SHOCK PROCESSES WITH AFTEREFFECTS AND
MULTIVARIATE LACK OF MEMORY

BY
S. G. GHURYE and ALBERT W. MARSHALL

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ABSTRACT

If the survival function \( \bar{F}(x_1, \ldots, x_n) = P(X_1 > x_1, \ldots, X_n > x_n) \)
satisfies the functional equation

\[
\bar{F}(0^-) = 1 \text{ and } \bar{F}(x + t_\infty) = \bar{F}(x)\bar{F}(t_\infty) \text{ for all } x \in [0, \infty)^n \text{ and } t \geq 0,
\]

where \( \mathbf{e} = (1, \ldots, 1) \), and if the marginal distributions are exponential,
then \( \bar{F} \) is the multivariate exponential distribution of Marshall and Olkin. The functional equation has many solutions if the requirement
of exponential marginals is not imposed, but the class of possible marginals
is somewhat limited (e.g., marginals must be absolutely continuous). The
class of possible solutions of the equation is characterized in this paper,
and several examples are obtained from models for dependence that may be
of practical interest.

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2/ Research sponsored in part by the Natural Sciences and Engineering Research
Council of Canada, and in part by the National Science Foundation at
Stanford University.
1. **INTRODUCTION.** For any random vector \( X = (X_1, \ldots, X_n) \) with distribution function \( F \), the function \( \bar{F} \) defined on \( \mathbb{R}^n \) by

\[
\bar{F}(x_1, \ldots, x_n) = P(X_1 > x_1, \ldots, X_n > x_n)
\]

is called the **survival function** of \( X \) or of \( F \). In case \( n = 1 \), the exponential distribution is well known to be characterized in terms of the survival function by the functional equation

\[
(1.1) \quad \bar{F}(0-) = 1 \quad \text{and} \quad \bar{F}(x+y) = \bar{F}(x)\bar{F}(y), \quad x, y \geq 0.
\]

For \( n > 1 \), Marshall and Olkin (1967) notice that (1.1) implies \( \bar{F} \) is a product of marginal exponential survival functions. They also consider the weaker functional equation

\[
(1.2) \quad \bar{F}(0-) = 1 \quad \text{and} \quad \bar{F}(\chi + t\xi) = \bar{F}(\chi)\bar{F}(t\xi) \quad \text{for all} \quad \chi \in [0, \infty)^n \quad \text{and} \quad t \geq 0,
\]

where \( \xi = (1, \ldots, 1) \). This equation has sometimes been called the "lack of memory" property. With \( n = 2 \), Marshall and Olkin (1967) show that the only solutions of (1.2) with exponential marginals have the form

\[
(1.3) \quad \bar{F}(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)}, \quad x_1, x_2 \geq 0.
\]

The functional equation (1.2) has many solutions if the requirement of exponential marginals is not imposed, but the class of possible marginals is somewhat limited as is shown in Section 2. Section 3 contains a representation of the general solution of (1.2) and some examples.
The most interesting examples of solutions to (1.2) arise from models for dependence. The different models introduced in Section 4 have independent interests; some can be viewed as multivariate competing risk models, or they can be viewed as shock models with aftereffects.

2. THE POSSIBLE MARGINAL DISTRIBUTIONS. Take \( n = 2 \) and denote the marginal distributions of \( F \) by \( G \) and \( H \).

2.1. Lemma (Marshall and Olkin, 1967). If \( F \) satisfies (1.2) then either (i) \( F \) concentrates its mass on the set \( \{(x, y) : xy = 0, x \geq 0, y \geq 0\} \), or (ii) for some \( \theta \in (0, \infty) \),

\[
\bar{F}(x, y) = e^{-\theta y}G(x-y), \quad x \geq y \geq 0,
\]

\[
= e^{-\theta x}H(y-x), \quad y \geq x \geq 0.
\]

Thus any pair \( G, H \) of marginals determines a solution of (1.2), but the solution they determine via (2.1) is not necessarily a survival function.

2.2. Theorem. Suppose that (2.1) holds for some \( \theta \in (0, \infty) \). Then \( \bar{F} \) is a survival function if and only if
(i) $G$ and $H$ are both degenerate at 0,

or

(ii) $G$ and $H$ are both absolutely continuous with right-hand derivatives

$$g(u) = \lim_{\delta \to 0} \frac{G(u) - G(u + \delta)}{\delta}, \quad h(u) = \lim_{\delta \to 0} \frac{H(u) - H(u + \delta)}{\delta}$$

which are right-continuous, are of bounded variation and have at most a countable number of discontinuities; further,

$$e^{\theta u} g(u) \text{ is nondecreasing in } u \geq 0,$$

$$e^{\theta u} h(u) \text{ is nondecreasing in } u \geq 0,$$

$$G(u) + H(u) \geq 1 - e^{-\theta u} \text{ for all } u \geq 0.$$ 

The proof of this theorem is obtained by combining the results of several propositions.

2.3. **Proposition.** If (1.2) holds and $F$ given by (2.1) is a survival function, then either

(i) $G$ and $H$ are both degenerate at 0,

or

(ii) neither $G$ nor $H$ place positive mass at 0.

**Proof:** From (2.1) it follows that $\overline{F}(x,x) = e^{-\theta x} \overline{G}(0) = e^{-\theta x} \overline{H}(0)$. Thus $\overline{G}(0) = \overline{H}(0)$. Now from (1.2), it follows that $\overline{F}(x+t, x+t) = \overline{F}(t, t) \overline{F}(x, x)$, i.e.,
\( e^{-\theta(x+t)}G(0) = e^{-\theta t}G(0)e^{-\theta x}G(0) \),

so \( \overline{G}(0) = [\overline{G}(0)]^2 \). Thus \( \overline{G}(0) = 0 \) or \( \overline{G}(0) = 1 \).

Notice that if \( \overline{G}(0) = 0 \), case (i) of Lemma 2.1 occurs, and \( \theta \) is arbitrary.

2.4. **Proposition.** If (2.1) holds and \( \overline{F} \) is a survival function, then

\[
\overline{G}(u) - \overline{G}(u+a) \leq e^{\theta b} \overline{G}(u+b) - \overline{G}(u+b+a), \quad a, b, u \geq 0 ,
\]

(2.5) \[
\overline{G}(u) - \overline{G}(u+b) \leq (e^{\theta b} - 1)\overline{G}(u+b), \quad u, b \geq 0 .
\]

**Proof:** (2.5) is equivalent to the fact that \( F \) places nonnegative mass on the rectangle

\[
\{(r,s): x \leq r \leq x + a, \quad y \leq s \leq y + b\}
\]

where \( x - y = u + b \). By letting \( a \to \infty \) in (2.5), (2.6) is obtained.

2.5. **Proposition.** If \( \overline{G}(0) = 1 \) and (2.6), then \( G \) is absolutely continuous.

**Proof:** It is necessary to show that for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( 0 \leq u_1 - b_1 < u_1 \leq u_2 - b_2 < u_2 \leq \ldots \leq u_m - b_m < u_m \) together with \( \sum_{1}^{m} |u_i - (u_i - b_i)| = \sum_{1}^{m} b_i \leq \delta \) implies \( \sum_{1}^{m} |\overline{G}(u_i - b_i) - \overline{G}(u_i)| \leq \varepsilon \). Take

\[
\delta = \frac{1}{\varepsilon \theta} \log(1+\varepsilon).
\]

Then from (2.6), \( |\overline{G}(u_i - b_i) - \overline{G}(u_i)| \leq e^{\theta b_i} - 1 \), so

\[
\sum_{1}^{m} |\overline{G}(u_i - b_i) - \overline{G}(u_i)| \leq \sum_{1}^{m} \left( e^{\theta b_i} - 1 \right) \leq e^{\theta b_i} - 1 \leq e^{\theta \delta} - 1 = \varepsilon .
\]

2.6. **Proposition.** If \( F \) is given by (2.1) and is a survival function, then (2.4) holds.
Proof: (2.4) is equivalent to the fact that there is nonnegative mass in the square \( \{(r,s): x < r < x+u, x < s < x+u\} \). It is also equivalent to the fact that
\[
P(X \leq u) + P(Y \leq u) \geq P(\min(X,Y) \leq u).
\]

2.7. Proposition. If (2.5) holds and \( G \) is not degenerate at 0, then

\[
G(u) - g_+(u)/\theta \text{ is nonincreasing in } u \geq 0,
\]
where \( g_+(u) = \liminf_{\delta \to 0} \frac{G(u) - G(u+\delta)}{\delta} \) is the lower right derivative of \( G \).

Proof: Rewrite (2.5) in the form
\[
[G(u) - G(u+b)] - [G(u+a) - G(u+a+b)] \leq (e^{\theta b} - 1)[G(u+b) - G(u+a+b)].
\]
Upon division by $b$ and letting $b \to 0$, it follows that

$$g_+(u) - g_+(u+a) \leq 0 \{ \overline{G}(u) - \overline{G}(u+a) \}, \quad a, u \geq 0,$$

which yields (2.7).

The following proposition helps clarify Proposition 2.7. With the assumption that $g$ is differentiable, the result is contained in Lemma 8.1 of Block and Basu (1974).

2.8. Proposition. If (2.1) holds, $\overline{F}$ is a survival function for $(X,Y)$ and $G$ is not degenerate at 0, then

$$P(X-Y > u) = \overline{G}(u) - g(u)/\theta, \quad u \geq 0. \quad (2.8)$$

Proof: Let

$$\Omega_\delta = \bigcup_{j=1}^{\infty} \{(x,y): u+j\delta < x, (j-1)\delta < y \leq j\delta\},$$

and note that $\Omega_\delta \subset \{(x,y): x-y > u\}$. Thus

$$P(X-Y > u) \geq \sum_{j=1}^{\infty} \left[ \overline{F}(u+j\delta, (j-1)\delta) - \overline{F}(u+j\delta, j\delta) \right]$$

$$= \sum_{j=1}^{\infty} \left[ e^{-\theta(j-1)\delta} \overline{G}(u+j\delta) - e^{-\theta j\delta} \overline{G}(u) \right]$$

$$= \overline{G}(u) - e^{-\theta\delta} \overline{G}(u) = \overline{G}(u) + \frac{\overline{G}(u+\delta) - \overline{G}(u)}{1 - e^{-\theta\delta}} \cdot \frac{\delta}{1 - e^{-\theta\delta}}$$

Consequently, if $D(u)$ is any limit point of $\frac{\overline{G}(u)-\overline{G}(u+\delta)}{\delta}$ as $\delta \to 0$, then
\[
P(X-Y > u) \geq \bar{G}(u) - \frac{D(u)}{\theta}.
\]

Notice that \( \{ \cap_{k=1}^{\infty} \Omega_{2^{-k}} = \{ (x,y): x-y > u \} \) forms an increasing sequence of sets such that \( \lim_{k \to \infty} \frac{\bar{G}(u) - \bar{G}(u+2^{-k})}{2^{-k}} \cdot \frac{2^{-k}}{1-e^{-\theta 2^{-k}}} \)

\[
= \bar{G}(u) - g^*(u)/\theta,
\]

where \( g^*(u) = \lim_{k \to \infty} \frac{\bar{G}(u) - \bar{G}(u+2^{-k})}{2^{-k}} \) exists. Thus,

\[
D(u) \geq g^*(u) \text{ for all } u \geq 0,
\]

and hence

\[
g^*(u) = \lim_{\delta \to 0} \inf_{\delta} \frac{\bar{G}(u) - \bar{G}(u+\delta)}{\delta} = g_*(u).
\]

Upon combining (2.9) and (2.10), (2.8) is obtained.

2.9. Proposition. If (2.1) holds and \( \bar{F} \) is a survival function, then the right-hand derivative \( g \) (and \( h \)) exists, is right-continuous, is of bounded variation, and has, at most, a countable number of discontinuities.

Proof. The right-continuity and bounded variation of \( g_* \) follow immediately from (2.8), since

\[
g_*(u) = \theta[\bar{G}(u) - P(X-Y > u)],
\]
and both terms on the right-hand side are bounded, monotone and right continuous.

Now, upon division by \( a \) in (2.5) and letting \( a \to 0 \), the following inequality between the right upper and lower derivatives is obtained.

\[
(2.12) \quad g^+(u) \leq e^{\theta b} g_+(u+b), \quad u, b \geq 0.
\]

Let \( b \to 0 \); the right-continuity of \( g_+ \) then yields the inequality \( g^+(u) \leq g_+(u) \), which establishes the existence of a unique right-hand derivative \( g \) (which is right-continuous and of bounded variation).

Consequently, (2.8) now becomes

\[
(2.13) \quad P(X-Y > u) = \overline{G}(u) - g(u)/\theta, \quad u \geq 0,
\]

and yields

\[
(2.14) \quad P(X-Y \geq u) = \lim_{\varepsilon \to 0} P(X-Y > u-\varepsilon) = \overline{G}(u) - g(u-)/\theta, \quad u > 0,
\]

and hence,

\[
(2.15) \quad P(X-Y = u) = [g(u)-g(u-)]/\theta, \quad u > 0;
\]

and similarly,

\[
(2.16) \quad P(Y-X = u) = [h(u)-h(u-)]/\theta, \quad u > 0.
\]

Thus, the discontinuity points of \( g \) and \( h \) are precisely the points (other than 0) of positive probability of \( (X-Y) \), and the corresponding point masses are proportional to the respective jumps in the values of \( g \) and \( h \).
2.10. **Remark.** If, in the proof of Proposition 2.8, \( \Omega_\delta \) is replaced by
\[
\Gamma_\delta = \bigcup_{j=0}^{\infty} \{(x,y): u + j\delta < x, j\delta < y \leq (j+1)\delta\},
\]
then it follows in a similar way that
\[
\text{P}(X\sim Y\sim u) = \overline{G}(u) - g^-(u)/\delta, \tag{2.17}
\]
where \( g^-(u) = \limsup_{\delta \to 0} \frac{[\overline{G}(u-\delta) - \overline{G}(u)]}{\delta} \) is the left upper derivative.

Replacing \( u \) by \( u-a \) in (2.5), and following a line of reasoning similar to that of the proof of Proposition 2.9, leads to the conclusion that there is a unique left-hand derivative of \( G \) at every point, and that it equals \( g(u-) \).

2.11. **Proposition.** If \( \overline{G}(0) = 1 \), then (2.5) is equivalent to (2.2).

**Proof:** Suppose (2.5) holds. In (2.5) divide by \( a \) and let \( a \to 0 \) to obtain
\[
g(u) \leq e^{\theta b} g(u+b),
\]
i.e.,
\[
e^{\theta u} g(u) \leq e^{\theta (u+b)} g(u+b), \quad u, b \geq 0.
\]

Next suppose that (2.2) holds. By Propositions 2.5 and 2.9, \( G \) is absolutely continuous and \( g \) is a density for \( G \). In the inequality
\[
e^{-\theta b} g(x) \leq g(x+b),
\]
integrate on \( x \) from \( u \) to \( u+a \) to obtain
\[
e^{-\theta b} [G(u+a) - G(u)] \leq G(u+a+b) - G(u+b),
\]
i.e.,
\[
e^{-\theta b} [\overline{G}(u) - \overline{G}(u+a)] \leq \overline{G}(u+b) - \overline{G}(u+a+b).
\]
Upon multiplication by \( e^{\theta b} \), this yields (2.5). \( \square \)
2.12. **Proof of Theorem 2.2:** Suppose first that \( \overline{F} \) is a survival function satisfying (2.1). Then by Propositions 2.3 and 2.5, either \( G \) and \( H \) are both degenerate at 0 or both are absolutely continuous. In the absolutely continuous case, \( g \) and \( h \) exist by Proposition 2.9, and (2.2) holds by Propositions 2.4 and 2.11. Interchange of \( X \) and \( Y \) yields (2.5). By Proposition 2.6, (2.4) holds.

Next suppose that (ii) holds. Then (2.5) holds by Proposition 2.11, and similarly (2.5) holds with \( \overline{H} \) in place of \( \overline{G} \) (interchange \( X \) and \( Y \)). Then, \( \overline{F} \) gives nonnegative mass to all rectangles lying entirely on one side of the diagonal \( x = y \). By (2.4), \( \overline{F} \) gives nonnegative mass to all squares centered on the diagonal \( x = y \). Thus, \( \overline{F} \) gives nonnegative mass to all rectangles, and hence is a survival function.

2.13. **Miscellanea.** Let

\[
\begin{align*}
q_1 &= P(X-Y > 0) = 1 - g(0)/\theta, \quad q_2 = P(Y-X > 0) = 1 - h(0)/\theta, \\
p &= P(X=Y) = 1 - q_1 - q_2 = \frac{g(0) + h(0)}{\theta} - 1.
\end{align*}
\]  

Since these are probabilities,

\[
(2.19) \quad g(0) \leq \theta, \quad h(0) \leq \theta, \quad g(0) + h(0) \geq \theta.
\]

These inequalities refine part (i) of Theorem 5.1 of Marshall and Olkin (1967).

If \( W = \max(X,Y) - \min(X,Y) \), then it follows from (2.13) that, for \( w \geq 0 \),

\[
(2.20) \quad P(W > w) = P(X-Y > w) + P(Y-X > w) = \overline{G}(w) + \overline{H}(w) - [g(w) + h(w)]/\theta.
\]

Since \( P(W=0) = P(X=Y) \), this probability is equal to \( p \), given in (2.18).

It is shown in Section 3 that \( W \) and \( \min(X,Y) \) are independent.
2.14 Proposition. Suppose that \( \overline{F} \) given by (2.1) is a survival function and \( G \) is not degenerate at 0. Let \( q_1 = P(X > Y) \). Then \( G \) has hazard (failure) rate \( r_G = \frac{g}{\overline{G}} \) which satisfies

\[
(2.21) \quad \theta [1 - \frac{q_1}{\overline{G}(u)}] \leq r_G(u) \leq \theta, \quad u > 0.
\]

Proof: This follows from (2.7) and (2.8) which show that

\[
q_1 = P(X-Y > 0) \geq \frac{\overline{G}(u)}{g(u)} \
geq 0.
\]

\( \square \)

3. SOLUTION OF THE FUNCTIONAL EQUATION. In this section, a representation for the general solution of (1.2) is obtained. The key to this representation is the following lemma, which was obtained by Black and Basu (1974) with restrictive assumptions and for \( n = 2 \).

3.1. Lemma. If the survival function \( \overline{F} \) of \( \chi = (X_1, \ldots, X_n) \) satisfies (1.2), then \( U = \min(X_1, \ldots, X_n) \) and \( \chi' = \chi - U \) are independent.

Proof: For real vectors \( \xi = (a_1, \ldots, a_n) \) and \( \eta = (b_1, \ldots, b_n) \) let

\[
\xi \vee \eta = (\max(a_1, b_1), \ldots, \max(a_n, b_n)).
\]

For any \( u > 0 \) and any \( \chi' \), let

\[
\Omega_k = \cup_{j=0}^{\infty} (\chi' > \chi + (u + j2^{-k})\xi), \quad u + j2^{-k} < U \leq u + (j+1)2^{-k}.
\]
Then $\Omega_k$ in $k$, $\bigcap_{k=1}^{\infty} \Omega_k = \{\xi \geq \omega, \ U > u\}$, so $\lim_{k \to \infty} P(\Omega_k) = P(\xi \geq \omega, \ U > u)$; and

$$P(\Omega_k) = \sum_{j=0}^{\infty} \left[ P\left\{ \frac{\chi}{1+j} \geq \omega + (u+j2^{-k}) \epsilon, \ U > u+j2^{-k} \right\} - P\left\{ \frac{\chi}{1+j} > \omega + (u+j2^{-k}) \epsilon, \ U > u+(j+1)2^{-k} \right\} \right]$$

$$= \sum_{j=0}^{\infty} \left[ \frac{\omega}{\omega + (u+j2^{-k}) \epsilon} \vee (u+j2^{-k}) \epsilon) - \frac{\omega}{\omega + (u+(j+1)2^{-k}) \epsilon} \vee (u+(j+1)2^{-k}) \epsilon) \right]$$

$$= \sum_{j=0}^{\infty} \left[ \frac{-\theta(u+j2^{-k})}{\omega \vee \omega} - e^{-\theta(u+j2^{-k})} \right. \left. \frac{-\theta(u+j2^{-k})}{\omega \vee 2^{-k} \epsilon} \right]$$

$$= e^{-\theta u} \frac{\overrightarrow{\omega \vee \omega} - \overrightarrow{\omega \vee 2^{-k} \epsilon}}{1 - e^{-\theta 2^{-k}}}$$

Therefore,

$$(3.1) \quad P(\omega \geq \omega, \ U > u) = e^{-\theta u} \lim_{k \to \infty} \frac{\overrightarrow{\omega \vee \omega} - \overrightarrow{\omega \vee 2^{-k} \epsilon}}{1 - e^{-\theta 2^{-k}}}$$

If $u < 0$, then $P(\omega \geq \omega, \ U > u) = P(\omega \geq \omega) = P(\omega \geq \omega)P(U > u)$.

This, together with (3.1), shows that $\omega$ and $U$ are independent.

\[ \square \]

**Remark.** From (*), it is easy to see that

$$P(\omega \geq \omega) = 0, \ \omega > 0; \ \text{and} \ \ P(\omega \geq \omega) = 1, \ \omega < 0,$$
which also follow from the definition of \( W \).

It is clear that the procedure of 2.9 and 2.10, which is carried out there in detail for the two-dimensional case, can be applied in a similar manner to an arbitrary finite number of dimensions, with the following results: the limit

\[
(3.2) \quad f_1(w) = \lim_{\delta \to 0} \frac{\bar{F}(w + \delta e) - \bar{F}(w)}{\delta}
\]

exists for all \( w \geq 0 \), and is right continuous. Further,

\[
(3.3) \quad \lim_{\delta \to 0} \frac{[\bar{F}(w) - \delta e) - \bar{F}(w)]/\delta}{\delta} \quad \text{exists for all} \quad w \geq 0,
\]

and equals \( \lim_{\delta \to 0} f_1(w - \delta e) \), denoted by \( f_1(w - 0e) \).

\[
(3.4) \quad \text{For all} \quad w \geq 0, \quad P(W = w) = [f_1(w) - f_1(w - 0e)]/\theta.
\]

which is (2.14).

3.2. Theorem. The survival function \( \bar{F} \) of \( X = (X_1, \ldots, X_n) \) satisfies (1.2) if and only if there exist random variables \( U \) and \( W = (W_1, \ldots, W_n) \) such that

(i) \( X = U + W \),

(ii) \( U \) and \( W \) are independent,

(iii) \( \min(W_1, \ldots, W_n) = 0 \) with probability 1,

(iv) Either \( U \) has an exponential distribution, say with parameter \( \theta \), or \( U \) is degenerate at 0.

The above theorem is equivalent to
3.3. Theorem. The survival function $\bar{F}$ satisfies (1.2) if and only if $\bar{F}$ has a representation of the form

$$\bar{F}(\chi) = \int_0^\infty \bar{F}_W(\chi-u\theta)e^{-\theta u}du, \quad 0 < \theta < \infty, \quad \chi \geq \underline{\chi},$$

or corresponding to the limiting case $\theta = \infty$,

$$\bar{F}(\chi) = \bar{F}_W(\chi), \quad \chi \geq \underline{\chi}$$

where $\bar{F}_W$ is a survival function with the properties that $\bar{F}_W(0^-) = 1$, and

$$\bar{F}_W(\chi) = 0 \text{ if } \min x_i > 0.$$ 

Proof: Suppose (3.5). Then by using (3.7) it follows that for $\chi \geq \underline{\chi}$, $t > 0$,

$$\bar{F}(\chi + t\theta) = \int_0^\infty \bar{F}_W(\chi-(u-t)\theta)e^{-\theta u}du = \int_0^{\min x_i} \bar{F}_W(\chi-(u-t)\theta)e^{-\theta u}du$$

$$= \int_0^{\min x_i} \bar{F}_W(\chi-z\theta)e^{-\theta(t+z)}dz = e^{-\theta t} \int_0^{\min x_i} \bar{F}_W(\chi-z\theta)e^{-\theta z}dz$$

$$= \bar{F}(te)\bar{F}(\chi),$$

Next suppose that (3.6) holds. Then because of (3.7), (1.2) reduces to $0 = 0$ unless $\chi = \underline{\chi}$, $t = 0$, in which case (1.2) reduces to $1 = 1$. Thus if either (3.5) or (3.6) hold, then (1.2) holds.

Now suppose that (1.2) holds, and make use of Lemma 3.1. Denote the survival function of $\mathcal{W}$ by $\bar{F}_\mathcal{W}$ and suppose $U$ has an exponential distribution with parameter $\theta$. Then $\chi = U\theta + \mathcal{W}$ has the distribution (3.5). If $U$ is degenerate at 0, then $\chi = \mathcal{W}$ has the distribution $\bar{F}_\mathcal{W}$. \qed
The special case that \( n = 2 \) is of particular interest. Let

\[
N = \begin{cases} 
0, & \text{if } X_1 = X_2, \\
1, & \text{if } X_1 > X_2, \\
-1, & \text{if } X_2 > X_1,
\end{cases}
\]

so that

\[
X_1 = U + WN(N+1)/2, \\
X_2 = U + WN(N-1)/2.
\]

Then Lemma 3.1 can be rephrased as follows: If the survival function of \((X_1, X_2)\) satisfies (1.2), then

\[(3.8) \quad \min(X_1, X_2) \quad \text{and} \quad [N, \max(X_1, X_2) - \min(X_1, X_2)] \quad \text{are independent.}\]

In particular,

\[(3.9) \quad \min(X_1, X_2) \quad \text{and} \quad \max(X_1, X_2) - \min(X_1, X_2) \quad \text{are independent.}\]

Note that in the notation of (2.18), \( p = P(W_1=W_2=0), \)
\( q_1 = P(W_1 > W_2 = 0), \) \( q_2 = P(W_2 > W_1 = 0). \) Let \( X_i \) be the conditional distribution of \( W_i \) given \( W_i > 0, i = 1, 2. \) Then (3.5) takes the form
(3.10) \( \overline{F}(x_1, x_2) = pe^{-\theta \max(x_1, x_2)} + q_1 \left[ \int_{(0, \max(x_1 - x_2, 0)]} e^{-\theta (x_1 - w_1)} dK_1(w_1) \right. \\
+ e^{-\theta x_2 \overline{K}_1(\max(x_1 - x_2, 0))} \\
\left. + q_2 \left[ \int_{(0, \max(x_2 - x_1, 0)]} e^{-\theta (x_2 - w_2)} dK_2(w_2) \right. \\
+ e^{-\theta x_1 \overline{K}_2(\max(x_2 - x_1, 0))} \right]. \)

For \( x_1 > x_2 \),

(3.11) \( \overline{F}(x_1, x_2) = (p + q_2) e^{-\theta x_1} + q_1 \left[ \int_{(0, x_1 - x_2]} e^{-\theta (x_1 - w_1)} dK_1(w_1) + e^{-\theta x_2 \overline{K}_1(x_1 - x_2)} \right] \\
= e^{-\theta x_2 \overline{G}(x_1 - x_2)} \\

where

(3.12) \( \overline{G}(x) = \overline{F}(x, 0) = (p + q_2) e^{-\theta x} + q_1 \left[ \int_{(0, x]} e^{-\theta (x - w_1)} dK_1(w_1) + \overline{K}_1(x) \right] \\
= (p + q_2) e^{-\theta x} + q_1 \int_0^\infty P(U > x - w_1) dK_1(w_1). \)

From (3.12) it follows, by computing \( \lim_{\delta \to 0} [\overline{G}(x) - \overline{G}(x + \delta)]/\delta = g(x) \),

that

(3.13) \( g(x) = \theta (p + q_2) e^{-\theta x} + \theta q_1 \int_{(0, x]} e^{-\theta (x - w_1)} dK_1(w_1) \).

Thus from (3.12) and (3.13), it follows that
which is (2.13). Upon taking \( x = 0 \), part of (2.18) is obtained. Thus, equations (3.14) and (2.18) indicate how to obtain \( K_1, K_2, q_1, p \) from \( \theta \) and the marginals \( G, H \). Notice that (2.2) is an immediate consequence of (3.13).

3.4. **Example.** Let \( p = \lambda_1 / \theta, q_1 = \lambda_2 / \theta, q_2 = \lambda_1 / \theta, \theta = \lambda_1 + \lambda_2 + \lambda_{12}, \)

\[
\bar{K}_1(w) = e^{-(\lambda_1 + \lambda_{12})w}, \quad \bar{K}_2(w) = e^{-(\lambda_2 + \lambda_{12})w}, \quad w \geq 0.
\]

Then from (3.9) it follows that for \( x_1 > x_2 > 0, \)

\[
P(X_1 > x_1, X_2 > x_2) = e^{-(\lambda_1 + \lambda_{12})x_1 - \lambda_2 x_2} = e^{-\lambda_1 x_1 - \lambda_2 x_2 + \lambda_{12} \max(x_1, x_2)}.
\]

Here, the bivariate exponential distribution of Marshall and Olkin (1967) is obtained. Because this is the only bivariate distribution with exponential marginals satisfying (1.2), it follows from Lemma 3.1 that

(i) \( \min(X_1, X_2) \) and \( (N, \max(X_1, X_2)) \) are independent

and

(ii) \( X_1, X_2 \) are both exponentially distributed,

if and only if \( X_1, X_2 \) have the bivariate exponential distribution of Marshall and Olkin (1967).

3.5. **Example.** Let \( p = 0, q_1 = \beta / \theta, q_2 = \alpha / \theta, \theta = \alpha + \beta, \)

\[
\bar{K}_1(w) = e^{-\alpha'w}, \quad \bar{K}_2(w) = e^{-\beta'w}, \quad w \geq 0.
\]
Then from (3.9) it follows that

(3.15) \[ P(X_1 > x_1, X_2 > x_2) = \begin{cases} \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} e^{-(\alpha + \beta) x_1} + \frac{\beta}{\alpha + \beta - \alpha'} e^{-(\alpha + \beta - \alpha') x_2 - \alpha' x_1}, & x_1 > x_2 \geq 0, \\ \frac{\beta - \beta'}{\alpha + \beta - \beta'} e^{-(\alpha + \beta) x_2} + \frac{\alpha}{\alpha + \beta - \beta'} e^{-(\alpha + \beta - \beta') x_1 - \beta' x_2}, & x_2 > x_1 \geq 0. \end{cases} \]

This is the bivariate extension of the exponential distribution obtained by Freund (1961).

By allowing \( p, q_1, q_2 \) to be arbitrary probabilities summing to one, and by allowing \( \tilde{K}_1, \tilde{K}_2 \) to be general exponential distributions, it is easy to obtain a bivariate distribution satisfying (1.2) that contains both Examples 3.4 and 3.5 as special cases. This shows a close relationship between the two examples.

With \( x_2 = 0 \) in (3.15) the marginal survival function \( \tilde{G} \) of \( X_1 \) is found to be

(3.16) \[ \tilde{G}(x) = \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} e^{-(\alpha + \beta) x} + \frac{\beta}{\alpha + \beta - \alpha'} e^{-\alpha' x}, \quad x \geq 0. \]

From (3.16) it is easy to check that the hazard rate \( r_G(x) = -\frac{d}{dx} \log \tilde{G}(x) \) is strictly increasing (decreasing) if \( \alpha < \alpha' \) (\( \alpha > \alpha' \)). Consequently,

\[ \tilde{G}(x + t) < \tilde{G}(x) \tilde{G}(t), \quad x, t \geq 0 \text{ if } \alpha < \alpha', \]

and similarly the marginal survival function \( \tilde{H} \) of \( X_2 \) satisfies

\[ \tilde{H}(x + t) < \tilde{H}(x) \tilde{H}(t), \quad x, t \geq 0 \text{ if } \beta < \beta'. \]
This example shows that even though the functional equation (1.2) is satisfied, the marginal survival functions can fail to satisfy the corresponding univariate functional equation (1.1). In fact equality in (1.1) can be replaced by strict inequality in either direction.

3.6 Example. In (3.6) with \( n = 2 \), suppose that \( F_W \) places mass \( \frac{1}{2} \) at \( (1,0) \) and at \( (0,1) \). Then

\[
P(X_1=1, X_2=0) = P(X_1=0, X_2=1) = \frac{1}{2}.
\]

Here \( (X_1, X_2) \) has a discrete distribution with correlation \(-1\). On the other hand, if \( q_1 = q_2 = 0 \). Then (3.8) yields a distribution with correlation \( 1 \).

4. SHOCK MODELS WITH AFTереFFECTS. Marshall and Olkin(1967) derive the bivariate exponential distribution (1.3) from "shock" models, the simplest of which can be described as follows: Consider a system with two components having respective life lengths \( X_1, X_2 \). Suppose that failures result only from fatal shocks which occur as events in one of the independent Poisson processes \( N_1, N_2 \) or \( N_{12} \) with respective rates \( \lambda_1, \lambda_2, \lambda_{12} \). Events in process \( i \) corresponds to fatal shocks only to component \( i, \ i = 1, 2 \), and the process \( N_{12} \) governs simultaneous shocks fatal to both components. If \( Z_1, Z_2, Z_{12} \) represent the waiting times for an event in the corresponding processes; then clearly

\[
X_1 = \min(Z_1, Z_{12}),
\]
\[
X_2 = \min(Z_2, Z_{12}),
\]

and \( X_1, X_2 \) have the joint survival function (1.3).
Now, suppose that there are two additional independent processes \( M_1 \) and \( M_2 \) with corresponding waiting times \( U_1, U_2 \) to an event. An event in the process \( M_1 \) (\( M_2 \)) causes an immediate fatal shock to the first (second) component, and an aftershock to the second (first) component with a random delay \( V_2(V_1) \). Then

\[
\begin{align*}
X_1 &= \min(Z_1, Z_{12}, U_1, U_2 + V_1) \\
X_2 &= \min(Z_2, Z_{12}, U_2, U_1 + V_2)
\end{align*}
\]

(4.2)

Of course, one can also consider the model in which only the processes \( M_1 \) and \( M_2 \) are present, and then

\[
\begin{align*}
X_1 &= \min(U_1, U_2 + V_1) \\
X_2 &= \min(U_2, U_1 + V_2)
\end{align*}
\]

(4.3)

These models are also of interest when the Poisson processes are replaced by other point processes. In the following discussion of these models, \( Z_1, Z_2, Z_{12}, U_1, U_2, V_1, V_2 \) are assumed to be independent nonnegative random variables, but are assumed to be exponentially distributed only where this assumption is explicitly stated.

Notice that the survival function generated by the model (4.2), is the product of survival functions generated by models (4.1) and (4.3). Because of this, separate discussions of (4.1) and (4.3) sometimes combine to apply to (4.2).

The survival function generated by (4.1) has the particularly simple form

\[
F(x_1, x_2) = F_1(x_1)F_2(x_2)F_{12}(\max(x_1, x_2)),
\]

(4.4)
where \( Z_1, Z_2 \) and \( Z_{12} \) have respective survival functions \( \overline{F}_1, \overline{F}_2 \) and \( \overline{F}_{12} \).

The model (4.3) is intended to allow for the possibility that failure of one component imposes a new stress upon the remaining component that need not result in immediate failure. Thus the model is a kind of competing risk model, where failure of one component imposes a new risk upon an unfailed component. This model is related to one due to Freund (1961). Freund assumes that both components behave as though they had independent and exponentially distributed life lengths until one fails. The unfailed component then continues to behave in an exponential manner, but with a new parameter.

A survival function resulting from model (4.3) has the form

\[
\overline{F}(x_1, x_2) = \int_0^\infty \overline{A}_2(\max(x_1 - \xi, x_2))dL_1(\xi) \int_0^\infty \overline{A}_1(\max(x_1, x_2 - \xi))dL_2(\xi)
\]

where \( U_i \) has survival function \( \overline{A}_i \) and \( V_i \) has survival function \( \overline{L}_i \), \( i = 1, 2 \).

4.1. Example. Suppose \( U_1, U_2, V_1 \) and \( V_2 \) are all exponentially distributed with respective parameters \( \alpha > 0, \beta > 0, \alpha' - \alpha > 0, \beta' - \beta > 0 \). Then (4.5) reduces to (3.10), Freund's bivariate extension of the exponential distribution, with the added requirements that \( \alpha' \geq \alpha \) and \( \beta' \geq \beta \).

4.2. Proposition. If \( U_i \) has an exponential survival function, \( i = 1, 2 \), then the survival function \( \overline{F} \) given by (4.5) satisfies the functional equation (1.2).

The proof of this proposition is straightforward.
4.3. **Proposition.** If \((X_1, X_2)\) has a representation of the form (4.1), (4.2) or (4.3) then \(X_1\) and \(X_2\) are associated in the sense of Esary, Proschan and Walkup (1967).

This proposition follows from the fact that in each case, \(X_1\) and \(X_2\) are increasing functions of independent random variables.

An immediate consequence of Proposition 4.3 is that

\[
\text{cov}(X_1, X_2) \geq 0.
\]

Consequently, not all distributions satisfying (1.2) arise from a representation (4.1), (4.2) or (4.3) (See Example 3.6).

4.4. **Example.** If in (4.3), \(V_1 = V_2 = 0\) with probability 1, then \(X_1 = X_2 = \min(U_1, U_2)\) with probability one so \(X_1\) and \(X_2\) have correlation 1. If \(V_1 = V_2 = \infty\) with probability one, then \(X_1\) and \(X_2\) are independent.

4.5. **Proposition.** If \((X_1, X_2)\) has a representation of the form (4.1), (4.2) or (4.3) and the relevant random variables among \(U_1, U_2, V_1, V_2, Z_1, Z_2, Z_{12}\) each have an increasing hazard rate, an increasing hazard rate average, or a new-better-than-used distribution, then also \(X_1\) and \(X_2\) have that property.

**Proof:** This follows from the fact that the three classes of distributions considered here have the property that if \(W_1\) and \(W_2\) are independent and in the class, then \(\min(W_1, W_2)\) and \(W_1 + W_2\) are in the class.
4.6. Example. Suppose that in (4.3),

\[ P(U_i > t) = e^{-\alpha_i t}, \quad P(V_i > t) = \sum_{n=0}^{\infty} pq^n \overline{G}_n(t), \quad i = 1, 2, \quad t > 0, \]

where

\[ \overline{G}_0(t) = 0, \quad t > 0, \]

\[ \overline{G}_n(t) = \sum_{k=0}^{n-1} \frac{-\alpha_i t (\alpha_i t)^k}{k!}, \quad n = 1, 2, \ldots, \quad t > 0. \]

Then \( V_i \) can be written in the form \( V_i = Z_{1,i} + \ldots + Z_{N,i} \) where \( Z_{1,i}, Z_{2,i}, \ldots \) are independent and distributed as \( U_i, \ i = 1, 2, \) and \( N, \) independent of the \( Z_{j,i}, \) has the geometric distribution

\[ P(N=j) = p(1-p)^j, \quad j = 0, 1, \ldots. \]

If \( p = \lambda_1/\theta, \ \alpha_1 = \lambda_1 \theta/(\lambda_1 + \lambda_2), \ \alpha_2 = \lambda_2 \theta/(\lambda_1 + \lambda_2) \) where

\[ \theta = \lambda_1 + \lambda_2 + \lambda_1 \lambda_2, \]

then the distribution (4.5) reduces to the bivariate exponential distribution (1.3) of Marshall and Olkin (1967). It is interesting to see that this distribution, which Marshall and Olkin obtained from model (4.1), can also arise from model (4.3).

In this example, model (4.3) is written in new notation as

\[ X_1 = \min(S_1, T_1 + \ldots + T_{M+1}), \quad X_2 = \min(T_1, S_1 + \ldots + S_{M+1}) \]

where \( S_1, S_2, \ldots, T_1, T_2, \ldots, M \) are all independent; \( S_1, S_2, \ldots \) are exponentially distributed with parameter \( \alpha_1, \ T_1, T_2, \ldots \) are exponentially distributed with parameter \( \alpha_2, \) and \( M \) has the geometric distribution.
The model (4.7), a special case of (4.3), can be interpreted in terms of shock processes as follows: there are two independent Poisson processes, say \( N_1 \) and \( N_2 \), with respective parameters \( \alpha_1 \) and \( \alpha_2 \). An event in the process \( N_1 \) corresponds to a fatal shock to the first component. It causes failure to the second component with probability \( p \) and does not affect the second component with probability \( 1-p \). The effects of successive shocks to the second component are independent. An event in the process \( N_2 \) acts similarly but with components 1 and 2 interchanged.

When written in the form (4.7) it is easy to see that the distribution of \( X_1, X_2 \) satisfies the functional equation (1.2) and has exponential marginals, hence must be of the form (1.3).

4.7. Example. Suppose that in (4.3), \( U_i \) has an exponential distribution with parameter \( \lambda_i \), \( i = 1,2 \), and \( P(V_i=0) = p_i \), \( P(V_i=\infty) = 1 - p_i \), \( i = 1,2 \). Then (4.5) takes the form

\[
\overline{F}(x_1, x_2) = [p_2 e^{-\lambda_1 \max(x_1, x_2)} + (1-p_2)e^{-\lambda_1 x_1}][p_1 e^{-\lambda_2 \max(x_1, x_2)} + (1-p_1)e^{-\lambda_2 x_2}].
\]

This survival function has marginals

\[
\overline{G}(x_1) = \overline{F}(x_1, 0) = p_1 e^{-(\lambda_1 + \lambda_2)x_1} + (1-p_1)e^{-\lambda_1 x_1}
\]

\[
\Pi(x_2) = \overline{F}(0, x_2) = p_2 e^{-(\lambda_1 + \lambda_2)x_2} + (1-p_2)e^{-\lambda_2 x_2}.
\]
These marginal distributions have completely monotone densities, hence

\[(4.8) \quad \overline{G}(s+t) \geq \overline{G}(s)\overline{G}(t), \quad \overline{H}(s+t) \geq \overline{H}(s)\overline{H}(t), \quad s, t \geq 0.\]

Distributions with this property are said to be "new worse than used". Since \( \overline{G} \) and \( \overline{H} \) are not exponential survival functions, (4.8) does not hold with equality replacing the inequalities.
References


