THE ASYMPTOTIC DISTRIBUTION OF COMMONALITY COMPONENTS

BY

INGRAM OLKIN and LARRY V. HEDGES

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1. Introduction.

Analytic studies in several areas of education are frequently concerned with the problem of assessing the individual and collective effects of several variables on a single dependent variable. One example is the problem of assessing school effects and social background effects on educational achievement. The report, *Equality of Educational Opportunity* (abbreviated EEO), by Coleman, Campbell, Hobson, McPartland, Mood, Weinfeld and York (1966), was addressed primarily to the assessment of the contribution of school effects and environmental effects to student achievement. The controversy generated by the EEO report suggested that there was no generally accepted method of assessing the relative importance of predictor variables when they are correlated.

The school-effects controversy produced a flurry of methodological papers dealing with methods of assessing or describing the relative contribution of multicollinear predictor variables. Pedhazur (1975) in a review of the literature on analytic methods in school-effects studies classified methods into two broad categories. One of those categories can be described as the "partitioning of variance" due to predictor variables. The method of variance-partitioning that is most commonly advocated is called *commonality analysis*. It was developed and used extensively in the U.S. Office of Education reanalyses of the data from the EEO report (Maveske, 1970; Maveske, Wisler, Beaton, Weinfeld, Cohen, Okada, Proshak and Tabler, 1972).

Commonality analysis has been advocated in the literature (Mood, 1971) as a useful tool for the development of learning models and in the search
for reasonably specific indicators of input variables for these models. It has been applied extensively to school-effects studies (Mood, 1969; Mayeske, Wisler, Beaton, Weinfeld, Cohen, Okada, Proshak and Tabler, 1972; Pedhazur, 1973). It has also been used in other areas of educational research such as research on teaching (Dunkin, 1978). This widespread use resulted in the inclusion of commonality analysis into at least one textbook on regression analysis in the social sciences (Kerlinger and Pedhazur, 1973). Although first used in the social sciences, the technique was developed independently by Newton and Spurrell (1969a,b) who called it "elements analysis" and demonstrated its application to problems in industrial prediction. They were concerned mainly with selecting the best set of predictor variables without running all $2^{k-1}$ possible regressions. Newton and Spurrell suggest essentially the same rules as Mood (1971) for selection of predictors; namely, the selection of independent variables should be made on the basis of large unique components and small commonalities.

2. **The Method of Commonality Analysis.**

The objective of commonality analysis is to partition the variance accounted for in the dependent variable into portions attributable (i) uniquely to independent variables and (ii) to various combinations of independent variables. More precisely, suppose that two variables $x_1$ and $x_2$ are used to predict a third $x_0$. Then $\rho^2_{0,12}$ can be partitioned into three components:
\[ U_1 = \text{the unique contribution of } x_1 \text{ to } \rho_{0.12}^2 , \]

\[ U_2 = \text{the unique contribution of } x_2 \text{ to } \rho_{0.12}^2 , \]

\[ C_{12} = \text{the common contribution of } x_1 \text{ and } x_2 \text{ to } \rho_{0.12}^2 . \]

The common contribution is called the commonality of \( x_1 \) and \( x_2 \). The above definitions lead to the following system of equations:

\[ \rho_{0.12}^2 = U_1 + U_2 + C_{12} , \]

\[ \rho_{01}^2 = U_1 + C_{12} , \]

\[ \rho_{02}^2 = U_2 + C_{12} , \]

whose solutions are:

\[ U_1 = \rho_{0.12}^2 - \rho_{02}^2 , \]

\[ U_2 = \rho_{0.12}^2 - \rho_{01}^2 , \]

\[ C_{12} = \rho_{01}^2 + \rho_{02}^2 - \rho_{0.12}^2 . \]

The general problem can be formulated in terms of \( k \) independent variables \( x_1, \ldots, x_k \) used to predict \( x_0 \). In this case there are \((2^k - 1)\) components, \( k \) unique components, and \((2^k - k - 1)\) components shared between two or more variables. Explicit formulae are available to express unique components and commonalities for any number of variables (see e.g., Newton and Spurrell, 1969a,b; Kerlinger and Pedhazur, 1973). The unique components can always be expressed as:
\[ U_i = \rho_{0,1,\ldots,k}^2 - \rho_{0,1,\ldots,i-1,i+1,\ldots,k}^2. \]

One application of commonality analysis is to help determine which predictor variable can be deleted with the smallest loss of variance accounted for. Researchers also frequently want to compare the magnitudes of various components to assess their relative importance.

Unfortunately, the published literature has not addressed the problem of establishing confidence intervals for the components of the commonality analysis or the differences between them. Hence statistical tests are currently not available to test for differences in magnitude of the commonality components. The exact sampling distributions of functions of multiple correlations appear to be intractable, suggesting that small sample confidence intervals would be difficult to obtain. However, some large sample results are far more tractable and do yield approximate confidence intervals for unique and common components.

3. An Application.

Mood (1969) used commonality analysis as a principal analytic tool for building his model of school effects. He presents data from a reanalysis of EEO at four grade levels with a sample of 4000. One of his models for school effects is designed to examine two composite predictors of school achievement: peer quality and school quality. Commonality analysis is used to quantify the substantial overlap between these two predictors of achievement. Mood's third-grade achievement data has the following variables:

\[ x_0 = \text{achievement, } x_1 = \text{peer quality, } x_2 = \text{school quality,} \]

with sample correlation matrix
\[ R = \begin{pmatrix} 1 & 0.7248 & 0.7001 \\ 0 & 1 & 0.8050 \\ 0 & 0 & 1 \end{pmatrix} \]

Mood used these data to obtain estimates of the unique and common contributions of variables \(x_1\) and \(x_2\) to the explained variance in \(x_0\). His commonality estimates are:

\[ \hat{U}_1 = 0.0745, \quad \hat{U}_2 = 0.0392, \quad \hat{C}_{12} = 0.4509. \]

The methods of this paper (Section 4) can be used to calculate approximate (asymptotic) confidence intervals for each of these components. The result of those calculations for 95 percent (nonsimultaneous) confidence intervals for Mood's data are:

\[ 0.0659 \leq U_1 \leq 0.0851, \quad 0.0310 \leq U_2 \leq 0.0474, \quad 0.384 \leq C_{12} \leq 0.4634. \]

An asymptotic confidence ellipse (with confidence coefficient .95) for the two unique components is given by

\[ 197.740(U_1 - 0.0745)^2 + 254.237(U_2 - 0.0392)^2 + 112.994(U_1 - 0.0745)(U_2 - 0.0392) = 5.991. \]

An asymptotic confidence ellipsoid (with confidence coefficient .95) for all three commonality components is given by

\[ 198.798(U_1 - 0.0745)^2 + 254.601(U_2 - 0.0392)^2 + 157.614(C_{12} - 0.4509)^2 + 111.754(U_1 - 0.0745)(U_2 - 0.0392) + 25.824(U_1 - 0.0745)(C_{12} - 0.4509) - 15.138(U_2 - 0.0392)(C_{12} - 0.4509) = 7.815. \]
4. Commonality Analysis with Two Predictors.

Multiple correlations are functions of sample moments and hence are asymptotically normal. The asymptotic distribution of any linear combination of multiple correlations can thus be obtained if the asymptotic variance is available. The variance of a linear combination can be obtained by repeated application of the identity:

\[(4.1) \quad \text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X,Y).\]

Hence, the asymptotic distribution of each of the commonality components (or any linear combination of them) can be obtained from the asymptotic variance of each multiple correlation and the asymptotic covariances of any pair of the multiple correlations.

The asymptotic joint distribution of \((r_{01}, r_{02}, r_{12})\) is well known (e.g., see Anderson, 1958). The asymptotic joint distributions of various functions of the \(r_{ij}\) have been obtained by Olkin and Siotani (1964, 1976). For example, they obtained the asymptotic joint distribution of the step-down multiple correlation coefficients. Their results can be used to obtain the necessary asymptotic variances and covariances of \(\hat{\rho}_1, \hat{\rho}_2\) and \(\hat{\rho}_{12}\). Henceforth in this paper the notation \(r_{0.12...p}\) and \(\rho_{0.12...p}\) are used for the sample and population correlation coefficients, respectively. The asymptotic variance of the squared multiple correlation coefficient is well-known:

\[(4.2) \quad \text{Var}(r_{0.12...p}^2) = \frac{1}{n} \rho_{0.12...p}^2 (1 - \rho_{0.12...p}^2)^2.\]

The results of Olkin and Siotani (1976) and an application of their Theorem 2.1 yield the following asymptotic covariances:
\begin{align}
(4.3) \quad \text{Cov}\left[r_{0,12}^2, r_{01}^2\right] &= -4\rho_{01} \left[ \text{Var}(r_{01}) \frac{\rho_{12}^3}{\rho_{11}} + \text{Cov}(r_{01}, r_{02}) \frac{\rho_{12}^2}{\rho_{11}^2} + \text{Cov}(r_{01}, r_{12}) \frac{\rho_{12}^2}{\rho_{11}^2} \right] \\
\text{where} \\
(4.4) \quad \text{Cov}(r_{01}, r_{02}) &= \frac{1}{n} \left[ \frac{1}{2} (2\rho_{02} - \rho_{01}\rho_{02})(1 - \rho_{01}^2 - \rho_{02}^2 - \rho_{12}^2) + \rho_{12}^3 \right], \\
(4.5) \quad \text{Cov}(r_{01}, r_{12}) &= \frac{1}{n} \left[ \frac{1}{2} (2\rho_{02} - \rho_{01}\rho_{12})(1 - \rho_{01}^2 - \rho_{02}^2 - \rho_{12}^2) + \rho_{12}^3 \right], \\
\text{and} \quad R = (r_{ij}), \quad P = (\rho_{ij}), \quad P^{-1} = (\rho_{ij}^{-1}). \quad \text{These computations yield the following result.}
\end{align}

4.1 Confidence Intervals for Commonality Components.

Theorem 1: The asymptotic distribution of the standardized unique component $U_1$:

\[ \frac{\sqrt{n}(\hat{U}_1 - U_1)}{\hat{\sigma}_\infty(U_1)} = \frac{\sqrt{n}([r_{0,12}^2 - r_{02}^2] - (\rho_{0,12}^2 - \rho_{02}^2))}{\hat{\sigma}_\infty(U_1)} \]

is a standard normal distribution, where

\[ \hat{\sigma}_\infty^2(U_1) = \text{Var}(r_{0,12}^2 - r_{02}^2) = \text{Var}(r_{0,12}^2) + \text{Var}(r_{02}^2) - 2 \text{Cov}(r_{0,12}^2, r_{02}^2), \]
and $\hat{\sigma}_\infty(U_1)$ is obtained by replacing the population parameters $\rho_{ij}$ by the consistent estimators $r_{ij}$.

The 100 $\gamma$ percent confidence interval for $U_1 = (\rho_{012}^2 - \rho_{02}^2)$ is therefore given by:

$$\frac{z_\gamma/\hat{\sigma}_\infty(U_1)}{\sqrt{n}} \leq U_1 \leq \frac{z_\gamma/\hat{\sigma}_\infty(U_1)}{\sqrt{n}} + \frac{z_\gamma/\hat{\sigma}_\infty(U_1)}{\sqrt{n}},$$

where $z_\gamma$ is obtained from the standard normal table.

Confidence intervals for $U_2$ are calculated similarly by interchanging the subscripts 1 and 2.

The asymptotic confidence intervals for the commonality of variates 1 and 2 are computed analogously, but require the additional asymptotic covariance:

$$(4.6) \quad \text{Cov}(r_{01}^2, r_{02}^2) = \frac{k_0102}{n} \left[ \frac{1}{2} (2\rho_{12} - \rho_{01}^2 \rho_{02}^2) (1 - \rho_{01}^2 - \rho_{02}^2 - \rho_{12}^2) + \rho_{12}^3 \right].$$

The asymptotic variance of $C_{12}$ is given by:

$$\sigma_\infty^2(C_{12}) = \text{Var}(r_{01}^2 + r_{02}^2 - r_{0(12)}^2),$$

and is given in detail in the appendix.

**Theorem 2:** The asymptotic distribution of the standardized commonality, $C_{12}$, of variates 1 and 2:
\[
\sqrt{n} \left( \hat{\zeta}_{12} - \zeta_{12} \right) \overset{\sigma_{\infty}(\zeta_{12})}{\sim} \sqrt{n} \left( r_{01}^2 + r_{02}^2 - r_{0,12}^2 - (\rho_{01}^2 + \rho_{02}^2 - \rho_{0,12}^2) \right) \]

is a standard normal distribution, where

\[
\sigma_{\infty}^2(\zeta_{12}) = \text{Var}(r_{01}^2 + r_{02}^2 - r_{0,12}^2)
\]

is given by (A.1.1), and \( \hat{\sigma}_{\infty}(\zeta_{12}) \) is obtained by replacing the population parameters \( \rho_{ij} \) by the consistent estimators \( \hat{r}_{ij} \).

The 100\( \gamma \) confidence interval for the commonality of variates 1 and 2 is given by:

\[
\hat{\zeta}_{12} - \frac{z_{\gamma/2} \hat{\sigma}_{\infty}(\zeta_{12})}{\sqrt{n}} \leq \zeta_{12} \leq \hat{\zeta}_{12} + \frac{z_{\gamma/2} \hat{\sigma}_{\infty}(\zeta_{12})}{\sqrt{n}}.
\]

\[4.2 \quad \text{The Joint Distribution of Commonality Components.}\]

The computations above give the marginal distributions of the commonality components. The same methods can be used to obtain the joint distribution of these components.

**Theorem 3:** The limiting joint distribution of

\[
\sqrt{n} \left[ (\hat{u}_1, \hat{u}_2, \hat{\zeta}_{12}) - (u_1, u_2, \zeta_{12}) \right]
\]

is normal with mean vector \((0,0,0)\) and covariance matrix \(\Sigma = (\Phi_{ij}), i,j = 1,2,3\), where
\[ \psi_{11} = \sigma_{\infty}^2(\hat{u}_1), \quad \psi_{22} = \sigma_{\infty}^2(\hat{u}_2), \quad \psi_{33} = \sigma_{\infty}^2(\hat{c}_{12}), \]

\[ \psi_{12} = \text{Cov}(\hat{u}_1, \hat{u}_2), \quad \psi_{13} = \text{Cov}(\hat{u}_1, \hat{c}_{12}), \quad \psi_{23} = \text{Cov}(\hat{u}_2, \hat{c}_{12}). \]

The expressions for \( \psi_{1j} \) are given in (A.1.2)-(A.1.4).

The joint distribution of the commonality components can be used to obtain asymptotic confidence ellipsoids for the commonality components (see e.g., Anderson, 1978), namely

\[ n(\hat{u}_1 - \mu_1, \hat{u}_2 - \mu_2, \hat{c}_{12} - c_{12}) \psi^{-1}(\hat{u}_1 - \mu_1, \hat{u}_2 - \mu_2, \hat{c}_{12} - c_{12})'' = c_\alpha, \]

where \( c_\alpha \) is the critical value for the upper tail of a chi-square distribution with 3 degrees of freedom.

5. **Commonality Analysis with Three Predictors.**

The case with three predictor variables is more complex than the case of two predictor variables. With three predictors, there are \( 2^3 - 1 = 7 \) components in the analysis. The estimators of the components are obtained by solving the following system of equations:

\[ \rho_{0.123}^2 = U_1 + U_2 + U_3 + C_{12} + C_{13} + C_{23} + C_{123}, \]

\[ \rho_{0.12}^2 = U_1 + U_2 + C_{12} + C_{13} + C_{23} + C_{123}, \]

\[ \rho_{0.13}^2 = U_1 + U_3 + C_{12} + C_{13} + C_{23} + C_{123}, \]

\[ \rho_{0.23}^2 = U_2 + U_3 + C_{12} + C_{13} + C_{23} + C_{123}, \]

\[ \rho_{01}^2 = U_1 + C_{12} + C_{13} + C_{123}, \]

\[ \rho_{02}^2 = U_2 + C_{12} + C_{23} + C_{123}, \]

\[ \rho_{03}^2 = U_3 + C_{13} + C_{23} + C_{123}. \]

These equations can be solved simultaneously to give the following expressions for the commonality components:
\[ U_1 = \rho_{0.123}^2 - \rho_{0.23}^2, \]
\[ U_2 = \rho_{0.123}^2 - \rho_{0.13}^2, \]
\[ U_3 = \rho_{0.125}^2 - \rho_{0.12}^2, \]
\[ C_{12} = \rho_{0.13}^2 + \rho_{0.23}^2 - \rho_{02}^2 - \rho_{0.123}^2, \]
\[ C_{13} = \rho_{0.23}^2 + \rho_{0.13}^2 - \rho_{02}^2 - \rho_{0.123}^2, \]
\[ C_{23} = \rho_{0.12}^2 + \rho_{0.13}^2 - \rho_{02}^2 - \rho_{0.123}^2, \]
\[ C_{123} = \rho_{01}^2 + \rho_{02}^2 + \rho_{03}^2 - \rho_{0.12}^2 - \rho_{0.13}^2 - \rho_{0.23}^2 + \rho_{0.123}^2. \]

The estimates of the commonality components are obtained by substituting sample correlations for the population correlations throughout.

**Theorem 4:** The asymptotic distribution of the standardized first, second and third order commonality components:

\[
\frac{\sqrt{n}(\hat{U}_1 - U_1)}{\hat{\sigma}(U_1)}, \frac{\sqrt{n}(\hat{C}_{12} - C_{12})}{\hat{\sigma}(C_{12})}, \frac{\sqrt{n}(\hat{C}_{123} - C_{123})}{\hat{\sigma}(C_{123})}
\]

are standard normal, where \( \hat{\sigma}(U_1), \hat{\sigma}(C_{12}) \) and \( \hat{\sigma}(C_{123}) \) are given by (A.2.3), (A.2.4) and (A.2.5) respectively.

The asymptotic distribution of the other commonality components can be obtained by substituting the appropriate subscripts in the results presented above. Confidence intervals are obtained in a manner completely
analogous to the two predictor case. The asymptotic joint distribution of the commonality components can also be obtained from the results presented here by a (tedious) repeated application of the identity (4.1).

6. Commonality Analysis with k Predictors.

Let us examine the general case of k predictor variables predicting the criterion variable. In this case there are $2^k - 1$ components to the analysis, k unique components and $2^k - k - 1$ commonalities. To find the distribution of any component or linear combination of components, it is sufficient to find the variances and covariances of all pairs of squared multiple correlations predicting the criterion variate. The asymptotic variances can then be calculated by repeated application of the identity (4.1).

The asymptotic variances of the squared multiple correlations have been given earlier. The general asymptotic covariances can be obtained from a result of Olkin and Siotani (1976), who derive the asymptotic covariances of determinants of correlation matrices.

Their result can be used to obtain the asymptotic covariance between the multiple correlations of any two subsets of predictor variables with the dependent variable. Let "a" be a subset of the set of predictor variables $\{x_1, x_2, \ldots, x_k\}$. Let "c" be another subset of the set of the set of predictor variables. Then the notation $r_{0.a}$ and $r_{0.c}$ is used for the multiple correlation of $x_0$ with the elements of the sets a and c respectively. Define the set b as $a \cup \{x_0\}$ and the set
d as c U \{x_0\}. If "x" is a subset of the set \{x_0, x_1, \ldots, x_k\},
then \( R_{xx} \) denotes the matrix of sample correlations between elements
of the set x. Therefore \( R_{aa} \) is used to indicate the matrix of
sample correlations of elements of the set a. The symbols \( R_{bb}, R_{cc} \)
and \( R_{dd} \) are used similarly. In this notation the general problem is
to find an expression for the covariances of \( r_{0,a}^2 \) and \( r_{0,c}^2 \). These
squared multiple correlations can be expressed as

\[
\begin{align*}
r_{0,a}^2 &= 1 - \frac{|R_{bb}|}{|R_{aa}|} \quad \text{and} \quad r_{0,c}^2 = 1 - \frac{|R_{dd}|}{|R_{cc}|},
\end{align*}
\]

where a, b, c, and d are defined as above.

Form the full correlation matrix \( R \) (with some variables possibly
repeated) defined by the diagonal submatrices \( R_{aa}, R_{bb}, R_{cc}, R_{dd} \). This
matrix can be represented as

\[
R = \begin{pmatrix}
R_{aa} & R_{ab} & R_{ac} & R_{ad} \\
R_{ab} & R_{bb} & R_{bc} & R_{bd} \\
R_{ac} & R_{bc} & R_{cc} & R_{cd} \\
R_{ad} & R_{bd} & R_{cd} & R_{dd}
\end{pmatrix},
\]

where the off diagonal submatrices are defined by the diagonal.

The result of Olkin and Siotani and an application of their
Theorem 2.1 gives:
\( \text{Cov}(r_{0,a}^2, r_{0,c}^2) = \varphi_{bd} \left( \frac{1}{R_{aa}} \right) + \varphi_{ac} \left( \frac{|R_{bb}|}{|R_{aa}|^2 |R_{cc}|} \right) \)
\[- \varphi_{bc} \left( \frac{|R_{dd}|}{|R_{aa}|^2 |R_{cc}|} \right) - \varphi_{ad} \left( \frac{|R_{bb}|}{|R_{aa}|^2 |R_{cc}|} \right),\]

where
\[ \varphi_{xy} = \frac{2}{n} \left( |R_{xx}| |R_{yy}| \text{ trace } [(R_{xx}^{-1} - I)R_{xy}(R_{yy}^{-1} - I)R_{xy}'] \right), \quad x, y \in \{a, b, c, d\}.\]

**Appendix**

This appendix contains the details of the derivations and computations for this paper.

A.1 Two predictors.

The asymptotic variance of the estimate of the commonality of variates \( x_1 \) and \( x_2 \) is obtained directly from the definition of the commonality and an application of identity (6.1). The result is:

\[ \sigma^2_{x_1x_2}(C_{12}) = \text{Var}(r_{01}^2) + \text{Var}(r_{02}^2) + \text{Var}(r_{0,12}^2) + 2\text{Cov}(r_{01}^2, r_{02}^2) \]
\[ - 2\text{Cov}(r_{01}^2, r_{0,12}^2) - 2\text{Cov}(r_{02}^2, r_{0,12}^2). \]

The asymptotic covariances for the joint distribution of the estimates of the commonality components are also obtained from the definitions and an application of the identity (6.1). As before, all the required
covariances between correlations are given in the text. The asymptotic covariances between commonality components are obtained from:

\[(A.1.2) \quad \text{Cov}(\hat{U}_1, \hat{U}_2) = \text{Var}(r_{0.12}^2) + \text{Cov}(r_{01}^2, r_{02}^2) - \text{Cov}(r_{01}^2, r_{0.12}^2) - \text{Cov}(r_{02}^2, r_{0.12}^2),\]

\[(A.1.3) \quad \text{Cov}(\hat{U}_1, \hat{C}_{12}) = 2\text{Cov}(r_{02}^2, r_{0.12}^2) + \text{Cov}(r_{01}^2, r_{0.12}^2) - \text{Cov}(r_{01}^2, r_{02}^2) - \text{Var}(r_{02}^2) - \text{Var}(r_{0.12}^2),\]

\[(A.1.4) \quad \text{Cov}(\hat{U}_2, \hat{C}_{12}) = 2\text{Cov}(r_{01}^2, r_{0.12}^2) + \text{Cov}(r_{02}^2, r_{0.12}^2) - \text{Cov}(r_{01}^2, r_{02}^2) - \text{Var}(r_{01}^2) - \text{Var}(r_{0.12}^2).\]

A.2 Three predictors.

The estimates of the commonality components are linear combinations of seven squared multiple correlations \(r_{01}^2, r_{02}^2, r_{03}^2, r_{0.12}^2, r_{0.13}^2, r_{0.23}^2, \text{ and } r_{0.123}^2\). Hence a total of seven variances and 21 covariances are necessary to calculate the asymptotic joint distribution of the commonality components. The asymptotic variances of the squared multiple correlations are given by (4.2). The asymptotic covariances of pairs of multiple correlations can be obtained from (6.1).

More explicitly, from (6.1) we obtain
(A.2.1) \[ \text{Cov}(r_{03}^2, r_{0.12}^2) = -4r_{03} \left[ \frac{\rho^{12}}{\rho^{11}} \text{Cov}(r_{01}, r_{03}) + \frac{\rho^{13}}{\rho^{11}} \text{Cov}(r_{02}, r_{03}) + \frac{\rho^{12} \rho^{13}}{\rho^{11} \rho^{11}} \text{Cov}(r_{03}, r_{12}) \right], \]

(A.2.2) \[ \text{Cov}(r_{0.12}^2, r_{0.23}^2) = -4 \left[ \text{Cov}(r_{02}, r_{01}) \frac{a_{12} b_{12}}{a_{11} b_{11}} + \text{Cov}(r_{02}, r_{12}) \frac{a_{12} b_{12}}{a_{11} b_{11}} + \text{Var}(r_{02}) \frac{a_{12} b_{12}}{a_{11} b_{11}} + \text{Cov}(r_{01}, r_{03}) \frac{a_{12} b_{13}}{a_{11} b_{11}} + \text{Cov}(r_{02}, r_{03}) \frac{a_{12} b_{13}}{a_{11} b_{11}} + \text{Cov}(r_{02}, r_{13}) \frac{a_{12} b_{13}}{a_{11} b_{11}} + \text{Cov}(r_{03}, r_{12}) \frac{a_{12} b_{13}}{a_{11} b_{11}} + \text{Cov}(r_{12}, r_{23}) \frac{a_{12} b_{13}}{a_{11} b_{11}} \right], \]

where

\[ A = \begin{pmatrix} 1 & r_{01} & r_{02} \\ r_{01} & 1 & r_{12} \\ r_{02} & r_{12} & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & r_{02} & r_{03} \\ r_{02} & 1 & r_{23} \\ r_{03} & r_{23} & 1 \end{pmatrix}, \]

\[ A^{-1} = (a_{ij}) \quad \text{and} \quad B^{-1} = (b_{ij}). \]

From (5.2)

(A.2.3) \[ \sigma^2_{\infty}(U_1) = \text{Var}(r_{0.123}^2 - r_{0.23}^2) = \text{Var}(r_{0.123}^2) + \text{Var}(r_{0.23}^2) - 2\text{Cov}(r_{0.123}, r_{0.23}^2), \]

(A.2.4) \[ \sigma^2_{\infty}(C_{12}) = \text{Var}(r_{0.13}^2 + r_{0.23}^2 - r_{0.3}^2 - r_{0.123}^2), \]
(A.2.5) \( \sigma_{\infty}^2(C_{123}) = \text{Var}(r_{0.1}^2 + r_{0.2}^2 + r_{0.3}^2 - r_{0.12}^2 - r_{0.13}^2 - r_{0.23}^2 + r_{0.123}^2) \).

To obtain estimates \( \hat{\sigma}_{\infty} \) of the standard error \( \sigma_{\infty} \), replace population parameters in the above expressions with the consistent estimators \( r_{12} \), and then use (A.2.1) and (A.2.2).
References


