x} y e^{y+1-e^{-y}} dy
\]

\[
- x^{e} e^{x} \int_{0}^{x} \xi_3 e^{2\xi_2} e^{-2\xi_3} d\xi_3 \int_{0}^{x} y e^{y+1-e^{-y}} dy d\xi_3
\]

\[
= 2 \int_{0}^{x} y e^{\xi_3} e^{-y} dy \int_{0}^{x} \xi_3 e^{-\xi_3} d\xi_3 - 2 \int_{0}^{x} \xi_3 e^{2\xi_2} e^{-2\xi_3} d\xi_3 \int_{0}^{x} y e^{y+1-e^{-y}} dy
\]

\[
- x^{e} e^{x} (2 \int_{0}^{x} \xi_3 e^{2\xi_1} e^{-2\xi_3} d\xi_3 - x^{e} e^{x} \int_{0}^{x} \xi_3 e^{2\xi_3} d\xi_3)
\]

Combining these results and changing the \( \xi_3 \) and \( y \) to \( u \) yields the theorem.

\[ \square \]

3.6 Linear Case Result: Limiting Joint Distribution of the \( \{g_1\} \)

To obtain the limiting joint distribution of the roots of \( |X-\beta E| = 0 \) in the non-null (linear) case, we begin by changing the parameters of the joint density of the \( \{g_1\} \), as presented in Theorems 1.1 and 1.2, to
(3.11) \( s = \min(n_1, p), \quad \ell = \frac{1}{2}(|p-n_1|-1) \) and \( m = \frac{1}{2}(n_1+n_2) \)

This change in parameters enables us to combine the expressions for \( n_1 \leq p \) and \( n_1 > p \) into a single expression given by

\[
p_{\gamma_1}(\phi_1, \ldots, \phi_s) = e^{\frac{-y_1}{2}} \cdot C^s \cdot \prod_{i=1}^{s} (\phi_i^{(1+\phi_1)^{-m}}) \cdot \prod_{i<j} (\phi_i - \phi_j^s) \\
\left\{ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(m+k_0)\Gamma(\ell+\frac{s}{2}+\frac{1}{2})\Gamma(\frac{s}{2})}{\Gamma(m+k_0)\Gamma(\ell+\frac{s}{2}+\frac{k_0}{2})\Gamma(\frac{s}{2}+k_0)} \cdot \left( \frac{\gamma_1}{2} \right)^{k_0} \right\}.
\]

(3.12)

\[
\sum_{k_1=0}^{k_0} \sum_{k_{s-1}=0}^{k_{s-2}} \sum_{i=1}^{s-1} \left( \prod_{i=1}^{k_i-1} \frac{\Gamma(k_i-1-k_1)^{1/2}}{\Gamma^{1/2}(k_i-k_1+1)^{1/2}} \right) \cdot \frac{\Gamma(k_{s-1}+1/2)}{\Gamma(k_{s-1}+1)^{1/2}} \cdot \\
\left( \prod_{i=1}^{s-1} \frac{\phi_i^{k_i-1-k_1}}{1+\phi_i} \right) \cdot \left( \frac{\phi_s^{k_{s-1}}}{1+\phi_s} \right),
\]

where

(3.13) \( C^s = \frac{\pi^{s^2}}{\prod_{i=1}^{s} \Gamma(m+\frac{1}{2}-\frac{i}{2}) \cdot \Gamma(m+\frac{1}{2}-\frac{i}{2})} \right). \]

Using this density we proceed as in Section 3.3 to obtain the limiting joint density of the \( \{\phi_i\} \) as given in

**Theorem 3.4:** Under the assumptions of Theorems 1.1 (ii) and 1.2 (ii), if we let \( \psi_i = m\phi_i, \ i = 1, \ldots, s, \) then as \( m \to \infty \) (i.e., \( n_2 \to \infty \), \( n_1 \) fixed, since \( m = \frac{1}{2}(n_1+n_2) \)), the limiting distribution of the
\( \{ \psi_1 \} \) is given by

\[
\mathcal{P}_{\gamma_1} (\psi_1, \ldots, \psi_s) = e^{-\frac{1}{2} \gamma_1} \cdot C^\infty \cdot \left( \prod_{i=1}^{s} \psi_i^{\ell} \right) \cdot \prod_{i=1}^{s} \psi_i \cdot \prod_{i<j} (\psi_i - \psi_j).
\]

\[
\left\{ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(\frac{1}{2} s) \Gamma(\frac{1}{2} s + \frac{1}{2})}{\Gamma(\frac{1}{2} s + k_0) \Gamma(\frac{1}{2} s + \frac{1}{2} + k_0)} \cdot \left( \frac{\gamma_1}{2} \right)^{k_0} \right\}.
\]

\[
\sum_{k_1=0}^{k_0} \cdots \sum_{k_{s-2}=0}^{k_{s-1}} \left( \prod_{i=1}^{s-1} \frac{\Gamma(k_{i-1} - k_i + \frac{1}{2})}{\Gamma(k_{i-1} - k_i + 1 + \frac{1}{2})} \right) \cdot \frac{\Gamma(k_{s-1} + \frac{1}{2})}{\Gamma(k_{s-1} + 1 + \frac{1}{2})},
\]

where \( C^\infty = \pi^{\frac{1}{2} s} \prod_{i=1}^{s} \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2})^{-1} \) for \( 0 \leq \psi_s \leq \cdots \leq \psi_1 \).

**Proof:** We begin by making the transformation

\[
\phi_i = \frac{\psi_i}{m}, \quad i = 1, \ldots, s,
\]

which has Jacobian \( J = m^{-s} \), in (3.12) to get

\[
\mathcal{P}_{\gamma_1} (\psi_1, \ldots, \psi_s) = e^{-\frac{1}{2} \gamma_1} \cdot C^* \cdot m^{-s} \cdot \left\{ \prod_{i=1}^{s} \left( \frac{\psi_i}{m} \right)^{\ell} \cdot \prod_{i=1}^{s} \left( \frac{1}{m} \right) \right\}.
\]

\[
\prod_{i<j} \left( \frac{\psi_i}{m} - \frac{\psi_j}{m} \right) \cdot \left\{ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(m + k_0) \Gamma(\frac{1}{2} s + \frac{1}{2})}{\Gamma(m) \Gamma(\frac{1}{2} s + \frac{1}{2} + k_0) \Gamma(\frac{1}{2} s + k_0)} \right\}.
\]

\[
\left( \frac{\gamma_1}{2} \right)^{k_0} \cdot \sum_{k_1=0}^{k_0} \cdots \sum_{k_{s-2}=0}^{k_{s-1}} \left( \prod_{i=1}^{s-1} \frac{\Gamma(k_{i-1} - k_i + \frac{1}{2})}{\Gamma(k_{i-1} - k_i + 1 + \frac{1}{2})} \right) \cdot \frac{\Gamma(k_{s-1} + \frac{1}{2})}{\Gamma(k_{s-1} + 1 + \frac{1}{2})}.
\]
\[
\left( \prod_{i=1}^{s-1} \left( \frac{\psi_i/m}{1 + \psi_i/m} \right)^{k_{i-1} - k_i} \right) \cdot \left( \prod_{i=1}^{s} \frac{\psi_s/m}{1 + \psi_s/m} \right)^{k_{s-1}}
\]

\[
e^{\frac{1}{2} \gamma_1} \cdot C^* \cdot \left( m^{s+k_s + \frac{1}{2} s(s-1)} \cdot \prod_{i=1}^{s} \left( \psi_i (1 + \frac{1}{m}) \right)^{-m} \right).
\]

(3.14)

\[
\prod_{1 \leq i \neq j} \left( \psi_i - \psi_j \right) \cdot \left[ 1 + \sum_{k_0=1}^{\infty} \frac{\Gamma(m+k_0)\Gamma(\frac{1}{2} s + \frac{1}{2} k_0)}{\Gamma(m)\Gamma(\frac{1}{2} s + \frac{1}{2} k_0)\Gamma(\frac{1}{2} s + k_0)} \right].
\]

\[
m^{-k_0} \cdot \left( \frac{\gamma_1}{2} \right)^{k_0} \cdot \sum_{k_1=0}^{k_0} \sum_{k_{s-2}} \left( \prod_{i=1}^{s-1} \frac{\Gamma(k_{i-1} - k_i + \frac{1}{2})}{\Gamma(k_{i-1} + k_i + \frac{1}{2})} \right) \cdot \Gamma(k_{s-1} + \frac{1}{2}) \cdot \left( \prod_{i=1}^{s-1} \frac{\psi_i}{1 + \psi_i/m} \right)^{k_{i-1} - k_i} \cdot \left( \prod_{i=1}^{s} \frac{\psi_s}{1 + \psi_s/m} \right)^{k_{s-1}},
\]

where \( C^* \) is given in (3.13). We now let \( m \to \infty \) (i.e., \( n_2 \to \infty \))

and observe that

\[
(1 + \frac{\psi_i}{m})^{-m} \to e^{-\psi_i}
\]

so that

\[
\prod_{i=1}^{s} \left( 1 + \frac{\psi_i}{m} \right)^{-m} \to e^{i=1 \psi_i}.
\]

In addition, using the formula in (3.1) we see that

\[
\frac{\Gamma(m+k_0)}{\Gamma(m) \cdot k_0} \to 1
\]
and also that

\[ C^* \cdot m \left( \frac{1}{2}s^2 + \frac{1}{2}s + \lambda s \right) \rightarrow \frac{s^{\frac{3}{2}S}}{\prod_{i=1}^{S} \left\{ \Gamma\left(\frac{1}{2}(s+1-i)\right)\Gamma\left(l+1+i\frac{S}{S} - \frac{1}{2}\right) \right\}} \]

\[ = \frac{s^{\frac{3}{2}S}}{\prod_{i=1}^{S} \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(2l+1+i)\right)} = C^\infty, \]

The remaining terms are

\[ \frac{\psi_{\lambda}}{1 + \psi_{\lambda}/m} \rightarrow \psi_{\lambda} \]

which, when substituted into (3.14) together with the above results, yield the theorem.

3.7 Summary

In this chapter we have the limiting joint distributions of the \( \{\xi_{1} = n\lambda_{1}\} \) and of the \( \{\psi_{1} = m\phi_{1}\} \), where \( n = \frac{1}{2}(n_{1} - p - 1) \) and \( m = \frac{1}{2}(n_{1} + n_{2}) \) are functions of the sample size. We also derive the limiting distributions of \( \xi_{1} \) for \( s = 2, 3 \), which are the same as the limiting distributions of \( \psi_{1} \) for \( s = 2, 3 \) since the \( \{\xi_{1}\} \) and the \( \{\psi_{1}\} \) have identical limiting distributions.

For large values of \( n \) (or \( m \)) the limiting distributions can be used as approximations to the distributions of the roots. For example, for large \( n \),

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\[ \Pr_{\gamma_1} \{ n \lambda_1 \leq x \} \approx \Pr_{\gamma_1}^{\infty} \{ \rho_1 \leq x \} . \]

The limiting distributions of the largest root also provide a bound on the power function as the sample size becomes infinite since the power for given values of \( \gamma_1 \) and \( x \) is given by

\[ 1 - \Pr_{\gamma_1}(\lambda_1 \leq x) . \]
4. POWER STUDIES FOR THE LARGEST ROOT TEST WITH SOME POWER COMPARISONS

4.1 Introduction

In this chapter we present the results of numerical work on the power of the largest root test for $s = 2$ and $3$. The power function of the largest root, $\lambda_1$, is obtained from the results of Chapter 2 as

$$\Pr_{\lambda_1}(\lambda_1 > x) = 1 - \Pr_{\lambda_1}(\lambda_1 \leq x).$$

This function was programmed and run on an IBM 370 system. The incomplete Beta functions were computed using the IBM Scientific Subroutine Package program BDTR, which computes the Beta functions using a continued fraction expansion. We include in the Appendix listings of the programs used.

To simplify the computations, the constants in the distribution of $\lambda_1$ were reduced to convenient recursive forms by use of the duplication formula

$$(4.1) \quad \Gamma(z + \frac{1}{2}) = \frac{\Gamma(2z + 1) \sqrt{\pi}}{\Gamma(z + 1) 2^{2z}}.$$

Using (4.1) we get that

$$\frac{\Gamma(k_i - l - k_i + \frac{1}{2})}{\Gamma(k_i - l - k_i + 1) \pi^2} = \prod_{j=1}^{k_i-l-k_i} \left(1 - \frac{1}{\varepsilon_j^2}\right)$$
and
\[
\frac{\Gamma(k_{s-1}+\frac{1}{2})}{\Gamma(k_{s-1}+1)\pi^{\frac{k}{2}}} = \prod_{j=1}^{k_{s-1}} \left(1 - \frac{1}{2j}\right).
\]

To find the values of x for the 5\% and 1\% significance level, we use the tables in Harris (1975) to get initial values and then interpolate to obtain the desired accuracy in \(\alpha\)-levels. We note that the power function is very sensitive to slight changes in the value of x and therefore found it necessary to obtain x values precisely to at least eight places. This yields accurate \(\alpha\)-levels to at least five places.

In Section 4.2 we have some comparisons of the power of the largest root as calculated by Pillai and Jayachandran (1967) and as computed using the results of Chapter 2 for a variety of parameter values; Section 4.3 contains power comparisons of the largest root test with other tests based on the roots and finally, in Section 4.4, we present extensive power tables for the largest root test.

4.2 Power Comparisons for the Largest Root

We begin by presenting a comparison of the power of the largest root test under the rank one alternative as calculated using the zonal polynomial expansion by Pillai and Jayachandran (1967) and using the distributions in Chapter 2. Table 1a contains the results for \(s = 2\) and Table 1b contains the results for \(s = 3\). It can be seen that the numbers are nearly identical for all cases. We recall that
\[
n = \frac{1}{2}(n_2-p-1), \quad \ell = \frac{1}{2}(|n_1-p|-1) \quad \text{and} \quad s = \min(n_1,p).
\]
Table 1a: Power Comparisons for the Largest Root

\[ s = 2 \]

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<td>Zonal</td>
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*Values in "Zonal" column are from Pillai and Jayachandran (1967), Table 5.

\[ n = \frac{1}{2}(n_2-p-1) \; ; \; \ell = \frac{1}{2}(|r_1-p|-1) \; ; \; s = \min(n_1,p) \]

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Table 1b: Power Comparisons for the Largest Root*  

\[ s = 3 \]

\[
\begin{array}{ccc}
\text{ } & \text{ } & \text{Zonal} & \text{Exact} \\
\hline
n = 15 & \lambda = 0 & n = 40 & \lambda = 0 \\
\hline
\gamma_1 & Zonal & Exact & Zonal & Exact \\
\hline
.01 & .050184 & .050184 & .050210 & .0502096 \\
.05 & .05093 & .0509265 & .05105 & .0510548 \\
.10 & .05187 & .0518656 & .05213 & .0521261 \\
.50 & .0598 & .0598517 & .0613 & .0612881 \\
1.00 & .0709 & .0709146 & .0742 & .0742108 \\
\hline
n = 15 & \lambda = 1 & n = 40 & \lambda = 1 \\
\hline
.01 & .050135 & .0501348 & .050157 & .0501572 \\
.05 & .05068 & .0506774 & .05079 & .0507913 \\
.10 & .05136 & .0513631 & .05159 & .0515940 \\
.50 & .0571 & .0571459 & .0584 & .0584263 \\
1.00 & .0651 & .0651238 & .0680 & .0680038 \\
\hline
n = 15 & \lambda = 2 & n = 40 & \lambda = 2 \\
\hline
.01 & .050108 & .0501076 & .050128 & .0501282 \\
.05 & .05054 & .0505405 & .05065 & .0506450 \\
.10 & .05109 & .0510869 & .05130 & .0512986 \\
.50 & .0557 & .0556707 & .0568 & .0568342 \\
1.00 & .0619 & .0619595 & .0645 & .0645361 \\
\hline
n = 15 & \lambda = 5 & n = 40 & \lambda = 5 \\
\hline
.01 & .050069 & .050068 & .050086 & .050086 \\
.05 & .05034 & .0503447 & .05043 & .0504319 \\
.10 & .05069 & .0506920 & .05087 & .0508684 \\
.50 & .0536 & .0535727 & .0545 & .0545242 \\
1.00 & .0574 & .0574338 & .0595 & .0595150 \\
\hline
\end{array}
\]

*Values in "Zonal" column are from Pillai and Jayachandran (1967), Table 4.

\[ n = \frac{1}{2}(n_2 - p - 1); \quad \lambda = \frac{1}{2}(|n_1 - p| - 1); \quad s = \text{min}(n_1, p). \]

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4.3 Power Comparisons of MANOVA Tests

In this section we compare the power of the largest root test to the power of other tests based on the roots under the rank one alternative for \( s = 2, 3 \). The statistics compared are

1) \( \lambda_1 \), the largest root

2) \( W = \prod_{i=1}^{s} (1 + \phi_i)^{-1} \), the Wilks' likelihood ratio

3) \( T_0^2 = n_2 \sum_{i=1}^{s} \phi_i \), the Lawley-Hotelling trace

4) \( V = \sum_{i=1}^{s} \lambda_i \), the Pillai trace.

The values of power for \( W, T_0^2 \) and \( V \) are obtained from papers by Ito (1962), Pillai and Jayachandran (1967), and Lee (1971). The values computed by Pillai and Jayachandran are based on the zonal polynomial expansion developed by Constantine (1963) and James (1964) using zonal polynomials through the sixth degree. The values computed by Lee are based on asymptotic formulae to the order \( n_2^{-2} \). The values computed by Ito are based on asymptotic expansions of the cumulative distribution functions to the order of \( n_2^{-1} \) in terms of non-central chi-square distributions.

In Tables 2a and 2b we have power comparisons of \( T_0^2, W \) and \( \lambda_1 \) for \( \gamma_1 = 0, 1, \ldots, 8 \). It can be seen that as \( \gamma_1 \) gets large, the power of the largest root test increases. For values of \( \gamma_1 \) greater than 3 or 4, when \( s = 2 \), and 4 or 5, when \( s = 3 \), the largest root has greater power than the other tests bearing out Monte Carlo
results that under the rank one alternative the largest root is the most powerful test (see e.g., Schatzoff (1966), Olson (1974), or Harris (1975)). For smaller values of $\gamma_1$ the power of the three tests is very close.

Table 3a contains power values for $W$, $T_0^2$, $V$, and $\lambda_1$ when $s = 2$. There are two values for $W$, $T_0^2$ and $V$: the left columns are from Lee (1971) and the right columns are from Pillai and Jayachandran (1967). Again we see that for small values of $\gamma_1$ the four tests have comparable powers and for larger values of $\gamma_1$, the largest root test has the greatest power.

Similarly, for $s = 3$, we see in Table 3b that the largest root test has higher power, as the noncentrality parameter gets large, than the other three tests.

Finally, in Table 4 we have some more comparisons with $W$, $T_0^2$, and $V$ for $s = 2$. In this case, the values of $\gamma_1$ are all fairly small and the power of the largest root test does not exceed that of the other tests.

It should be noted that the power function is very sensitive to small deviations in the true $\alpha$-level of the test especially for values of the noncentrality near the null hypothesis (i.e., for small values of $\gamma_1$). Therefore, although the nominal $\alpha$-level is 5%, for example, a slightly smaller actual significance level will result in lower power over a range of values of the noncentrality and may lead to erroneous conclusions. In this study, the significance levels for the largest root test are accurate to at least five places.
Table 2a: Power Comparisons of MANOVA Tests Ia

\( s = 2 \)

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<th>( \lambda_1 )</th>
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<th>( W )</th>
<th>( \lambda_1 )</th>
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<td>.1393</td>
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<td>.498</td>
<td>.5146</td>
<td>.512</td>
<td>.515</td>
<td>.5280</td>
</tr>
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</table>

*Values of power for $W$ and $T_0^2$ are from Ito (1962), Table 3.

$W = \prod (1+\phi_i)^{-1}$ = Wilks' likelihood ratio;

$T_0^2 = n_2 V = n_2 \Sigma \phi_i$ = Lawley-Hotelling trace;

$V = a$ Pillai trace.

$n = \frac{1}{2}(n_2-p-1)$; $\ell = \frac{1}{2}(n_1-p-1)$; $s = \min(p, n_1)$.
Table 2b: Power Comparisons of MANOVA Tests \( T^2 \)

\[ s = 3 \]

\[
\begin{array}{cccccc}
\gamma_1 & \tau^2_0 & W & \lambda_1 & \tau^2_0 & W & \lambda_1 \\
0 & 0.050 & 0.050 & 0.500 & 0.050 & 0.050 & 0.500 \\
1 & 0.074 & 0.082 & 0.080 & 0.082 & 0.082 & 0.081 \\
2 & 0.119 & 0.121 & 0.118 & 0.121 & 0.121 & 0.121 \\
3 & 0.164 & 0.164 & 0.163 & 0.163 & 0.163 & 0.163 \\
4 & 0.212 & 0.212 & 0.212 & 0.212 & 0.212 & 0.212 \\
5 & 0.263 & 0.263 & 0.263 & 0.263 & 0.263 & 0.263 \\
6 & 0.317 & 0.315 & 0.323 & 0.323 & 0.323 & 0.323 \\
7 & 0.317 & 0.368 & 0.368 & 0.368 & 0.368 & 0.368 \\
8 & 0.425 & 0.421 & 0.438 & 0.438 & 0.438 & 0.438 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\gamma_1 & \tau^2_0 & W & \lambda_1 & \tau^2_0 & W & \lambda_1 \\
0 & 0.050 & 0.050 & 0.500 & 0.050 & 0.050 & 0.500 \\
1 & 0.074 & 0.077 & 0.074 & 0.077 & 0.077 & 0.075 \\
2 & 0.107 & 0.109 & 0.107 & 0.107 & 0.107 & 0.107 \\
3 & 0.134 & 0.146 & 0.142 & 0.142 & 0.142 & 0.142 \\
4 & 0.165 & 0.186 & 0.185 & 0.185 & 0.185 & 0.185 \\
5 & 0.209 & 0.229 & 0.231 & 0.231 & 0.231 & 0.231 \\
6 & 0.256 & 0.275 & 0.281 & 0.281 & 0.281 & 0.281 \\
7 & 0.324 & 0.321 & 0.326 & 0.326 & 0.326 & 0.326 \\
8 & 0.374 & 0.368 & 0.385 & 0.385 & 0.385 & 0.385 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\gamma_1 & \tau^2_0 & W & \lambda_1 & \tau^2_0 & W & \lambda_1 \\
0 & 0.050 & 0.050 & 0.500 & 0.050 & 0.050 & 0.500 \\
1 & 0.074 & 0.077 & 0.074 & 0.077 & 0.077 & 0.075 \\
2 & 0.107 & 0.109 & 0.107 & 0.107 & 0.107 & 0.107 \\
3 & 0.134 & 0.146 & 0.142 & 0.142 & 0.142 & 0.142 \\
4 & 0.165 & 0.186 & 0.185 & 0.185 & 0.185 & 0.185 \\
5 & 0.209 & 0.229 & 0.231 & 0.231 & 0.231 & 0.231 \\
6 & 0.256 & 0.275 & 0.281 & 0.281 & 0.281 & 0.281 \\
7 & 0.324 & 0.321 & 0.326 & 0.326 & 0.326 & 0.326 \\
8 & 0.374 & 0.368 & 0.385 & 0.385 & 0.385 & 0.385 \\
\end{array}
\]

\[ n = 47.5 \quad \ell = 0 \]

*Values of power for \( W \) and \( \tau^2_0 \) are from Ito (1962), Table 3.

\[
W = \prod (1+\varphi_1)^{-1} \quad \tau^2_0 = n_2 \sum \varphi_1 \\
\begin{align*}
n &= \frac{1}{2} (n_2 - p - 1) \\
\ell &= \frac{1}{2} (|n_1 - p| - 1) \\
s &= \min(n_1, p)
\end{align*}
\]

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Table 3a: Power Comparisons of MANOVA Tests IIa

\[ s = 2 \]

\[ n = 30.0 \quad \ell = 0 \]

<table>
<thead>
<tr>
<th>W</th>
<th>( T_0^2 )</th>
<th>V</th>
<th>( \lambda_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>.3097</td>
<td>.308</td>
<td>.5113</td>
</tr>
<tr>
<td>8</td>
<td>.4900</td>
<td>.487</td>
<td>.4931</td>
</tr>
<tr>
<td>10</td>
<td>.5988</td>
<td>.589</td>
<td>.6026</td>
</tr>
</tbody>
</table>

\[ n = 30.0 \quad \ell = 1 \]

<table>
<thead>
<tr>
<th>W</th>
<th>( T_0^2 )</th>
<th>V</th>
<th>( \lambda_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>.2404</td>
<td>.240</td>
<td>.2427</td>
</tr>
<tr>
<td>8</td>
<td>.3872</td>
<td>.386</td>
<td>.3922</td>
</tr>
<tr>
<td>10</td>
<td>.4848</td>
<td>.478</td>
<td>.4915</td>
</tr>
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</table>

\[ n = 30.0 \quad \ell = 2 \]

<table>
<thead>
<tr>
<th>W</th>
<th>( T_0^2 )</th>
<th>V</th>
<th>( \lambda_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>.3244</td>
<td>.324</td>
<td>.3294</td>
</tr>
<tr>
<td>10</td>
<td>.4098</td>
<td>.408</td>
<td>.4171</td>
</tr>
</tbody>
</table>

\[ n = 30.0 \quad \ell = 5 \]

<table>
<thead>
<tr>
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<th>( T_0^2 )</th>
<th>V</th>
<th>( \lambda_1 )</th>
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<tr>
<td>5</td>
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<tr>
<td>8</td>
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<tr>
<td>10</td>
<td>-</td>
<td>.284</td>
<td>-</td>
</tr>
</tbody>
</table>

*Values of power for \( W, T_0^2, V \) are from Lee (1971), Table 1 and from Pillai and Jayachandran (1967), Table 7a.

\[ W = \prod (1 + \phi_1)^{-1} \quad T_0^2 = n_2 \Sigma \phi_4 \quad V = \Sigma \lambda_1 \]

\[ n = \frac{1}{2}(n_2 - p + 1) \quad \ell = \frac{1}{2}(|n_1 - p| - 1) \quad s = \min(n_1, p) \]

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Table 3b: Power Comparisons of MANOVA Tests \( II_b^* \)

\[ s = 3 \]

\[ n = 38, \quad \ell = -0.5 \]

<table>
<thead>
<tr>
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<th>( W )</th>
<th>( T_0^2 )</th>
<th>( V )</th>
<th>( \lambda_1 )</th>
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</thead>
<tbody>
<tr>
<td>3</td>
<td>0.162</td>
<td>0.163</td>
<td>0.161</td>
<td>0.1607</td>
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<td>6</td>
<td>0.312</td>
<td>0.315</td>
<td>0.309</td>
<td>0.3182</td>
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<tr>
<td>9</td>
<td>0.471</td>
<td>0.476</td>
<td>0.467</td>
<td>0.4862</td>
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</tbody>
</table>

\[ n = 58, \quad \ell = -0.5 \]

<table>
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<th>( \lambda_1 )</th>
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<tbody>
<tr>
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<td>0.165</td>
<td>0.1646</td>
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<td>6</td>
<td>0.321</td>
<td>0.325</td>
<td>0.319</td>
<td>0.3277</td>
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<tr>
<td>9</td>
<td>0.484</td>
<td>0.487</td>
<td>0.481</td>
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\[ n = 48, \quad \ell = 0 \]

<table>
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<th>( V )</th>
<th>( \lambda_1 )</th>
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<tbody>
<tr>
<td>3</td>
<td>0.145</td>
<td>0.146</td>
<td>0.144</td>
<td>0.1430</td>
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<tr>
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<td>0.275</td>
<td>0.278</td>
<td>0.272</td>
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</tr>
<tr>
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<td>0.419</td>
<td>0.425</td>
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\[ n = 73, \quad \ell = 0 \]

<table>
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<th>( T_0^2 )</th>
<th>( V )</th>
<th>( \lambda_1 )</th>
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</thead>
<tbody>
<tr>
<td>3</td>
<td>0.148</td>
<td>0.148</td>
<td>0.147</td>
<td>0.1460</td>
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<tr>
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<td>0.282</td>
<td>0.284</td>
<td>0.280</td>
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<tr>
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<td>0.430</td>
<td>0.434</td>
<td>0.428</td>
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</table>

*Values of power for \( W, T_0^2, V \) are from Lee (1971), Table 2.

\[ W = \prod (1 + \phi_1)^{-1}; \quad T_0^2 = n_2 \Sigma \phi_1; \quad V = \Sigma \lambda_1. \]

\[ n = \frac{1}{2}(n_2 - p - 1); \quad \ell = \frac{1}{2}(|n_1 - p| - 1); \quad s = \min(n_1, p). \]
Table 4: Power Comparisons of MANOVA Tests III*

\[ s = 2 \]

\[ n = 5 \quad \lambda = 0 \]

<table>
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<tr>
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<td>.0794634</td>
<td>.0772267</td>
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<tr>
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\[ n = 5 \quad \lambda = 1 \]

<table>
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</table>

\[ n = 5 \quad \lambda = 2 \]

<table>
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</table>

\[ n = 5 \quad \lambda = 5 \]

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<th>( V )</th>
<th>( \lambda )</th>
</tr>
</thead>
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<td>.0547795</td>
<td>.0535545</td>
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<tr>
<td>1</td>
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<td>.0584539</td>
<td>.0596835</td>
<td>.0573063</td>
</tr>
<tr>
<td>2</td>
<td>.06937</td>
<td>.06768</td>
<td>.06983</td>
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<tr>
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<td>.0742989</td>
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</table>

*Values of power for \( W, T_0^2 \) and \( V \) are from Pillai and Jayachandran (1967), Table 10.

\[ W = \prod (1+\phi_i)^{-1} \quad T_0^2 = n_2 \sum \phi_i \quad V = \sum \lambda_i \]

\[ n = \frac{1}{2} (n_2 - p - 1) \quad \lambda = \frac{1}{2} (|n_1 - p| - 1) \quad s = \min(n_1, p) \]
Table 4: continued

\[
\begin{array}{|c|c|c|c|c|}
\hline
\gamma & W & T_0^2 & V & \lambda_1 \\
\hline
\end{array}
\]

\[s = 2, \quad \ell = 0\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\gamma & W & T_0^2 & V & \lambda_1 \\
\hline
0.5 & 0.067644 & 0.067587 & 0.067618 & 0.066646 \\
1 & 0.087101 & 0.087084 & 0.086912 & 0.085407 \\
2 & 0.13080 & 0.13112 & 0.12992 & 0.128512 \\
3 & 0.1797 & 0.1807 & 0.1777 & 0.177987 \\
\hline
\end{array}
\]

\[n = 15, \quad \ell = 1\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\gamma & W & T_0^2 & V & \lambda_1 \\
\hline
0.5 & 0.062511 & 0.062412 & 0.062506 & 0.061367 \\
1 & 0.076185 & 0.076078 & 0.076039 & 0.074136 \\
2 & 0.10677 & 0.10695 & 0.10596 & 0.103687 \\
3 & 0.1413 & 0.1420 & 0.1392 & 0.138139 \\
\hline
\end{array}
\]

\[n = 15, \quad \ell = 2\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\gamma & W & T_0^2 & V & \lambda_1 \\
\hline
0.5 & 0.059920 & 0.059798 & 0.059933 & 0.058772 \\
1 & 0.070665 & 0.070495 & 0.070579 & 0.068546 \\
2 & 0.09453 & 0.09430 & 0.09389 & 0.091049 \\
3 & 0.1214 & 0.1218 & 0.1197 & 0.117337 \\
\hline
\end{array}
\]

\[n = 15, \quad \ell = 5\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\gamma & W & T_0^2 & V & \lambda_1 \\
\hline
0.5 & 0.056371 & 0.056227 & 0.056406 & 0.055346 \\
1 & 0.065140 & 0.062894 & 0.063153 & 0.061185 \\
2 & 0.07786 & 0.07754 & 0.07763 & 0.074305 \\
3 & 0.0941 & 0.0940 & 0.0934 & 0.089387 \\
\hline
\end{array}
\]

\[n = 40, \quad \ell = 0\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\gamma & W & T_0^2 & V & \lambda_1 \\
\hline
0.5 & 0.069298 & 0.069304 & 0.069271 & 0.068432 \\
1 & 0.090823 & 0.090873 & 0.090717 & 0.089435 \\
2 & 0.13965 & 0.13994 & 0.13926 & 0.138172 \\
3 & 0.1946 & 0.1952 & 0.1938 & 0.194150 \\
\hline
\end{array}
\]

\[n = 40, \quad \ell = 1\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\gamma & W & T_0^2 & V & \lambda_1 \\
\hline
0.5 & 0.063964 & 0.0639615 & 0.063944 & 0.062964 \\
1 & 0.079435 & 0.079481 & 0.069336 & 0.077733 \\
2 & 0.1146 & 0.11489 & 0.11416 & 0.112400 \\
3 & 0.1548 & 0.1555 & 0.1538 & 0.153222 \\
\hline
\end{array}
\]

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Table 4: continued

<table>
<thead>
<tr>
<th>$\gamma_1$</th>
<th>$W$</th>
<th>$T_0^2$</th>
<th>$V$</th>
<th>$\lambda_1$</th>
</tr>
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$n = 40$  $k = 2$

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$n = 40$  $k = 5$
4.4 Power Tables for the Largest Root Test

Before presenting the power tables for the largest root, we make a few observations. The monotonic character of the power function as the non-centrality parameter, $\gamma_1$, increases is obvious. It should also be noted that as the sample size increases (i.e., $n_2$ increases), the power increases.

In addition, we observe that for $n_2$ and $\gamma_1$ fixed, the power decreases monotonically as the value of the parameter $\ell$ increases. We recall that $\ell = \frac{1}{2}(|n_1-p| - 1)$, where $n_1$ is the hypothesis degrees of freedom and $p$ is the number of variates. In terms of a one-way analysis of variance, $n_1 + 1$ is the number of populations whose means are being compared. Therefore, we find that the power of the largest root is greatest when the number of variates observed equals the number of populations and decreases as one becomes larger than the other.

The tables which follow are for both 1% (Tables 5a, 5b) and 5% (Table 6a, 6b) levels of significance as noted. If it is desired to obtain the power for intermediate values of $\ell$ or $n_2$, linear interpolation is possible since the values change slowly. For greater accuracy, the original computer programs are in the appendix and can be used to generate additional values.
| GAMMA | L | ALPHA | S = 2 | N = 50 | 100 | 900 | 800 | 700 | 600 | 500 | 400 | 300 | 200 | 100 | 50 | 20 | 10 | 5 | 2 | 1 | 0.5 | 0.25 | 0.125 | 0.0625 | 0.03125 | 0.015625 | 0.0078125 | 0.00390625 |
|-------|---|-------|-------|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 0.00  | 0.0100 | 0.0100 | 1.00  | 0.0100 | 0.0100 | 0.0100 | 0.0100 | 0.0100 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.05  | 0.0133 | 0.0113 | 0.0102 | 0.0100 | 0.0095 | 0.0091 | 0.0087 | 0.0083 | 0.0079 | 0.0074 | 0.0069 | 0.0064 | 0.0059 | 0.0054 | 0.0049 | 0.0044 | 0.0039 | 0.0034 | 0.0029 | 0.0024 | 0.0019 | 0.0014 | 0.0009 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.10  | 0.0167 | 0.0147 | 0.0126 | 0.0109 | 0.0096 | 0.0087 | 0.0079 | 0.0070 | 0.0060 | 0.0052 | 0.0043 | 0.0035 | 0.0028 | 0.0022 | 0.0016 | 0.0011 | 0.0007 | 0.0004 | 0.0002 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.15  | 0.0200 | 0.0180 | 0.0155 | 0.0126 | 0.0101 | 0.0083 | 0.0067 | 0.0051 | 0.0039 | 0.0029 | 0.0019 | 0.0012 | 0.0007 | 0.0005 | 0.0003 | 0.0002 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.20  | 0.0233 | 0.0213 | 0.0190 | 0.0157 | 0.0126 | 0.0099 | 0.0074 | 0.0052 | 0.0032 | 0.0019 | 0.0010 | 0.0005 | 0.0003 | 0.0002 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.25  | 0.0266 | 0.0246 | 0.0220 | 0.0182 | 0.0149 | 0.0114 | 0.0080 | 0.0055 | 0.0029 | 0.0013 | 0.0006 | 0.0002 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.30  | 0.0300 | 0.0279 | 0.0250 | 0.0206 | 0.0171 | 0.0132 | 0.0093 | 0.0059 | 0.0025 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.35  | 0.0333 | 0.0311 | 0.0279 | 0.0231 | 0.0186 | 0.0140 | 0.0102 | 0.0058 | 0.0014 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.40  | 0.0366 | 0.0339 | 0.0296 | 0.0244 | 0.0192 | 0.0145 | 0.0107 | 0.0054 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

Table 5a: Power of the Largest Root Test

\[ s = \min(n_{1}, p) \]

\[ n = \left( \frac{1}{2} \right) (n_{2} - p - 1) \]

\[ k = \left( \frac{1}{2} \right) |n_{1} - p| - 1 \]
| TABLE 5b: Power of the Largest Root Test* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| GAMMA | L = -0.5 | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
| 0.00 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.25 | 0.0013 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.50 | 0.0024 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.75 | 0.0032 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.00 | 0.0039 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.25 | 0.0045 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.50 | 0.0050 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.75 | 0.0055 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.00 | 0.0059 | 0.0008 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| ALPHAS | S = 0.5 | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
| 0.00 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.25 | 0.0013 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.50 | 0.0024 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.75 | 0.0032 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.00 | 0.0039 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.25 | 0.0045 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.50 | 0.0050 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.75 | 0.0055 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.00 | 0.0059 | 0.0008 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

*\( s = \min(n_1, p) \); \( n = \frac{1}{2}(n_2-p-1) \); \( \ell = \frac{1}{2}(n_1-p-1) \).
Table 6a: Power of the Largest Root Test

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\[ *s = \min(n_1, n_2); \quad n = \frac{1}{2}(n_2 - n_1); \quad \beta = \frac{1}{2}\left| n_1 - n_2 \right| \]
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Table 6b: Power of the Large Root Test

\[ *s = \min(n_1, p) \]

\[ n = \frac{1}{2} (n_2 - p - 1) \]

\[ \ell = \frac{1}{2} (|n_1 - p| - 1) \]
4.5 Summary

This chapter contains a number of tables comparing the power of the largest root test to other tests based on all the roots under the rank one alternative. It can be seen that for large values of the non-centrality parameter, \( \gamma_1 \), the largest root test is the most powerful.

In addition, the monotonic character of the power function can be seen in the tables of Section 4.4. The power increases with \( \gamma_1 \) and \( n_2 \) and decreases with \( \ell \). We recall that \( \ell = \frac{1}{2} (|n_1 - p| - 1) \) and therefore observe that it would be desirable to have \( n_1 = p \) for maximum power. However, we note that these results do not provide a procedure for determining which of the observable variates from a given population should be chosen to achieve highest power.
APPENDIX

Included in this appendix are listings of the FORTRAN programs used to obtain the tabled values in Chapter 4. All of these programs make use of the subroutines BETTA (which is listed last) and BCDF (which is not listed). The subroutine BCDF is the same as the IBM Scientific Subroutine Package (SSP) Subroutine EDTR. As the programs are now structured, it is necessary to make some changes to use EDTR with these programs. There are two possible ways to proceed:

1) change all the statements of the form CALL BCDF ( ) to CALL EDTR ( );

2) change the name of SUBROUTINE EDTR to SUBROUTINE BCDF.

A brief description of each program follows.

ALPHA2

This program computes accurate critical values for the power calculations with $s = 2$ by linear interpolation between close initial values $x_1$ and $x_2$. These initial values can be obtained from the literature (e.g., Harris (1975) has tables of critical values to 3 places). The data cards needed for this program are:

1) a card with the desired alpha level (e.g., .05 or .01 in FL0.4 format;

2) a card with the number of data cards following it (for the variable NCARD) in 13 format;
3) the NCARD data cards each with the values \( n, \ell, x_1, x_2 \) in \( 4F10.4 \) format and with \( x_1 < x_2 \).

The program iterates until either the actual alpha level corresponding to the critical point \( x \) is within \( 10^{-10} \) of the nominal alpha level or it has gone through 100 iterations. The results are printed as output where TRIAL1 and TRIAL2 are the actual alpha levels corresponding to \( x_1 \) and \( x_2 \), respectively, and TRIAL is the actual alpha level corresponding to the critical value \( x \). In addition, data cards for POWER2 are punched, each with \( n, \ell, x, s \) in \( 2F10.4, F20.10, F10.4 \) format, where \( n, \ell, s \) are defined in Chapter 4 and \( x \) is the critical value such that

\[
\Pr\{\lambda_1 > x\} = \alpha
\]

under the null hypothesis.

**POWER2**

This program computes the power function of the largest root test with \( s = 2 \) for different values of the noncentrality parameter \( \gamma_1 \geq 0 \) using the distribution function derived in Chapter 2. The program has several large arrays and needs approximately 130K of core while executing. The data cards needed are:

1) a card with the number of cards with different values for \( n, \ell, x \) (for the variable NCARD) and with the number of different values of \( \gamma_1 \) (NGAMMA) in 2I3 format;
2) the NGAMMA data cards, each with a value of $\gamma_1 \geq 0$ in F10.4 format;

3) the NCARD data cards each with values for $n$, $\ell$, $x$, $s$
(e.g., as punched by ALPHA2) in 2F10.4, F20.10, F10.4 format.

The output gives the values of $\gamma_1$, $n$, $\ell$ and $x$. The column
headed "CDF" gives the values of

$$\Pr_{\gamma_1} \{\lambda_1 \leq x\}$$

and the column labeled "POWER" gives the values of the power

$$1 - \Pr_{\gamma_1} \{\lambda_1 \leq x\} = \Pr_{\gamma_1} \{\lambda_1 > x\}$$

for the corresponding value of $\gamma_1$ with $s = 2$. The column headed
"NTERMS" gives the number of terms in the sum over $k_0$ needed to
achieve the desired level of convergence. The program adds terms
until the new term is both less than 0.00001 of the sum over $k_0$
accumulated so far and contributes less than $10^{-10}$ to the distribution
function.

ALPHA3

This program computes accurate critical values for the power
calculations with $s = 3$ by linear interpolation between close
initial values $x_1$ and $x_2$. The data cards and output formats are the
same as those for ALPHA2 above.
This program computes the power function of the largest root with $s = 3$ for different values of the noncentrality parameter $\gamma_1 \geq 0$ using the distribution function derived in Chapter 2. The data cards and output format are the same as those for POWER2 above.
C THIS IS JOB ALPHA2
C THIS PROGRAM COMPUTES CRITICAL POINTS FOR S=2
C BY INTERPOLATING BETWEEN CLOSE VALUES XI AND X2 WHICH
C CAN BE OBTAINED FROM THE LITERATURE.
C
C ALPHAZ USES SUBROUTINES VALUE2, BETA (SEE END) AND
C BCF (NOT LISTED).
C THE CALL FOR PROGRAM BCF CORRESPONDS TO THE IBM SSP
C ROUTINE DBFR. SUBROUTINE DBFR CAN BE USED BY CHANGING
C THE CALL TO CALL BDTR OR BY CHANGING THE NAME OF BDTR TO
C SUBROUTINE BCF.
C
C IMPLICIT REAL*8 (A-H,O-P)
C REAL*4 A,B,A1,A2,B1,B2,RE,RL
C COMMON BETA1,BETA2,FAC1,FAC2,CONST,
C COMMON A1,A2,B1,B2,RL,RE
C
C S = 2
C PI = 3.141592653589793
C TOLER = 1.E-10
C READ (5,100) ALPHA
C 100 FORMAT (F10.4)
C NCARD = 5
C 1000 FORMAT (E)
C DO 1001 J12 = 1,NCARD
C CALL VALUE2(X1,J12)
C X1 IS ALWAYS TO BE SMALLER THAN X2 AND IN GENERAL,
C TRIAL1 SHOULD TURN OUT LARGER THAN TRIAL2 OR AN
C ERROR HAS OCCURRED
C
C READ (5,101) RN,RL,X1,X2
C 101 FORMAT (F10.4,F10.4)
C ICNT = 0
C C CALCULATE CONSTANT
C A = 2.*RN + 2.
C B = 2.*RL + 2.
C IR = B - 1
C CALL BETA(A,B,BET)
C CONST = (2.*RL+2.*RN+4.)/BET
C CONST = CONST / (4.*PI)
C C
C SET PARAMETERS FOR BETA1 AND BETA2
C RI = RN+1
C A1 = RL+1
C CALL BETA(A1,R1,BET1)
C A2 = 2.*RL + 2.
C B2 = 2.*RN + 2.
C CALL BETA(A2,B2,BET2)
C FACT = PI/(RL*RN+2.)
C C WRITE (6,201) BET,BET1,BET2,CONST,FAC
C 201 FORMAT (* BET, BET1, BET2, CONST, FAC
C *)
C C END
I CONTINUE
I AT THIS POINT CONST=(2(L+1)+1)*(2L+1)/(2N+1)*Pi4
    CONST = CONST * ((2*RL+3)/Pi) * ((2*RL+2)/Pi)
I CALCULATE SOME INITIAL VALUES
    BET(1) = (X**(PL+1) + (1-X)**(RN+1))
I WRITE (6,500)
500 FORMAT(1,E8.1,GAMMA,T1(X)**T2,N**T3)**L**T4**K**T5,CDF,F12.X, + TERMS,8X,ALPHA,8X,FACT)
I DO 600 JJ = 1,NGAMMA
I SET NONCENTRALITY
GAM = GAM(JJ)
HGM = GAM / 2.
EGM = DEXP(-HGM)
PACT = EGM * CONST
    TOLER = 1.E-10 / FACT
I CALCULATE SUM ON K0
SUNKO = 0.
GCODEF = 1.
    G = 1.
I SET PARAMETERS FOR BETA1 AND BETA2
B1 = RN + 1.
B2 = 2.*RN + 2.
I CALCULATE TERM FOR K0 = 0
I SET PARAMETERS ANS CALL BDTR
A1 = RL + 1.
A2 = 2.*RL + 2.
CALL BDTR(KK,AL,RL,PL,DER)
CALL BETTA(A1,B1,BET1)
BET1 = P * BET1
CALL ECODEF(A2,B2,DER)
BEE2 = 02.
CALL BETTA(A2,B2,BET2)
BET2 = P * BET2
I TERM = 2.*BET2(1) - BET1(BET1(1))
    TERM = TERM * PI/(RL+RN+2)
I SUNK = SUNK + TERM
ALPHA = 1. + SUMK*CONST
I DO 3 KOP1 = 2,101
    K0 = KOP1 - 1.
    RK0 = K0
I G = GAMMA/2*K0
    G = GAMHGM
I SET GCODEF=((2(L+1)+2)*(L+1))/((L+K0+2)*S+((L+1)*S)*K0)
GCODEF = GCODEF * ((RL+RN+2+RK0)/(RL+RN+RK0+RK0))
C
C EVALUATE NEW VALUES FOR TERMS
C GAMA(KOP1) = GAMA(K) + (S) / GAMA(K) + (1)
GAMA(KOP1) = GAMA(K)(K) + (1) = 1/(2*K*K)
C
C BET(KOP1) = (X***(RL+K0+1))*(1-X)**(RN+1)
BET(KOP1) = BET(K) + X
C
C COEF(KOP1) = PROD(((RL+K0+2) - J) / (RL+K0+RN+2 - J))
COEF(KOP1) = COEF(KOP1) + (J) / (RL+K0+RN+1)
C
C NOTE THAT FIRST TIME THRU LOOP, J = 2, I BLT YOU GO THRU
C ONCE ANYWAY BUT THIS DOES NOT AFFECT PROGRAM IN ANY WAY
DO 9 I = 2, KO
9 PROD = (RL+K0+2 - J) / (RL+K0+RN+2 - J)
COEF(KOP1) = COEF(KOP1) + PROD
CONTINUE
C
C CALL SUBROUTINE GDRT TO CALCULATE INCOMPLETE BETAS
C BETA1(KOP1) = BETAL(K0+1)
C BET(A2(KOP1)) = BETAL(K) + (2)
C SET PARAMETERS AND CALL GDRT
A1 = RL + K0 + 1
CALL GDRT(A1,B1,B2,B3,B4,B5,B6,B7,B8,B9)
IF (12 <= 0) GO TO 600
WRITE (6,502) GAMAX, RN, RL, IER
502 FORMAT(1,1,F15.7,IER = 1,14)
GO TO 600
600 CONTINUE
C
C CALL SUBROUTINE SSRT2 TO CALCULATE COMPLETE BETAS
C BET(A2(KOP1)) = P + BET
C
A2 = 2*RL + K0 + 2
CALL GDRT(A2,B2,B3,B4,B5,B6,B7,B8,B9,B10)
IF (IER.EQ.0) GO TO 607
WRITE (6,503) GAMAX, RN, RL, IER
503 FORMAT(1,1,F15.7,IER = 1,14)
GO TO 600
607 CONTINUE
C
C CALL SUBROUTINE TO CALCULATE S*(K0,K1) = ESTAR
CALL SST2
C
C A TERM IN K0 SUM IS CALLED TERM
TERM = GCOEF = 0 + ESTAR
SUNKO = SUNKO + TERN
NTERMS = K0 + I
CHECK2 = 0.0001 / SUNKO
IF (TERM.LE.TOLER.AND.TERM.LE.CHECK2) GO TO 5
C
C 5 CONTINUE
C
C CALCULATE COF = PROB(LARGEST ROOT <LT,X>)
S
S
CONTINUE
C
C COF = EGM + CONST + SUM0
POWER = 1.0 - COF
POWER(I) = POWER
WRITE (6,501) GAMAX, RN, RL, COF, POWER, NTERMS, ALPHA, FACT
501 FORMAT(1,1,F15.7,15,F15.7,15,F15.7)
600 CONTINUE
605 CONTINUE
S
STOP
S
END
C
C
C SUBROUTINE SST2
C DIMENSION GAMA(100),BETA(100),BETA1(100),BETA2(100),
C XCOEF(100,100)
C REAL*8 GAMMA,BETAL,BET(A2),COEF,ESTAR
C REAL*8 SUM1,TERM1,SSUM1,GAMA,GFACT,SUNK1,TERM1
C COMMON ESTAR,RL,RR,RA,K0,K1,KOP1
C SUM1 = 0
C
C CALL BETAIL(KOP1) = BETAL(K0+1)
C CALL BET(A2,KOP1) = BETAL(K) + (2)
C CALL GDRT(A1,B1,B2,B3,B4,B5,B6,B7,B8,B9)
C IF (IER.EQ.0) GO TO 600
C WRITE (6,502) GAMAX, RN, RL, IER
C 502 FORMAT(1,1,F15.7,IER = 1,14)
C GO TO 600
C
C CONTINUE
C
C ADD ON FIRST PART -- FOR I = 0
C TERM = 2*BETAIL(KOP1) - BETAIL(KOP1)*BETAIL(1)
C TERM = TERM1*COEF(KOP1,1)
C SUM1 = SUM1 + TERM1
100 CONTINUE
C
C SUM = SUM + SUM1
C SUNK1 = 0
C KHALF = K0 / 2
C IF (KHALF.EQ.0) GO TO 104
C DO 101 K1 = 1,KHALF
C RK1 = K1
C KIP1 = K1
C
C CALL GSUM(SUM) ON K1
S
C SUNK1 = 0
C LIM = K0 - 2*K1
C IF (LIM.EQ.0) GO TO 103
C DO 102 I = 1,LIM
C TERM = 2*BETAIL(KOP1) - BETAIL(KOP1)*BETAIL(KIP1)
C TERM = TERM1*COEF(KOP1,KIP1)
C SSUM1 = SSUM1 + TERM1
102 CONTINUE
C
C CALL SUM = SUM + SSUM1
C 103 CONTINUE
C CALL I = 0 PART OF SUBSUM
C TERM = 2*BETAIL(KOP1) - BETAIL(KOP1)*BETAIL(KIP1)
C
SSUM1 = (SSUM1+TERM1) * 1./(RL+RO-RK1+RN+2.1)

C
GFACT = (RK1-5)/RK1 - (ROK-RK1+2)/(ROK-RK1+1.)
GAMA = GAMMA(KSP1-11) + GAMMA(A1) + GFACT
TERM1 = SSUM1 * GAMA
SUMK1 = SSUM1 + TERM1

101 CONTINUE
104 ESTAR = SUMK1 + SUMK1
RETURN
END
C
C
C
******************************************************************************
C
C
THIS IS JOE ALPHA3
C
C
THIS PROGRAM COMPUTES CRITICAL POINTS FOR S = 3.
C
BY INTERPOLATING BETWEEN CLOSE VALUES X1 AND X2 WHICH
C
CAN BE OBTAINED FROM THE LITERATURE.
C
C
ALPHA3 USES SUBROUTINES VALUE3, BETTA (SEE END) AND
C
BCOF (NOT LISTED).
C
THE CALL FOR PROGRAM BCOF CORRESPONDS TO THE IBM SEP
C
ROUTINE BDFR. SUBROUTINE BDFR CAN BE USED BY CHANGING
C
THE CALL TO CALL BDR OR BY CHANGING THE NAME OF BDR TO
C
SUBROUTINE BCOF.
C
C
C
IMPLICIT REAL*8 (A-H,O-P)
DREAL# BET1(3),BET2(3),TRIAL1,TRIAL2,TRIAL
DREAL# PI,BET1,CONST1,ALPHA,FACT1,FACT2,X1,X2,X
COMMON BET1,BET2,FACT1,FACT2,CONST1
COMMON A1,A2,B1,B2,RL,RN
C
C
S = 3.
PI = 3.141592653589793
TOLER = 1E-10
C
READ(5,100) ALPHA
100 FORMAT(F10.4)
READ(5,301) NCARD
301 FORMAT(13)
C
DO 302 J=1,NCARD
READ (5,10) JRN,RL,X1,X2
302 FORMAT(4F10.4)
C
ICNT = 0
C
C
CALCULATE CONSTANT
C
A = RL+2.
C
B = RN+2.
CALL BETTA (ALPHA,BETTA)
CONST = 1./BETTA
A = 2.*RL + RN
B = 2.*RN + 2.
CALL BETTA (ALPHA,BETTA)
CONST = CONST/BETTA * (RN+RL+2.S) / (RL+RN+2.S)
C
C
C
SET PARAMETERS FOR BETA1 AND BETA2
B1 = RN+1.
B2 = 2.*RN + 2.
C
C
SET NORMALIZING PARAMETERS FOR BETA1 AND BETA2
DD = 300 I = 1,3
RI = FLOAT(I)
A1 = RL + RI
A2 = 2.*RL + 1.*RI
CALL BETTA (A1,B1,BET1(I))
CALL BETTA (A2,B2,BET2(I))
300 CONTINUE
C
C
C
SET PARAMETERS BAKED ONLY ON L AND N
FACT1 = 1./(RL+RN+3.1)
FACT2 = 1./(RL+RN+2.1)
C
C
C
CALL VALUE3(X1,TRIAL1)
CALL VALUE3(X2,TRIAL2)
WRITE (6,200) X1,TRIAL1,X2,TRIAL2,RN,RL
200 FORMAT(13,13,1F10.4)
TRIAL1 = "",TRIAL2 = "",RN = "",RL = ""
C
IF (DABS(TRIAL1-ALPHA)+LE,TOLER) GO TO 105
IF (DABS(TRIAL2-ALPHA)+LE,TOLER) GO TO 105
C
104 IF (DABS(TRIAL1-TRIAL2) .LE. 1.E-10) GO TO 105
X = ((ALPHA-TRIAL2) / (TRIAL1-TRIAL2)) * (X1-X2)
X = X2 + X
ICNT = ICNT + 1
CALL VALUE3(X,TRIAL1)
C
IF (DABS(TRIAL-ALPHA)+LE,TOLER) GO TO 105
IF (ICNT.GE.100) GO TO 105
IF (TRIAL .EQ. ALPHA) 102,102,103
102 X2 = X
CALL VALUE3(X2,TRIAL2)
GO TO 104
C
103 X1 = X
CALL VALUE3(X1,TRIAL1)
GO TO 104
105 CONTINUE
WRITE (6,106) ICNT,ALPHA,TRIAL1,X1,RN,RL
106 FORMAT(*ICNT,ALPHA,"",TRIAL1="",X1="",RN="",RL=""
WRITE (7,400) RN,RL,X1
400 FORMAT(13)
REALS GGAM,20, TERM, POWER, FACT
REALS BETA, COND, PI, HGAM, GAMMA, COEF, G, X, ALPHA, COF
REALS BET, BET, PART, PART, BETA, PART, T, PROD, GAMMA
COMMON GAMMA, BET, BETA, BETA, COEF, ESTAR
COMMON RL, RN, NOK, RKL, RKL
COMMON K0, K1, K2, K0P, K1P, K2P

C
C PI = 3.141592653589793
C GAMMA (1) = DSQRT (PI)
C READ (5, 607) NCARD, NAGAMMA
C
C FORMAT (2I13)
C READ VALUES OF GAMMA
C DO 620 I = 1, NAGAMMA
C READ (5, 621) GAMMA(I)
C 621 FORMAT (F10.4)
C 620 CONTINUE
C
C DD 609 J1 = 1, NCARD
C C READ VALUES OF N, L, X
C C READ (5, 608) RN, RL, XS
C 608 FORMAT (2F10.4, F20.10, F10.4)
C RLPL = RL + RN
C NSTAR = 0
C WRITE (6, 500)
C 500 FORMAT (*, 8X, 'GAMMA', I10, X, 'X', I10, X, 'N', I10, X, 'L', I10, X,
C 'COEF', I12, X, 'POWER', I10, X, 'INTERPS', I8, X, 'ALPHA')
C C C CALCULATE CONSTANT
C A = RL + 2
C B = RN + 2
C I = B + 1
C CALL BET(A, B, BETT)
C CONST = BET (1) - BET (I)
C RETURN
C END
C
C THIS IS JOB POWER
C THIS PROGRAM COMPUTES THE CDF OF THE LARGEST ROOT FOR S=3
C POWER3 USES SUBROUTINES TFCN, STAR3, BETTA (SEE END)
C BCDP NOT LISTED.
C THE CALL FOR PROGRAM BCDP CORRESPONDS TO THE IBM SSP
C ROUTINE BDT. SUBROUTINE BDT CAN BE USED BY CHANGING
C THE CALL TO CALL BDT OR BY CHANGING THE NAME OF BDT TO
C SUBROUTINE BCDP.
C
DIMENSION REALS (30), REALX (30)
REALS BETA (100), BETA (100), BETA (100), COEF (100, 100)
REALS DSQRT, EXP
REALS ESTAR (100), POWER (30, 30), GAMMA (100)
C C CALCULATE SUM ON K0
C SUMK0 = 0
C G = 1.
C C CALCULATE FIRST VALUES FOR K0 = 0
C C SET PARAMETERS AND CALL BCDF
C C NEED THREE VALUES FOR FIRST TERM
DO 602 I = 1, 3
   P1 = FLOAT(I)
   A1 = RL + RI
   A2 = 2. * RL + 1. + RI
   CALL BCDF(XA1,B1,P,DLIER)
   BEE1 = B1
   CALL BETTA(A1,B1,BET1)
   BETAI(I) = P * BET1
   CALL BCDF(XA2,B1,P,DRIER)
   BEE2 = B2
   CALL BETTA(A2,B2,BET2)
   BETAI(I) = P * BET2
602 CONTINUE
C C COMPUTE TERM FOR K0 = 0, K0P1 = 1
K0 = 0
K1 = 0
K2 = 0
K0P1 = 1
K0P2 = 1
R0 = 0
QK = 0
RK = 0
DST1 = 2. * BETAI(1) * BETAI(1)
DST2 = 2. * BETAI(1) * BETAI(1)
DST3 = BETAI(1) * BETAI(1)
TVAL = DST1 + DST2 + DST3
C C TERMK0 = (PART1 - PART2 - PART3) * (BETAI(1) + 3)
TERMK0 = TERMK0 + (PRPH + 3)
C C SUMK0 = SUMK0 + TERMK0
C C CALCULATE SUM ON K0
DD 603 K0P1 = 2, 100
K0 = K0P1 - 1
RK0 = FLOAT(K0)
GAM = GAM(JJ)
WRITE(*,501) GAM,AA,RL,CDP,POWER,PTERM,ALPHA
501 FORMAT (**,*4F10.7,I10,F20.7)
600 CONTINUE
60V CONTINUE
STOP
END
C
C SUBROUTINE TFNC(TVAL,INTUP,INTON)
REAL BET(100),BETA1(100),BETA2(100),CDEF(100,100)
REAL ESTAR(100), GAMMA(100)
REAL TVAL,SUMI,TERMI,PROD
COMMON GAMMA,BET,BETA1,BETA2,CDEF,ESTAR
COMMON RL,RA,K0,K1,K2,KPI,K2P1
IF (INTUP.LT.101) I = 1, INTON = 1
INTON IS USUALLY K = K1
C DO CASE OF INTUP.GT.INTON
109 SUMI = 0.
C DO CASE OF I = 0 FIRST
TERMI = 2.00*BETA2(K0P1+K2+K2)-BET(K0-K1)*BETA1(K2P1+1)
SUMI = SUMI + TERMI
C
C 104 TVAL = SUMI/(RL+RA+K0+RA+K0+RA)*K2
GO TO 108
C
C 108 RETURN
END
C
C SUBROUTINE SSTAR3
REAL BET(100),BETA1(100),BETA2(100),CDEF(100,100)
REAL ESTAR(100), GAMMA(100), TVAL
REAL PART1,PART2,PART3,GPACT,GAMA
COMMON GAMMA,BET,BETA1,BETA2,CDEF,ESTAR
COMMON RL,RA,K0,K1,K2
COMMON K0,K1,K2,K0P1,K1P1,K2P1
C CAN BE USED FOR VALUES OF K0,GE,1
SUNKI = 0.
DO 101 KIP1 = 1,K0P1
K1 = KIP1-1
PK1 = FLOG(K1)
SUMI = 0.
C CALCULATE K2 = 0 TERM
K2 = 0
K2P1 = 1
RK2 = 0.
KOMK1 = K0-K1
CALL TFNC(TVAL,KOMK1,1)
C C CALCULATE SUM ON I=0..K1
DO 100 IPI = 1,KIP1
I = IPI-1
PART1 = 2.00*BETA2(K1P1+I-2)*BETA1(K0P1+1)
PART2 = 2.00*BETA2(K1P1+I-1)*BETA1(2)
PART3 = BET(K1P1-1+1) * TVAL
TERM1 = (PART1-PART2-PART3) * COEF(K1P1, 1P1)

SUM1 = SUM1 + TERM1
100 CONTINUE

SUM1 = SUM1 * GAMMA(K1P1) * GAMMA(1) / (RLRK1+3.3+RN)

CALCULATE SUM ON K2 = 1. * (K1/2)
SUM2 = 0.00
KHALF = K1/2
IF (KHALF,E0.0) GO TO 103
DO 101 K2 = 1,KHALF

K3P1 = K2 + 1
RK2 = FLOAT (K2)
CALL TFCN (TVNLK0MK1K2P1)

CALCULATE SSUM ON I = 0,K1-2*K2
SSUM1 = 0.
LIMP1 = K1-2*K2 + 1
DO 102 IPI = 1,LIMP1
I = IPI-1
PART1 = 2.*BETA2(K1P1+2-I)*BETA1(KP1-K1)
PART2 = 2.*BETA2(KP1-K2+1-I)*BETA1(K2P1+1)
PART3 = BETA2(KKP1-K1+1) * TVAL

TERM1 = (PART1-PART2-PART3) * COEF(K1P1-K2+1P1)

SUM1 = SSUM1 + TERM1
102 CONTINUE

GFACT = (RK2-.5)/(RK2-4.0) / (RK1-RK2+5)/(RK1-RK2+1)
GAMA = GAMMA(K1P1-K2) * GAMMA(K2) * GFACT

TERM2 = SSUM1 * GAMA / (RLRK1-RK2+3.3+RN)

SUM2 = SUM2 + TERM2
103 CONTINUE

TERM1 = SUM1 + SUM2
TERM1 = TERM1 * GAMMA(KP1-K1)

SUM1 = SUM1 + TERM1
104 CONTINUE

ESTAA(K0) = SUM1
RETURN
END

******************************************************************************

SUBROUTINE BETTA(AA,BB,BETT)

C C

!BETTA COMPUTES GAMMA(A)*GAMMA(B)/GAMMA(A+B)

REAL*8 DSQRT
REAL*8 BETT,PI
PI = 3.141592653589793
BETT = 1.00

DO CASE OF A AND B INTEGERS. B.GE.1, A.GE.1
AND CASE OF A=AA + .5, B AND A=INTEGERS. B.GE.1,
A+.5,GE.0 IN 400 LOOP
CASE OF B = B+.5, FOR A AND B INTEGERS IS DONE
IN 403
AND THEN RETURNED TO 400 AFTER A AND B ARE EXCHANGED
FINALLY DO CASE OF A AND B BOTH HALF INTEGERS

USE OF AA AND BB IS TO PROTECT AGAINST INTERCHANGE
IN 403
A=AA
BB=BB
IA = A
IB = B
IF ((A=IA),EQ.0,.AND.(B=IB),EQ.0.) GO TO 400
IF ((A=IA),EQ.0,.AND.(B=IB),EQ.0.) GO TO 400
IF ((A=IA),EQ.0,.AND.(B=IB),EQ.0.) GO TO 403

IF (IB,EQ.0) GO TO 405
DO 404 I = 1,IB
RI = FLOAT(I)
404 BETT = BETT * ((B-RI)/(A+B-RI))
405 BETT = BETT * DSQRT(PI)

IF (IA,EQ.0) GO TO 406
DO 407 I = 1,IA
RI = FLOAT(I)
407 BETT = BETT * ((A-RI)/(A+S-RI))
406 BETT = BETT * DSQRT(PI)
GO TO 409
400 IWN = IN - 1
IF (IWN,EQ.0) GO TO 402
DO 401 I = 1,W,1
RI = FLOAT(I)
401 BETT = BETT * ((B-RI)/(A+B-RI))
402 BETT = BETT/AA
GO TO 409
403 AYE = A
B = B
AYE = A
IA = A
IB = B
GO TO 400
400 RETURN
END
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