DUAL PAIRS OF STOPPING TIMES FOR RANDOM WALK

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PRISCILLA GREENWOOD and MOSHE SHAKED

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Summary. A definition of duality for pairs of stopping times of any random walk is motivated by the duality relation of ascending and descending ladder epochs $N, \overline{N}$ of random walk in $\mathbb{R}^1$. Dual pairs share several of the properties of the pair $N, \overline{N}$.

I. Introduction. Sparre-Andersen [8] introduced a combinatorial method for studying the fluctuations of a random walk. A main idea was that certain sets of paths, viewed with the direction of time reversed, are more easily described or counted, while a set of paths viewed in either direction has the same probability. The relation between a set of paths and the reversed set is called duality. Lindley [6] found that the virtual waiting time in a one-server queue has the same distribution as the maximum of a random walk because of the duality relation. Spitzer used duality to obtain a transform formula for the maximum of a random walk [9] and to give a probabilistic solution of the Wiener-Hopf integral equation on a half-line [10]. The duality relation is the key to Spitzer and Pollaczek's factorization formula for a distribution on $\mathbb{R}^1$ (see, e.g., Feller [1]). The duality relation which was exploited in these works and in the large literature which followed can be stated in terms of the first ascending and first descending ladder epochs of a random
walk \( S_n \) in \( \mathbb{R}^1 \),

\[
N = \min(n > 0 : S_n > 0), \quad \bar{N} = \min(n > 0 : S_n \leq 0).
\]

A time \( n \) is an ascending ladder epoch if \( S_n \) reaches a new maximum at \( n \), which is to say that \( n \) is obtained by some number of repetitions of \( N \). The duality relation is: the set of paths for which \( n \) is an ascending ladder epoch, viewed in reverse, is the set of paths such that \( n < \bar{N} \).

In this paper a definition of duality for a pair of stopping times of any random walk is stated, and some properties of dual pairs are explored. In a following paper [4] duality for particular pairs of stopping times for multivariate random walks is used to begin a fluctuation theory for these processes, somewhat analogous to the well-known theory for random walk in \( \mathbb{R}^1 \). The multivariate fluctuation theory will have applications for certain storage and queueing systems.

II. Definitions and Notations. Let \( S_n = \sum_{i=1}^{n} X_i \), \( S_0 = 0 \), where \( X_i, i = 1,2,3 \ldots \) is a sequence of independent identically distributed random elements on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). For most of what follows the range of the \( X_i \), denoted by \( R \), may be any topological group. We have in mind \( \mathbb{R}^d \). The words 'for each \( n \)' will mean for each non-negative integer. For each \( n \), \( \mathcal{F}_n \) is the sub \( \sigma \)-field of \( \mathcal{F} \) generated by \((S_0, S_1, \ldots, S_n)\).

For each \( n \) let \( r_n \) denote the map on \( \mathcal{F}_n \) induced by mapping all subsets of \((X_1, \ldots, X_n)\) into the corresponding subsets of
\((X_1, \ldots, X_n)\). We call \(r_n\) time reversal from \(n\). The mapping \(r_n\) preserves set operations and measure, since \(X_1, \ldots, X_n\) are exchangeable.

For each \(k\) we denote by \(\theta_k\) the map on \(\Omega\) induced by
\[(X_{k+1}, X_{k+2}, \ldots) \to (X_1, X_2, \ldots)\]. If, for instance, \(\tau\) is a stopping time for \(S_n\), \(\tau \circ \theta_k\) is the same stopping time for \(S_{n+k} - S_k\).

Let \(\tau\) be a stopping time for \(S_n\), \(\tau_0 = 0, \tau_1 = \tau, \tau_2 = \tau_1 + \tau'\) where \(\tau' = \tau \circ \theta_1\), and so on. We denote by \(\mathcal{M}_{\tau}\) the random set
\[(\tau_0, \tau_1, \tau_2, \ldots)\]. The set \(\mathcal{M}_{\tau}\) is almost surely finite if \(P(\tau < \infty) < 1\). By an occurrence of \(\tau\) at \(n\) we mean an \(n \in \mathcal{M}_{\tau}\). By \(\mathcal{M}_{\tau} \circ \theta_k\) we will mean the random set \(\mathcal{M}_{\tau}\) for \(\theta_k(\omega)\).

Let \(\tau\) and \(\eta\) be stopping times for \(S_n\). We say that \(\tau\) is dual to \(\eta\) if
\[(1) \quad r_n(n \in \mathcal{M}_{\tau}) = (n < \eta) \quad \text{for each } n.\]

If \(\tau, \tau'\) are dual to \(\eta\), then \(\mathcal{M}_{\tau} = \mathcal{M}_{\tau'}\), and \(\tau = \tau'\). Also \(\tau\) is dual to at most one \(\eta\). Since (1) is not symmetric in \(\tau\) and \(\eta\), it is not evident that duality is reflexive, that if (1) holds then \(\eta\) is also dual to \(\tau\). Accordingly, we must preserve the order of \(\tau\) and \(\eta\) in duality statements until it is proved in Theorem 1 that duality is reflexive.

For each \(n\) let \(L_{\tau, n} = \max(1 \leq i < n : i \in \mathcal{M}_{\tau})\). The notation \(\overset{d}{=}\) means that the two random variables have the same distribution.

III. Reflexivity and Related Properties. If \(\tau\) is dual to \(\eta\), then both \(\tau\) and \(\eta\) have the property that a backward shift from any occurrence leaves us no further from the next occurrence than the length
of the shift. Typically, stopping times do not have this property. Often an easy computation will show that a certain stopping time is not a member of a dual pair. Some examples are in Section V.

**Lemma 1.** If \( \tau \) is dual to \( \eta \), then

\[
\begin{align*}
(2) \quad n \in \mathcal{M}_\tau \implies \tau \circ \theta_{n-j} \leq j, & \quad j = 1, \ldots, n, \text{ for each } n > 0, \\
(3) \quad n \in \mathcal{M}_\eta \implies \eta \circ \theta_{n-j} \leq j, & \quad j = 1, \ldots, n, \text{ for each } n > 0.
\end{align*}
\]

**Proof.** If \( n \in \mathcal{M}_\tau \), then \( j \leq n < \eta \) for the step sequence

\[
r_n(X_1, \ldots, X_n) = (X_n, \ldots, X_1),
\]

and \( r_j(j < \eta) = (j \in \mathcal{M}_\tau) \) for this sequence. Now \( r_j \) takes \( X_n, \ldots, X_{n-j+1} \) to \( X_{n-j+1}, \ldots, X_n \), so \( j \in \mathcal{M}_\tau \) for the sequence \( X_{n-j+1}, \ldots, X_n \). Consequently, \( \tau \circ \theta_{n-j} \leq j \).

To prove (3) we will show that if \( \eta \circ \theta_{n-j} > j \) for some \( 1 \leq j \leq n - 1 \) then \( n \notin \mathcal{M}_\eta \). Note that \( \omega \in \Omega \) and \( n \) are fixed. Let \( L_{\eta, n-1} = k \). First suppose \( n - j > k \). Then \( \eta \circ \theta_k > n - k \) and we wish to show that \( > \) holds. From the assumption \( \eta \circ \theta_{n-j} > j \), by duality,

\[
\omega \in r_n(j \in \mathcal{M}_\tau).
\]

Also, \( \eta \circ \theta_k > n - k > (n-j) - k \), so

\[
\omega \in r_{n-j}(n-k-j \in \mathcal{M}_\tau).
\]

Together, (4) and (5) imply \( \omega \in \tau_n(n-k \in \mathcal{M}_\tau) \). By duality, \( \eta \circ \theta_k > n - k \).

It is not possible that \( n - j \leq k \). Otherwise let

\[
\theta_0 = \min(i \geq n-j : i \in \mathcal{M}_\eta), \quad k_0 = L_{\eta, n-n-1}.
\]

Write \( \eta \circ \theta_{n-j} > j \) as
\[ \eta \circ \theta_{n_0 - (n_0 - n - j)} > j > j - (n - n_0). \] Use \( n_0, k_0 \) in the argument for the first case to see \( n_0 \notin \mathcal{M}_\eta \). But this is a contradiction.

The apparently stronger statement, \( j \in \mathcal{M}_\tau \circ \theta_{n - j} \), \( j = 1, \ldots, n \) is in fact equivalent to \( \tau \circ \theta_{n - j} \leq j, j = 1, \ldots, n \) in (2) as can be seen, for instance, by induction on \( j \) for fixed \( n \).

Statement (6) in Theorem 1 is the basis for an arcsine law for certain dual stopping times, Corollary 7, and for a splitting of geometrically stopped random walk into independent terms, the part before the last occurrence of a \( \tau \) and the rest, Corollary 5, (18). The proof of (6) also shows that duality is reflexive.

**Theorem 1.** If \( \tau \) is dual to \( \eta \), then \( \eta \) is dual to \( \tau \), and for each \( n \)

\[ (6) \quad L_{\tau, n} \overset{d}{=} n - L_{\eta, n} \quad \text{and} \quad \sum_{i=1}^{L_{\tau, n}} X_i \overset{d}{=} \sum_{i=L_{\eta, n} + 1}^{n} X_i. \]

**Proof.** If \( \tau \) is dual to \( \eta \), then (1) and (3) hold. For fixed \( k \) consider the \( \omega \)-set.

\[ (7) \quad (L_{\tau, n} = k) = (k = \max i \leq n : i \in \mathcal{M}_\tau). \]

Condition (1) implies that (7) is the same set as

\[ (k = \max i \leq n : \omega \in r_i(i < \eta)). \]

Apply \( r_n \) to (7),

\[ (8) \quad r_n(L_{\tau, n} = k) = r_n(k = \max i \leq n : \omega \in r_i(i < \eta)) \]

\[ = (n - k = \min i \leq n : \eta \circ \theta_i > n - i). \]
Now consider $L_{\eta,n}$. For each $k$ the $\omega$-set

$$(9) \quad (L_{\eta,n} = k) = (k = \min i : i \in \mathcal{M}_\eta \text{ and } \eta \circ \theta_i > n-i).$$

For each $\omega$, $k \leq n$. Using (3) we will justify dropping the condition $i \in \mathcal{M}_\eta$ from (9).

Fix $\omega$. Let $k_1 = \min (i : \eta \circ \theta_i > n-i, i \in \mathcal{M}_\eta)$,

$$k_2 = \min (i : \eta \circ \theta_i > n-i).$$

Then $k_2 \leq k_1 \leq n$.

For this $\omega$, $k_1 \in \mathcal{M}_\eta$. By (3),

$$\tau \circ \theta_{k_1-j} \leq j \leq n - (k_1-j), \text{ so } k_2 \neq k_1 - j, \text{ for } j = 1, \ldots, k,$$

and

$$(10) \quad (L_{\eta,n} = k) = (k = \min i : \eta \circ \theta_i > n-i).$$

From (8) and (10) we see that $r_n$ maps $(L_{\tau,n} = k)$ into $(L_{\eta,n} = n-k)$. The sum on the left in (6) is mapped by $r_n$ to the sum on the right, and $r_n$ preserves measure. In particular, $(n \in \mathcal{M}_\eta)$

$$(L_{\eta,n} = n) \text{ is mapped by } r_n \text{ to } (L_{\tau,n} = 0) = (n < \tau), \text{ so } \eta \text{ is dual to } \tau.$$

Here is a way to construct new dual pairs from known ones.

**Corollary 1.** Suppose that $\gamma, \bar{\gamma}$ are dual stopping times and that $\eta, \bar{\eta}$ are dual stopping times. Then $\tau = \min n \in \mathcal{M}_\gamma \cap M_\eta$ is dual to $\bar{\tau} = \bar{\gamma} \wedge \bar{\eta}$.
Proof. From its definition

\[ \mathcal{M}_\tau = \mathcal{M}_\gamma \cap \mathcal{M}_\eta. \]

Then

\[ r_n (n \in \mathcal{M}_\tau) = (n < \min i \in \mathcal{M}_\gamma) \cap (n < \min i \in \mathcal{M}_\eta) \]

\[ = (n < \min i \in \mathcal{M}_\gamma - \mathcal{M}_\eta) \cup (n < \mathcal{M}_\eta - \mathcal{M}_\gamma) = (n < \gamma \cup \eta). \]

Some examples are constructed using Corollary 1 in Section V.

IV. A Spitzer-Pollaczek factorization and consequences. A formula like (12) for a given distribution F, if the distributions \( H_\tau \) and \( H_\eta \) have disjoint supports, is sometimes called a Wiener-Hopf factorization. Pollaczek [7] found such a formula in the context of queueing theory, by analytic methods. Spitzer found (14) in developing combinatorial fluctuation theory for random walk in \( \mathbb{R}^1 \). The extensions and variations form a large literature, essentially related to the dual pair \( N, \overline{N} \) as defined in Section I.

It is useful to include the possibility that the step distribution \( F \) is a defective distribution. If \( P(X_1 \in \mathbb{R}) = u < 1 \), add a state \( \Delta \) to the range of the random walk. Define \( S_n = \Delta \) if \( X_i \notin \mathbb{R} \), some \( i \leq n \). This device kills the process \( S_n \) at \( T + 1 \) where \( T \) is an independent, geometrically distributed time, \( P(T \geq n) = u^n \). The proofs of Lemma 2 and Theorem 2, adapted from Feller [1], XII.3, are valid for defective \( F \) with this interpretation.

We use \( dx \) to denote a measurable set in the range \( \mathbb{R} \).

**Lemma 2.** If \( \eta, \tau \) are dual stopping times, and

\[ H_\eta (dx) = P(S_\eta \in dx), C_\tau (dx) = \sum_{n=0}^{\infty} P(S_n \in dx, n < \tau), \]

then
(11) \[ G_\tau = \sum_{n=0}^{\infty} H^{n*}_\eta. \]

**Proof.** The map \( r_n \) takes the \( \omega \)-set \( (S_n \in dx) \) to itself and preserves intersections as well as measure. We have for each \( n \),

\[ P(S_n \in dx, n = \eta_m) = P(S_n \in dx, n < \tau) \]

Sum on \( n \),

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(S_n \in dx, n = \eta_m) = G_\tau(dx). \]

The left side is \[ \sum_{m=0}^{\infty} P(S_m \in dx) = \sum_{m=0}^{\infty} H^{n*}_\eta(dx). \]

**Theorem 2.** If \( \eta \) and \( \tau \) are dual stopping times, then

(12) \[ \delta - F = (\delta - H_\tau) * (\delta - H_\eta) \]

where \( H_\tau, H_\eta \) are the distributions of \( S_\tau \) and \( S_\eta \), and \( \delta \) is a unit mass at 0.

**Proof.** First assume \( F \) is defective. Then all sums which follow are finite.

\[ H_n(dx) = P(S_n \in dx, n = \eta), H_\eta = \sum_{n=1}^{\infty} H_n \]

\[ \overline{H}_n(dx) = P(S_n \in dx, n = \tau), H_\tau = \sum_{n=1}^{\infty} \overline{H}_n \]

\[ G_n(dx) = P(S_n \in dx, n < \tau), G_\tau = \sum_{n=0}^{\infty} G_n. \]

The distribution of \( S_{n+1} \) restricted to the event \( n < \tau \) can be
expressed by either the right or the left side of the following which are therefore equal:

\[ G_n \ast F = G_{n+1} + H_{n+1}. \]

Summation on \( n \) produces

\[ G_\tau \ast F = G_\tau + H_\tau - \delta, \]

i.e.,

\[ G_\tau \ast (\delta - F) = \delta - H_\tau. \]

Now convolve with \( \delta - H_\eta \), which according to Lemma 2 is a convolution inverse of \( G_\tau \) when \( G_\tau \) is finite, to obtain (12).

If \( F \) is proper, let \( S_n = \Delta \) if \( n \geq T + 1 \) where \( T \) is independent of the random walk and \( P(T \geq n) = u^n, \ 0 < u < 1. \) By the above argument

\[(13) \quad \delta - uF = (\delta - H_\tau, u) \ast (\delta - H_\eta, u) \]

where \( H_{\tau, u}(dx) = P(S_\tau \in dx, \tau \leq T) = \sum_{n=0}^{\infty} u^n P(S_n \in dx, \tau = n), \)

similarly \( H_{\eta, u}. \) Let \( u \to 1 \) to obtain (12).

Applying transforms to (13) gives

\[(14) \quad (1-u\phi(\xi)) = (1-\psi_\tau(u, \xi))(1-\psi_\eta(u, \xi)) \]

where \( \phi(\xi) = \text{E} \exp i\xi \cdot X, \ \psi_\tau(u, \xi) = \text{E} u^\tau \exp i\xi \cdot S_\tau, \ \psi_\eta(u, \xi) = \text{E} u^\eta \exp i\xi \cdot S_\eta. \)

Corollary 2. Let \( g_\eta(u) = \text{E} u^\eta, \ g_\tau(u) = \text{E} u^\tau. \) Then for \( 0 \leq u \leq 1, \)

\[(15) \quad 1 - u = (1-g_\eta(u))(1-g_\tau(u)). \]
Proof. Let $\xi = 0$ in (14).

A number of properties of the pair $N, \bar{N}$ follow from identities (13) or (14). Such properties are shared by any dual pair $\tau, \eta$. The following three corollaries list some of these. Relation (18) also uses Theorem 1.

**Corollary 3.** Either $\tau, \eta$ are both proper and $E\eta = E\tau = \infty$, or $\eta$ is defective and $E\tau = 1/(1 - P(\eta < \infty))$ or $\tau$ is defective and $E\eta = 1/(1 - P(\tau < \infty))$.

**Proof.** From (15), $(1 - g_\tau(u))/(1 - u) = (1 - g_\eta(u))^{-1}$. Let $u \to 1$.

**Corollary 4.** If $S_\eta$ is random walk in $\mathbb{R}^1$ and if $E\tau, E\eta$ are finite, then $E X_1 = 0$ and $E X_1^2 = -2 E \tau E \eta$.

**Proof.** See Feller XVIII.4, Lemma 3.

**Corollary 5.** The following are equivalent independent decompositions of the distribution of $S_\tau$:

(16) \[ S_\tau \overset{d}{=} Y_\tau(T_\tau) + Y_\eta(T_\eta) \]

(17) \[ S_\tau \overset{d}{=} S(L_\tau, T) + S(L_\eta, T) \]

(18) \[ S_\tau \overset{d}{=} \sum_{i=1}^{T_\tau} X_i + \sum_{i=L_\tau, T+1}^{T} X_i \]

In (16), $T_\tau, T_\eta$ are independent, $P(T_\tau \geq n) = P(\tau \leq T)^n$, $P(T_\eta \geq n) = P(\eta \leq T)^n$, and $Y_\tau, Y_\eta$ are random walks with step distributions $P(S_\tau \in dx | \tau \leq T), P(S_\eta \in dx | \eta \leq T)$.
Proof. Relation (16) is equivalent to (13) as for $N, \tilde{N}$ in Greenwood [2], equation 5.4. Relation (17) is also equivalent as in Greenwood [3]. Relation (18) differs from (17) only in the second term on the right. Since the terms are added independently, the replacement is justified if $S(L_{T,1}) \sim \sum_{i=L_{T,1}+1}^{T} X_i$. This follows from (6) by a routine conditioning argument, since $T$ is independent of the random walk.

The factorization (12) can be repeated. Consider the random walk $Y_n$ which is $S_n$ restricted to the time set $\mathcal{H}_n$, i.e., $Y_n = S_{\tau_n}$. The killing time for $Y_n$, if $F$ is defective, is $T_Y$. If $\gamma_1, \gamma_2$ are dual stopping times for $Y_n$, (12) is

$$\delta - H_\tau = (\delta - H_\tau, \gamma_1) \ast (\delta - H_\tau, \gamma_2)$$

where

$$H_{\tau, \gamma_1}(dx) = P(y_{\gamma_1} \in dx, \gamma_1 < T_\tau)$$

$$= P(S(\tau_{\gamma_1}) \in dx, \tau_{\gamma_1} < T).$$

Repeated factorization is used in [4] to obtain $F$ in terms of distributions concentrated on each element of a partition of $R^2$ into convex cones.

If $\tau$ is defective, the random walk can be evaluated at the last occurrence of $\tau$. Let $L_\tau = \max n \in \mathcal{H}_n$. Then

$$S(L_\tau) = \sum_{\tau_i \in \mathcal{H}_\tau} S(\tau_i+1) - S(\tau_i).$$

The terms of this sum are independent and distributed like $S_\tau$. 

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The distribution of $S(L_\tau)$ is $P(\tau = \infty) \sum_{n=0}^{\infty} H_\tau^n$, where $H_\tau$ is the distribution of $S_\tau$. If $\eta$ is dual to $\tau$, this distribution can be written in terms of $\eta$, using Lemma 2 and Corollary 3.

**Corollary 6.** If $\tau$ is defective and has dual $\eta$, the distribution of $S(L_\tau)$ is $(E\eta)^{-1} G_\eta$ where $G_\eta(dx) = \sum_{n=0}^{\infty} P(S_n \in dx, \eta < \eta)$.

Suppose now that $\tau$ is proper. Recall the notation $L_{\tau,n} = \max(i \leq n : i \in \Lambda_{\tau,n})$. It is known from renewal theory that $(n-L_{\tau,n})/n$ has a limiting distribution if and only if the distribution of $\tau$ is regularly varying, that is, $P(\tau > n) \sim n^{-\alpha} L(n)$, $0 < \alpha < 1$, where $L(nn)/L(n) \rightarrow 1$ as $n \rightarrow \infty$ (see, e.g., Feller [1], XIV.3). Then the limiting distribution has arcsine density

$$q_\alpha(x) = \frac{\sin \pi \alpha}{\pi} x^{-\alpha}(1-x)^{\alpha-1}, \quad 0 \leq x \leq 1.$$ 

If $\tau$ has a dual, the same kind of statement can be made about $L_{\tau,n}/n$.

**Corollary 7.** If $\tau$ and $\eta$ are dual and the distribution of either has a regularly varying tail, then $L_{\tau,n}/n$ has a limiting arcsine distribution.

**Proof.** If one of $\tau, \eta$ has a regularly varying tail with exponent $\alpha$, the other is regularly varying with exponent $1 - \alpha$. This follows from (15) and a Tauberian theorem, e.g., Feller [1], XIII.5, Theorem 5.

By the result quoted above, $(n-L_{\eta,n})/n$ has a limiting arcsine distribution. But $n - L_{\eta,n} \overset{d}{=} L_{\tau,n}$, according to (6).

Some properties of the pair $N, \overline{N}$ are not shared by all dual pairs of stopping times. For instance if $S_n$ is in $R^d$, Baxter's
equation, (21) below, is true only when the supports of \( S_\tau, S_\eta \) lie in complementary half-spaces. Another form of (14), obtained by expanding logarithms is

\[
\sum_{n=1}^{\infty} \frac{u^n}{n} \phi^n/n = \sum_{n=1}^{\infty} \frac{\psi^n}{n} + \sum_{n=1}^{\infty} \frac{\psi^n}{n}.
\]

(20)

Now, \( \phi^n(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \mathbb{P}^n(dx) \),

and \( \psi^n(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \mathbb{H}^n(dx) \), where \( \cdot \) is \( \tau \) or \( \eta \).

If \( \mathbb{H}_\tau, \mathbb{H}_\eta \) have support on the convex cones, with vertex at \( 0, C_\tau, C_\eta \), respectively, so do \( \mathbb{H}_\tau^n, \mathbb{H}_\eta^n \) for each \( n \). By the uniqueness theorem for inverses of Fourier transforms of measures,

\[
\log(1-\psi(s,\xi))^{-1} = \sum_{n=1}^{\infty} \frac{\psi^n}{n} = \sum_{n=1}^{\infty} \frac{u^n}{n} \int_{C_\tau \cup C_\eta} e^{i\xi \cdot x} \mathbb{P}^n(dx),
\]

\( \cdot \) is \( \tau \) or \( \eta \).

If \( F \) has support in a convex cone with vertex at \( 0 \), then \( C_\tau \cup C_\eta \) is this cone. In general \( C_\tau \cup C_\eta = \mathbb{R}^d \) and since \( C_\tau, C_\eta \) are both convex, each must be a half-space.

Kingman [5] observed that Baxter's equation (21) cannot be generalized in any essential way. The restriction that \( C_\tau \) and \( C_\eta \) lie in complementary half-spaces is an example of Kingman's more encompassing statement which is derived from an operator identity of Baxter.

By allowing more than two factors in (14) hence more than two cones, relation (21) is generalized in [4].

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V. **Examples.** Let $S_n$ be random walk in $\mathbb{R}^1$.

1. Let $b > 0$. Let $\eta = \min(n : X_n > b)$. Let $\tau = 1$ if $X_1 \leq b$, $\tau = \infty$ if $X_1 > b$. Then $(n \in \mathcal{M}_\eta) = (X_n > b)$, $r_n (n \in \mathcal{M}_\eta) = (X_1 > b) = (n < \tau)$ for each $n > 0$, so $\tau$ and $\eta$ are dual.

2. Let $\eta$ and $\tau$ be as in example 1. Then $\tau$ and $\eta$ are dual, also $N$ and $\overline{N}$ are dual. By Corollary 1, $\gamma = \min n \in \mathcal{J}_N, \tau \cap \mathcal{M}_N$ is dual to $\overline{\gamma} = \eta \wedge \overline{N}$. The stopping time $\overline{\gamma}$ is the first time $S_n$ is $\leq 0$ or has a step $> b$. Its dual is $\gamma = \min(n : n = N, X_1 \leq b, i = 1, \ldots, N)$, $\gamma = \infty$ otherwise.

3. With the same $\eta$ and $\tau$, Corollary 1 gives the additional dual pairs $\beta = \min n \in \mathcal{M}_\tau \cap \mathcal{J}_N$, $\overline{\beta} = \eta \wedge N$ and $\alpha = \min n \in \mathcal{J}_N \cap \mathcal{M}_N$, $\overline{\alpha} = \tau \wedge \overline{N}$. The remaining possibility is not interesting.

4. Let $\eta = N$ restricted to the $\omega$-set $(S_i \geq S_{i-N} - b, i = 0, 1, \ldots, N)$. Let $N_b = \min(n : \text{there exists } i \leq n \text{ such that } S_i - S_i \geq b)$, where $m_i = \max \{j < i : S_m \text{ is minimal over } (S_m, \ldots, S_n)\}$. Let $\tau = N \wedge N_b$. Then $r_n (n \in \mathcal{M}_\eta) = (n < \tau)$. The verification is notationally, but not conceptually difficult.

The following are some stopping times without duals.

5. Let $\eta > 1$ a.s., e.g., $\eta = \min(n > 0 : S_n = 0)$ for random walk with steps of 1 or $-1$. If $\eta$ has a dual $\tau$, then $(\eta > 1) = (\tau = 1)$. But $\tau = 1$ a.s., with duality, implies $n < \eta$ a.s. for all $n$, so $\eta = \infty$ a.s.
6. Let $N_b = \min(n : S_n > b)$. It happens, with positive probability, that $n \in \mathcal{M}_{N_b}$ but $X_n < b$ so that $N_b \circ \theta_{n-1} > 1$. By Lemma 1, $N_b$ does not have a dual.

Examples for random walks in $\mathbb{R}^2$ are in [4].

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