MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

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Multivariate Exponential Distributions

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The bivariate exponential distribution (BVE) introduced by Marshall and Olkin (1967a) is given by

\[ F(x, y) = P[X > x, Y > y] = \exp{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)}, \quad x, y \geq 0, \]

where \( \lambda_1, \lambda_2, \lambda_{12} \) are nonnegative parameters such that \( \lambda_1 + \lambda_{12} > 0, \lambda_2 + \lambda_{12} > 0 \). This distribution has the following origins:

a. A "fatal shock" model. Independent Poisson processes \( Z_1(t; \lambda_1) \), \( Z_2(t; \lambda_2) \), and \( Z_{12}(t; \lambda_{12}) \) govern the occurrence of fatal shocks, respectively, to component 1, to component 2, and to components 1 and 2 simultaneously. If \( X \) and \( Y \) denote the respective life lengths of components 1 and 2, their joint distribution is given by (1).

b. Minimums. If

\[ X = \min(U, W), \quad Y = \min(V, W), \]

where \( U, V \) and \( W \) are independently and exponentially distributed with respective parameters \( \lambda_1, \lambda_2 \) and \( \lambda_{12} \), then the joint distribution of \( X \) and \( Y \) is given by (1). This fact is essentially the same as that given in (a) above.
c. A "non-fatal shock" model. Events in the Poisson process \( Z(t; \theta) \) cause failure to the \( i \)-th component (but not the other) with probability \( p_i, i = 1, 2 \) and they cause failure to both components with probability \( p_{12} \), where \( 1 - p_1 - p_2 - p_{12} \geq 0 \). If \( \lambda_i = p_i \theta, i = 1, 2 \) and \( \lambda_{12} = p_{12} \theta \), then the respective times \( X \) and \( Y \) to failure of components 1 and 2 have a joint distribution given by (1).

d. A random sums model. Let \( X = U_1 + \ldots + U_N \), \( Y = U_1 + \ldots + U_M \), where \( \{U_i\} \) are independently and exponentially distributed with parameter \( \theta \) and where \((N, M)\) have the bivariate geometric distribution described by Hawkes (1972):

\[
P[N > n, M > m] = p_0^n (p_{01} + p_{11})^{m-n}, \quad \text{if } n \leq m,
\]

\[
p_0^m (p_{10} + p_{11})^{n-m}, \quad \text{if } n > m.
\]

If \( \{U_i\} \) and \((N, M)\) are independent, \( \lambda_1 = (p_{10} + p_{11}) \theta \), \( \lambda_2 = (p_{01} + p_{11}) \theta \) and \( \lambda_{12} = p_{11} \theta \), then \( X \) and \( Y \) have a joint distribution given by (1). This fact, the same as that given in (c) above, was noticed by Esary and Marshall (1974) and by Arnold (1975). The modification \( X = U_1 + \ldots + U_N \) and \( Y = V_1 + \ldots + V_M \), where \( \{U_i\} \) and \( \{V_i\} \) are independently and exponentially distributed leads to the bivariate exponential distribution of Hawkes (1972) which generalizes that of Downton (1970).

e. Lack of memory property. The univariate exponential distribution is characterized by the functional equation

\[
F(s + t) = F(s) F(t), \quad s, t \geq 0, \quad \text{where } F(x) = P[X > x].
\]
The same equation in vectors \( s, t \) leads to the case of independence. The less stringent functional equation

\[
(4) \quad \bar{F}(s_1 + \delta, s_2 + \delta) = \bar{F}(s_1, s_2) \bar{F}(\delta, \delta), \quad s_1, s_2, \delta \geq 0
\]

has many solutions, but the only solutions with exponential marginals are given by (1). Other solutions of (4) include the bivariate distribution of Friday and Putil (1977), of Freund (1961) and the distribution \( F_a \) defined below by (6) and studied by Block and Basu (1974). The functional equation (4) is also studied by Block (1977), and Fermann (1981). The class of all solutions of (4) is characterized by Ghurye and Marshall (1982).

**Properties.** From (1) or form (b) it is clear that \( \lambda_{12} > 0 \) implies \( P[X = Y] > 0 \) so that the distribution (1) is not absolutely continuous. The singular and absolutely continuous parts \( F_s \) and \( F_a \) are given by

\[
(5) \quad \bar{F}_s(x,y) = \exp\{-\lambda \max(x,y)\},
\]

\[
(6) \quad \bar{F}_a(x,y) = [\lambda/(\lambda_1 + \lambda_2)] \bar{F}(x,y) - [\lambda_{12}/(\lambda_1 + \lambda_2)] \bar{F}_s(x,y),
\]

where \( \lambda = \lambda_1 + \lambda_2 + \lambda_{12} \).

The moment generating function \( \psi \) is given by

\[
(7) \quad \psi(s,t) = \frac{(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})(\lambda + s + t) + st \lambda_{12}}{(\lambda_1 + \lambda_{12} + s)(\lambda_2 + \lambda_{12} + t)(\lambda + s + t)}
\]

It follows that \( \text{Cov}(X,Y) = \lambda_{12}/[\lambda(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})] \), \( \text{Corr}(X,Y) = \lambda_{12}/\lambda \geq 0 \).
The nonnegativity of the correlation follows from the fact that (2) implies that \( X \) and \( Y \) are associated in the sense of Esary, Proschan and Walkup (1967).

Statistics. In a sample \((X_1,Y_1), \ldots, (X_n,Y_n)\) from the bivariate exponential distribution (1), let \( N_{10}, N_{01}, N_{11} = n - N_{10} - N_{01} \) be, respectively, the number of \( X_i > Y_i \), of \( X_i < Y_i \) and of \( X_i = Y_i \). A sufficient statistic is \([N_{10}, N_{01}, \Sigma X_i, \Sigma Y_i, \Sigma \min(X_i, Y_i)]\).

To estimate the parameters of the distribution, an iterative method for solving the nonlinear maximum likelihood equations is given by Proschan and Sullo (1976), who also show that the MLE are consistent. Method of moment estimators are obtained by Bemis, Bain and Higgins (1972), and an intuitive estimator is proposed and studied by Proschan and Sullo (1976).

An estimator of the correlation \( \rho \) is proposed by Bemis, Bain and Higgins (1972), who also propose a test of the hypothesis \( \rho = 0 \) (equivalently, \( P[X=Y]=0 \), or \( \lambda_{12}=0 \), or \( X \) and \( Y \) are independent). They show that in case \( \lambda_1 = \lambda_2 \), the test is uniformly most powerful against the alternative \( \rho > 0 \).

If only \( T_i = \min(X_i, Y_i), i=1, \ldots, n \) and \( N_{10}, N_{01} \) are observed, then MLE of the parameters are obtained explicitly by George (1977), who also studies some tests of hypotheses for this type of data.

Applications. Applications of the BVE appear most notably in the literature on nuclear reactor safety, competing risks and reliability. These applications are tied together by the possibility of common causes of failure, an intrinsic feature of the BVE. For references on competing risks and on life lengths see Gail (1975), Prentice et al (1978), Tolley, Manton and
and Poss (1978), Langberg, Proschans and Quinzi (1981); for references in the context of nuclear risk see Vesely (1977), Hagen (1980); and in the context of reliability see Sarkar (1971), Apostolakis (1976).

The multivariate case. Let \( \{U_J, J \in \mathcal{J}\} \) be a set of independent exponentially distributed random variables indexed by the set \( \mathcal{J} \) of all non-empty subsets of \( \{1, \ldots, n\} \) and let \( U_J \) have parameter \( \lambda_J > 0 \), \( J \in \mathcal{J} \).

Assume that \( \sum_{J \in \mathcal{J}} \lambda_J > 0 \) and as an extension of (2), let

\[
X_i = \min_{J \in \mathcal{J}} U_{J,i} \quad i = 1, \ldots, K.
\]

Then \( (X_1, \ldots, X_k) \) has the multivariate exponential distribution given by

\[
F(x_1, \ldots, x_k) = P[X_1 > x_1, \ldots, X_k > x_k] = \exp\{-\sum_{J \in \mathcal{J}} \lambda_J (\max x_i)\},
\]

\( x_1, \ldots, x_k > 0 \). The \((k-1)\)-dimensional marginal distributions of (8) have the same structure, and the two-dimensional marginal distributions are BVE of the form (1). Moreover, the functional equation

\[
F(s_1 + \delta, \ldots, s_k + \delta) = F(s_1, \ldots, s_k) F(\delta, \ldots, \delta)
\]

is satisfied, and the only distributions with exponential marginal distributions that satisfy (9) have the form (8).

Generalizations. Random variables with the BVE distributions (1) can be regarded as joint waiting times in a bivariate Poisson process. A more general definition of "waiting times" leads to the more general BVE distribution of Marshall and Olkin (1967b).

The multivariate exponential distributions given by (8) are examples
of distributions with exponential minimums, i.e., $\min_{i \in I} X_i$ has an exponential distribution for each nonempty $I \subset \{1, \ldots, k\}$. Pickands (1982) has shown that distributions with exponential minimums have the form

$$\log \tilde{F}(x_1, \ldots, x_k) = \int_S (\max_{1 \leq i \leq k} q_i x_i) \mu(q),$$

where $\mu$ is a finite measure on the unit simplex $S = \{q: q_i \geq 0, i=1, \ldots, k, \sum q_i = 1\}$. If $(X_1, \ldots, X_n)$ has a distribution given by (10), then there exist random variables $Y_1, \ldots, Y_k$ with a distribution of the form (8) such that $\min_{i \in I} X_i$ has the same distribution as $\min_{i \in I} Y_i$ for all nonempty $I \subset \{1, \ldots, k\}$. Thus, for many purposes, one can assume (8) whenever (10) holds (Esary and Marshall, 1974; Langberg, Proschan and Quinzé, 1977).

Distributions of the form (8) arise as boundary cases in certain non-parametric classes of life distributions important in reliability theory (see e.g., Marshall and Shaked, 1981).
REFERENCES


