ESTIMATION AND TESTING OF HYPOTHESES FOR CORRELATION MODELS

BY

INGRAM OLKIN

TECHNICAL REPORT NO. 19
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1. Introduction.

The development of correlation analysis dates back in part to the work of Galton, K. Pearson, and Yule during the period 1895-1910 in connection with the study of physiological and genetic characteristics. However, the inferential theory began in 1915 when R. A. Fisher [4] obtained the distribution of the sample correlation coefficient, r. When the population correlation coefficient, \( \rho \), is zero, a simple function of r has Student's t-distribution; but when \( \rho \) is not zero, the distribution of r cannot be expressed in a closed form but only as an infinite series or as an integral. Thus, the test of the hypothesis \( H: \rho = 0 \) can be carried out; but the test that \( \rho \) is equal to a non-zero value, expressions for the moments of r, or unbiased estimators of r, are more troublesome to come by. Since large-sample results frequently are of a simpler form than the small-sample counterparts, one approach is to consider asymptotic theory with the hope that some of the difficulties mentioned can be overcome. For large samples, the distribution of r approaches a normal distribution. Unfortunately, the variance of this asymptotic distribution depends on the population.

* This paper was presented at the 7th Annual Symposium of Phi Delta Kappa on Educational Research, held at Madison, Wisconsin, August 9-11, 1965.
correlation coefficient, $\rho$, which means that it varies depending on what value $\rho$ takes. This led Fisher [5] in 1921 to consider a transformation of $r$ to a new variable, $z$, which has the effect of stabilizing the variance, i.e., it transforms $r$ in such a way that the variance of the asymptotic distribution of $z$ does not depend on $\rho$.

In 1916 the distribution of the product-moment correlation was studied extensively in a cooperative study [15]. Tables of the distribution of the correlation coefficient were prepared by F. N. David [1] in 1938; this book also includes confidence intervals for $\rho$.

The distribution theory for partial and multiple correlations was developed by Fisher [6] in 1924. The partial correlation behaves as a product-moment correlation for a conditional normal distribution, so that properties for the partial correlation are obtained almost by analogy. A somewhat different analysis is required for the multiple correlation coefficient. Its distribution, when the population multiple correlation coefficient is not zero, involves an infinite series or integral (quite different from the infinite series for the product-moment correlation coefficient). This distribution simplifies when the population multiple correlation coefficient is zero and enables us to test that the multiple correlation coefficient is zero. A satisfactory solution to the problem of testing that the multiple correlation is a specified value (not zero) appears to be unavailable at this time.

There have been various other papers on this subject; one that stands out is a paper by H. Hotelling [9] in 1933 in which there is an extensive and detailed analysis of large-sample results for the
product-moment correlation coefficient and its transforms.

By and large, the procedures discussed in any elementary or moderately advanced text are essentially those developed 30-50 years ago. My concern in this paper is to review some of the procedures and problems in tests of hypotheses and estimation for one and more than one correlation coefficient. Of particular concern are vintage problems of correlation analysis which have been reconsidered recently and upon which we can shed some new light. An attempt is made to consider alternative models for some of these problems.

The review may be outlined as follows:

ONE POPULATION

1. Bivariate normal distribution (Section 3)
   1.1 Confidence limits and tests for $\rho$
   1.2 Minimum variance unbiased estimation of $\rho$

2. Multivariate normal distribution
   2.1 Confidence limits and tests for the difference between the correlation of each of two predictor variables with a single criterion (Section 4)
   2.2 Confidence limits and tests for the difference between two correlations (Section 5)

TWO OR MORE BIVARIATE NORMAL POPULATIONS

1. Case of common correlation coefficients
   1.1 Estimation of the common correlation: averaging of correlations for various models (Sections 6.1, 6.2, 6.3)
   1.2 A test for the equality of variances (Section 7)

2. Case of different correlation coefficients
   2.1 Test for the difference between two correlations (Section 7)
2.2 Estimation of the difference between two correlations (Section 7)

**INTRACLASS CORRELATION MODEL**

1. One population
   1.1 A test for the intraclass correlation coefficient (Section 8.1)  
   1.2 Confidence limits for the intraclass correlation coefficient (Section 8.2)

2. Two or more populations
   2.1 Averaging correlations (Section 8.3)

2. Notational Preliminaries.

Before beginning our discussion, we note certain notational conventions. By $\text{BNV}(\mu, \Sigma)$, we mean the bivariate normal distribution with mean vector $\mu = (\mu_1, \mu_2)$ and covariance matrix $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$, i.e., variances $\sigma_{11}$ and $\sigma_{22}$, and covariance $\sigma_{12}$. The correlation is defined by $\rho_{12} = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$. (Subscripts on $\rho_{12}$ are omitted when there is no ambiguity.) The multivariate (usually $p$-variate) normal distribution is denoted by $\text{MVN}(\mu, \Sigma)$, where $\mu = (\mu_1, \ldots, \mu_p)$ is the mean vector and $\Sigma = (\sigma_{ij})$ is the matrix of variances and covariances. The correlations are then defined by $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$. Samples will generally be of size $N$, and we denote $N - 1$ by $n$.


Suppose now that $X$ denotes the score on a prognostic test and $Y$ denotes a measure of success, perhaps average grade, at the end of a semester. Further, assume that $X$ and $Y$ have a bivariate normal
distribution, \( \text{BIV} (\mu, \Sigma) \). We are given a sample of size \( N \) from the population:

\[
(X_1, Y_1), (X_2, Y_2), \ldots, (X_N, Y_N),
\]

from which we wish to estimate \( \rho \).

Since \( r \), the sample correlation coefficient, is the maximum likelihood estimator (MLE) of \( \rho \), we may wish to use \( r \) to estimate \( \rho \). Exact two-sided confidence limits may easily be obtained from the charts by F. N. David [1] for confidence coefficients .99, .98, .95, and .90, and for sample sizes 3-8, 10, 12, 15, 20, 25, 50, 100, 200, 400. A sample chart giving 95\% confidence limits for \( \rho \) is appended as Chart 1. For example, if \( r = .65 \), \( N = 40 \), we find by interpolation that the 95\% confidence limits are \(.43 \leq \rho \leq .80\).

For large samples, it is known that \( \sqrt{N} (r-\rho) \) is approximately normally distributed with mean 0 and variance \((1-\rho^2)^2\). By using Fisher's z-transformation for \( r \) and \( \rho \); which is

\[
z = \frac{1}{2} \log \frac{1+r}{1-r}, \quad \zeta = \frac{1}{2} \log \frac{1+\rho}{1-\rho},
\]

approximate confidence limits may be obtained from the fact that \( \sqrt{N-3} (z-\zeta) \) is approximately normally distributed with mean 0 and variance 1. Thus, a 95\% confidence interval for \( \zeta \) is given by

\[
z - \frac{K_{\alpha/2}}{\sqrt{N-3}} < \zeta < z + \frac{K_{\alpha/2}}{\sqrt{N-3}},
\]

where \( K_\gamma \) is the point such that the probability of exceeding \( K_\gamma \) (when the distribution is standard normal) is \( \gamma \).

As an aid in the conversion from \( r \) to \( z \), see Table 1.
Thus, for our example, if \( r = .65 \) and \( N = 40 \), and we wish 95% confidence limits for \( \rho \), we find that an \( r \) of .65 corresponds to \( z = .7753 \). The value \( k_{.025} = 1.96 \) is obtained from the standard normal probability table so that the limits for \( \zeta \) are 
\[
[.775 - 1.96/\sqrt{37} , .775 + 1.96/\sqrt{37} ] \quad \text{or} \quad .453 \leq \zeta \leq 1.097.
\] These limits may be converted to confidence limits for \( \rho \) using Table 2. Thus, \( \zeta = .453 \) is transformed into \( \rho = .424 \); \( \zeta = 1.097 \) is transformed into \( \rho = .799 \), so that 95% confidence limits for \( \rho \) are 
\[
.42 \leq \rho \leq .80.
\] These approximate limits virtually coincide with the exact limits.

Although the closeness of the approximation is perhaps unexpected, the fact that the approximation is "good" for \( N = 40 \) and \( \rho \) not extreme is not unexpected.

We note that the confidence limits are not symmetric about \( r \); this is due to the skewness of the distribution of \( r \). It is also known that \( r \) is a biased estimator of \( \rho \), and that for large \( N \), the bias term is approximately \( -\rho(1-\rho^2)/(2n) \). Various unbiased estimators of \( \rho \) can be found; however, there is a unique minimum variance unbiased estimator, \( G(r) \), which is a function of \( r \). The range of \( G(r) \) is \([-.1, 1]\); it differs from \( r \) only by terms of order \( 1/n \), and has the same asymptotic distribution as \( r \). (This result was obtained by Olkin and Pratt [12].) The form of \( G(r) \) is in terms of an infinite series, so that a conversion from \( r \) to the unbiased estimator, \( G(r) \), cannot easily be made. Table 3 has been appended to facilitate translating from \( r \) to \( G(r) \).
Remark. The main differences between $r$ and $G(r)$ occur for small samples. For $n \geq 8$, an approximation (accurate to within .01) for $G(r)$ is given by

$$G(r) = r \left[ 1 + \frac{1-r^2}{2(n-3)} \right].$$

For samples of size at least 18, this approximation is accurate to within .001.

4. **Correlation Analysis: trivariate normal distribution.**

Consider now an extension to the case where $(X_0', X_1', X_2')$ are three correlated variables, and we wish to find confidence limits for $\rho_{01} - \rho_{02}$, or to test the hypothesis $H: \rho_{01} = \rho_{02}$. For example, $X_1$ and $X_2$ may be two predictor variables, of a single criterion, and because of expense we wish to choose only one of the predictor variables.

The assumption is that $(X_0', X_1', X_2')$ has a trivariate normal distribution with means $\mu_0$, $\mu_1$, $\mu_2$, variances $\sigma_{00}'$, $\sigma_{11}'$, $\sigma_{22}'$, covariances $\sigma_{01}'$, $\sigma_{02}'$, $\sigma_{12}'$, and correlations $\rho_{01}'$, $\rho_{02}'$, $\rho_{12}'$.

The available data is a sample of size $N$,

$$(X_{01}', X_{11}', X_{21}'), (X_{02}', X_{12}', X_{22}'), \ldots, (X_{0N}', X_{1N}', X_{2N}')$$

from which we may determine sample estimates of the parameters. In particular, $r_{01}'$, $r_{02}'$, and $r_{12}'$ are the MLE of $\rho_{01}'$, $\rho_{02}'$, and $\rho_{12}'$, respectively.

One procedure that comes to mind is to determine a Fisher's $z$-transformation for $r_{01}$ and $r_{02}'$ (and the corresponding $\rho_{01}$, $\rho_{02}$) i.e.,
\[
\begin{align*}
    z_1 &= \frac{1}{2} \log \frac{1 + r_{01}}{1 - r_{01}}, \\
    z_2 &= \frac{1}{2} \log \frac{1 + r_{02}}{1 - r_{02}}, \\
    \zeta_1 &= \frac{1}{2} \log \frac{1 + p_{01}}{1 - p_{01}}, \\
    \zeta_2 &= \frac{1}{2} \log \frac{1 + p_{02}}{1 - p_{02}},
\end{align*}
\]

and argue that since \( z_1 \) and \( z_2 \) are approximately normally distributed, \( z_1 - z_2 \) can be used to estimate or test for \( \zeta_1 - \zeta_2 \), which can subsequently be converted to an estimate or test for \( \rho_{01} - \rho_{02} \). Although it is true that \( z_1 \) and \( z_2 \) are approximately normally distributed, they are correlated so that the statement: for large samples

\[
\sqrt{N-3} \left[(z_1 - z_2) - (\zeta_1 - \zeta_2)\right]
\]

has a standard normal distribution is in general incorrect. As a matter of fact, the correlation between \( z_1 \) and \( z_2 \) involves not only \( \rho_{01} \) and \( \rho_{02} \), but also \( \rho_{12} \). If the above expression is divided by an estimate of the variance of \( z_1 - z_2 \), the approach to a standard normal distribution is correct. But now the effect of the transformation in stabilizing the variance is lost. Instead, it can be shown (Olkin and Siotani [14]) that the approximate distribution of

\[
\frac{\sqrt{n} \left[(r_{01} - r_{02}) - (\rho_{01} - \rho_{02})\right]}{\sqrt{(1-r_{01}^2)^2 + (1-r_{02}^2)^2 - 2r_{12}^2 + (2r_{12}^2 - r_{01}r_{02})(1-r_{01}^2)(1-r_{02}^2)^2}}
\]

is standard normal. Consequently, the above statistic may be used to test for differences \( (\rho_{01} - \rho_{02}) \). If we denote the denominator by \( \hat{\sigma}_{r_{01} - r_{02}} \), i.e., an estimate of the standard deviation of \( (r_{01} - r_{02}) \), then, for large samples,
\[
\hat{\sigma}^{2}_{r_{01}-r_{02}} = \frac{K_{\alpha/2}}{\sqrt{n}} \left( r_{01}-r_{02} \right) \leq \rho_{01} - \rho_{02} \leq r_{01}-r_{02} + \frac{K_{\alpha/2}}{\sqrt{n}} \left( r_{01}-r_{02} \right)
\]
provides confidence limits for \((\rho_{01}-\rho_{02})\) with confidence coefficient \(1 - \alpha\).

For example, if the sample correlation matrix is
\[
\begin{pmatrix}
1 & r_{01} & r_{02} \\
r_{01} & 1 & r_{12} \\
r_{02} & r_{12} & 1
\end{pmatrix} = \begin{pmatrix}
1 & .56 & .43 \\
.56 & 1 & .52 \\
.43 & .52 & 1
\end{pmatrix},
\]
and \(n = 100\), then
\[
\hat{\sigma}^{2}_{r_{01}-r_{02}} = (1-.56^2)^2 + (1-.43)^2 - 2(.52^2)
\]
\[
+ [2(.52)(.56)(.43)][1-.56^2-.43^2-.52^2]
\]
\[
= .7794,
\]
and a 95% confidence interval for \(\rho_{01} - \rho_{02}\) is given by
\[
\left[ (.56-.43) - \frac{\sqrt{.7794}(1.96)}{\sqrt{100}}, (.56-.43) + \frac{\sqrt{.7794}(1.96)}{\sqrt{100}} \right],
\]
or
\[
- .04 \leq \rho_{01} - \rho_{02} \leq .30.
\]

Since zero falls in the confidence interval for \(\rho_{01} - \rho_{02}\), there is no significant difference between \(\rho_{01}\) and \(\rho_{02}\) (two-sided alternative) at the 5% level of significance.


The previous model was concerned with a comparison of two variables...
vis-à-vis a third. Now we consider a comparison between two pairs of variables with respect to their correlations. Suppose we have a pre- and post-test at time $t_1$, denoted by $(X_1, X_2)$, and at time $t_2$, denoted by $(X_3, X_4)$. A question of interest may be whether there is any difference between $\rho_{12}$ and $\rho_{34}$. We again assume that $(X_1, X_2, X_3, X_4)$ has a $MVN(\mu, \Sigma)$ distribution. As in the case for the trivariate distribution, for large samples, the approximate distribution of

$$\frac{\sqrt{n} \left[ (r_{12} - r_{34}) - (\rho_{12} - \rho_{34}) \right]}{\sigma_{12 - 34}},$$

where

$$\sigma_{12 - 34}^2 = (1-r_{12}^2)^2 + (1-r_{34}^2)^2 + r_{12}r_{34}[r_{13}^2 + r_{14}^2 + r_{23}^2 + r_{24}^2] + 2[r_{13}r_{24} + r_{14}r_{23} - 2r_{12}r_{34}]$$

is standard normal. Confidence limits for the difference between $\rho_{12}$ and $\rho_{34}$ are given by

$$r_{12} - r_{34} - \frac{K_{\alpha/2}}{\sqrt{n}} \sigma_{12 - 34} \leq \rho_{12} - \rho_{34} \leq r_{12} - r_{34} + \frac{K_{\alpha/2}}{\sqrt{n}} \sigma_{12 - 34},$$

with confidence coefficient $1 - \alpha$.

We now illustrate this analysis. In a recent doctoral dissertation by A. H. Yee [18], written under the direction of Professor N. L. Gage, the following four variates are considered:

$F = \text{Pre-test score on pupils' opinions concerning teachers' effectiveness, measured during the first week of the school year.}$

$F' = \text{Post-test score on pupils' opinions concerning teachers' effectiveness, measured approximately after four months of the school year.}$
$C$ = Pre-test score of teachers' attitudes towards class on a semantic differential scale, measured during the first week of the school year.

$C'$ = Post-test score of teachers' attitudes toward class on a semantic differential scale, measured approximately after four months of the school year.

Yee considered the problem of whether there is a difference between $\rho_{F,C'}$ and $\rho_{F',C'}$, which in the context of the problem has a certain interpretation. By relabeling $F$, $F'$, $C$, $C'$ so that $(F,C')$ and $(F',C)$ become $(X_1,X_2)$ and $(X_3,X_4)$, respectively, we can immediately use the test procedure described above.

If the sample correlation matrix based on a sample of 35 observations is

\[
\begin{array}{cccc}
X_1 & X_2 & X_3 & X_4 \\
X_1(=F) & 1 & .29 & .74 & .47 \\
X_2(=C') & .29 & 1 & .55 & .32 \\
X_3(=F') & .74 & .55 & 1 & .56 \\
X_4(=C) & .47 & .32 & .56 & 1 \\
\end{array}
\]

then

\[
\sigma^2_{r_{12} - r_{34}} = (1-.29^2)^2 + (1-.56^2)^2 + (.29)(.56)(.74^2 + .47^2 + .55^2 + .32^2)
\]

\[
+ 2[(.74)(.32) + (.47)(.55)]
\]

\[
- 2[(.29)(.74)(.47) + (.29)(.53)(.32) + (.74)(.55)(.56) + (.47)(.32)(.56)],
\]

\[
= 1.563,
\]

so that 95% confidence limits for $\rho_{12} - \rho_{34}$ are given by

\[
[.29 - .56 - \frac{(1.96)}{\sqrt{34}} \sqrt{1.563}, \quad .29 - .56 + \frac{(1.96)}{\sqrt{34}} \sqrt{1.563}],
\]

11
i.e.,

\[-0.69 < \rho_{12} - \rho_{34} < 0.15.\]

Since zero falls in the confidence interval for \( \rho_{12} - \rho_{34} \), there is no significant difference between \( \rho_{12} \) and \( \rho_{02} \) (two-sided alternative) at the 5% level of significance.

6. Correlation Analysis: two or more bivariate normal distributions.

When independent samples are drawn from two (or more) bivariate normal populations with a common correlation, \( \rho \), there are various procedures for averaging the sample correlation coefficients as an estimator of \( \rho \). For example, we may use the arithmetic or geometric means of the \( r \)'s. More frequently, it is suggested (see, e.g., Guilford [8, p. 348], McNemar [11, p. 140]) that the \( r \)'s be converted to \( z \)'s, from which a weighted average is determined:

\[
z_{av} = \frac{(N_1-3)z_1 + (N_2-3)z_2}{(N_1-3) + (N_2-3)},
\]

and then converted back to an average \( r \), denoted by \( r_{av} \). It is straightforward to show that, in terms of the original \( r \)'s,

\[
r_{av} = \frac{(1+r_1)^c (1+r_2)^{1-c} - (1-r_1)^c (1-r_2)^{1-c}}{(1+r_1)^c (1+r_2)^{1-c} + (1-r_1)^c (1-r_2)^{1-c}},
\]

where \( c = (N_1-3)/[(N_1-3) + (N_2-3)] \).

Although this estimator appears reasonable, it does not take into account any underlying assumptions concerning the variances. Consequently, it would be surprising if this estimator provides a satisfactory averaging procedure in all cases.
The following are possible alternative assumptions for the variances. For each, the population correlation coefficients for the two populations are identical, and the problem is to estimate the common $\rho$.

<table>
<thead>
<tr>
<th></th>
<th>Populations</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>II</td>
</tr>
<tr>
<td></td>
<td>Var(X)</td>
<td>Var(Y)</td>
</tr>
<tr>
<td>A</td>
<td>$\sigma_{11}$</td>
<td>$\sigma_{22}$</td>
</tr>
<tr>
<td>B</td>
<td>$\sigma_{11}$</td>
<td>$\sigma_{22}$</td>
</tr>
<tr>
<td>C</td>
<td>$\sigma^2$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>D</td>
<td>$\sigma^2$</td>
<td>$\sigma^2$</td>
</tr>
</tbody>
</table>

Independent samples of size $N_1$ and $N_2$ from populations I and II are taken

I: \( (X_1^{(1)}, Y_1^{(1)}), \ldots, (X_1^{(N_1)}, Y_1^{(N_1)}) \),

II: \( (X_2^{(1)}, Y_2^{(1)}), \ldots, (X_2^{(N_2)}, Y_2^{(N_2)}) \).

The sample variances and correlations are denoted by

<table>
<thead>
<tr>
<th></th>
<th>Populations</th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>II</td>
</tr>
<tr>
<td></td>
<td>$s_{11}^{(1)}$, $s_{22}^{(1)}$, $r_1$</td>
<td>$s_{11}^{(2)}$, $s_{22}^{(2)}$, $r_2$</td>
</tr>
</tbody>
</table>

For Models A and B we consider the MLE of $\rho$ and its relation to $r_{av}$. Models C and D are special cases of a more general model which we call the "intraclass correlation" model. The discussion of these models is postponed to Section 8.
6.1 Model A.

In this case both populations are identical with respect to the covariances, so that all the estimators should be based on pooled data. If the means in the two populations are also identical, then the populations are indistinguishable; and both samples may be considered as one sample of size \( N = N_1 + N_2 \) from a single population. The ordinary \( r \) from this single population is the MLE of \( \rho \) and possesses the properties previously discussed.

When the means of the two populations differ, we pool variances by weighting according to sample size:

\[
\begin{align*}
    s_{11} &= \frac{n_1 s_{11}^{(1)} + n_2 s_{11}^{(2)}}{n_1 + n_2}, \\
    s_{22} &= \frac{n_1 s_{22}^{(1)} + n_2 s_{22}^{(2)}}{n_1 + n_2}, \\
    s_{12} &= \frac{n_1 s_{12}^{(1)} + n_2 s_{12}^{(2)}}{n_1 + n_2}.
\end{align*}
\]

Then

\[
\hat{r} = \frac{s_{12}}{\sqrt{s_{11}s_{22}}}
\]

\[
= r_1 \frac{\sqrt{s_{11}^{(1)}s_{22}^{(1)}}}{\sqrt{(n_1 s_{11}^{(1)} + n_2 s_{11}^{(2)})(n_1 s_{22}^{(1)} + n_2 s_{22}^{(2)})}} + r_2 \frac{\sqrt{s_{11}^{(2)}s_{22}^{(2)}}}{\sqrt{(n_1 s_{11}^{(1)} + n_2 s_{11}^{(2)})(n_1 s_{22}^{(1)} + n_2 s_{22}^{(2)})}},
\]

is the MLE of \( \rho \). Although \( \hat{r} \) is a weighted average of the individual \( r \)'s, the weights are not constants but depend on all the sample variances. Therefore, it is clear that examples can be obtained whereby this estimator differs radically from \( r_{av} \).

To see this, suppose \( n_1 = n_2 \) and the sample variances and
covariances are:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s_{11}$, $s_{22}$, $s_{12}$, $r_1$</td>
<td>$s_{11}$, $s_{22}$, $s_{12}$, $r_2$</td>
</tr>
<tr>
<td>Case (i)</td>
<td>1, 16, 2, .5</td>
<td>2, 9, 3, .7</td>
</tr>
<tr>
<td>Case (ii)</td>
<td>4, 4, 2, .5</td>
<td>2, 9, 3, .7</td>
</tr>
</tbody>
</table>

Then $r_{av} = .61$ is the same for (i) and (ii), whereas $r = .36$ for (i) and $r = .19$ for (ii).

5.2 Model B.

This model is of particular interest, since it is probably the structure most frequently encountered in practice. The MLE of the parameters can be shown (Olkin and Siotani [13]) to be

$$
\hat{\rho} = \frac{(n_1+n_2)(1+r_1r_2) - \sqrt{(n_1+n_2)^2(1-r_1r_2)^2 - 4n_1n_2(1-r_2)^2}}{2(n_1r_1+n_2r_2)}
$$

Note that when $N_1 = N_2$ (or equivalently, $n_1 = n_2$) we obtain

$$
\hat{\rho} = \frac{1 + r_1r_2 - \sqrt{(1-r_1^2)(1-r_2^2)}}{r_1 + r_2}
$$

as the MLE of $\rho$. Recall the definition of $r_{av}$: if $N_1 = N_2$,

$$
\frac{r_{av} = \sqrt{(1+r_1)(1+r_2)} - \sqrt{(1-r_1)(1-r_2)}}{\sqrt{(1+r_1)(1+r_2)} + \sqrt{(1-r_1)(1-r_2)}} = \frac{1 + r_1r_2 - \sqrt{(1-r_1^2)(1-r_2^2)}}{r_1 + r_2}
$$

Thus, the two estimators $r_{av}$ and the MLE $\hat{\rho}$ are identical for equal
sample sizes. Both \( \hat{\rho} \) and \( r_{av} \) are consistent estimators of \( \rho \), and both are asymptotically normal, namely,

\[
\frac{\sqrt{n_1 + n_2}}{\sqrt{1-\rho^2}} (\hat{\rho} - \rho) \quad \text{and} \quad \frac{\sqrt{N_1 + N_2 - 6}}{\sqrt{1-\rho^2}} (r_{av} - \rho)
\]

are each approximately normally distributed with mean zero and unit variance. Consequently, the most extreme differences between \( r_{av} \) and \( \hat{\rho} \) occur for small samples. To illustrate the order of magnitude of these differences, if

\[
\begin{align*}
n_1 &= 20, \quad r_1 = .50, \quad z_1 = .55, \\
n_2 &= 80, \quad r_2 = .60, \quad z_2 = .69,
\end{align*}
\]

then \( z_{av} = .66, \quad r_{av} = .58 \), whereas \( \hat{\rho} = .52 \). If

\[
\begin{align*}
n_1 &= 10, \quad r_1 = .50, \quad z_1 = .55, \\
n_2 &= 20, \quad r_2 = .60, \quad z_2 = .69,
\end{align*}
\]

then \( z_{av} = .65, \quad r_{av} = .57 \), whereas \( \hat{\rho} = .54 \).

The definition of \( r_{av} \) can easily be extended to more than two populations, e.g., for three populations,

\[
r_{av} = \frac{(1+r_1)^{c_1} (1+r_2)^{c_2} (1+r_3)^{c_3} - (1-r_1)^{c_1} (1-r_2)^{c_2} (1-r_3)^{c_3}}{(1+r_1)^{c_1} (1+r_2)^{c_2} (1+r_3)^{c_3} + (1-r_1)^{c_1} (1-r_2)^{c_2} (1-r_3)^{c_3}}
\]

where

\[
\begin{align*}
c_1 &= \frac{N_1 - 3}{N - 9}, \quad c_2 = \frac{N_2 - 3}{N - 9}, \quad c_3 = \frac{N_3 - 3}{N - 9}, \\
N &= N_1 + N_2 + N_3.
\end{align*}
\]

However, the MLE of \( \rho \) for three (or more) populations cannot be
determined in a simple fashion. It involves obtaining a solution for $\rho$ of

$$\frac{n_1 + n_2 + n_3}{1 - \rho^2} = \frac{n_1 r_1}{1 - \rho r_1} + \frac{n_2 r_2}{1 - \rho r_2} + \frac{n_3 r_3}{1 - \rho r_3}.$$

Although the solution to this equation cannot be obtained in closed form, it can be determined numerically.

7. Test for the Equality of Correlation Coefficients and of Variances from Different Bivariate Normal Populations.

In this model we have two bivariate normal populations:

<table>
<thead>
<tr>
<th>parameters</th>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{11}^{(1)}$, $\sigma_{22}^{(1)}$, $\rho_1$</td>
<td>$\sigma_{11}^{(2)}$, $\sigma_{22}^{(2)}$, $\rho_2$</td>
</tr>
</tbody>
</table>

Independent samples from each, and we wish to test $H: \rho_1 = \rho_2$. Now if we convert each sample correlation coefficient by the $z$-transformation, then $z_1$ and $z_2$ are independently distributed, so that, for large samples,

$$\frac{(z_1 - z_2) - (\hat{z}_1 - \hat{z}_2)}{\sqrt{\frac{1}{N_1 - 3} + \frac{1}{N_2 - 3}}}$$

has a standard normal distribution. Confidence limits for $(\hat{z}_1 - \hat{z}_2)$, and hence for $\rho_1 - \rho_2$, can then easily be obtained. As before, there is no cognizance taken of the variances in the test procedure. An alternative procedure is based on the likelihood ratio statistic

$$L = \frac{\left(1 - \rho^2\right)^{n_1} \left(1 - r_1^2\right)^{\frac{n_1}{2}} \left(1 - r_2^2\right)^{\frac{n_2}{2}}}{\left(1 - \hat{\rho}_1\right)^{n_1} \left(1 - \hat{\rho}_2\right)^{n_2}},$$

\[17\]
where \( n = n_1 + n_2 \), and \( \hat{\rho} \) is the MLE of \( \rho \) assuming \( \rho_1 = \rho_2 \),
given previously in Section 6.2. Note that the sample variances do not
explicitly appear in the expression for \( L \). However, \( L \) has the present
form after taking advantage of the special relations between the
variances, and this expression would change if a different model were
assumed. On the other hand, the test based on the \( z \)'s is the same
regardless of the underlying model assumed!

To carry out the test, the large sample distribution of \( L \) may be
employed by noting that \(-2 \log L\) is approximately distributed as a
chi-square variate with 1 degree of freedom. The critical region of
the test is given by \(-2 \log L < c_{\gamma,1}\), where \( c_{\gamma,1} \) is the point such
that the probability of exceeding it (for the chi-square distribution
with 1 degree of freedom) is \( \gamma \).

collecting activities of 50 boys and 50 girls between the ages of 10-14
were studied. Among the correlations obtained was a correlation between
mental age and the average rating of the quality of the child's collections.
For boys, this yielded \( r_1 = .31 \), and for girls, \( r_2 = .06 \). The analysis
using the z-transformation gives \( z_1 = .3205 \), \( z_2 = .0601 \), and
\[
\frac{z_1 - z_2}{\sqrt{\frac{1}{N_1-3} + \frac{1}{N_2-3}}} = 1.26.
\]

Using \( L \), we have \( \hat{\rho} = .19 \), \(-2 \log L = 1.58\). To make a comparison
between the \( z \)-test and the \( L \)-test, we must compare a normal deviate with
a chi-square deviate with 1 degree of freedom. Since the square of a
normal deviate is such a chi-square deviate, we have the comparison
\[
z^2 = (1.26)^2 = 1.588 \quad \text{versus} \quad -2 \log L = 1.58.
\]
Thus, for this set of
data, the two tests are virtually the same. A detailed analysis of
virtues of each test is unavailable and would be a welcome study.

For the same model, if one wishes to test for the equality of
variances, when the correlations are known to be equal, the LR statistic becomes

$$\lambda = \frac{n\left(s_{11}^{(1)}s_{22}^{(1)}\right)^{\frac{n_1}{2}} \left(s_{11}^{(2)}s_{22}^{(2)}\right)^{\frac{n_2}{2}}} {\sqrt{n_1 n_2} \left|s_{11}^{(1)} + s_{22}^{(2)}\right|^2 (1-\rho)^2},$$

where $n = n_1 + n_2$,

$$\left|s_{11}^{(1)} + s_{22}^{(2)}\right| = \left(s_{11}^{(1)} + s_{22}^{(2)}\right) - \left(s_{12}^{(1)} + s_{12}^{(2)}\right)^2,$$

and where $s_{ij}^{(1)}$ and $s_{ij}^{(2)}$ are the sample covariances for each
population.

The asymptotic distribution of $-2 \log \lambda$ is that of a chi-square
variate with 1 degree of freedom.

8. **Intraclass Correlation Model.**

By an intraclass correlation model we mean that we have a p-variate
normal distribution with homogeneous variances and homogeneous covariances,
i.e., $\sigma_{11} = \sigma_{22} = \ldots = \sigma_{pp} = \sigma^2$, and $\sigma_{ij} = \sigma^2 \rho (i \neq j)$. Thus, the
covariance matrix has the form

$$\Sigma_0 = \sigma^2 \begin{pmatrix}
1 & \rho & \rho & \ldots & \rho \\
\rho & 1 & \rho & \ldots & \rho \\
\rho & \rho & 1 & \ldots & \rho \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho & \rho & \rho & \ldots & 1
\end{pmatrix},$$
and the parameters of interest are $\sigma^2$ and $\rho$. We call the common correlation, $\rho$, the intraclass correlation coefficient. This model is perhaps the fundamental model in test construction in that the $p$ items $x_1, \ldots, x_p$ have the same variance, and the correlation between any two items is the same corresponding to the fact that items on a test are homogeneous.

Various assumptions might be made about the population means, $(\mu_1, \ldots, \mu_p)$. For example, these might be unrestricted, all equal to $\mu$, or all known to be zero. Although the theory of estimation and testing hypotheses varies slightly with each of these assumptions, the modifications can easily be made, and only the case of unrestricted means is treated here.

A sample of size $N$ is available:

<table>
<thead>
<tr>
<th>$x_{11} \ldots x_{1N}$</th>
<th>$x_1.$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{21} \ldots x_{2N}$</td>
<td>$x_2.$</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>$x_{pl} \ldots x_{pN}$</td>
<td>$x_p.$</td>
</tr>
</tbody>
</table>

Means of Rows

<table>
<thead>
<tr>
<th>$x.p \ldots x.N$</th>
<th>$x.$</th>
</tr>
</thead>
</table>

Means of Columns Grand Mean

Here $x_{i0}$ denotes the $\alpha$-th observation on the $i$-th item. Determine
\[ u = p \sum_{\alpha=1}^{N} (x_{\alpha} - x)^2, \]

\[ v = \sum_{\alpha=1}^{N} \sum_{i=1}^{p} (x_{i\alpha} - x_i - x_{\alpha} + x)^2, \]

then \( u \) and \( v \) are independently distributed, each having a chi-square distribution, namely, \( u/(\sigma^2[1 + (p-1)\rho]) \) has a chi-square distribution with \( n \) degrees of freedom, \( v/(\sigma^2(1-\rho)) \) has a chi-square distribution with \( (p-1)n \) degrees of freedom.

### 3.1 A Test for the Intraclass Correlation Coefficient.

To test \( H: \rho = 0 \) versus \( A: \rho \neq 0 \), one can use the test statistic \((p-1)u/v\), which has an F-distribution with \( n \) and \((p-1)n\) degrees of freedom. This test can be viewed in terms of the analysis of variance model:

<table>
<thead>
<tr>
<th>Source</th>
<th>Sums of Squares</th>
<th>Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Items</td>
<td>( p \sum_{1}^{N} (x_{\alpha} - x)^2 )</td>
<td>( n )</td>
</tr>
<tr>
<td>Examinees</td>
<td>( n \sum_{1}^{p} (x_i - x)^2 )</td>
<td>( p )</td>
</tr>
<tr>
<td>Error</td>
<td>( \sum_{\alpha=1}^{N} \sum_{i=1}^{p} (x_{i\alpha} - x_i - x_{\alpha} + x)^2 )</td>
<td>((p-1)n)</td>
</tr>
<tr>
<td>Total</td>
<td>( \sum_{\alpha=1}^{N} \sum_{i=1}^{p} (x_{i\alpha} - x)^2 )</td>
<td>( pN )</td>
</tr>
</tbody>
</table>

The computations for this test can most conveniently be carried out using the standard ANOVA techniques.

21
8.2 Confidence Limits for the Intraclass Correlation.

Since

\[
\frac{(p-1)u}{v} \leq \frac{(1-\rho)}{1+(p-1)\rho}
\]

has an F-distribution with \( n \) and \((p-1)n\) degrees of freedom, 100(1-\(\alpha\)) percent confidence limits for \(\rho\) can easily be obtained from

\[
\frac{(p-1) u/v - F_{\alpha/2}}{(p-1) [u/v + F_{\alpha/2}]} \leq \rho \leq \frac{(p-1) u/v - F_{1-\alpha/2}}{(p-1) [u/v + F_{1-\alpha/2}]} ,
\]

\(F_{\gamma}\) is the cutoff point on the F-distribution with \(n\) and \((p-1)n\) degrees of freedom, such that the probability of exceeding \(F_{\gamma}\) is \(\gamma\).

The conventional estimate of \(\rho\) (e.g., DeLury [2], Kendall [10]) is given by

\[
r' = \frac{p}{p-1} \frac{\sum \sum (x_{1\alpha} - x_i)(x_{j\alpha} - x_j)}{\sum \sum (x_{1\alpha} - x_i)^2} .
\]

The idea here is that the sample covariances estimate \(\sigma^2\rho\), the sample variances estimate \(\sigma^2\), so that averages of the covariances divided by averages of the variances should yield an estimate of \(\rho\). The estimator \(r'\) is not unbiased, and a minimum variance unbiased estimator, \(H(r')\), is given by an infinite series in \(r'\) (Olkin and Pratt [12]). This series is different for each value of \(\rho\), i.e., there is a separate relation between \(r'\) and its unbiased estimator \(H(r')\) which depends on whether we have a bivariate, trivariate, or \(p\)-variate distribution. This necessitates a separate table for converting from \(r'\) to \(H(r')\) for each \(\rho\). At the present time the table for the bivariate case is available and is appended as Table 4.
Remark. For the bivariate case with moderate \( n \),

\[
H(r') = r' \left[ 1 + \frac{1-r'^2}{n-5/2} \right]
\]
yields a good approximation (to within \( .01 \) for \( n \geq 10 \) and to within \( .001 \) for \( n \geq 26 \)).

3.3 Averaging Correlations.

Consider two sets of tests \((X_1', \ldots, X_p')\) and \((Y_1', \ldots, Y_p')\), where the variances of the \( X \)'s are all equal to \( \sigma^2 \), the covariances between the \( X \)'s are all \( \sigma^2 \rho \); the variances of the \( Y \)'s are all equal to \( \tau^2 \), the covariances between the \( Y \)'s are all \( \tau^2 \rho \), and the \( X \)'s and \( Y \)'s are independent. The problem is to obtain the MLE of the common \( \rho \).

We assume that each population is normally distributed with unknown means. The available data are samples of size \( N_1 \) and \( N_2 \) on the \( X \)'s and \( Y \)'s, respectively. The statistics computed are \((u_1, v_1')\) and \((u_2, v_2')\) for each population using the definition of \( u \) and \( v \) given previously in Section 8.

The MLE of \( \rho \) is shown to be (Olkin and Siotani [13])

\[
\hat{\rho} = \frac{pg - 2h + p \sqrt{g^2 + 4h}}{2(p^2 + pg - h)},
\]

where

\[
g = \frac{(pn_2 - n)}{n} \left( 1 + \frac{v_1}{u_1} \right) + \frac{(pn_1 - n)}{n} \left( 1 + \frac{v_2}{u_2} \right) - p,
\]

\[
h = p - 1 - \frac{pn_1}{n} \left( 1 + \frac{v_2}{u_2} \right) - \frac{pn_2}{n} \left( 1 + \frac{v_1}{u_1} \right).
\]

The exact distribution of \( \hat{\rho} \) appears to be untractable, but large-sample results are available, namely, \( \sqrt{n} (\hat{\rho} - \rho) \) is approximately normally distributed with mean 0 and variance \( 2(1-\rho)^2 [1+(p-1)p]^2 / [p(p-1)] \). As
in the case of the product moment correlation coefficient, the variance of the asymptotic distribution depends on \( \rho \). For this distribution, the transformation which stabilizes the variance is given by

\[
Z_I = \frac{1}{p} \log \frac{1 + (p-1)\hat{\rho}}{1 - \hat{\rho}},
\]

where the subscript refers to the intraclass correlation model to distinguish it from the usual Fisher's z-transformation. If we define the corresponding transformation of \( \rho \),

\[
\zeta_I = \frac{1}{p} \log \frac{1 + (p-1)\rho}{1 - \rho},
\]

then \( \sqrt{np/(p-1)/2} (Z_I - \zeta_I) \) is approximately normally distributed with mean 0 and variance 1. (As in the case of the Fisher's z-transformation, a more refined constant instead of \( N \) can probably be obtained.)

In the theory of test construction, various functions of the intraclass correlation coefficient are of interest, e.g., \( \gamma = \frac{1 + k\hat{\rho}}{1 + (p-1)\hat{\rho}} \). Whenever these functions are monotone functions of \( \rho \) (this is usually the case), then the MLE of the function is obtained immediately by inserting the MLE of \( \rho \). Thus,

\[
\hat{\gamma} = \frac{1 + k\hat{\rho}}{1 + (p-1)\hat{\rho}}
\]

is the MLE of \( \gamma \). However, one cannot obtain minimum variance unbiased estimators in this way, and each problem must be treated afresh. Minimum variance unbiased estimators of the class of functions \( (a+b\rho)/(c+d\rho) \) can be obtained, and are generally in the form of an infinite series.

More extensive models involving intraclass correlations in blocks
were studied by Votaw [16] under the name "compound symmetry." Tests of hypotheses for means, means and covariances were developed in this paper.


The intent of this paper has been twofold:

(1) To present some new techniques for resolving a number of canonical problems in correlation analysis of concern in educational or psychological research.

(2) To emphasize the fact that there is no universal or simplified methodology which governs one's approach to problems concerning correlation coefficients. Rather, alternative assumptions concerning where and how the data arose (i.e., on the parameters of the problem) lead to alternative techniques.

If, indeed, one decides to use a reasonable universal technique without regard to the underlying assumptions, what are the risks involved? For the present set of problems quantitative answers to this question are unavailable. One might hazard the guess that if the sample sizes are large, the loss in efficiency in using a gross method rather than one refined to the particular problem may be negligible. However, this loss may become appreciable for samples of small size. Since most educational research studies consist of a number of simultaneous analyses, the cumulative effect of small losses in efficiency may become significant in terms of the experiment as a unit.

Finally, the recognition of the existence of alternative underlying models will serve to point out to the researcher where caution in drawing conclusions is warranted. Such considerations prompted the study of
solutions to these problems and continue to motivate further research in this area.

10. Acknowledgement.

The author is grateful to L. Gleser for his helpful comments and suggestions.
TABLE 1

Values for Transforming $r$ into $z$ from $z = \frac{1}{2} \log \frac{1+r}{1-r}$

<table>
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<tr>
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### TABLE 2

Values for Transforming \( z \) into \( r \) from \( z = \frac{1}{2} \log_e \frac{1+r}{1-r} \).

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CHART 1

CONFIDENCE BELT FOR THE CORRELATION COEFFICIENT $\rho$ WHEN $\alpha = .05$

Scale of $\rho$

Scale of $r$ (= Sample Correlation Coefficient)
The numbers on the curves indicate sample size
REFERENCES


