ON THE BIAS OF THE CHARACTERISTIC ROOTS OF A RANDOM SYMMETRIC MATRIX

BY
THEOPHILOS CACOULLOS AND INGRAM OLKIN

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1. Summary and Introduction.

Let $Z$ be a random symmetric $p \times p$ matrix with $EZ = A$. An application to the theory of response surface estimation led van der Vaart [3] to consider the expectation-bias and median-bias of the characteristic roots of $Z$ as estimators of the characteristic roots of $A$. Denote the (real) characteristic roots of a matrix $X$ by $\lambda(X)$ with the ordering $\lambda_1(X) \geq \cdots \geq \lambda_p(X)$. Van der Vaart proves that if $Z$ is symmetric, then $E\lambda_1(Z) \geq \lambda_1(A)$ and $E\lambda_p(Z) \leq \lambda_p(A)$. These inequalities become strict if absolute continuity is assumed. With the additional assumption that the distribution of $Z$ is symmetric about $A$, i.e., $P[\sum c_{ij}(z_{ij} - a_{ij}) > 0] = \frac{1}{2}$, for all matrices $C = (c_{ij}) \neq 0$, similar inequalities hold for median-bias, namely, $P(\lambda_1(Z) \geq \lambda_1(A)) > \frac{1}{2}$ and $P(\lambda_p(Z) \leq \lambda_p(A)) > \frac{1}{2}$.

In this paper we show how to generate a wide class of inequalities as direct consequences of known results concerning the convexity of scalar or matrix functions of a matrix (Section 2). In particular, the results of [3] can be strengthened by a weakening of the assumption that

1/ Dedicated to Professor Charles Loewner on the occasion of his 70-th birthday.
Z be symmetric. When Z is symmetric, expectation-bias and median-bias may also be obtained for partial sums of the roots, and when Z is positive definite, the bias is obtained for more general functions of the roots. In Section 3 we consider the roots of the determinantal equation \(|Z_1 - \Theta Z_2| = 0\), as well as the canonical correlations. Inequalities for median-bias are developed in Section 4.


We first consider convex scalar and matrix functions with matrix argument (Section 2.1), then obtain inequalities when Z has real roots (Section 2.2), when Z is symmetric (Section 2.3), and when Z is positive definite (Section 2.4).

2.1 Convex Functions with Matrix Argument.

A scalar function, \(g(X)\) of a \(p \times n\) matrix \(X\), in some set \(\mathcal{X}\), is convex if for any two matrices \(X, Y \in \mathcal{X}\) and \(0 \leq \lambda \leq 1\),

\[
g(\lambda X + (1-\lambda) Y) \leq \lambda g(X) + (1-\lambda) g(Y)
\]

(2.1)

For example, \(g(X) = \text{tr} \ XX'\) is such a function.

If \(EX < \infty\) and \(g(X)\) is convex, we can assert by Jensen's inequality,

\[
g(Ex) \leq Exg(X)
\]

(2.2)

This development may be extended to matrix functions of a matrix if an ordering in (2.1) is properly defined. Such an ordering was considered
by Loewner [1, p. 86], namely, for any symmetric matrices $X$ and $Y$, write $X \geq Y$ iff $X - Y$ is positive semi-definite. Consequently, if $G(X)$ is a real symmetric matrix function of a $p \times n$ matrix $X$ defined on some set $\mathcal{X}$, such that for $X$ and $Y \in \mathcal{X}$, $0 \leq \lambda \leq 1$,

$$G(\lambda X + (1-\lambda)Y) \leq \lambda G(X) + (1-\lambda) G(Y),$$

in the sense of Loewner, then $G(X)$ is called convex. We note that $X$ need not be symmetric or positive definite, though the domain of $X$ will be restricted in the usual applications. It is immediate that $G(X)$ is convex iff $g_u(X) = u G(X) u'$ is a scalar convex function of $X$ for all $u \neq 0$.

The above definition permits an extension of Jensen's inequality.

**Theorem 2.1.** If $X \in \mathcal{X}$ is a random matrix, $EX = A$, and $G(X)$ is a convex symmetric matrix function with $\lambda_1[Eg(X)] < \infty$, then

$$G(EX) \leq EG(X).$$

**Proof.** For every real vector $u$ of unit length, $g_u(X) = u G(X) u'$ is a convex scalar function of $X$. The result then follows from (2.2). ||

Thus, for example, if $X : p \times n$, then $G(X) = XX'$ is convex, and $EXX' \geq (EX)(EX')$. A more interesting example is the inverse matrix function.

**Lemma 2.2.** If $\mathcal{X} = \{X : X > 0\}$, then $G(X) = X^{-1}$ is convex.

---

For convenience, we refer to [1] where most of the results needed and related references are given.
Proof. For any \( X, Y \in \mathcal{X}, 0 \leq \lambda \leq 1 \), we must show that

\[
(2.3) \quad (\lambda X + (1-\lambda)Y)^{-1} \leq \lambda X^{-1} + (1-\lambda)(Y^{-1}).
\]

There exists a non-singular matrix \( W \) such that \( X = WW', \ Y = WDW' \), where \( D = \text{diag}(d_1, \ldots, d_p) \). Hence, (2.3) is equivalent to

\[
[\lambda I + (1-\lambda)D]^{-1} \leq \lambda I + (1-\lambda)D^{-1}.
\]

But this is an inequality between diagonal matrices, so that the result follows from the scalar case. We note that (2.3) is strict for \( 0 < \lambda < 1 \). \|

In some instances an alternative proof based on the following Lemma may be simpler.

**Lemma 2.3.** If \( g(X) \) is a scalar or matrix function of \( X \in \mathcal{X} \) and \( h(\alpha) = g(\alpha X + (1-\alpha) Y) \) for \( X, Y \in \mathcal{X}, 0 \leq \alpha \leq 1 \), then \( g(X) \) is convex in \( X \) if and only if \( h(\alpha) \) is convex in \( \alpha \).

Thus an alternative proof for Lemma 2.2 is to consider

\[
h(\lambda) = (\lambda X + (1-\lambda)Y)^{-1},
\]

from which \( \frac{dh}{d\lambda} = -h(\lambda)(X-Y)h(\lambda) \) and since \( h(\lambda) > 0 \),

\[
\frac{d^2h(\lambda)}{d\lambda^2} = 2h(\lambda)(X-Y)h(\lambda)(X-Y)h(\lambda) \geq 0.
\]

As a consequence, we have Jensen's inequality for the inverse matrix function, namely: If \( Z \) is a random \( p \times p \) symmetric matrix, \( Z > 0, \ EZ = A > 0 \), then

\[
(\text{EZ})^{-1} \leq \text{EZ}^{-1}.
\]
The inequality is strict unless \( P(Z = A) = 1 \).

An interesting example of a concave matrix function which arose in a time series regression problem was considered by Ylvisaker [4].

**Lemma 2.4.** If \( V: p \times p, M: k \times p, k \leq p \), are such that \( V > 0 \), \( M \) is of rank \( k \), then \( G(V) = (MV^{-1}M')^{-1} \) is a concave function of \( V \).

**Proof.** The present proof based on Lemma 2.3 offers an alternative to that of [4]. Define

\[
g(\alpha) = (M[\alpha V + (1-\alpha)W]^{-1}M')^{-1} = (M h(\alpha)M')^{-1},
\]

then

\[
\frac{dg}{d\alpha} = g(\alpha) M h(\alpha)(V-W) h(\alpha) M' g(\alpha),
\]

\[
\frac{d^2g}{d\alpha^2} = 2 g(\alpha) M h(\alpha)(V-W) \frac{1}{2} h^{\frac{1}{2}}(\alpha) [f(\alpha)-I] h^{\frac{1}{2}}(\alpha)(V-W) h(\alpha) M' g(\alpha),
\]

where

\[
f(\alpha) = h^{\frac{1}{2}}(\alpha) M' g(\alpha) M h^{\frac{1}{2}}(\alpha).
\]

It is easily checked that \( f^2(\alpha) = f(\alpha) \), so that \( f(\alpha) \) is idempotent, and hence \( I - f(\alpha) \geq 0 \).

2.2 **Inequalities for Matrices with Real Roots.**

We obtain two inequalities by first exhibiting convexity and then using Jensen's inequality.

**Theorem 2.5.** (Lax [1, p. 85]). If \( \mathcal{X} \) is a linear space of \( p \)-dimensional real matrices such that every \( X \) has only real roots, then \( \lambda_1(X) \) is a convex function, \( \lambda_p(X) \) is a concave function, of \( X \).
Corollary 2.6. If \( Z \) is a real random matrix with only real roots, \( \mathbb{E} Z = A \), then

\[
\mathbb{E} \lambda_1(Z) \geq \lambda_1(A), \quad \mathbb{E} \lambda_p(Z) \leq \lambda_p(A).
\]

Remark. If the distribution of \( Z \) is absolutely continuous with respect to \( \mathbb{P} \)-dimensional (the dimensionality of \( Z \)) Lebesgue measure, an argument similar to that in [3] shows that the inequalities become strict.

2.3 Inequalities for Symmetric Matrices.

Let \( X : p \times p \) be a symmetric matrix, and define for \( k = 1, \ldots, p \),

\[
(i) \quad S_k(X) = \lambda_p(X) + \lambda_{p-1}(X) + \cdots + \lambda_{p-k+1}(X), \\
(ii) \quad T_k(X) = \lambda_1(X) + \lambda_2(X) + \cdots + \lambda_k(X).
\]

For symmetric matrices, one can assert the convexity or concavity of \( S_k(X) \) and \( T_k(X) \), which is a stronger result than Theorem 2.5.

Theorem 2.7 (Fan [1, p. 75]). If \( X \) is a symmetric matrix, then \( S_k(X) \) is a concave function, \( T_k(X) \) is a convex function of \( X \), \( k = 1, 2, \ldots, p \).

Using Jensen's inequality, we have

Corollary 2.8. If \( Z \) is a \( p \times p \) random symmetric matrix, \( \mathbb{E} Z = A \), then

\[
\mathbb{E} S_k(Z) \leq S_k(A), \quad \mathbb{E} T_k(Z) \geq T_k(A), \quad k = 1, \ldots, p.
\]
Remark. Absolute continuity with respect to \( p(p+1)/2 \) dimensional Lebesgue measure yields strict inequalities. The results of [3] are contained in the case \( k = 1 \).

We note that when \( k = p \), \( S_p(X) = T_p(X) = \text{tr } X \), and hence

\[
\sum_{j=1}^{p} \lambda_j(Z) = \sum_{j=1}^{p} \lambda_j(A).
\]

This suggests the use of the Muirhead condition [2, p. 44]: If

\[
\begin{align*}
    u_1 &\geq u_2 \geq \cdots \geq u_p, \\
    v_1 &\geq v_2 \geq \cdots \geq v_p,
\end{align*}
\]

we say that \( u = (u_1, \ldots, u_p) \) majorizes \( v = (v_1, \ldots, v_p) \), and write \( u \triangleright v \).

Another characterization of (2.4) is given by the Schur transformation [1, p. 31], namely: A necessary and sufficient condition that \( u \triangleright v \) is that there exist a doubly-stochastic matrix \( Q = (q_{ij}) \), i.e., \( q_{ij} \geq 0 \), \( \sum_{j} q_{ij} = \sum_{i} q_{ij} = 1 \), \( i, j = 1, \ldots, p \), such that

\[
(2.5) \quad v = uQ.
\]

Moreover, if \( F(t_1, \ldots, t_p) \) satisfies \( \frac{\partial F}{\partial t_i} \geq \frac{\partial F}{\partial t_j} \) whenever \( t_i \geq t_j \), and \( u \triangleright v \), then \( F(u) \geq F(v) \) (Ostrowski, [1, p. 32]). A special case is Karamata’s result [1, p. 30], that if \( u \triangleright v \), and \( g(t) \) is a convex scalar function, then
\[ \sum_{1}^{P} g(u_i) \geq \sum_{1}^{P} g(v_i) . \]

The main point is that Corollary 2.8 asserts that

\[ E\lambda(Z) \equiv (E\lambda_1(Z), \ldots, E\lambda_p(Z)) \geq (\lambda_1(Z), \ldots, \lambda_p(A)) \equiv \lambda(A) . \]

Consequently, there exists a doubly-stochastic matrix \( Q \) such that

\[ (\lambda_1(A), \ldots, \lambda_p(A)) = (E\lambda_1(Z), \ldots, E\lambda_p(Z)) Q \ , \]

and if \( g(t) \) is any convex function, then

\[ \sum_{1}^{P} g(E\lambda_1(Z)) \geq \sum_{1}^{P} g(\lambda_1(A)) . \]

Another situation in which the majorization condition arises naturally is the following. Let \( R = (r_{ij}) \) be a correlation matrix, i.e., \( R > 0, \ r_{ii} = 1, \) and let \( \rho_1 \geq \cdots \geq \rho_p \) denote the characteristic roots of \( R \). Then \( (\rho_1, \ldots, \rho_p) \geq (1, \ldots, 1) \). This is a consequence of the fact that \( \text{tr} R = p = \sum_{1}^{P} \rho_1 \). Suppose \( \rho_1 < 1 \), then \( \sum_{1}^{P} \rho_1 < p \), which is a contradiction. Hence \( \rho_1 \geq 1 \). Suppose \( \rho_1 + \rho_2 < 2 \), then \( \rho_2 < 1 \), so that again \( \sum_{1}^{P} \rho_1 < p \), and hence \( \rho_1 + \rho_2 \geq 2 \). The argument is then repeated.

Thus from (2.5) we obtain the curious result that if \( \rho_1, \ldots, \rho_p \) are the characteristic roots of a correlation matrix, then there exists a doubly stochastic matrix \( Q \) such that

\[ (\rho_1, \ldots, \rho_p) Q = (1, \ldots, 1) . \]
A more general result follows from a theorem of Fan [1, p. 77], namely that if $A$ is a real symmetric matrix, then

$$(\lambda_1(A), \lambda_2(A), \ldots, \lambda_p(A)) \succ (a_{11}, a_{22}, \ldots, a_{pp})$$

As a consequence, if $Z$ is a real random symmetric matrix with $EZ = A$, then

$$E(\lambda_1(Z) + \cdots + \lambda_k(Z)) \geq E(Z_{11} + \cdots + Z_{kk}) = a_{11} + \cdots + a_{kk},$$

$k = 1, 2, \ldots, p - 1$; equality holds for $k = p$.

2.4 Inequalities for Positive Definite Matrices.

Let $s_r(a)$ denote the $r$-th elementary symmetric function of the set $a = \{a_1, \ldots, a_p\}$. For any matrix $X$ define $\text{tr}_j X$ to be the sum of the principal minors of order $j$. It is well known that $s_j(\lambda(X)) = \text{tr}_j X$.

**Theorem 2.9.** (Marcus and Lopes [1, p. 33]). If $X > 0$, then for $1 \leq r \leq j \leq p$,

$$\frac{1}{s_r(\lambda_p(X), \ldots, \lambda_{p-j+1}(X))},$$

and

$$\frac{s_r(\lambda_p(X), \ldots, \lambda_{p-j+1}(X))}{s_{r-1}(\lambda_p(X), \ldots, \lambda_{p-j+1}(X))}$$

are concave functions of $X$.  

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Theorem 2.10. (Oppenheim [1, p. 71]). If $X > 0$,

\[
\left[ \prod_{l}^{r} \lambda_{p-l+1}(X) \right]^{1/r}, \quad r = 1, \ldots, p
\]

is a concave function of $X$.

The inequalities using expectations are then immediate.

Corollary 2.11. Let $Z$ be a random positive definite matrix, $EZ = A$. Then

\[
E s_r^r(\lambda_p(Z), \ldots, \lambda_{p-j+1}(Z)) \leq s_r^r(\lambda_p(A), \ldots, \lambda_{p-j+1}(A)),
\]

$r \leq j \leq p$;

\[
E \left[ \prod_{l}^{r} \lambda_{p-j+1}(Z) \right]^{1/r} \leq \left[ \prod_{l}^{r} \lambda_{p-j+1}(A) \right]^{1/r},
\]

$r = 1, \ldots, p$.

3. Inequalities for Roots of a Determinantal Equation.

Suppose we have two random positive definite matrices $Y$ and $Z$, with $EY = A$, $EZ = B$, and we are interested in the roots of $|Y-\lambda Z| = 0$, or equivalently of $|YZ^{-1}-\lambda I| = 0$. If we use the results of Section 2.2, we obtain

\[
E T_k(YZ^{-1}) = E T_k(\frac{1}{2}Z^{-1}YZ^{-1}\frac{1}{2}) \geq T_k(E (\frac{1}{2}Z^{-1}Y^2))
\]

so that if

(3.1) \[
E \frac{1}{2}Z^{-1}Y^2 = A^{-\frac{1}{2}}B^{-\frac{1}{2}}A^{-\frac{1}{2}}
\]
we have that

\[ E T_k(YZ^{-1}) \geq T_k(AB^{-1}), \quad E S_k(YZ^{-1}) \leq S_k(AB^{-1}) \, . \]

Since the assumption (3.1) is somewhat unnatural, it is of interest to see whether we can still obtain the result with the simpler and more natural assumption \( EY = A \) and \( EZ = B \).

We first need the following Lemma.

**Lemma 3.1.** If \( U \geq V \), then \( T_k(U) \geq T_k(V) \), \( S_k(U) \geq S_k(V) \).

**Proof.** \( U \geq V \) implies that \( \lambda_j(U) \geq \lambda_j(V) \), \( j = 1, \ldots, p \), from which the result follows.

**Theorem 3.2.** If \( Y \) and \( Z \) are independent random positive definite matrices, \( EY = A \), \( EZ = B \), then

\[ E T_k(YZ^{-1}) \geq T_k(AB^{-1}) \, , \quad k = 1, \ldots, p \, . \]

**Proof.** Using conditional expectation and convexity,

\[ E T_k(YZ^{-1}Y^2) = E_Y[E_{Z} T_k(Y^2Z^{-1}Y^2)|Y] \]

\[ \leq E_Y T_k[E_{Z}(Y^2Z^{-1}Y^2)|Y] = E_Y T_k(Y^2(EZ^{-1})Y^2) \, . \]

But from Jensen's inequality, \( EZ^{-1} \geq (EZ)^{-1} = B^{-1} \), and hence

\[ T_k(Y^2(EZ^{-1})Y^2) \geq T_k(Y^2B^{-1}Y^2) = T_k(B^{-1}Y) \, , \]

from Lemma 3.1. The result then follows from Corollary 2.8.
Remark. The method of proof does not carry through for $S_k(YZ^{-1})$. It is clear that $ES_k(YZ^{-1}) \leq S_k(AEZ^{-1})$, but $S_k(AEZ^{-1}) \geq S_k(A^2)$.

Consider a random matrix $Z > 0$, $EZ = A$, with the partition

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$ 

The roots of the determinantal equations

$$|Z_{12}Z_{22}^{-1}Z_{11} - \rho^2 Z_{11}| = 0, \quad |A_{12}A_{22}^{-1}A_{11} - \sigma^2 A_{11}| = 0,$$

are the canonical correlations. We now show that

$$(3.3) \quad E T_k(Z^{-1}_{11}Z_{12}^{-1}Z_{22}^{-1}Z_{21}) \geq T_k(A^{-1}_{11}A_{12}^{-1}A_{22}A_{21}).$$

Using Corollary 2.8 for the conditional expectation, we have

$$E_{Z_{11}} E_{Z_{12}} E_{Z_{22}} \{ T_k(Z^{-1}_{12}Z^{-1}_{22}Z^{-1}_{12}Z^{-1}_{22}) | Z_{11}, Z_{12} \}$$

$$\geq E_{Z_{11}} \{ E_{Z_{12}} T_k(Z^{-1}_{12}A^{-1}_{12}Z^{-1}_{22}A^{-1}_{22}) | Z_{11} \}$$

$$\geq E_{Z_{11}} T_k(Z^{-1}_{11} E(Z_{12}A^{-1}_{12}Z_{22}A^{-1}_{22}) Z^{-1}_{11}).$$

But $Z_{12}A^{-1}_{12}Z_{22}$ is a convex (matrix) function of $Z_{12}$, so that $E(Z_{12}A^{-1}_{12}Z_{22}) \geq A_{12}A^{-1}_{12}A_{22}$, and by Lemma 3.1,

$$E_{Z_{11}} T_k(Z^{-1}_{11} E(Z_{12}A^{-1}_{12}Z_{22}A^{-1}_{22}) Z^{-1}_{11}) \geq E_{Z_{11}} T_k(Z^{-1}_{11} A_{12}A^{-1}_{12}A_{22}).$$
The proof is completed by another application of Corollary 2.8.

As in the case of Theorem 3.2, the proof does not carry over to the $S_k$ function.

4. **Median Bias of the Roots of a Random Matrix.**

In this section we assume that $Z$ is a symmetric random matrix, with an absolutely continuous distribution which is symmetric about a matrix $A$, i.e.,

\[(4.1) \quad P(\text{tr } C(Z-A) \geq 0) = P(\text{tr } C(Z-A) \leq 0) = \frac{1}{2}, \quad \text{for all } C \neq 0.\]

**Theorem 4.1.** If $Z$ is a random symmetric matrix, $EZ = A$, and the distribution of $Z$ is symmetric about $A$, then

\[P(S_k(Z) \leq S_k(A)) \geq \frac{1}{2},\]

\[P(T_k(Z) \geq T_k(A)) \geq \frac{1}{2}, \quad k = 1, \ldots, p.\]

**Proof.** We use a characterization of Fan [1, p. 77],

\[S_k(Z) = \min_{\Omega} \text{tr } GZG', \quad T_k(Z) = \max_{\Omega} \text{tr } GZG',\]

\[\Omega = \{G : k \times p \mid GG' = I\}.\]

Let $S_k(A) = \min_{\Omega} \text{tr } GAG' = \text{tr } \Gamma A \Gamma'$. Then, for all $G \in \Omega$,

\[P(S_k(Z) \leq S_k(A)) \geq P(\text{tr } GZG' \leq \text{tr } \Gamma A \Gamma').\]
Choose \( G = \Gamma \), use the definition of symmetry, \((4.1)\), and the result follows.

The result for \( T_k(Z) \) is proved similarly. ||

Remark. Theorem 4.1 generalizes the result of [3] in which the case \( k = 1 \) was considered.

For the case of two matrices \( Y \) and \( Z \), \( EY = A \), \( EZ = B \), \( Y \) symmetric about \( A \), \( Z \) symmetric about \( B \), we have for all \( G \in \Omega \),

\[
P(S_k(YZ^{-1}) \leq S_k(AB^{-1})) \geq P(\text{tr } GZ^{-\frac{1}{2}}YZ^{-\frac{1}{2}}G' \leq \text{tr } GB^{-\frac{1}{2}}AB^{-\frac{1}{2}}G') ,
\]

where \( \min \text{tr } GB^{-\frac{1}{2}}AB^{-\frac{1}{2}}G' = GB^{-\frac{1}{2}}AB^{-\frac{1}{2}}G' \). Choose \( G = \Gamma \), and consider the conditional probability for \( Z = B \). Since

\[
P(\text{tr } GB^{-\frac{1}{2}}YB^{-\frac{1}{2}} \leq \text{tr } GB^{-\frac{1}{2}}AB^{-\frac{1}{2}}G') \geq \frac{1}{2} ,
\]

the unconditional probability is \( \geq \frac{1}{2} \).

We obtain the same inequality if we assume \( EZ^{-\frac{1}{2}}YZ^{-\frac{1}{2}} = B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \), and \( Z^{-\frac{1}{2}}YZ^{-\frac{1}{2}} \) is symmetric about \( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \).

For canonical correlations we obtain a similar result:

\[
P(S_k(Z_{11}^{-1}Z_{12}^{-1}Z_{22}^{-1}Z_{21}^{-1}) \leq S_k(A^{-1}A_{11}^{-1}A_{12}^{-1}A_{22}^{-1}A_{21}^{-1})) \geq \frac{1}{2} ,
\]

\[
P(T_k(Z_{11}^{-1}Z_{12}^{-1}Z_{22}^{-1}Z_{21}^{-1}) \geq T_k(A^{-1}A_{11}^{-1}A_{12}^{-1}A_{22}^{-1}A_{21}^{-1})) \geq \frac{1}{2} , ~ k = 1,\ldots,p .
\]

By using conditional probability with \( Z_{11} = A_{11} \), \( Z_{12} = B_{12} \), the proof follows that of Theorem 4.1.
REFERENCES


