BAYESIAN PARAMETER AND RELIABILITY ESTIMATION
FOR A BIVARIATE EXPONENTIAL DISTRIBUTION:
PARALLEL SAMPLING

BY

A. A. SHAMSELDIN and S. JAMES PRESS

TECHNICAL REPORT NO. 170
AUGUST 1981

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT
MCS 78-07736

Ingram Olkin, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
BAYESIAN PARAMETER AND RELIABILITY ESTIMATION
FOR A BIVARIATE EXPONENTIAL DISTRIBUTION:
PARALLEL SAMPLING

By

A. A. Shamseldin and S. James Press

TECHNICAL REPORT NO. 170
AUGUST 1981

Prepared Under the Auspices
of
National Science Foundation Grant MCS78-07736

Ingram Olkin, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
BAYESIAN PARAMETER AND RELIABILITY ESTIMATION
FOR A BIVARIATE EXPONENTIAL DISTRIBUTION:
PARALLEL SAMPLING

by

A. A. Shamseldin* and S. James Press
Department of Statistics
Stanford University and University of California, Riverside

Abstract and Summary

The present investigation is concerned with deriving Bayesian statistical inferences for the bivariate exponential (BVE) distribution of Marshall and Olkin (1967) applied as a failure model for a two-component parallel system. In this paper joint posterior distributions for the BVE parameters and marginal posterior densities for individual parameters are developed. The posterior distributions are derived for the case of informative prior knowledge.

Bayesian estimators for the BVE parameters and the corresponding reliability are derived in a closed form. Bayesian approximated credibility intervals ("confidence" intervals) for parameters are derived by utilizing a gamma approximation to the marginal posterior densities.

*Dr. A. A. Shamseldin is now a postdoctoral scholar at Stanford University.
1. **Introduction**

The use of redundant elements in a system is a well-known strategy for improving system reliability. However, in some circumstances the common (simultaneous) failure might dominate the chance (individual) failure. Therefore, a quantitative analysis of common cause failures is encouraged in order to eliminate or decrease the chances of such failures (Apostolakis 1976).

For example, in an electric power distribution, it is a common practice to use two generators to supply energy. Similarly, in the case of data processing of vital information the redundancy principle is applied and two (or more) computers are employed in parallel. In both cases an extra unit may pay for itself by allowing service to continue while a single failed component (e.g., power line, or computer) is repaired.

The bivariate exponential (BVE) distribution of Marshall and Olkin (1967) is applicable as a failure model for such systems when there exists positive probability of simultaneous failure of exponential type, for both components, in addition to the individual exponential failure of each component.

Sinha and Guttman, 1976, considered the problem of Bayesian inference about the reliability function in the two parameter exponential distribution. Draper and Guttman, 1978, studied the problem of Bayesian Reliability function estimation in multicomponent systems using independent exponential distributions. In our problem, the failure times in multicomponent systems will be permitted to be
jointly dependent following multivariate exponential distributions. The estimation problem, in the BVE parallel model, has been considered only from a classical point of view. Bemis (1971) and Bemis, Bain and Higgins (1972) have derived the method of moments, estimators, and discussed the maximum likelihood method for estimating the three parameters of the BVE distribution. They also considered the problem of testing hypotheses for the correlation between marginal failure times. The same problem was considered by Bhattacharyya and Johnson (1971). They investigated the maximum likelihood estimates of the BVE parameters and included the study of the special case of systems having identical marginals.

Proshan and Sullo (1974) and Sullo (1973) have also studied the maximum likelihood method and have suggested another estimator derived from an intuitive principle and called it (INT). The INT estimator has proved to be more efficient than the method-of-moments estimator. It is also simpler than the maximum likelihood estimator since it is the first iterate solution of the nonlinear maximum likelihood equations.

It is well known that one of the major difficulties in estimating the parameters of the BVE distribution is the lack of absolute continuity with respect to the usual two-dimensional Lebesgue measure. The density function has been derived, however, with respect to a dominating measure of a mixture of one- and two-dimensional Lebesgue measures. The non-avoidable complicated reference measure used in deriving the density function causes the existence of
a nonidentifiable subset of the sample space. This subset leads to an apparent difficulty in estimating the parameter vector of the BVE distribution. This difficulty will be discussed in some detail in section (3.1). The difficulties encountered in the classical estimation provide motivations for using the Bayesian approach in this problem.

In addition to the general plausibility features of the Bayesian approach, many other positive reasons exist in applying this approach to parameter and reliability estimation of the BVE parallel model. First, since the BVE distribution is a failure distribution which can be applied to technological systems, it is argued that some sort of prior data or experience about the failure behavior of such systems must exist. In a serious applied statistical analysis there is merit in expressing mathematically this additional information and including it in the formal analysis. Secondly, in highly reliable technological systems, experimental (or sampling) data are often very limited; and we have to appeal to all other possible relevant information to draw meaningful conclusions about reliability properties of such systems. Third, in deriving classical estimators for the BVE parameters, it is realized that in some sampling circumstances these estimators do not exist or are not uniquely determined. Finally, when the classical estimators do exist, confidence intervals for parameters might not be amenable to derivation in small or even moderate sample sizes, since classical sampling distributions for these estimators are quite difficult to derive in closed form.
2. **Problem Statement**

A major system consists of two parallel subsystems (or components) which are subjected to failure that can be characterized by the BVE random failure time vector $\mathbf{T} = (T_1, T_2)$ of Marshall and Olkin. This failure distribution is characterized by the conditional survival probability function given by

$$
P(T_1 > t_1, T_2 > t_2 | \lambda) = \bar{F}(t_1, t_2 | \lambda)
$$

$$
= \begin{cases} 
\exp\left[-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_0 \max(t_1, t_2), \lambda_i + \lambda_0 > 0, t_i > 0, \quad i = 1, 2 \right], \\
0 & \text{otherwise}
\end{cases}
$$

(2.1)

where $(\lambda_1, \lambda_2, \lambda_0)$ is a random determination of the random vector parameter $(\Lambda_1, \Lambda_2, \Lambda_2)$ that assumes the values in the set $\Omega$ given by

$$
\Omega = \{(\lambda_1, \lambda_2, \lambda_0) : 0 < \lambda_i + \lambda_0 < \infty, \quad i = 1, 2\}.
$$

The problem is how to use a Bayesian oriented methodology to estimate both the realization vector $\hat{\lambda} = (\lambda_1, \lambda_2, \lambda_0)$ that indexes the BVE distribution and the corresponding reliability function at a fixed but arbitrary mission time $t$.

3. **Likelihood Function**

3.1. **Background**

Let $(T_1, T_2)$ have the bivariate exponential distribution given by (1.2.1) and denote this distribution by $\text{BVE}|\lambda_1, \lambda_2, \lambda_0$. Let $\{T_{ij} = (T_{1j}, T_{2j}) : j = 1, 2, \ldots, n\}$ be a random sample of size $n$ from the $\text{BVE}|\lambda_1, \lambda_2, \lambda_0$ and $\{t_{ij}\}_{j=1}^n$ be an arbitrary sample point. In a parallel
sampling situation, the systems on test are observed until all 2n 
components have failed. The random variables \( \min(T_1, T_2) \) and 
\( \max(T_1, T_2) \) can be observed. Also observable is the source of the 
first failure (fatal Poisson shock). The cause of the second failure 
cannot be determined exactly when the first failure is an individual 
failure.

This difficulty in identifying the second failure is transferred into the likelihood function in the form of a superposition of 
two Poisson processes. One is a selective process to produce fatal 
shocks to a particular component and the other process is for 
destroying both components simultaneously. For example, if the first 
failure is component labeled 1, the cause of the first failure eventu-
ally would be identified by a Poisson process of intensity 
parameter \( \lambda_1 \). However, the second failure cannot be identified 
exactly; it could be caused by a Poisson process of failure rate \( \lambda_2 \) 
or from the catastrophic Poisson process of failure rate \( \lambda_0 \). To get 
around this difficulty, it has been assumed that the second failure 
is caused by the superposition of the two Poisson processes, which is 
again a Poisson process of failure rate \( \gamma_2 = \lambda_2 + \lambda_0 \). This super-
position principle takes care of the non-identifiability problem, but 
a price is paid in finding efficient estimators for \( \lambda_1 \), \( \lambda_2 \), and \( \lambda_0 \) 
and maximum likelihood (ML) estimators in a closed form.

3.2. Density Function

Let \( \mathcal{B} \) be the Borel \( \sigma \)-field in \( \mathbb{R}^+ \), let \( \mu_2 \) be the two-
dimensional Lebesgue measure on \( (\mathbb{R}^+, \mathcal{B}) \), let \( C = \{(t_1, t_1) : 0 < t_1 < \infty\} \),
and for $B \in \mathcal{B}$, let $\nu(B)$ defined by

$$\nu(B) = \mu_1(\{t_1 : (t_1,t_1) \in B \cap C\}) ,$$

where $\mu_1$ is the Lebesgue measure on the real line.

As noted by Bemis, et al. (1972) and Bhattacharyya and Johnson (1973), the probability density function of (1.2.1) with respect to the dominating measure $\mu = \mu_2 + \nu$ is given by

$$f(t_1,t_2|x) = \begin{cases} \lambda_1(\lambda_2 + \lambda_0) \bar{F}(t_1,t_2|x), & t_2 > t_1 > 0 \\ \lambda_2(\lambda_1 + \lambda_0) \bar{F}(t_1,t_2|x), & t_1 > t_2 > 0 \\ \lambda_0 \bar{F}(t_1,t_2|x), & t_1 = t_2 > 0 \end{cases} \quad (3.1)$$

For a redundant system of two similar components, likely to experience the same types of shocks, individually, in addition to occasionally being subjected to catastrophic shocks concurrently, the Marshall-Olkin bivariate distribution for this case of identical marginals is given by the conditional survival function

$$P(T_1 > t_1, T_2 > t_2|\lambda_1,\lambda_0) = \exp[-\lambda_1(t_1 + t_2) - \lambda_0 \max(t_1,t_2)] \quad (3.2)$$

where

$$(t_1,t_2) \in \mathbb{R}_+^2, (\lambda_1,\lambda_0) \in \tilde{\Omega}, \tilde{\Omega} = \{(\lambda_1,\lambda_0) : 0 < \lambda_1 < \infty, 0 < \lambda_0 < \infty\}$$

is the parameter space from which $(\lambda_1,\lambda_0)$ take on values. During our investigation, we designate the BVE failure distribution with two parameters by BVE$|\lambda_1,\lambda_0$. 

6
The density function with respect to the reference dominating measure \( \mu \) is expressed in the form
\[
f(t_1, t_2 \mid \lambda) = \lambda_0 \exp[-(2\lambda_1 + \lambda_0)t_1] I_C(t_1, t_2)
+ \lambda_1(\lambda_1 + \lambda_0) \exp[-\lambda_1(t_1 + t_2) - \lambda_0 \max(t_1, t_2)]
\cdot [1 - I_C(t_1, t_2)], (t_1, t_2) \in \mathbb{R}^+, (\lambda_1, \lambda_0) \in \Omega,
\]
where \( I_C \) is the indicator of the set \( C \) (characteristic function).

### 3.3. Likelihood Function

(A) For BVE \( \lambda_1, \lambda_2, \lambda_0 \): let \( \{T_j = (T_{1j}, T_{2j})\}_{j=1}^n \) denote a random sample of size \( n \) from BVE \( \lambda_1, \lambda_2, \lambda_0 \), and \( \{t_j\}_{j=1}^n \) the corresponding set of sample values.

Let \( n_1, n_2, \) and \( n_0 \) be the number of observations in regions \( \{t_2 > t_1\} \), \( \{t_2 < t_1\} \), and \( \{t_1 = t_2\} \), respectively. It has been shown by Sullo (1973) that the set \( (N_1, N_2, \sum_{j=1}^n T_{1j}, \sum_{j=1}^n T_{2j}, \sum_{j=1}^n \min(T_{1j}, T_{2j})) \) constitutes a set of minimal sufficient statistics for the BVE family of three parameters. Therefore, the likelihood function for a BVE \( \lambda_1, \lambda_2, \lambda_0 \) sample of size \( n \) is given by
\[
\ell(\lambda_1, \lambda_2, \lambda_0) = \prod_{j=1}^n f(t_{1j}, t_{2j})
= [\lambda_1(\lambda_2 + \lambda_0)]^{n_1}[\lambda_2(\lambda_1 + \lambda_0)]^{n_2} \lambda_0^{n_0}
\cdot \exp(-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_0 t_0), \lambda \in \Omega
\]
where
\[ \tau_1 = \sum_{j=1}^{n} t_{1j}, \quad \tau_2 = \sum_{j=1}^{n} t_{2j}, \quad \text{and} \quad \tau_0 = \sum_{j=1}^{n} \max(t_{1j}, t_{2j}) . \]

(B) For BVE|\lambda_1, \lambda_0: \text{let } \{T_j = (T_{1j}, T_{2j})\}_{j=1}^{n} \text{ denote a random sample of size } n \text{ from BVE|}\lambda_1, \lambda_0, \text{ and } \{t\}_{j=1}^{n} \text{ the corresponding set of sample values. As it is noted by Bhattacharyya and Johnson (1973), the set } (N_0, \Sigma \max(T_{1j}, T_{2j}), \Sigma(T_{1j} + T_{2j})) \text{ constitutes a set of minimal sufficient statistics for the BVE|\lambda_1, \lambda_0 family. The likelihood function is then given by}

\[ \ell(\lambda_1, \lambda_0) = [\lambda_1(\lambda_1 + \lambda_0)]^{n-n_0} \lambda_0^{n_0} \exp[-\lambda_1 \tau_1 - \lambda_0 \tau_0], \lambda \in \tilde{\Omega} \quad (3.5) \]

where

\[ \tau_1 = \sum_{j=1}^{n} (t_{1j} + t_{2j}) . \]

Notice that because the induced family of distributions of the minimal sufficient statistics for both BVE|\lambda_1, \lambda_0 and BVE|\lambda_1, \lambda_2, \lambda_0 families is not complete, some difficulties are present in establishing whether or not an M.V.U.E. for \( \lambda \) exists. This is still an open problem.

4. Bayesian Inferences for the Two Parameter BVE Distribution

We devote our efforts in this paper to derive Bayesian statistical inferences for the situation where the BVE distribution is assumed to have identical marginals (for non-identical marginals see Shamseldin (1979)). The identical marginal failure model can be
applied to a redundant parallel system of two similar components. These components are likely to experience the same types of shocks, individually, in addition to occasionally being subjected to catastrophic shocks which destroy both components simultaneously.

The treatment of this simplified case first offered us the key for the whole theory of Bayesian statistical inference in the MVE-parallel systems. In the following we shall derive estimators for the two parameters, as well as the corresponding reliability in the case of informative prior information.

4.1. Joint Posterior Density

In this paper we shall be interested in the case where the analyst has a substantial amount of prior information about the individual parameters and is able to quantify them in terms of gamma distribution family. Also, we restrict our discussion here to the two-parameter case. Other cases may be found in Shamseldin (1979).

In this case we adopt the following joint prior density for the two parameters, $\lambda_1$ and $\lambda_0$:

$$g(\lambda_1, \lambda_0) = g_1(\lambda_1)g_0(\lambda_0)$$  \hspace{1cm} (4.1)

where

$$g_i(\lambda_i) = \frac{\nu_i^{-1}}{\lambda_i^{1-\nu_i}} e^{-\lambda_i\alpha_i}$$;

while

$$0 < \lambda_i < \infty; \quad \nu_i > 0, \alpha_i > 0; \quad i = 0,1.$$
Note that the hyperparameters $\nu_i$ and $\alpha_i$, $i=0,1$, have to be assessed a priori. This assessment assumes that $\Lambda_1$ and $\Lambda_0$ are independent a priori, and each has a gamma prior density.

With respect to the prior density (4.1) the joint density for $\Lambda_1$ and $\Lambda_0$ is proportional to the product of this prior density times the likelihood function given by (3.5). Let $\pi(\lambda|d)$ denote the posterior density function, where $d = (\tilde{d}, h)$, while

$$\tilde{d} = (n_0, \Sigma \max(t_{1j}, t_{2j}), \Sigma(t_{1j} + t_{2j}))$$

is a realization of the vector $(N_0, \Sigma_j=1 \max(T_{1j}, T_{2j}), \Sigma_j=1(T_{1j} + T_{2j}))$ of minimal sufficient statistics for the BVE $\lambda_1, \lambda_0$ family and $h = (\nu_i, \alpha_i; i=0,1)$ is the vector of the hyperparameters. Therefore, the joint posterior density function takes the form

$$\pi(\lambda_1, \lambda_0|d) \propto \lambda_1^{n_1 + \nu_1 - 1} \lambda_0^{n_0 + \nu_0 - 1} (\lambda_1 + \lambda_0)^{n_1}$$

$$\quad \cdot \exp[-(\tau_1 + \alpha_1)\lambda_1 - (\tau_0 + \alpha_0)\lambda_0],$$

where $n_1 = n - n_0$. (Note that this $n_1$ is different than $n_1$ used in the likelihood function (3.4).)

The missing constant of proportionality (say $C(d)$) can easily be found from the requirement that $\pi(\lambda|d)$ must be a density and therefore integrate to one. It is clear that the constant will be in terms of the value $\tilde{d}$ of the joint sufficient statistic.

Let us now compute the value of $C(d)$. Integrating $\pi$ with respect to $\lambda_0$ gives
\[ f_0^\infty \pi(\lambda_1, \lambda_0 | d) d\lambda_0 = f_0^\infty \lambda_1^{n_1 + \nu_1 - 1} \lambda_0^{n_0 + \nu_0 - 1} (\lambda_1 + \lambda_0)^{n_1} \\
\cdot e^{-(\tau_1 + \alpha_1)\lambda_1} e^{-(\tau_0 + \alpha_0)\lambda_0} d\lambda_0 \]

\[ = \lambda_1^{n_1 + \nu_1 - 1} e^{-(\tau_1 + \alpha_1)\lambda_1} f_0^\infty \lambda_0^{n_0 + \nu_0 - 1} (\lambda_1 + \lambda_0)^{n_1} \]

\[ \cdot e^{-(\tau_0 + \alpha_0)\lambda_0} d\lambda_0 . \]

Let us denote the last integral by \( I \) and manipulate it first

\[ I = f_0^\infty e^{-(\tau_0 + \alpha_0)\lambda_0} \lambda_0^{n_0 + \nu_0 - 1} \frac{n_1}{\Sigma} \left( \begin{array}{c} n_1 \\ i \end{array} \right) \lambda_1^{n_1 - i} \lambda_0^i d\lambda_0 \]

\[ = \frac{n_1}{\Sigma} \left( \begin{array}{c} n_1 \\ i \end{array} \right) \lambda_1^{n_1 - i} \frac{\Gamma(n_0 + \nu_0 + i)}{(\tau_0 + \alpha_0)^{n_0 + \nu_0 + i}} . \]

Therefore,

\[ \pi(\lambda_1 | d) = C(d) \frac{n_1}{\Sigma} \left( \begin{array}{c} n_1 \\ i \end{array} \right) \lambda_1^{2n_1 + \nu_1 - i - 1} e^{-(\tau_1 + \alpha_1)\lambda_1} \frac{\Gamma(n_0 + \nu_0 + i)}{(\tau_0 + \alpha_0)^{n_0 + \nu_0 + i}} . \]

(4.3)

Now we integrate \( \pi \) with respect to \( \lambda_1 \) and equate the result to one.

This leads to the identity

\[ [C(d)]^{-1} = \frac{n_1}{\Sigma} \frac{\Gamma(2n_1 + \nu_1 - 1)}{(\tau_1 + \alpha_1)^{2n_1 + \nu_1 - 1}} \frac{\Gamma(n_0 + \nu_0 + i)}{(\tau_0 + \alpha_0)^{n_0 + \nu_0 + i}} , \]

(4.4)
which is a linear combination of double products of gamma functions. The joint posterior density, given the value $d$ of the joint sufficient statistics and the hyperparameters, can be written explicitly in the form

$$
\pi(\lambda_1, \lambda_0 | d) = \begin{cases} 
\frac{\Gamma(n_i)}{\Gamma(n_{i+1})} \lambda_i^{2n_i + \nu_i - i - 1} \lambda_0^{n_0 + \nu_0 + i - 1} \exp \left[ - (\tau_i + \alpha_i) \lambda_i - (\tau_0 + \alpha_0) \lambda_0 \right] & \text{if } 0 < \lambda_i < \infty; \ i = 0, 1 \\
0 & \text{otherwise}
\end{cases} 
$$

(4.5)

where $C(d)$ is given by (4.4).

It is clear that the joint posterior function (4.5) has the form of a convex combination of double products of gamma densities. The interpretation of this is that this posterior density reflects the type of dependency of $\Lambda_1$ and $\Lambda_0$ that exists in the likelihood function. This dependency is caused by adopting the dominating reference measure, $\mu = \mu_2 + \nu$, which was necessary for defining a density function.

4.2. **Marginal Posterior Density and Bayes Estimators**

Integrating the joint posterior density function (4.5) with respect to $\lambda_1$, we obtain the marginal posterior density of $\Lambda_0$. Let $\pi(\lambda_0 | d)$ denote the posterior density of $\Lambda_0$. That is,
\[
\pi(\lambda_0 | \bar{d}) = \begin{cases} 
\frac{n_1 \sum_{i=0}^{n_1} \binom{n_1}{i} \frac{\Gamma(2n_1 + \nu_1 - i)}{(\tau_1 + \alpha_1)^{2n_1 + \nu_1 - i}} \frac{\lambda_0^{n_0 + \nu_0 + i - 1}}{(\tau_0 + \alpha_0)^{n_0 + \nu_0 + i - 1}} e^{-(\tau_0 + \alpha_0)\lambda_0}}{\tau_0 + \alpha_0} & \text{if } 0 < \lambda_0 < \infty \\
0 & \text{otherwise} 
\end{cases} 
\]

When the squared-error loss function given by

\[
L(\tilde{\lambda}_i, \lambda_i) = (\lambda_i - \tilde{\lambda}_i)^2, \quad i = 0, 1
\]

is used to approximate the consequences of erroneous estimation, the Bayes estimator for \(\lambda_0\) is given by

\[
\tilde{\lambda}_0 = E(\lambda_0 | \bar{d}) = \int_0^\infty \lambda_0 \pi(\lambda_0 | \bar{d}) d\lambda_0
\]

\[
\tilde{\lambda}_0 = \frac{n_1 \sum_{i=0}^{n_1} \binom{n_1}{i} \frac{\Gamma(2n_1 + \nu_1 - i)}{(\tau_1 + \alpha_1)^{2n_1 + \nu_1 - i}} \frac{\lambda_0^{n_0 + \nu_0 + 1 + i}}{(\tau_0 + \alpha_0)^{n_0 + \nu_0 + 1 + i}}}{n_1 \sum_{i=0}^{n_1} \binom{n_1}{i} \frac{\Gamma(2n_1 + \nu_1 - i)}{(\tau_1 + \alpha_1)^{2n_1 + \nu_1 - i}} \frac{\lambda_0^{n_0 + \nu_0 + i}}{(\tau_0 + \alpha_0)^{n_0 + \nu_0 + i}}}
\]

Similarly, Bayes estimator for \(\lambda_1\) can be derived, which is given by

\[
\tilde{\lambda}_1 = \frac{n_1 \sum_{i=0}^{n_1} \binom{n_1}{i} \frac{\Gamma(2n_1 + \nu_1 + 1 - i)}{(\tau_1 + \alpha_1)^{2n_1 + \nu_1 + 1 - i}} \frac{\lambda_0^{n_0 + \nu_0 + i}}{(\tau_0 + \alpha_0)^{n_0 + \nu_0 + i}}}{n_1 \sum_{i=0}^{n_1} \binom{n_1}{i} \frac{\Gamma(2n_1 + \nu_1 + 1 - i)}{(\tau_1 + \alpha_1)^{2n_1 + \nu_1 + 1 - i}} \frac{\lambda_0^{n_0 + \nu_0 + i}}{(\tau_0 + \alpha_0)^{n_0 + \nu_0 + i}}}
\]
It is noticed that both marginals (4.3) and (4.6) have the form of a convex combination of gamma densities, and the Bayes estimators can be easily computed using hand calculators.

From the marginal posterior density (4.6), the variance of $\Lambda_0$ is derived; the result is given by

$$\text{Var}(\Lambda_0 | d) = C(d) \sum_{i=1}^{n_1} \left( \frac{\Gamma(2n_1 + \nu_1 - 1)}{\Gamma(\nu_1)} \cdot \frac{\Gamma(n_0 + \nu_0 + 2 + i)}{\Gamma(\nu_0 + 2 + i) \cdot n_0^{\nu_0 + 2 + i}} \right) \left( \tilde{\lambda}_0 \right)^2 .$$

(4.10)

The posterior variance of $\Lambda_1$ is similarly computed and is given by

$$\text{Var}(\Lambda_1 | d) = C(d) \sum_{i=1}^{n_1} \left( \frac{\Gamma(2n_1 + \nu_1 - 2 + i)}{\Gamma(\nu_1)} \cdot \frac{\Gamma(n_0 + \nu_0 + 1)}{\Gamma(\nu_0 + 1) \cdot n_0^{\nu_0 + 1}} \right) \left( \tilde{\lambda}_1 \right)^2 .$$

(4.11)

4.3. Reliability Estimation

For a parallel redundant system, the system is in an unfailed state at time $t$ if at least one component survives at the time $t$. Hence the reliability of a two component parallel system $R(t | \lambda)$ given the realization $\lambda = (\lambda_1, \lambda_0)$ is

$$R(t | \lambda) = P(T_1 > t \text{ or } T_2 > t | \lambda)$$

$$= 1 - P(T_1 < t, T_2 < t | \lambda)$$

$$= 1 - F(t, t | \lambda) .$$

We now develop an analysis for the two parameter system, although results may readily be found for the three parameter system as well (see Shamseldin, 1979 for details).
By using the identity
\[ F(t_1, t_2 | \lambda) = \bar{F}(t_1, t_2 | \lambda) + \bar{F}(0, 0 | \lambda) - \bar{F}(0, t_2 | \lambda) - \bar{F}(t_1, 0 | \lambda) \]
the system reliability at a given but arbitrary mission time \( t \) takes the form
\[ R(t | \lambda) = 2 \exp[-(\lambda_1 + \lambda_0)t] - \exp[-(2\lambda_1 + \lambda_0)t], 0 < t < \infty. \quad (4.12) \]
The objective now is to estimate \( R(t | \lambda) \) with a squared-error loss function defined by
\[ L(\tilde{R}, R) = (\tilde{R} - R)^2. \quad (4.13) \]
We assume that \( (\Lambda_1, \Lambda_0) \) has the joint prior distribution given by (4.1). Thus the Bayes estimator of \( R \) for the squared-error loss is its posterior expectation (the expectation is with respect to the joint posterior density (4.5)).

\[ \tilde{R}(t) = \int_0^\infty \int_0^\infty (2e^{-(\lambda_1 + \lambda_0)t} - e^{-(2\lambda_1 + \lambda_0)t}) \pi_1(\lambda_1, \lambda_0) | \bar{d} | d\lambda_1 d\lambda_0 \]

\[ = c(\bar{d}) \sum_{i=0}^{n_1} \binom{n_1}{i} \frac{\Gamma(2n_1 + \nu_1 - i) \Gamma(n_0 + \nu_0 + i)}{(\tau_0 + \alpha_0 + t)^{n_0 + \nu_0 + i}} \]

\[ \cdot \left\{ \frac{2}{(\tau_1 + \alpha_1 + t)^{2n_1 + \nu_1 - i}} - \frac{1}{(\tau_1 + \alpha_1 + 2t)^{2n_1 + \nu_1 - i}} \right\}. \quad (4.14) \]
4.4. Approximate Bayesian Credibility Intervals

Approximate Bayesian Credibility (confidence) intervals for \( \Lambda_1 \) and \( \Lambda_0 \) will be developed by approximating the marginal posterior distributions of \( (\Lambda_1 | d) \) and \( (\Lambda_0 | d) \). Note from (4.3) and (4.6) that the two marginal densities \( \Lambda_1 | d \) and \( \Lambda_0 | d \) can be expressed as a convex linear combination of gamma densities. We shall now be concerned with establishing approximated confidence intervals for \( \Lambda_0 | d \); similarly, confidence intervals for \( \Lambda_1 | d \) can be developed. The marginal posterior density for \( \Lambda_0 \) is a weighted finite sum of gamma densities, given by

\[
\pi(\lambda_0 | d) d\lambda_0 = \sum_{i=0}^{n_1} A_i(d) dG(\lambda_0 | n_0 + \nu_0 + i, \tau_0 + \alpha_0),
\]

(4.15)

where

\[
A_i(d) = \frac{\left(\begin{array}{c} n_1 \\ i \end{array}\right) \Gamma(2n_1 + \nu_1 - i) \Gamma(n_0 + \nu_0 + i)}{\Gamma(2n_1 + \nu_1 - i) \Gamma(n_0 + \nu_0 + i)} \frac{\Gamma(2n_1 + \nu_1 - i) \Gamma(n_0 + \nu_0 + i)}{\Gamma(2n_1 + \nu_1 - i) \Gamma(n_0 + \nu_0 + i)} \frac{\Gamma(2n_1 + \nu_1 - i) \Gamma(n_0 + \nu_0 + i)}{\Gamma(2n_1 + \nu_1 - i) \Gamma(n_0 + \nu_0 + i)}
\]

(4.16)

and

\[
dG(\lambda_0 | n_0 + \nu_0 + i, \tau_0 + \alpha_0) = \frac{(\tau_0 + \alpha_0)^{n_0 + \nu_0 + i + 1}}{\Gamma(n_0 + \nu_0 + i + 1)} \frac{\lambda_0^{n_0 + \nu_0 + i + 1}}{\lambda_0^{n_0 + \nu_0 + i + 1}} e^{-(\tau_0 + \alpha_0)\lambda_0} d\lambda_0,
\]

(4.17)
where $G(\lambda_0 | n_0 + \nu_0 + 1, \tau_0 + \alpha_0)$ is a gamma distribution function of $\Lambda_0$ for each $i$, $i = 0, 1, \ldots, n_i$.

Although only a few seconds of computer time may be needed for computing the marginal posterior density function of $\Lambda_0$, we found that when sample sizes are not too small, it can be well approximated (for the purpose of deriving credibility intervals) by a gamma density function giving the same mean and variance as the exact posterior density function.

Let $Y_0$ be a gamma distributed random variable $G(\nu_0', \alpha_0')$, a candidate for approximating $\Lambda_0|d$, where $\nu_0'$ and $\alpha_0'$ have to be evaluated so that the corresponding means and variances of the two distributions are equal. Since

$$E(Y_0) = \frac{\nu_0'}{\alpha_0'}, \quad \text{var}(Y_0) = \frac{\nu_0'}{\alpha_0'^2},$$

we should find $\nu_0'$ and $\alpha_0'$ such that

$$E(\Lambda_0|d) = E(Y_0) \quad \text{and} \quad \text{var}(\Lambda_0|d) = \text{var}(Y_0).$$

Utilizing the formulae (4.8) and (4.10) for the mean and variance of $\Lambda_0|d$ gives

$$\alpha_0' = \frac{E(\Lambda_0|d)}{\text{var}(\Lambda_0|d)} \quad \text{and} \quad \nu_0' = \frac{[E(\Lambda_0|d)]^2}{\text{var}(\Lambda_0|d)}.$$

For an upper bound credibility interval and a level of confidence $1 - \alpha$, we have
\[ P_r(0 \leq \lambda_0 \leq C_0|d) \approx 1 - \alpha \, , \quad (4.21) \]

where \( C_0 \) is a constant determined uniquely for the preassigned \( \alpha \) and the computed \( \nu_0' \) and \( \alpha_0' \).

5. A Numerical Example

A simulated complete sample life testing experiment was conducted on a random sample of size 10 of two-component parallel systems in which the failure times \((T_1, T_2)\) in each system is the BVE\(\lambda_1, \lambda_0\). The failure times \(T_1\) and \(T_2\) are recorded and listed in the following table:

\begin{table}[h]
\centering
\caption{Observed Values of Failure Times \((T_1, T_2)\) for 10 Simulated BVE\(\lambda_1, \lambda_0\) Parallel Systems: \(\lambda_1 = 2.5, \lambda_0 = 2\)}
\begin{tabular}{c c c}
\hline
Sample & \(T_1\) & \(T_2\) \\
\hline
1 & 0.05843 & 0.31203 \\
2 & 1.15382 & 0.33266 \\
3 & 0.05597 & 0.05597 \\
4 & 0.05816 & 0.05816 \\
5 & 0.14310 & 0.26255 \\
6 & 0.17246 & 0.57087 \\
7 & 0.09113 & 0.77613 \\
8 & 0.03660 & 0.03660 \\
9 & 0.05688 & 0.29577 \\
10 & 0.21697 & 0.06951 \\
\hline
\end{tabular}
\end{table}
From Table 1, the jointly minimal sufficient statistic
\[ D = \left( \sum_{j=1}^{10} (T_{1j} + T_{2j}), \max_{j=1}^{10} \max(T_{1j},T_{2j}), N_1 \right) \]
takes the value
\[ d = (4.81377, 2.9177, 7) \]. In this example we shall derive Bayesian
statistical inferences for the case of a vague prior knowledge
\[ (\alpha_i = 0, \nu_i = 0, i = 0, 1) \].

For the parameter vector \( \lambda = (\lambda_0, \lambda_1) \), the Bayes point estima-
tor with respect to the quadratic loss function is given by
\[ \tilde{\lambda} = (\tilde{\lambda}_0, \tilde{\lambda}_0) \],
where \( \tilde{\lambda}_0 \) is given by (4.8) and \( \tilde{\lambda}_1 \) by equation (4.9). Thus
\[ \tilde{\lambda}_0 = 2.1977 \]
\[ \tilde{\lambda}_1 = 2.199 \] .

The marginal posterior variances of \( \Lambda_0 \) and \( \Lambda_1 \) are computed from for-
mulae (4.10) and (4.11). The result is
\[ \text{Var}(\Lambda_0|d) = 1.087 \quad \text{and} \quad \text{Var}(\Lambda_1|d) = 0.578 \] .

Bayes estimates of the system reliability \( R \), given the
mission time \( t \), are computed from equation (4.14). The result is
shown in Table 2.
TABLE 2
BAYES ESTIMATES OF SYSTEM RELIABILITY
AT 16 MISSION TIMES

<table>
<thead>
<tr>
<th>t</th>
<th>( \tilde{R} )</th>
<th>t</th>
<th>( \tilde{R} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.97779</td>
<td>0.09</td>
<td>0.79514</td>
</tr>
<tr>
<td>0.02</td>
<td>0.95523</td>
<td>0.1</td>
<td>0.77282</td>
</tr>
<tr>
<td>0.03</td>
<td>0.93241</td>
<td>0.11</td>
<td>0.75078</td>
</tr>
<tr>
<td>0.04</td>
<td>0.90945</td>
<td>0.12</td>
<td>0.72906</td>
</tr>
<tr>
<td>0.05</td>
<td>0.88642</td>
<td>0.13</td>
<td>0.70770</td>
</tr>
<tr>
<td>0.06</td>
<td>0.86341</td>
<td>0.14</td>
<td>0.68671</td>
</tr>
<tr>
<td>0.07</td>
<td>0.84049</td>
<td>0.15</td>
<td>0.66612</td>
</tr>
<tr>
<td>0.08</td>
<td>0.81771</td>
<td>0.16</td>
<td>0.64593</td>
</tr>
</tbody>
</table>

Using (4.3) and (4.5), the marginal posterior densities of \( \Lambda_1 \) and \( \Lambda_0 \) are calculated. The resulting forms are shown in Figures 1 and 2 together with gamma densities that approximate the corresponding marginal posterior densities. These gamma densities can be used to develop credibility intervals for \( \Lambda_0 \) and \( \Lambda_1 \) conditional on the sampling data.

6. Conclusions

We have shown that Bayes estimators can be derived even in cases where the corresponding maximum likelihood estimators cannot be found. Moreover, the above efforts can be extended to derive Bayes estimators with respect to a more complicated loss structures than the squared-error loss that is assumed in our analysis, by using numerical analysis methods. The Bayes estimators derived above were
Figure 1. Marginal Posterior Density Function for $A_0$ and its Gamma Approximation.
Figure 2. Marginal Posterior Density Function for $\Lambda_1$ and its Gamma Approximation.
given in a closed form which can be computed using only hand calculators (maximum likelihood estimators have to be found, in general, by complex iterative methods). It is well known that Bayes estimators with respect to proper priors are optimal even in small sampling situations, a result which is not available for the existing classical estimators of the BVE model, since these have optimal properties only asymptotically.
References


