NEW BETTER THAN USED PROCESSES

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ALBERT W. MARSHALL and MOSHE SHAKED

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Abstract

A stochastic process \( \{Z(t), t > 0\} \), such that \( P(Z(0) = 0) = 1 \), is said to be new better than used (NEU) if, for every \( x \), the first passage time \( T_x = \inf \{ t : Z(t) > x \} \) satisfies \( P(T_x > s+t) < P(T_x > s) \) \( P(T_x > t) \) for every \( s > 0, t > 0 \). In this paper it is shown that many useful processes are NEU. Examples of such processes include processes with shocks and recovery, processes with random repair-times, various Gaver-Miller processes and some strong Markov processes. Applications in reliability theory, queueing, dams, inventory and electrical activity of neurons are indicated. It is shown that various waiting times for clusters of events and for short and wide gaps in some renewal processes are NEU random variables.

The NEU property of processes and random variables can be used to obtain bounds on various probabilistic quantities of interest; this is illustrated numerically.

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1. Introduction

Let \( \{Z(t), t > 0\} \) be a stochastic process such that \( Z(0) = 0 \) and \( Z(t) > 0 \) for all \( t > 0 \) with probability one. The process is said to be \textbf{new better than used (NEU)} if the first passage times

\[ T_x = \inf\{t > 0: Z(t) > x\} \]

have NEU distributions for all \( x > 0 \), i.e., if

\[
(1.1) \quad P(T_x > s+t|T_x > t) < P(T_x > s) \quad \text{for all} \quad s > 0, t > 0, x > 0
\]

such that \( P(T_x > t) > 0 \) (even if \( P(T_x = \infty) > 0 \)).

The purpose of this paper is to show that certain kinds of processes are NEU so that known facts about NEU distributions can be applied in the study of first passage times. Such results are obtained for a number of processes that arise in applications; for example, \( Z(t) \) might represent the value at time \( t \) of the virtual waiting time in a single server queue, the content of a dam, an inventory, or the level of electrical activity in a neuron. However, throughout this paper the terminology of reliability theory is used: \( Z(t) \) is referred to as the "wear of an item at time \( t \)" and the item fails when the wear exceeds a fixed threshold \( x \), so that \( T_x \) is the lifelength of the item.

Ross (1979) has introduced a different notion of an NEU process which requires \( Z(t) \) to be monotone in \( t \) and also requires

\[
P(T_x > s+t|Z(u), 0 < u < t) < P(T_x > s) \quad \text{for all} \quad s, t > 0.
\]

Our condition, which does not require monotone sample paths, is in fact an NEU analog of an increasing failure rate average (IFRA) process, defined by Ross (1979) as a process for which each \( T_x \) has an increasing failure rate average.
distribution. El-Neweihi, Proschan and Sethursman (1978) discuss processes
which are NEU in our sense but they restrict themselves to the case that
sample paths are monotone and each $Z(t)$ is integer-valued.

Ross (1979, 1981) has shown that some processes arising in reliability
theory are IFRA processes, so that these processes are NEU processes in our
sense. But there are a number of interesting NEU processes that are not IFRA
processes.

In this paper, four kinds of processes are considered. Three of these
(Sections 2–4) have sample paths that move in a deterministic manner (given
the present or past) between random points in time. At such points, the
sample paths possibly have random jumps. A fourth class of processes (Section
5) consists of Markov processes. An application is given in Section 6.

As usual, we write "increasing" for "nondecreasing" and "decreasing" for
"nonincreasing". Also, random variables which are identically zero are
regarded as being both NEU and NWU.

2. Processes with shocks and recovery.

The processes considered in this section unify several special cases that
have been previously studied. Some of these special cases are described here
to introduce the general case.

Let $A_1, A_2, \ldots$ be a sequence of independent, identically distributed
(iid) nonnegative random variables that represent times between shocks to a
device. Let $C_i$ be the damage inflicted by the $i$th shock and suppose that
$C_1, C_2, \ldots$ are iid and independent of $A_1, A_2, \ldots$. If

$$N(t) = \max\{n > 0 : \sum_{i=1}^{n} A_i < t\}$$

is the number of shocks experienced by time $t$ and damages accumulate additively then $Z(t) = \sum_{i=1}^{N(t)} C_i$ is the total
damage sustained by time $t$. In case the $C_i$ are nonnegative and the $A_i$
have an NEU distribution, \( \{Z(t), t > 0\} \) is an NEU process (Esary, Marshall and Proschan, 1973, together with A-Hameed and Proschan, 1975, Theorem 2.7, or Block and Savits, 1978, Section 3).

The more general case that wear is allowed to decrease between shocks ("recovery" takes place) in some deterministic fashion such as exponentially or linearly (but never below zero) has received considerable attention in the literature (see, e.g., Smith and Yeo, 1981). Many such processes are also NEU as a consequence of Theorem 2.1 below.

In general, let \( \{A_i\}_{i=1}^{\infty} \) and \( \{C_i\}_{i=1}^{\infty} \) be sequences of random variables and suppose that the \( A_i \) are positive. Let \( R_0 = 0 \), \( R_n = \sum_{i=1}^{n} A_i \), \( n = 1, 2, \ldots \). Then \( 0 = R_0 < R_1 < \ldots \). Our motivation is to define a process which jumps an amount \( C_n \) (possibly negative) at \( R_n \) and between jumps moves deterministically (given the magnitude and location of earlier jumps), all subject to the requirement that the process stays nonnegative.

Accordingly, for \( j = 0, 1, \ldots \), the deterministic behavior of the process in the interval \( (R_j, R_{j+1}) \) is to be governed by a function \( h_j \). Assume that for fixed \( 0 < r_1 < r_2 \ldots < r_j \) and \( c_1, \ldots, c_j \), \( h_j(r_1, \ldots, r_j; c_1, \ldots, c_j) \) is a measurable function defined on \( [r_1; r_2; \ldots; r_j; r_1] \) (\( h_o(\cdot) \) is a function of one nonnegative argument). Define \( \{Z(t), t > 0\} \) by

\[
Z(t) = h_j(R_1, \ldots, R_j; C_1, \ldots, C_j; t), \quad R_j < t < R_{j+1}, \quad j = 0, 1, \ldots.
\]

See Figure 2.3 for an example.

As indicated above, the motivating examples for this study also satisfy the condition

\[
[h_{j-1}(r_1, \ldots, r_{j-1}; c_1, \ldots, c_{j-1}; r_j) + c_j]^+ = h_j(r_1, \ldots, r_j; c_1, \ldots, c_j; r_j)
\]
so that the process $Z$ does indeed jump $C_j$ at $R_j$, subject to remaining nonnegative. However, this condition is not required in the following theorem, where the nonnegativity follows from (iii) and (v).

2.1 Theorem. Suppose that

(i) $A_1, A_2, \ldots$ are i.i.d. and NEU,

(ii) $C_1, C_2, \ldots$ are i.i.d. and independent of $\{A_1, A_2, \ldots\}$.

Suppose also that for every realization $(r_i, c_i)$, $i=1, 2, \ldots$, the functions $h_j$, $j=0, 1, \ldots$, satisfy

(iii) $h_0(t) = 0$, $t > r_0 = 0$,

(iv) $h_j(r_1, \ldots, r_j; c_1, \ldots, c_j; t) = h_j(r_1+\Delta, \ldots, r_j+\Delta; c_1, \ldots, c_j; t+\Delta)$, $\Delta > 0$, $t > r_j$,

(v) $h_j(r_1, \ldots, r_j; c_1, \ldots, c_j; t) > h_{j-1}(r_2-r_1, \ldots, r_j-r_1, c_2, \ldots, c_j; t-r_1)$, $t > r_j$.

Then $\{Z(t), t > 0\}$ is an NEU process.

Comment about (iv). Condition (iv) says that if $A_1 = R_1$ is replaced by $A_1 + \Delta$, then the resulting process $Z^*$ which develops according to the prescription of (2.1) has sample paths which satisfy

$$Z^*(t+\Delta) = Z(t), t > 0.$$

From this, it follows that

(2.3) $R_1$ and $T_x - R_1$ are independent.
Comment about (v). Condition (v) implies via an easy induction that

\( (v') h_j(r_1, \ldots, r_l, r_{l+1}, \ldots, r_j; c_1, \ldots, c_l, c_{l+1}, \ldots, c_j; t) \)

\( > h_{j-l}(r_{l+1-r_l}, \ldots, r_{j-r_l}; c_{l+1}, \ldots, c_j; t-r_l), t > r_j, j = l, l + 1, \ldots \)

This condition together with (iii) says that the process \( \tilde{Z} \) which develops from the sequences \( \{A_i\}_{i=l+1}^\infty, \{C_i\}_{i=l+1}^\infty \) according to the prescription of (2.1) has the property that

\( \tilde{Z}(t-R_x) < Z(t), t > R_x \)

---

Graph of \( Z(t) \)
---

Graph of \( \tilde{Z}(t-R_x) \)

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Figure 2.1. Comparison of \( Z \) and \( \tilde{Z} \).
(see Fig. 2.1). If \( \tilde{r}_x = \inf\{t > 0 : \tilde{Z}(t) > x\} \) and \( \tilde{r} = n - 1 \), this implies that for \( s > 0, \ t > 0 \),

\begin{equation}
(2.4) P(T_x > s + t | N(t) = n - 1, (R_i, C_i) = (r_i, c_i), i = 1, \ldots, n - 1) \nonumber
\end{equation}

\begin{equation}
< P(T_x > s + t - n - 1 | t - n - 1 < R_n) = P(T_x > s + t - n - 1 | t - n - 1 < R_1). \nonumber
\end{equation}

Proof of Theorem 2.1. From (2.4), it follows that

\[ p = P(T_x > s + u | N(t) = n - 1, (R_i, C_i) = (r_i, c_i), i = 1, \ldots, n - 1) \]

\[ < P(T_x > s + u | A_1 > u) \]

where \( u = t - n - 1 \). From (2.3), it follows that \( A_1 \) and \( T^* = T_x - R_1 \) are independent. Using the fact that \( A_1 \) is NEU, it follows that

\[ p < P(T^* + A_1 > s + u | A_1 > u) = \int_0^s P(A_1 > u + s - t^* | A_1 > u) dP(T^* < t^*) + \int_{s+}^\infty dP(T^* < t^*) \]

\[ < \int_0^s P(A_1 > s - t^*) dP(T^* < t^*) + \int_{s+}^\infty dP(T^* < t^*) \]

\[ = P(A_1 + T^* > s) = P(T_x > s). \]

From the inequality \( p < P(T_x > s) \), (1.1) follows by partially unconditioning in \( p \), but retaining the condition \( T_x > t \).

2.2 Example. Let \( h_j(r_1, \ldots, r_j; c_1, \ldots, c_j; t) = \sum_{i=1}^j [c_i]^t \), \( t > r_j \), be independent of \( t > r_j \), \( j = 0, 1, \ldots \). Then (iii), (iv) and (v) are immediate. This is the special case of A-Hameed and Proschan (1975, Theorem 2.7). On the other hand, \( h_j(r_1, \ldots, r_j; c_1, \ldots, c_j, t) = \sum_{i=1}^j [c_i]^t \) fails to satisfy (v) if some \( c_1 \) is negative.

If \( P(C_i > x) = 1 \), then \( T_x = A_1 \) (NEU by assumption). This shows that without additional restrictions on the common distribution of the \( A_1 \) in
Theorem 2.1, no conclusion about the distribution of $T_{x}$ stronger than NEU is possible.

2.3. Example. In many applications (2.2) holds and the deterministic development of the process between jumps is of the form: "the rate of change depends only on the height of the process". Then the functions $h_{j}$ must all be increasing or all decreasing in $t$. Suppose this monotonicity is strict where the function is positive, and suppose that the $h_{j}$ are continuous in $t$. Then there exists a nonnegative continuous function $g$ (see Fig. 2.2), strictly monotone on $(-\infty;0]$ or on $[0,\infty)$, such that

\[ h_{0}(t) = 0, \quad t > 0, \]
\[ h_{j}(r_{1}, \ldots, r_{j}; c_{1}, \ldots, c_{j}; t) = g(t - r_{j} + g^{-1}([h_{j-1}(r_{1}, \ldots, r_{j-1}, c_{1}, \ldots, c_{j-1}; r_{j}) + c_{j} t]), \quad t > r_{j}, j = 1, 2, \ldots. \]

Figure 2.2. A typical function $g$.

To check the conditions of Theorem 2.1 for this case, notice, by induction, that for $j = 0, 1, \ldots$, $h_{j}(r_{1}, \ldots, r_{j}; c_{1}, \ldots, c_{j}; t)$ depends on $r_{1}, \ldots, r_{j}$ and $t$ only through $t - r_{j} r_{j} - r_{j-1}, \ldots, r_{2} - r_{1}$, hence (iv) holds. Since $g$ is nonnegative,
\[ h_1(r_1; c_1; t) = g(t - r_1 + g^{-1}(c_1^+)) > h_0(t - r_1), \quad t > r_1, \]

(v) holds for \( j = 1 \). Assume (v) holds for \( j = n-1 \). By this induction hypothesis and the monotonicity (in the same direction) of \( g \) and \( g^{-1} \),

\[
\begin{align*}
\frac{h_n(r_1, \ldots, r_n; c_1, \ldots, c_n; t)}{g(t - r_1 + g^{-1}(h_{n-1}(r_1, \ldots, r_{n-1}; c_1, \ldots, c_{n-1}; r_n) + c_n^+))}
&< g((t - r_1) - (r_n - r_1) + g^{-1}(h_{n-2}(r_2 - r_1, \ldots, r_{n-1} - r_1; c_2, \ldots, c_{n-1}; r_n - r_1) + c_n^+)) \\
&= h_{n-1}(r_2 - r_1, \ldots, r_{n-1} - r_1; c_2, \ldots, c_{n-1}; t - r_1). 
\end{align*}
\]

Thus (v) holds.

For the case of Example 2.3, it is not difficult to see that the hypotheses of Theorem 2.1 can be weakened as follows. In place of the assumption that \( C_1, C_2, \ldots \) are identically distributed, assume that they are stochastically increasing, i.e. \( P(C_i > c) \) is increasing in \( i \) for all \( c \).

In case the \( C_i \) are all nonnegative, the assumption that \( A_1, A_2, \ldots \) are identically distributed can be replaced by the assumption that \( A_1, A_2, \ldots \) are stochastically decreasing, i.e., \( P(A_i > a) \) is decreasing in \( i \) for all \( a \). The essential point is that with these modified assumptions, the equality of (2.4) becomes an inequality.

Several particular special cases of Example 2.3 are of interest.

2.3a. Example (Linear Recovery). If \( g(s) = [-s]^+ \) then

\[
\begin{align*}
h_1(r_1; c_1; t) &= [c_1^+ - (t - r_1)]^+,
\end{align*}
\]

\[
\begin{align*}
h_j(r_1, \ldots, r_j; c_1, \ldots, c_j; t) &= [(h_{j-1}(r_1, \ldots, r_{j-1}; c_1, \ldots, c_{j-1}; r_j) + c_j]^+ - (t - r_j)]^+,
\end{align*}
\]

\( j = 2, 3, \ldots \). For the resulting process, a typical realization is given in Figure 2.3.
2.3b. **Example. (Short Gap in a Renewal Process).** Gilbert and Pollak (1957) considered the distribution of the waiting time for a "cluster of two events" in a Poisson process. This is the time of the first event in the process which follows the preceding event by less than $\alpha$ units of time (here, the origin is not counted as an event time). By taking $C_1, C_2$ to be degenerate at $\alpha$, and by taking $x = 3\alpha/2$, say, this waiting time is $T_x$ of Example 2.3a, so it has an NHU distribution. The particular case of Gilbert and Pollak (1957) is obtained when $A_1, A_2, \ldots$ are independent and have a common exponential distribution. Generalizations of this result is obtained in a different way in Examples 2.4 and 2.5.

2.3c. **Example. (Large gap in a renewal process).** Let $\{N(t), t > 0\}$ be the renewal process with interarrival times $A_1, A_2, \ldots$. Let $T^*$ be the first time after the first renewal for which there is no renewal in the interval $(T^*-\alpha, T^*], \text{i.e.},$

$$T^* = \inf(t > A_1: t-N(t) > \alpha).$$
Then $T^*$ is the waiting time for a large gap between renewals when the origin is not considered to be a renewal point. The case that the origin is regarded as a renewal point, so $T^*$ need not exceed $A_1$, is considered in Example 3.2.

Construct the process \( \{Z(t), t > 0\} \) according to (2.1) using (2.5) with \( g(t) = t, t > 0 \), and with \( P(C_i = -\alpha) = 1, i = 1, 2, \ldots \). Then $T^* - T_\alpha$ is NEU by Theorem 2.1. This result is obtained by a different method in Example 3.2.

If \( N(t) \) is a Poisson process recorded by a Type II counter (See Feller, 1971, p. 189) then the successive times that the counter becomes unblocked form a renewal process with waiting times between renewals that have the same distribution as $T^* = T_\alpha$, which we have just noted is NEU.

2.3d. Example. (Exponential Recovery). If \( g(s) = e^{-\theta s} \), then

$$h_1(r_1, c_1; t) = c_1 e^{-\theta (t-r_1)}$$

$$h_j(r_1, \ldots, r_j; c_1, \ldots, c_j; t) = [h_{j-1}(r_1, \ldots, r_{j-1}; c_1, \ldots, c_{j-1}; r_j; c_j)^{+} e^{-\theta (t-r_j)}]^{j-1} e^{-\theta (t-r_j)}, \quad j = 2, 3, \ldots$$

Here, recovery is at an exponential rate, rather than linear as in Example 2.3a.

2.4 Example. Leslie (1969) has considered the waiting time until the occurrence of a "cluster of size k", $k > 2$, in a Poisson process. For the purposes of this example, a cluster of size $k$ is said to occur at the $k$th of a group of $k$ renewals if no gap between successive renewals exceeds $\alpha$, a
prescribed positive number. More formally, a cluster of size $k$ is said to occur at $t$ if

(i) for some $m > k$, $t = R_m$

(ii) $R_{m-1} - R_{m-2-1} < a$, $t = 0, 1, \ldots, k - 2$.

The time of first occurrence of a cluster is the passage time $T_{k-1/2}$ of a process $Z$ which increases by one at renewal points $R_i$ and drops to 0 whenever $a$ units of time have elapsed since the last renewal. Formally, let $Z$ be defined by (2.1) with

$$h_j(r_1, \ldots, r_i, c_1, \ldots, c_j, t) = h_{j-1}(r_1, \ldots, r_{j-1}, c_1, \ldots, c_{j-1}, r_j) + c_j \quad \text{if} \quad t \in [r_j, r_j + a)$$

$$= 0 \quad \text{if} \quad t > r_j + a$$

and let $C_i$ be degenerate at 1, $i = 1, 2, \ldots$, (See Figure 2.4). If $A_1, A_2, \ldots$ are iid and NHU, then it follows from Theorem 2.1 that the waiting time $T_{k-1/2}$ has an NHU distribution.

---

Figure 2.4. A typical realization of the process of Example 2.4.
2.5. Example. A definition of a "cluster" of size $k > 2$ alternative to that of Example 2.4 says that a cluster of size $k$ occurs at time $t$ if

(i) for some $m > k$, $t = R_m$

(ii) $R_m - R_{m-k+1} < \alpha$.

Although he mentioned this definition, Leslie (1969) does not study the distribution of waiting time to this kind of cluster because of the mathematical difficulties involved. On the other hand it is probably the most interesting definition of a cluster for many applications.

The waiting time to this kind of cluster is a first passage time $T_{k-\frac{1}{2}}$ in the process $Z(t) = N(t) - N(t-\alpha)$ where $N$ is the renewal process generated by the interarrival times $A_1, A_2, \ldots$. This process takes the form of (2.1) if $C_i = 1$, $i = 1, 2, \ldots$, and $h_j(r_1, \ldots, r_j, c_1, \ldots, c_j, t)$ is the number of $r_1, \ldots, r_j$ in the interval $(t - \alpha, t]$, (see Figure 2.5). If $A_1, A_2, \ldots$ are NEU, then Theorem 2.1 shows that the waiting time $T_{k-\frac{1}{2}}$ to a cluster is NB

---

Figure 2.5. A typical realization of the process of Example 2.5.
Of course the definitions of a "cluster" used in Examples 2.4 and 2.5 coincide when \( k = 2 \). This is the case of Example 2.3b which was studied by Gilbert and Pollak (1957) when \( A_1, A_2, \ldots \) have a common exponential distribution (the underlying renewal process is Poisson). This special case is further discussed in Section 6. With \( k = 2 \), the passage times \( T_{k-1/2} \) of Examples 2.4 and 2.5 are not necessarily NEU if the interarrival times \( A_j \) are not NEU. To see this, suppose that \( P(A_j = a) = p \), \( P(A_j = b) = 1-p \), \( p \in (0,1) \), \( 0 < 3a < b \) and \( \alpha = 2a \). Then for \( s \) sufficiently small,

\[
1 = P(T_{3/2} > 2a+s \mid T_{3/2} > 2a) > P(T_{3/2} > s),
\]

so \( T_{3/2} \) is not NEU.

3. A random repair-times process.

A process of particular interest in storage theory [Moran, 1959, p. 80] and queueing theory [Prabhu, 1965, p. 102] again has successive times \( B_1, B_2, \ldots \) between shocks that are independent identically distributed and nonnegative. But at the occurrence of the \( i \)th shock the process has a nonpositive jump \( D_i \) (the process is set equal to zero if such a jump would carry it below zero). Again, \( D_1, D_2, \ldots \) are independent, identically distributed and independent of \( B_1, B_2, \ldots \). The process starts at zero; before the first shock and between successive shocks, the process increases in some deterministic fashion. In the context of reliability theory "shocks" might represent repairs, with continuous wear between repairs. Many such processes are NEU processes as a consequence of Theorem 3.1 below.
In contrast with the processes of Section 2, no restrictions on the \( B_i \) and \( D_i \) are imposed except \( P(B_i > 0) = 1, P(D_i < 0) = 1 \). However the deterministic increase of the process is of the form: "the rate of increase depends only on the height of the process", as in Example 2.3.

To formally define the process described above, let \( g: [0, \infty) \rightarrow [0, \infty) \) be a strictly increasing function such that \( g(0) = 0 \). For \( s > 0, u > 0 \), define a function \( h_{s,u} \) on \([s, \infty)\) by

\[
h_{u,s}(t) = g(t - s + g^{-1}(u)), \quad t > s.
\]

Let \( S_0 = 0 \), and \( S_n = \sum_{i=1}^{n} B_i \). Then the process \( \{U(t), t > 0\} \) is defined by

\[
U(t) = g(t) = h_{0,0}(t), \quad t < B_1 = S_1 = [g(s_1) + D_1]^+, \quad t = S_1 \]

\[= h_{U(S_1),S_1}(t), \quad S_j < t < S_{j+1} \]

\[= h_{U(S_j),S_j}(S_{j+1}) + D_{j+1})^+, \quad t = S_{j+1}, j = 1, 2, \ldots
\]

3.1. Theorem. Suppose that

\[(i) \quad B_1, B_2, \ldots \text{ are i.i.d. random variables such that } P(B_1 > 0) = 1, \]

\[(ii) \quad D_1, D_2, \ldots \text{ are i.i.d. random variables, independent of } B_1, B_2, \]

such that \( P(D_1 < 0) = 1 \).

Then the process \( \{U(t), t > 0\} \) is an NEU process.
Proof. For each \( x > 0 \), it is necessary to show that \( T_x = \inf \{ t : U(t) > x \} \) has an NEU distribution, i.e.,

\[
P(T_x > s + t | T_x > t) < P(T_x > s), \quad s, t > 0;
\]

this can be accomplished by conditioning on the process up to time \( t \).

Fix \( t \), let \( N(t) = \max \{ n : B_1 + \ldots + B_n < t \} \) and suppose that \( N(t) = n - 1 \).
To insure that \( T_x > t \), consider \( (s_i, d_i), i = 1, \ldots, n-1 \) with the property that the corresponding realization \( u \) of \( U \) satisfies \( U(v) < x \) for all \( v < t \). By definition of \( n \), there are no renewals in the time interval \( (s_{n-1}, t] \). Such a renewal could only decrease the sample path of the process, because \( B_n < 0 \) and the rate of increase of the process depends only on its height. Thus, such a renewal could only increase \( T_x \), and hence

\[
p = P(T_x > s + t | (S_1, D_1) = (s_1, d_1), \quad i = 1, \ldots, n-1, \quad B_n > t - s_{n-1})
\]

\[
< P(T_x > s + t | (S_1, D_1) = (s_1, d_1), \quad i = 1, \ldots, n-1).
\]

Let \( \tilde{U} \) be the process defined in the same way as \( U \) but in terms of the sequences \( B_n, B_{n+1}, \ldots \), and \( B_n, B_{n+1}, \ldots \), and let \( \tilde{T}_x = \inf \{ v : \tilde{U}(v) > x \} \).
Then

\[\tilde{U}(v - s_{n-1}) < U(v) \text{ for all } v > s_{n-1},\]

(see Figure 3.1) so that \( \tilde{T}_x + s_{n-1} > T_x \). Let \( \tilde{t} = t - s_{n-1} \). Then
\[ p < P(T_x > s + t | (S_i, D_i) = (s_i, d_i), i=1, \ldots, n-1) < P(T_{x^+ t} > s + t) \]

(3.4)

\[ = P(T_x > s + t) = P(T_x > s + t) < P(T_x > s) \]

The inequality (3.2) follows by partially unconditioning in the inequality \( p < P(T_x > s) \), retaining the condition \( T_x > t \).

\[ \text{Figure 3.1. Comparison of } U \text{ and } \hat{U}. \]

Remark. A simple modification of the above proof shows that if conditions of Theorem 3.1 are replaced by the conditions that \( B_1, B_2, \ldots, \) and \( D_1, D_2, \ldots \) are independent, \( P(B_i > 0) = 1, P(D_i < 0) = 1 \) and

\[ P(B_i > b) \text{ is increasing in } i = 1, 2, \ldots, b > 0, \]

\[ P(D_i > d) \text{ is decreasing in } i = 1, 2, \ldots, d < 0, \]

then the process \( \{U(t), t > 0\} \) is NEU. In this case, the second equality of (3.4) becomes an inequality.
3.1. **Example.** (Wide gap in a renewal process). The distribution of the waiting time, $T$, until a wide gap in a renewal process plays an important role in various applications (Feller, 1971, p. 189). In the sense of this example the origin is regarded as a renewal point. If $B_1, B_2, \ldots$ are the waiting times between renewals and $N(t) = \max\{n: B_1 + \ldots + B_n < t\}$ then

$$T = \inf\{t < 0: t - (B_1 + \ldots + B_{N(t)}) = \alpha\}.$$

To show that $T$ is NEU let $U$ be the process defined by (3.1) with $B_1, B_2, \ldots$ as already introduced here, with $g(t) = t^+$ and with $P(D_i = -\alpha) = 1, i = 1, 2, \ldots$ (See Figure 3.2). Then

$$T = \inf\{t > 0: U(t) > \alpha\} = T_\alpha$$

is NEU by Theorem 3.1.

---

**Figure 3.2.** A typical realization of the process of Example 3.1.
Suppose that $A_i = B_i, i = 1, 2, \ldots$ so that the renewal process of this example is the same as that of Example 2.3.c. Then $T^*$ of Example 2.3.c can be written as $T^* = T + B_1$. Since $T$ is NEU, it follows that $T^*$ is NEU whenever $B_1$ is NEU, because the convolution of NEU distributions is NEU. Thus the conclusion of Example 2.3.c is obtainable from Example 3.2.


Gaver and Miller (1962) consider a process \{Z(t), t > 0\} with continuous sample paths that alternately increase and decrease in a deterministic fashion (e.g., linearly with fixed slopes as long as they remain nonnegative). The random duration $B_1$ of the \textit{i\textsuperscript{th}} period of increase and the random duration $A_1$ of the following period of decrease have the property that $A_1, A_2, \ldots$ are identically distributed, $B_1, B_2, \ldots$ are identically distributed and all of these random variables are independent.

If the $A_i$ are NEU and if the rates of increase and decrease of the process (in the appropriate time intervals) depend only on the height of the process, then \{Z(t), t > 0\} is an NEU process. The proof of this fact uses arguments similar to those of Theorems 2.1 and 3.1, and is omitted.

5. Some Strong Markov NEU Processes

Let \{X(t), t > 0\} be a Wiener process, so that $X(0) = 0$. Intuitively, it is reasonable to expect that $|X(t)|$ is an NEU process because $|X(t)|$ is "as far from $x$" as possible when $t = 0$. More generally, let $(X_1(t), \ldots, X_n(t), t > 0)$ be a Brownian motion in $\mathbb{R}^n$, and for $t > 0$ let $Y(t) = [\sum_{i=1}^{n} X_i^2(t)]^{1/2}$ be a Bessel process. Then again it is reasonable to expect that $Y(t)$ is an NEU process. The next theorem shows that these processes are indeed NEU.
5.1 Theorem. Let \( \{Y(t), t > 0\} \) be a strong Markov process with state space \([0, \infty)\). If

(i) \( Y(0) = 0 \),

(ii) \( \{Y(t), t > 0\} \) has stationary transition probabilities,

(iii) with probability 1, the sample paths of \( Y \) have no positive jumps,

then \( \{Y(t), t > 0\} \) is an NEU process.

Proof. Fix \( x > 0 \). For each \( u, 0 < u < x \), let \( \{Y_u(t), t > 0\} \) be a process which has the same transition probabilities as \( \{Y(t), t > 0\} \), but such that \( Y_u(0) = u \). Let

\[
T_{u, x} = \inf\{t > 0: Y_u(t) > x\}.
\]

By (ii)

\[
(5.1) \quad P\{T_{0, x} > s + t | T_{0, x} > t, Y(t) = u\} = P\{T_{u, x} > s\}.
\]

By the strong Markov property and (iii), \( T_{0, u} \) and \( T_{u, x} = T_{0, x} - T_{0, u} \) are independent; and \( T_{u, x} \) has the same distribution as \( T_{u, x} \). Thus

\[
T_{0, x} = T_{0, u} + \tilde{T}_{u, x}
\]

is stochastically larger than \( T_{u, x} \), that is,

\[
(5.2) \quad P\{T_{u, x} > s\} < P\{T_{0, x} > s\}
\]
By combining (5.1) and (5.2) and partially unconditioning, it follows that

\[ P(T_{0,x} > s + t | T_{0,x} > t) < P(T_{0,x} > s), \]

where \( T_{0,x} \) is NEU.

There is an analog of Theorem 5.1 for processes \( \{Y(t), t > 0\} \) that are nonnegative and integer-valued. Here (iii) is replaced by the assumption that the process is free of positive skips, that is, sample paths cannot have positive jumps greater than one.

5.2. **Theorem.** Let \( \{Y(t), t > 0\} \) be a strong Markov process with state space \( \{0, 1, 2, \ldots\} \). If

1. \( Y(0) = 0, \)
2. \( \{Y(t), t > 0\} \) has stationary transition probabilities,
3. \( \{Y(t), t > 0\} \) is free of positive skips,

then \( \{Y(t), t > 0\} \) is an NEU process.

The proof of Theorem 5.2 is similar to the proof of Theorem 5.1 and is omitted.

Note that if (iii) of Theorem 5.1 or (iii) of Theorem 5.2 fails, the conclusion of these theorems may fail. To see this for Theorem 5.2, assume that (i) and (ii) hold and suppose that the process can jump from the state 0 to the state \( n_0(>2) \) with probability \( p \) or to state 1 with probability \( 1-p \). Further, suppose that the process jumps from state 1 to state \( i+1 \) with probability \( 1, i = 1, 2, \ldots, n_0 \). If \( x = n_0 - \frac{1}{2} \) and the sojourn times of the process in each state are exponential with mean 1, then
\[ P(T > t) = pe^{-t} + (1-p) e^{-t} \sum_{i=0}^{n-1} \frac{1}{i!} \] 

To see that this survival function is not NHU, take, e.g., \( p = .9 \) and \( n_0 = 3 \); then

\[ P(T_x > 2|T_x > 1) > P(T_x > 1). \]

Assumption (iii) of Theorem 5.1 and 5.2 can be replaced by the assumption that \( \{Y(t), t > 0\} \) is stochastically monotone to obtain the same conclusion (see Narlow and Proschan, 1976, Lemma 2.8). This result and the above theorems complement each other in the sense that each can be applicable when the other is not.

Birth and death processes which start at 0 are free of positive skips and satisfy the conditions of Theorem 5.2. For this case, Nelson (1979, p. 64) obtained that \( T_x \) is PP*, a much stronger conclusion than NHU. Related results have been obtained by Rosler (1980), Clarotti (1981), Brown and Rao (1980) and Derman, Ross and Schechner (1979).

7. An application.

Gilbert and Pollak (1957) obtained an explicit expression for the distribution of the waiting time until a cluster of two events (gap \( < \alpha \)) in a Poisson process with intensity \( \lambda \). This is the distribution of the random variable \( T_x \) of Example 2.3b. The survival function of \( T_x \) is

\[ \Phi(t) = e^{-\lambda t} \sum_{k=0}^{1+[t/\alpha]} \frac{(\lambda t)^k}{k!} \left( 1 - \frac{(k-1)\alpha}{t} \right)^k, t > 0. \]
When \( t \) is large the numerical computation of (6.1) can be tedious, because for large \( t \), the number of terms in the sum is large. Gilbert and Pollak (1957) derived the following bounds on \( \overline{F}(t) \),
\[
(6.2) \quad (1-\alpha/\lambda)e^{(s-\lambda)a}e^{-at} < \overline{F}(t) < e^{(a-\lambda)a}e^{-at} \equiv \overline{H}(t), \quad t > 0,
\]
where \( a = -s \) and \( s \) is the largest real root of
\[
s + \lambda = \lambda e^{-(s+\lambda)a}.
\]

Notice that \( \overline{H} \) is not necessarily a survival function.

The upper bound of (6.2) can be improved by using the fact, shown in Example 2.3b, that \( F \) is NEU.

For fixed \( t_0 > 0 \) and any NEU distribution \( F \), Marshall and Proschan (1972) show that
\[
\overline{F}(t) < \overline{C}_{t_0}^{-1}(t), \quad t > 0
\]
where
\[
\overline{C}_{t_0}(t) = (\overline{F}(t_0))^k, \quad k_0 < t < (k+1)t_0,
\]
\[
k = 0, 1, 2, \ldots
\]

Thus, an improvement of the upper bound in (6.2) is
\[
(6.3) \quad \overline{F}(t) < \min (\overline{H}(t), \overline{C}_{t_0}(t)), \quad t > 0.
\]

When \( t \) is large enough, then \( e^{(a-\lambda)a}e^{-at} < \overline{C}_{t_0}(t) \) and (6.3) is the same as the right hand side inequality in (6.2). However, when \( t \) is
moderate (6.3) can substantially improve (6.2). The idea is to choose \( t_0 \) small enough, so that the computation of \( \mathcal{H}(t_0) \) is relatively simple, and then use the upper bound \( \mathcal{H}_0(t_0) \), which may be smaller than \( \mathcal{H}(t) \) for moderate \( t \).

Table 6.1 gives some numerical comparisons of \( \mathcal{H}(t) \) and \( \mathcal{H}_0(t) \) for \( t_0 = 1,2,3,4 \). For purposes of the numerical illustration we set the parameters \( \lambda = 1 \) and \( \alpha = 1 \). Then \( a = 0.4328565 \), \( \mathcal{H}(t) = \exp\{0.5671435 - 0.4328565t\} \), \( \mathcal{F}(1) = 0.7357593 \), \( \mathcal{F}(2) = 0.4736738 \), \( \mathcal{F}(3) = 0.3070202 \) and \( \mathcal{F}(4) = 0.1991821 \).

### Table 6.1: Comparison of \( \mathcal{H}(t) \) and \( \mathcal{H}_0(t) \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \mathcal{H}(t) )</th>
<th>( \mathcal{H}_1(t) )</th>
<th>( \mathcal{H}_2(t) )</th>
<th>( \mathcal{H}_3(t) )</th>
<th>( \mathcal{H}_4(t) )</th>
<th>( \mathcal{F}(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.14372</td>
<td>0.7357593</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.7357593</td>
</tr>
<tr>
<td>2</td>
<td>0.7418793</td>
<td>0.5413415</td>
<td>0.4736738</td>
<td>1</td>
<td>1</td>
<td>0.4736738</td>
</tr>
<tr>
<td>3</td>
<td>0.4812226</td>
<td>0.3982967</td>
<td>0.4736738</td>
<td>0.3070202</td>
<td>1</td>
<td>0.3070202</td>
</tr>
<tr>
<td>4</td>
<td>0.3121468</td>
<td>0.2933050</td>
<td>0.2243668</td>
<td>0.3070202</td>
<td>1</td>
<td>0.1991821</td>
</tr>
<tr>
<td>5</td>
<td>0.2024751</td>
<td>0.2156144</td>
<td>0.2243668</td>
<td>0.3070202</td>
<td>1</td>
<td>0.1991821</td>
</tr>
<tr>
<td>6</td>
<td>0.1313362</td>
<td>0.1586402</td>
<td>0.1062766</td>
<td>0.0442614</td>
<td>0.0442614</td>
<td>0.1991821</td>
</tr>
<tr>
<td>7</td>
<td>0.0851917</td>
<td>0.1167209</td>
<td>0.1062766</td>
<td>0.0442614</td>
<td>0.0442614</td>
<td>0.1991821</td>
</tr>
<tr>
<td>8</td>
<td>0.0552599</td>
<td>0.0858784</td>
<td>0.0503404</td>
<td>0.0442614</td>
<td>0.0442614</td>
<td>0.0396735</td>
</tr>
<tr>
<td>9</td>
<td>0.0358445</td>
<td>0.0631858</td>
<td>0.0503404</td>
<td>0.02894014</td>
<td>0.02894014</td>
<td>0.0396735</td>
</tr>
<tr>
<td>10</td>
<td>0.0232507</td>
<td>0.0464893</td>
<td>0.0230449</td>
<td>0.02894014</td>
<td>0.02894014</td>
<td>0.0396735</td>
</tr>
</tbody>
</table>
As can be seen from Table 6.1, $\overline{G}_{t_0}(t)$ is a substantial improvement of $\overline{H}(t)$ when $t = kt_0$ for some moderate integer $k$. This observation suggests the following procedure of choosing $t_0$ illustrated with $\lambda = 1$.

Assume that bound on $\overline{F}(t)$ is needed for a fixed moderate $t$ (e.g., $t = 19$). Decide about the number $n$ of terms in (6.1) that you are willing to compute (e.g., $n = 6$). Find the largest $t_0 < (n-1)\alpha$ such that $t/t_0$ is an integer, that is, such that $t_0 = (t/(an\text{ integer})) < \alpha(n-1)$ (in the present illustration $t_0 = [19/(an\text{ integer})] < 5$, i.e., $t_0 = 19/4 = 4.75$). Denote the integer by $m$ (so $m = 4$). The upper bound on $\overline{F}(t)$ is $[\overline{F}(t_0)]^m$. (In the present illustration $\overline{G}_{4.75}(19) = [\overline{F}(4.75)]^4 = .0004296$; this is an improvement over $\overline{H}(19) = .004727$. The lower bound of (6.1) for $t = 19$ is $\mu = .002681$.)

Marshall and Proschan (1972) also give a lower bound,

$\overline{H}(t) < \overline{F}(t)$, where

$$\overline{H}(t) = \begin{cases} 
1 - t/\mu & t < \mu \\
0 & \mu \leq t
\end{cases},$$

and $\mu = ET = \frac{1-(1-e^{-\lambda \alpha})^2}{\lambda e^{-\lambda \alpha}}$.

Since usually $\mu$ is a small number, the bound $\overline{H}(t)$ is not useful here, because it is easy to compute $\overline{F}(t)$ explicitly for $t < \mu$. 
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