CHOOSING THE SUITABLE TYPE OF SEASONALITY

BY

ADI RAVEH

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Abstract

This paper presents a nonmetric technique for determining suitable type of seasonality of empirical time-series. A criterion using the "Parsimony Principle" is constructed for choosing the appropriate model from among the three following: Multiplicative, Additive or Mixed. For the purpose of demonstration, some known examples are analyzed.

Key words: Additive model, Coefficient of Goodness-of-fit, Mixed additive-multiplicative model, Multiplicative model, Parsimony Principle, Seasonality model.
1. Introduction

In the empirical study of time series we usually try to distinguish between three components: a trend, a seasonal component and an irregular (or random) component. A series is said to be decomposed when each of these three components is empirically identified. To do so, however, it is necessary to define the structure of the time series, which may be purely additive, purely multiplicative, or mixed additive-multiplicative in both seasonal and irregular components. The structure of the time series is called purely additive if the seasonality is independent of the trend, and is called purely multiplicative if the seasonality is proportional to the trend. When the trend level does not vary over time, the additive and multiplicative models may be virtually indistinguishable. The structure is called a mixed additive-multiplicative model when the seasonality is decomposed into two components: additive and multiplicative.

In this manuscript we relate only to the seasonality component and the criterion for choosing the suitable model. Of course, there exist cases when these three models are not appropriate. A criterion for choosing the suitable model for the irregular component will not be discussed here.

Let $Y_t$ denote the value of a quantitative time series at time $t$. One way to represent a decomposition of a mixed model of $Y_t$ into its components is given in eq. (1) below.

(1) \[ Y_t = T_t \cdot S_t \cdot I_t + s_t + i_t \quad t = 1, \ldots, N \]

where $T_t$ denotes the underlying trend at time $t$, $S_t$ and $s_t$ are the multiplicative and additive seasonal components, respectively. $I_t$ and
\( i_t \) are the multiplicative and additive irregular components, respectively. The purely multiplicative model is obtained by using the constraints

\[ s_t = i_t = 0 \text{ for all } t. \]  
A purely additive model is obtained by using the constraints

\[ S_t = I_t = 1 \text{ for all } t. \]

There are infinitely many ways to express irregularity by both \( I_t \) and \( i_t \) \( t = 1, \ldots, N \). Thus we limit ourselves to the simpler model

\[
(2) \quad Y_t = T_t \cdot I_t \cdot S_t + s_t \\
\quad = Z_t \cdot S_t + s_t
\]

where \( Z_t = T_t \cdot I_t \) is the periodicity-free series which is known in literature as a Seasonally Adjusted Series (S.A.D.).

In eq. (2) the original series \( Y_t \) is decomposed into a Seasonally Adjusted Series (S.A.D.) denoted by \( \{Z_t\} \) and a seasonal component combined from both multiplicative \( S_t \) and additive \( s_t \) components. In this manuscript we omit the type of irregularity and try to choose the suitable model for seasonality among the following possibilities: \( m \) - multiplicative; \( a \) - additive and \( mx \) - mixed. There is of course a possibility that none of these is appropriate and a (fixed) seasonality is not revealed in the data.

Every set of numbers \( \{Y_t\} \) can be decomposed into infinite representations as in eq. (2). In time series analysis one tries to do the decomposition using the principle of parsimony. This means that we use the minimum number of parameters that give maximum adaptation, namely, trade-off between the number of parameters that are involved and the goodness-of-fit for a given definitions.
In the next section some notes and definitions will be given and the technique will be discussed. For purposes of demonstration some known examples from the literature will be analysed.

The problem of choosing a model of seasonal behaviour was brought up by several authors in recent years. Durbin & Murphy (1975), and Durbin & Kenny (1976) pointed out Time-Series (especially some unemployment series) that behave at times in a way that is neither purely additive nor purely multiplicative. Durbin & Murphy (1975) seasonal-adjustment procedure based on regression methods permits the use of a mixture of additive and multiplicative components. They recommend using their procedure on monthly sub-series of approximately seven years' length in a moving fashion (p.387). Raveh (1981a) suggested a nonmetric procedure for seasonal adjustment of very short series, which may contain no more that 24 observations for multiplicative or additive model. In this manuscript we emphasize the criterion for choosing the suitable model using the nonmetric procedure.

2. Some Notations and Definitions

A numerical time-series is a set of numerical observation \( \{ Y_t \} \) ordered in time. We will consider discrete series with observations \( Y_t \) made at times \( t, t = 1, \ldots, N \).

A series \( Y_t \) will be called increasing (decreasing) monotone if \( Y_i \leq Y_j \) (\( Y_i \geq Y_j \)) for every \( 1 \leq i < j \leq N \). A series \( \{ Y_t \} \) is said to be polytone of order \( m \) if there are \( (m-1) \) turning points such that the series \( \{ Y_t \} \) is monotone between successive turning points, at which the series changes the type of monotonicity. The segments between the turning
points have at least three observations. Let us denote these segments by \( I_k, k = 1, \ldots, m \).

A series \( \{Y_t\} \) is said to be piecewise monotone of order \( m \) if there are \( m \) segments of indices within which \( \{Y_t\} \) is monotone with the same direction (either positive or negative). For example, the series \( \{t - \lfloor t \rfloor \}, t = \frac{1}{N}, \frac{2}{N}, \ldots, \frac{1}{N}, \ldots, 1, \ldots, 3 \) (where \( \lfloor t \rfloor \) is the greatest integer which is less or equal to the quotient) is a piecewise monotone of order \( m = 3 \) with \( 3 \cdot N \) observation. Clearly, if \( m = 1 \) the polytone series is a monotone one.

It will be useful to replace the observation index \( t \) by an index of the form \( i + pa \), where \( p \) is the proposed period length, \( i \) is the position of the observation within a period, and \( a \) is the period index in the sequence of periods, with the first indexed 0, the second 1, etc. Since each complete period has exactly \( p \) time points, \( i = 1, \ldots, p \). If \( N \) is divisible by \( p \), then \( a = N/p - 1 \) for the last period. However, here we shall not assume that the last period is necessarily complete. We denote the number of complete periods by \( n \), so that \( a = 0, 1, \ldots, n-1 \). Given this notation, a sequence \( \{Y_t\} (t = 1, \ldots, N) \) can be written as \( Y_{i+pa} \) \((i = 1, \ldots, p; \ a = 0, 1, \ldots, n-1)\).

We shall consider linear periodic transformation of \( \{Y_t\} \) with period-length \( p \) as the series \( \{Z_t\} \) of the form:

\[
(3) \quad Z_{i+pa} = \frac{Y_{i+pa} - s_i^{(p)}}{s_i^{(p)}} \quad (i = 1, \ldots, p; \ a = 0, 1, \ldots, n-1), \quad s_i^{(p)} > 0.
\]

where the transformation coefficients \( s_i^{(p)} \) and \( s_i^{(p)} \) represent multiplicative and additive periodic coefficients, respectively. When \( s_i^{(p)} \neq 1 \) and \( s_i^{(p)} = 0 \), equation (3) represents a mixed multiplicative-additive seasonality model. Equation (3) can be written in a different way:
(4) \[ Y_{i+pa} = Z_{i+pa} \cdot S_{i}^{(p)} + s_{i}^{(p)} \]

which is similar to eq.(2) above and to Durbin & Murphy's model, except for the irregular component. In this discussion the estimation of \( S_{i}^{(p)} \) and \( s_{i}^{(p)} \) is based on the observations which obviously involve an error term. The coefficients \( S_{i}^{(p)} \) and \( s_{i}^{(p)} \) depend on the period length \( p \) and on the observation position \( i \) but not on the particular period \( a \). Transformations such as (3) permit, therefore, the study of a periodic pattern of observations.

If \( \{Y_t\} \) is not a polytone (or monotone) series, it might be possible that a period length \( p \) and coefficients \( S_{i}^{(p)} \) and \( s_{i}^{(p)} \) exist for which the transformed series \( \{Z_t\} \) is polytone. Such \( \{Z_t\} \) can be regarded as an underlying polytone Periodicity-free series known as seasonally adjusted data and the \( p \) pairs of coefficients \( S_{i}^{(p)} \), \( s_{i}^{(p)} \) are the periodic pattern of observations, i.e., the seasonal components.

In empirical time series an irregular component usually exists, thus, the transformed series \( \{Z_t\} \) is a "trend and error" which means that it is only an approximately polytone one. In a second stage, a decomposition of \( \{Z_t\} \) into the components, trend and irregularity can be obtained (if desired). This stage of decomposition will be described in a subsequent article. In order to deal with empirical deviations from ideal polytonicity we use one of Guttman's regression-free coefficient of polytonicity (Raveh 1978, Appendix). Specifically, to assess the extent to which any series, say \( \{Y_t\} \) is polytone (of order \( m \)), we shall use the quantity

\[
\mu_m = \frac{\sum_{k=1}^{m} \sum_{i>j} (Y_i - Y_j) \cdot \delta_k}{\sum_{k=1}^{m} \sum_{i>j \in I_k} |Y_i - Y_j|}
\]

(5)
where \( I_k \) is the \( k \)-th monotone segment \((k=1, \ldots, m)\) and \( \delta_k = (-1)^{k-1} \) within \( I_k, \ k = 1, \ldots, m \). The inner summation is over all \((i, j) \in I_k\), such that \( i > j \). Obviously, \(-1 \leq \mu_m \leq 1\), and \( |\mu_m| = 1 \) if and only if the series is perfectly polytone, whether of increasing or decreasing slope interchangeably.

The coefficient of polytonicity for the transformed series \( Z_t \), \( t = 1, \ldots, N, \) will be designated by \( \mu_m^{(p)} \) where, as noted earlier, \( t = i + pa \). The lower index \( m \) indicates the order of polytonicity while the upper index \( p \) indicates the period length.

The series \( \{Y_t\} \) will be called a periodically-polytone time series of period-length \( p \) if there exists linear periodic transformations with period-length \( p \) which converts \( \{Y_t\} \) into a polytone series \( \{Z_t\} \). In practice, a perfect transformation will not be insisted upon, depending on the irregular components. Instead, only a "sufficiently large" value for the coefficient of fit \( |\mu_m^{(p)}| \) will be sought.

3. The Procedure for Seasonal Adjustment

We first assess the order of polytonicity of the original time series \( \{Y_t\} \) by computing its smallest order of polytonicity \( m \) \((m = 1, 2, \ldots)\). In other words, we search for the minimal number of segments \( I_k, \ K = 1, \ldots, m \) in order to obtain as parsimonious a model as is possible. The \( m \) segments are fixed by the \((m-1)\) turning points. In order to locate the proposed turning points, one can often use additional information like occurrence of a strike, wave, changing value of money or any other abrupt change. For such cases one has to take into consideration the content hidden in the analysed series. When information is not available one can try to locate the turning points just by looking at the graph of the original series.
It is always possible to try many combinations of \((m-1)\) turning points and choose the most appropriate, namely, that combination that involves as small \(m\) as possible yielding \(\left|\mu_m^\prime\right|\) quite high. In data analysis there are no magic numbers that one can use as critical values. There is a need for common sense while analyzing various series even with the same tool. Some light would be thrown later in the examples. If \(\left|\mu_m^\prime\right|\) departs substantially from 1, say \(\left|\mu_m^\prime\right| < .95\), such departure may be assumed to originate from periodic fluctuations when the seasonal components, modifying the polytonicity of the trend, or by an irregular component, or both. After fixing the \(m\) segments \(I_k\), \(k = 1, \ldots, m\) and the period-length \(p\) we search for linear periodic transformations with period-length \(p\) and coefficients \(s_i^{(p)}, s_i^{(p)}\) \((i = 1, \ldots, p)\) converting the original series \(\{Y_t\}\) into an approximately polytone series \(\{Z_t\}\) in an optimal manner. That is, bringing \(\left|\mu_m^{(p)}\right|\) as close to 1, the theoretical maximum, as possible. The closer \(\text{Max} \left|\mu_m^{(p)}\right|\) is to 1, the closer is the series \(\{Y_t\}\) to being Periodically-Polytone. The series \(\{Y_t\}\) is converted to be as close as possible to a polytone series \(\{Z_t\}\) in the sense of the loss function (5). This does not imply that the deseasonalized series \(\{Z_t\}\) should have as few turning points as possible. The coefficient of goodness-of-fit given below in (6), which is based on (5), might be close enough to 1 while there may be relatively many turning points, each having small deviations from an "ideal" basic polytone series.

The maximization of \(\left|\mu_m^{(p)}\right|\) as a function of the \(2p\) variables \((s_1^{(p)}, \ldots, s_p^{(p)}) = \underline{s}^{(p)}, (s_1^{(p)}, \ldots, s_p^{(p)}) = \underline{s}^{(p)}\) in the general mixed model (or only \(p\) variables in the simple purely multiplicative or purely additive model), may be obtained by known Quasi-Newton, Powell or Zangwill algorithms, see Zangwill (1967). These algorithms require first an initial
approximation for the \(2p\) coefficients of the transformations and by a successive procedure converge to optimal coefficients. We use as an initial guess the coefficients \(s_{(p)}(1) = 1, s_{(p)}(0) = 0\), i.e., we start with the assumption of no seasonal effects. For the multiplicative model we set the constraints \(s_{(p)}(0) = 0\) and for the additive model: \(s_{(p)}(1) = 1\).

For the usual case where the original series is not perfectly polytöne of order \(m\) and \(|\mu_m| < 1\), we define a measure \(M_{m}^{(p)}\) of the improvement toward polytonicity gained by the transformation (3), applied as a mixed model,

\[
M_{m,mx}^{(p)} = \frac{\text{Max}\ |\mu_{m}^{(p)}| - |\mu_{m}^{(1)}|}{1 - |\mu_{m}^{(1)}|}
\]

Clearly, \(0 \leq M_{m,mx}^{(p)} \leq 1\). Further, \(M_{m,mx}^{(p)} = 1\) if and only if the series \(\{Y_t\}\) is perfectly (without irregularities) periodically-polytöne of order \(m\) and \(M_{m,mx}^{(p)} = 0\), if and only if there is no periodic component. For the sake of simplicity, let us denote by \(M_{m,a}^{(p)}\) and \(M_{m,m}^{(p)}\) the appropriate coefficients of goodness-of-fit for purely additive and purely multiplicative models, respectively. The multiplicative and additive models use simpler linear periodic transformations of \(\{Y_t\}\) of the form:

\[
Z_i + pa = Y_i + pa / s_i^{(p)} \quad \text{and} \quad Z_i + pa = Y_i + pa - s_i^{(p)} \quad (i = 1, \ldots, p; \ a = 0, \ldots, n - 1),
\]

respectively. Obviously, each of these simpler models involve only \(p\) parameters. In order to produce periodicity-free series, namely, seasonally adjusted series, classical decomposition methods for economic time series employ one of the moving average techniques (filters) and then, fit a function (trigonometric, or any other function) to the periodic components (e.g., Fase et al. [1973] and Durbin & Murphy [1975]).
When the period-length \( p \) of a series is not known in advance then, our first step is to estimate it. This is done by computing \( M_{m,mx}^{(p)} \) for various periodic-lengths \( p = 2, 3, \ldots \) and plotting these coefficients of goodness-of-fit versus \( p \), as presented below in figure 1.

![Graph showing the function \( M_{m,mx}^{(p)} \) vs. \( p \) with peaks at \( p = 2, 12, 24, 30 \).

Figure 1: The graph \( M_{m,mx}^{(p)} \) versus \( p = 2, 3, \ldots, 30 \) for a typical periodic series that has period-length \( p = 12 \).

The length of period-\( p \) is estimated as the smallest \( p \) which generates a peak value of \( M_{m,mx}^{(p)} \) sufficiently close to 1.0. Thus a typical graph for a periodic series that has period-length \( p = 12 \) is presented in figure 1. Seasonal adjustment methods usually used a specific period-length that can not be changed like the X-11, Shiskin et al. (1967) that used \( p = 12 \).

4. The Criterion for Choosing the Appropriate Model

As was mentioned above the coefficient of goodness-of-fit \( M_{m}^{(p)} \) for the various models measure the amount of adaptation of a given series to an
ideal periodically-polytene series. The length of the period \( p \) and the order of polytonicity \( m \) are fixed through the stage of estimating the seasonality pattern. The following inequalities hold:

\[
0 < \left\{ \begin{array}{c}
M_{m,a}^{(p)} \\
M_{m,m}^{(p)}
\end{array} \right\} < M_{m,mx}^{(p)} < 1
\]

One of the three types of models may be appropriate if the respective coefficient is close enough to 1. If \( M_{m,mx}^{(p)} \) is as low as zero or nearly zero no such model is appropriate and the series might be decomposed into trend and error only.

In the case that \( M_{m,m}^{(p)} \) is greater (lower) than \( M_{m,a}^{(p)} \) the purely multiplicative model is better (worse) than the purely additive one.

In the case that \( M_{m,mx}^{(p)} \) is only slightly greater than \( M_{m,m}^{(p)} \) (or \( M_{m,a}^{(p)} \)) we choose the simpler model multiplicative or additive (depending on the coefficients) since they use only half the number of parameters for the model of seasonality. In the case that \( M_{m,mx}^{(p)} \) is "much greater" than \( M_{m,m}^{(p)} \) (or \( M_{m,a}^{(p)} \), the mixed model is chosen as the appropriate model.

There is no strict rules for computing the amount of difference that \( M_{m,mx}^{(p)} \) has to be greater than \( M_{m,m}^{(p)} \) for being chosen as the suitable model as well as the impossibility of estimating the exact number of components in principal component analysis.

5. **Examples**

In this section we use the procedure for choosing the appropriate seasonality model in some known empirical series used in literature in other connections.
Example 1: The Chatfield-Prothero case study

This monthly series of "Sales of Company X" consists of 77 observations. The series has a marked monotone trend and also a marked seasonal pattern. The data is given in Chatfield & Prothero (1973, p.297) and the plotted series is presented in figure 2. Chatfield and Prothero (1973), Wilson (1973) and Box-Jenkins (1973) assumed a multiplicative seasonal model. By using our proposed method with \( p = 12 \) as the period's length and \( m = 1 \) as the order of polytonicity, the computed coefficients given in Table 1 are obtained. Thus, in this case it seems that the appropriate model is the mixed one. The multiplicative model is substantially better than the additive one as assumed by Chatfield and Prothero and the other discussants. We apply our procedure for deseasonalizing the series using a mixed seasonality model. As a second stage, the Box-Jenkins approach was applied for periodicity-free series \( \{Z_t\} \) and forecasts for trend were obtained. By combining the forecasts for trend with the estimated mixed seasonal components, much better forecasts were obtained for this case-study, see, Raveh (1981b),

![Graph showing sales of Company X from 1965 to 1972](image)

Figure 2. The Chatfield-Prothero case-study of monthly series of "Sales of Company X" January 1965-May 1971 and initial forecasts from May 1971. This graph was presented in Chatfield & Prothero (1973, p.298).
Table 1. The three types of coefficients of goodness-of-fit $M^{(p)}_m$ for the examples. For these monthly series the period's length assumed to be $p = 12$.

<table>
<thead>
<tr>
<th>Example</th>
<th>Type of Model</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Additive-$M^{(12)}_{1,a}$</td>
<td>Multiplicative-$M^{(12)}_{1,m}$</td>
<td>Mixed-$M^{(12)}_{1,mx}$</td>
</tr>
<tr>
<td>1</td>
<td>.712</td>
<td>.855</td>
<td>.886</td>
</tr>
<tr>
<td>2</td>
<td>.784</td>
<td>.928</td>
<td>.948</td>
</tr>
<tr>
<td>3</td>
<td>.876</td>
<td>.879</td>
<td>.890</td>
</tr>
<tr>
<td>4</td>
<td>.867</td>
<td>.884</td>
<td>.889</td>
</tr>
<tr>
<td>5</td>
<td>$M^{(12)}_{5,a}$</td>
<td>$M^{(12)}_{5,m}$</td>
<td>$M^{(12)}_{5,mx}$</td>
</tr>
</tbody>
</table>


The first 102 observations of this series were analysed. A typical part of the series is plotted in figure 3 below. The trend is clearly monotone and $p = 12$ is assumed. By looking at Table 1 it is straightforward that the multiplicative model has much greater coefficient of goodness-of-fit than the additive model. The coefficient for the mixed model is slightly greater than that of the multiplicative model and thus it is not so clear which model is more appropriate using the principle of parsimony. We personally prefer the multiplicative model for this well-known series. Box-Jenkins assumed multiplicative model as well.
Figure 3. A typical part of the monthly International Airline passengers series from January 1952 until December 1954. This graph was taken from Box & Jenkins (1970, p.300).

Example 3: Passenger miles (millions flown on domestic services by U. K. Airlines)

This series was analyzed by Anderson (1976) assuming an additive model. The series includes 119 observations from July 1962 until May 1972. The order of polytonicity of the trend is $m = 1$, i.e., monotonicity. Coefficient of monotonicity of the original series $\mu_1 = .337$ and the coefficients of goodness-of-fit for the three seasonality models are given in Table 1. Thus it seems that both the multiplicative and additive models are equally good and "better" than the mixed model that has only very slightly greater goodness-of-fit but used double the number of parameters.
Example 4: U.S. total retail sales in millions of dollars (Shiskin et al. 1967)

This series of 144 observations was analysed by Shiskin et al. in their example demonstrating the X-11 method. They used the multiplicative version. The coefficient of monotonicity for the original series $\mu_1 = .868$ indicate an increasing monotone of the trend. From the coefficients presented in Table 1 it seems that the multiplicative and mixed models have nearly the same goodness-of-fit and so the preferred model is the more simple, i.e., the multiplicative one.

Example 5: Unemployed men in the U.S.A. in the years 1949-58.

This series of 120 observations was analysed by B.L.S. (1966) using a multiplicative model. A graph of this series is presented in figure 4. We estimated the order of polytonicity as $m = 5$ and the turning points are the 11, 53, 68 and 100 observations. The coefficient of monotonicity $\mu_1$ is about zero but $\mu_5 = .71$. The coefficients of goodness-of-fit for the three models are given in Table 1. These coefficients indicate that the mixed model is supposed to be an adequate model. A multiplicative model is much better than an additive model.

Conclusions

The recently developed nonmetric approach to data analysis of periodic series, see Raveh (1981a, 1981b) is used for determining the suitable type of seasonality model. For purposes of demonstration five known examples were analyzed and the appropriate models were chosen. The resultant are in part similar to previous discussions and in part not. The same models were estimated for short sub-series of only 35 observations. From the presented examples it
seems that sometimes it is difficult to choose the appropriate model in spite of using the principle of parsimony.

Figure 4. Unemployed men in U.S.A. in the years 1949-1958.
References


