JOINT DISTRIBUTIONS OF SOME INDICES
BASED ON CORRELATION COEFFICIENTS

BY

LARRY V. HEDGES and INGRAM OLKIN

TECHNICAL REPORT NO. 194
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I. INTRODUCTION

Functions of correlation coefficients have been used extensively in psychological research for much of this century. Because of the paucity of natural scales of measurement (non-arbitrary scale factors) for many areas of psychology, such as personality and mental ability, scale-free measures of association become more important. In these areas the search for absolute scales of measurement has been all but abandoned, which places a stronger emphasis on scale-free measures of association.

One of the most important and commonly used scale-free measures of association is the product-moment correlation coefficient. The asymptotic distribution of the sample product-

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moment correlation was known to Pearson and Filon (1898) and the exact sampling distribution was obtained by Fisher (1915).

All but the simplest data analyses involve more than two variables. Most data analytic problems require at least two generalizations of the bivariate product-moment correlation, namely, the partial and the multiple correlations. Fisher (1924) obtained the distribution of the partial correlation coefficient, which is that of the product-moment correlation with reduced degrees of freedom. Fisher (1928) also obtained the distribution of the squared multiple correlation. Other generalizations of the bivariate correlation coefficient have sometimes been suggested, such as the "partial-multiple correlation" in which one set of variates is used to predict a criterion variable while controlling for the values of a third set of variables (see, e.g., Rao, 1973, p. 268). The exact non-null distribution of the partial-multiple correlation was obtained by Das Gupta (1977).

In spite of the importance of these correlation coefficients as statistical tools in behavioral science research, there is relatively little work on distribution theory for combinations of indices. For each of the partial, multiple, and partial-multiple correlations, marginal distributions are known, so that tests that the corresponding population parameter is zero can be carried out. Yet in each case, the joint distribution of two or more of the correlations is unknown. One of the reasons for the lack of joint distribution theory is the complexity of the marginal distributions. For many problems, the mathematics involved in obtaining exact joint distributions appears intractable. The asymptotic distributions
are quite tractable, however. Olkin and Siotani (1976) obtained the asymptotic distribution of a number of functions of sample correlation matrices. For example, they obtained the asymptotic joint distribution of the determinants of a correlation matrix and its principal submatrices. Hedges and Olkin (1981) extend these results to include the asymptotic joint distribution of certain differences between squared multiple correlations.

The present paper was motivated by a number of queries concerning confidence intervals for functions of correlation matrices. To obtain such confidence intervals, we first obtain the asymptotic joint distribution of arbitrary sets of partial, multiple, or partial-multiple correlations. These results can be used, for example, to obtain an asymptotic confidence interval for changes in squared multiple correlations or for the difference between partial correlations that control for different variables. We first (Section II) present applications that motivated some of the results obtained in this paper, then (Section III) give a general theorem on the asymptotic joint distribution of the determinants of arbitrary correlation matrices of variables. This theorem is the fundamental tool used to obtain the joint distributions of partial, multiple, and partial-multiple correlations given in Section IV. Some special results for the trivariate normal distributions are given in Section V.
II. SOME MOTIVATING EXAMPLES

The general results presented in this paper are quite complicated and tend to involve rather heavy notation. The fundamental nature of the questions motivating this paper are best appreciated by an examination of some applications.

The Berkeley Growth Study (Tuddenham and Snyder, 1954) measured the height of a longitudinal sample of $n = 66$ boys once a year between the ages 2 and 18. The intercorrelations of some of those measurements (made at ages 2, 4, 10, and 18) are given in Table I.

A. Comparison of a Multiple and Product-Moment Correlation

The correlations in Table I corroborate the intuition that measurements made closer in time correlate more highly. For example, height at age 18 correlates more highly with height at age 10, than it does with height at age 2 or at age 4. One question that arises from such data is how much additional variance is explained when height at age 18 is predicted from a composite of heights at ages 10 and 4 as opposed to a prediction based only on height at age 10. If we note heights at ages 18, 10, 4, and 2 by $X_0$, $X_1$, $X_2$, and $X_3$, respectively, then the problem is to obtain a confidence interval for

<table>
<thead>
<tr>
<th>Age in years</th>
<th>2</th>
<th>4</th>
<th>10</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age in years</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>years 2</td>
<td>.83</td>
<td>.72</td>
<td>.60</td>
<td></td>
</tr>
<tr>
<td>years 4</td>
<td>.33</td>
<td>.88</td>
<td>.76</td>
<td></td>
</tr>
<tr>
<td>years 10</td>
<td>.33</td>
<td>.88</td>
<td>.88</td>
<td></td>
</tr>
<tr>
<td>years 18</td>
<td>.60</td>
<td>.75</td>
<td>.88</td>
<td>1.00</td>
</tr>
</tbody>
</table>
\[ \rho_{0(12)}^2 - \rho_{01}^2, \]

where \( \rho_{0(12)} \) is the population multiple correlation of \( X_0 \) and \( (X_1, X_2) \) and \( \rho_{01} \) is the population product-moment correlation between \( X_0 \) and \( X_1 \).

A point estimate of \( \rho_{0(12)}^2 - \rho_{01}^2 \) is given by the sample counterparts, namely, \( r_{0(12)}^2 - r_{01}^2 \), which for the data of Table 1 is \( r_{0(12)}^2 = 0.777, r_{01}^2 = 0.774 \).

Because the asymptotic distribution of \( (r_{01}^2, r_{12}^2, r_{0(12)}^2) \) is multivariate normal, we can use this fact to obtain a large sample confidence interval for \( \rho_{0(12)}^2 - \rho_{01}^2 \). In particular, in large samples,

\[ \sqrt{n} \{ (r_{0(12)}^2 - r_{01}^2) - (\rho_{0(12)}^2 - \rho_{01}^2) \} \sim N(0, \sigma_\infty^2), \]

where the asymptotic variance, \( \sigma_\infty^2 \), of \( r_{0(12)}^2 - r_{01}^2 \) is a function of the population correlations:

\[ \sigma_\infty^2 = \sigma_\infty^2(\rho_{01}, \rho_{02}, \rho_{12}) = 4a^2 \phi_\infty, \]

\[ a = \left( \begin{array}{ccc} \rho_{12} & \rho_{02} & \rho_{01} \\ \rho_{00} & \rho_{02} & \rho_{01} \\ \rho_{00} & \rho_{00} & \rho_{00} \end{array} \right), \]

\( p^{-1} = (\rho_{ij}) \) is the inverse of the matrix \( P = (\rho_{ij}) \) of population correlations, and \( \phi_\infty \) is the asymptotic covariance matrix of \( (r_{01}, r_{02}, r_{12}) \):

\[ \phi_\infty = (\phi_{st}) = \text{Cov}_\infty(r_{ij}, r_{ik}), \]

\[ \text{Cov}_\infty(r_{ij}, r_{ij}) = (1 - \rho_{ij}^2)^2, \]

\[ \text{Cov}_\infty(r_{ij}, r_{ik}) = \frac{1}{2} (2\rho_{jk} - \rho_{ij}^2 \rho_{ik})(1 - \rho_{ij}^2 - \rho_{ik}^2 - \rho_{jk}^2) + \rho_{jk}^3. \] (2)

Note that (2) reduces to (1) when \( j = k \).

Substitution of consistent estimators \( r_{01}, r_{02}, \) and \( r_{12} \) for \( \rho_{01}, \rho_{02}, \) and \( \rho_{12} \) in \( \sigma_\infty^2 \) and \( \phi_\infty \) then yields a
large sample confidence interval for $\rho_{01}^2 - \rho_{01}^2$:

$$r_{01}^2 - r_{01}^2 \pm z_{\alpha/2} \hat{\sigma}_{\infty}/\sqrt{n},$$

where $\hat{\sigma}_{\infty} \equiv \sigma_{\infty}(r_{01}, r_{02}, r_{12})$ and $z_{\alpha/2}$ is the 100(1-α) percentage point of the standard normal distribution.

For the data of Table I, $r_{01}^2 = 0.777$, $r_{01}^2 = 0.774$, so the estimates of $a$ and $\phi_{\infty}$ are $a = (0.190, -0.216, 0.211)$, and

$$\phi_{\infty} = \begin{pmatrix}
0.509 & 0.070 & 0.076 \\
0.070 & 0.191 & 0.202 \\
0.076 & 0.202 & 0.410
\end{pmatrix}$$

which yield $\hat{\sigma}_{\infty} = 0.0487$. Using $n = 66$, an approximate 95% confidence interval for $\rho_{01}^2 - \rho_{01}^2$ is

$$-0.009 \leq \rho_{01}^2 - \rho_{01}^2 \leq 0.015.$$

Therefore the prediction of height at age 18 is not improved when height at age 4 is used as a predictor in addition to the height at age 10.

B. Comparison of a Partial and a Product-Moment Correlation

Is the correlation between heights at ages 10 and 18 affected by the height at age 4? This question is translated as the difference between a partial and product-moment correlation:

$$\rho_{01 \cdot 2} - \rho_{01}.$$

A point estimate of this difference is

$$r_{01 \cdot 2} - r_{01} = 0.700 - 0.880 = -0.180.$$

To obtain a confidence interval for $\rho_{01 \cdot 2} - \rho_{01}$, we use the asymptotic joint distribution of $r_{01}$ and $r_{01 \cdot 2}$. In particular, in large samples,
\[ \sqrt{n} \left\{ (r_{01.2} - r_{01}) - (\rho_{01.2} - \rho_{01}) \right\} \sim N(0, \sigma^2_\phi), \]

where the asymptotic variance, \( \sigma^2_\phi \), of \( r_{01.2} - r_{01} \) is a function of the population correlations:

\[
\sigma^2_\phi = \frac{c^2_\phi(\rho_{01}, \rho_{02}, \rho_{12})}{(1 - \rho^2_{02})(1 - \rho^2_{12})},
\]

\[
a = (1 - \sqrt{(1 - \rho^2_{02})(1 - \rho^2_{12})}, \frac{\rho_{01}\rho_{02} - \rho_{12}}{1 - \rho^2_{02}}, \frac{\rho_{01}\rho_{02} - \rho_{02}}{1 - \rho^2_{02}}),
\]

and \( \phi_\phi \) is given by (1) and (2).

The data in Table (I) gives the sample estimates of \( a \) as \( a = (0.686, -0.503, 0.108) \). Using the estimated value of \( \phi_\phi \) given in (1) and (2) we obtain \( \hat{\sigma}_\phi = \sigma_\phi(r_{01}, r_{02}, r_{12}) = 0.4261 \), so that an approximate 95% confidence interval is

\[-0.283 \leq \rho_{01.2} - \rho_{01} \leq -0.077.\]

Thus the correlation between heights is significantly different from that obtained when the effect of height at age 4 is removed.

C. Comparison of Two Partial Correlations

In the context of the growth model, is there a difference in the correlations between the heights at ages 10 and 8 when we hold fixed the height at age 4 or the height at age 2.

That is, we wish to obtain a confidence interval for

\[ \rho_{01.2} - \rho_{01.3}. \]

A point estimate of this difference is \( r_{01.2} - r_{01.3} = 0.700 - 0.807 = -0.107. \)

To obtain a confidence interval for \( \rho_{01.2} - \rho_{01.3} \) we use the large sample result:

\[ \sqrt{n} \left\{ (r_{01.2} - r_{01.3}) - (\rho_{01.2} - \rho_{01.3}) \right\} \sim N(0, \sigma^2_\phi), \]
where the asymptotic variance \( \sigma^2 \) of \( r_{0.1.2} - r_{01.3} \) is now a function of the population correlations \( \rho_{01}, \rho_{02}, \rho_{03}, \rho_{12}, \rho_{13}, \rho_{23} \). In particular

\[
\sigma^2 = \sigma^2(\rho_{01}, \rho_{02}, \rho_{03}, \rho_{12}, \rho_{13}, \rho_{23}) = \phi_{\infty} a'^2,
\]

where \( \phi_{\infty} = (\phi_{st}) = \text{Cov}_\infty(r_{ij}, r_{kl}) \) and in addition to (1) and (2) we require

\[
\text{Cov}_\infty(r_{ij}, r_{kl}) = \frac{1}{2} \rho_{ij} \rho_{kl} (\rho_{ik}^2 + \rho_{il}^2 + \rho_{jk}^2 + \rho_{jl}^2) + \rho_{ik} \rho_{jl} + \rho_{il} \rho_{jk} - (\rho_{ij} \rho_{ik} \rho_{jl} + \rho_{ij} \rho_{jl} \rho_{ik} + \rho_{il} \rho_{jk} \rho_{kl} + \rho_{il} \rho_{jk} \rho_{il}) + \rho_{li} \rho_{lj} \rho_{kl}.
\]

The vector \( a \) equals \( (a_1, a_2, a_3, a_4, a_5) \) where

\[
a_1 = \frac{1}{|P_{(2)}| \sqrt{\rho_{(2)}^2 \rho_{(3)}^2}} - \frac{1}{|P_{(3)}| \sqrt{\rho_{(3)}^2 \rho_{(3)}^2}},
\]

\[
a_2 = \frac{\rho_{(2)}^{23}}{|P_{(2)}| \sqrt{\rho_{(2)}^{11} [\rho_{(2)}^{22}]^3}}, \quad a_3 = \frac{-\rho_{(3)}^{23}}{|P_{(3)}| \sqrt{\rho_{(3)}^{11} [\rho_{(3)}^{22}]^3}},
\]

\[
a_4 = \frac{\rho_{(2)}^{13}}{|P_{(2)}| \sqrt{[\rho_{(2)}^{11}]^3 \rho_{(2)}^{22}}}, \quad a_5 = \frac{-\rho_{(3)}^{13}}{|P_{(3)}| \sqrt{[\rho_{(3)}^{11}]^3 \rho_{(3)}^{22}}}.
\]

\( P_{(2)} \) is the matrix of population correlations of variates \( X_0, X_1, X_2 \); \( P_{(3)} \) is the matrix of population correlations of variates \( X_0, X_1, X_3 \); \( P^{-1}_{(k)} = (\rho_{(k)}^{ij}) \) is the inverse of \( P_{(k)} \), and \( \phi_{\infty} \) is given by (1), (2), and (4).

Evaluation of \( \sigma_{\infty}^2 = \sigma_{\infty}^2(r_{01}, r_{02}, r_{03}, r_{12}, r_{13}, r_{23}) \) using \( a = (1.382, -1.601, 0.540, 0.344, -0.126) \) and
\[
\phi_\infty = \begin{pmatrix}
0.051 & 0.070 & 0.076 & 0.019 & 0.031 \\
0.070 & 0.191 & 0.202 & 0.070 & 0.100 \\
0.076 & 0.202 & 0.410 & 0.082 & 0.248 \\
0.019 & 0.070 & 0.082 & 0.051 & 0.068 \\
0.031 & 0.100 & 0.248 & 0.068 & 0.232 \\
\end{pmatrix}
\]
(5)

gives \( \hat{\phi}_\infty = 0.3611 \) so that an approximate 95 percent confidence interval for \( \rho_{01 \cdot 2} - \rho_{01 \cdot 3} \) is

\[-0.194 \leq \rho_{01 \cdot 2} - \rho_{01 \cdot 3} \leq -0.017.\]

In this case there is a significant difference between the correlation of heights at ages 10 and 18 when the height at age 2 is held fixed versus when the height at age 4 is held fixed.

\[D. \text{ Comparisons of Two Multiple Correlations}\]

If we use the height at age 10 to predict the height at age 18, and wish to use an additional piece of information to improve our prediction, should we use the height at age 2 or at age 4? In terms of confidence intervals, this question translates into finding a confidence interval for

\[\rho_{02(12)} - \rho_{02(13)}^2.\]

A point estimate of this difference is obtained from

\[r_{0(12)}^2 - r_{0(13)}^2 = 0.7770 - 0.7767 = 0.0003. \]

To obtain a confidence interval for \( \rho_{02(12)}^2 - \rho_{02(13)}^2 \), we use the asymptotic joint distribution of \( r_{0(12)}^2 \) and \( r_{0(13)}^2 \). In large samples

\[\sqrt{n} \left( (\rho_{02(12)}^2 - \rho_{02(13)}^2) - (\rho_{02(12)}^2 - \rho_{02(13)}^2) \right) \sim N(0, \sigma_{2\infty}^2),\]

where the asymptotic variance, \( \sigma_{2\infty}^2 \), of \( r_{0(12)}^2 - r_{0(13)}^2 \) is a function of the population correlations:

\[\sigma_{2\infty}^2 = \phi_{\infty}(\rho_{01}, \rho_{02}, \rho_{03}, \rho_{12}, \rho_{13}, \rho_{23}) = 4n \phi_{\infty} a',\]

where
\[ a = \begin{pmatrix} \frac{-\rho_{12}^{(2)}}{\rho_{11}^{(2)}} + \frac{\rho_{12}^{(3)}}{\rho_{11}^{(3)}} & \frac{-\rho_{13}^{(2)}}{\rho_{11}^{(2)}} & \frac{\rho_{13}^{(3)}}{\rho_{11}^{(3)}} & \frac{-\rho_{12}^{(2)} \rho_{13}^{(2)}}{\rho_{11}^{(2)}} & \frac{\rho_{12}^{(3)} \rho_{13}^{(3)}}{\rho_{11}^{(3)}} \\ \frac{\rho_{12}^{(2)}}{\rho_{11}^{(2)}} & \frac{-\rho_{13}^{(2)}}{\rho_{11}^{(2)}} & \frac{\rho_{13}^{(3)}}{\rho_{11}^{(3)}} & \frac{-\rho_{12}^{(2)} \rho_{13}^{(2)}}{\rho_{11}^{(2)}} & \frac{\rho_{12}^{(3)} \rho_{13}^{(3)}}{\rho_{11}^{(3)}} \\ \frac{\rho_{12}^{(3)}}{\rho_{11}^{(3)}} & \frac{-\rho_{13}^{(3)}}{\rho_{11}^{(3)}} & \frac{\rho_{13}^{(3)}}{\rho_{11}^{(3)}} & \frac{-\rho_{12}^{(3)} \rho_{13}^{(3)}}{\rho_{11}^{(3)}} & \frac{\rho_{12}^{(3)} \rho_{13}^{(3)}}{\rho_{11}^{(3)}} \end{pmatrix} \]

\( P_{(2)} \) and \( P_{(3)} \) are matrices of population correlations of variates \( X_0, X_1, X_2, \) and \( X_0, X_1, X_3, \) respectively, \( P^{-1} = (\rho_{ij}^{(k)}) \), and \( \phi_\infty \) is given by (1), (2), and (4).

Evaluation of \( \hat{\sigma}_\infty = \sigma_\infty (r_{01}, r_{02}, r_{03}, r_{12}, r_{13}, r_{23}) \) using \( a = (0.0899, -0.2163, 0.1395, 0.2109, -0.1298) \), and \( \phi_\infty \) given in (5) yields \( \hat{\sigma}_\infty = 0.424 \) which yields an approximate 95 percent confidence interval for \( \rho_{0(12)}^2 - \rho_{0(13)}^2 \)

\[-0.0099 \leq \rho_{0(12)}^2 - \rho_{0(13)}^2 \leq 0.0105. \]

Thus the inclusion of heights at ages 2 or 4 to that at age 10 does not matter in the prediction of height at age 18.

III. A FUNDAMENTAL THEOREM

Determinants or elements of the inverse of correlation matrices are involved in the definitions of many indices based on correlations. Examples are multiple and partial correlations. Because correlation matrices of normal variates are functions of sample moments, they will have asymptotic normal distributions. Consequently, the distribution of the determinants of correlation matrices is therefore a starting point for obtaining the distributions of functions of such determinants.

In some cases the asymptotic covariance matrix has a simple, compact form. In others, the expressions are quite complicated. Rather than present very complicated expressions, we indicate how they may be determined numerically.
Consider a partition of \( p + 1 \) variables into \( k + 1 \)
vectors \( x_0, x_1, \ldots, x_k \), where \( x_i \) is a \( p_i \)-dimensional column
vector, \( i = 0, 1, \ldots, k \) with \( p_0 + p_1 + \ldots + p_k = p + 1 \) and
where \( p_0 = p_1 = 1 \). Assume that \( k \) is small compared to \( p \)
so that the number of subsets of variables is less than the
number of nonredundant correlations, that is, \( 2^{k+1} - 1 < p(p+1)/2 \).
Given a sample of size \( n \) from such a \((p+1)\)-variate normal distribution, the variables are partitioned
as above and the sample is partitioned conformably.

We require a notation to denote the determinants of a
correlation matrix made up of a subset of the vectors. The
symbols \( R(s_1, s_2, \ldots, s_m) \) and \( P(s_1, s_2, \ldots, s_m) \) are used to
denote the determinants of the sample and population corre-
lation matrices, respectively, made up of vectors \( x_{s_1}, \ldots, x_{s_m} \).
In some instances we write more simply \( R(S) \) where
\( S = \{s_1, \ldots, s_m\} \).

Denote by \( d \) and \( \delta \) the vectors of determinants of
sample and population correlation matrices listed in lexi-
cographic order, i.e.,
\[
\mathbf{d} = (R(0), R(1), \ldots, R(0, 1), \ldots, R(0, 1, \ldots, k)),
\]
\[
\mathbf{\delta} = (P(0), P(1), \ldots, P(0, 1), \ldots, P(0, 1, \ldots, k)).
\]
In this notation, \( R(0, 1, \ldots, k) \) represents the determinant
of the \((p + 1) \times (p + 1)\) matrix of sample correlations.

**Theorem 1.** For a sample of \( n \) observations from a
\((p + 1)\)-variate normal distribution, the asymptotic distribu-
tion of \( \mathbf{d} \) is given as
\[
\sqrt{n} (\mathbf{d} - \mathbf{\delta}) \sim N(0, \Psi_0),
\]
where the covariance matrix of the limiting distribution has
elements of the form
\[
\text{Cov}_\omega(R(U), R(V)) = 2P(U)P(V) \sum_{i,j \in U} \sum_{l, m \in V} \rho_{ij}^l \rho_{jm}^l \left( \frac{1}{2} \rho_{ij}^l \rho_{jm}^l (\rho_{il}^2 + \rho_{im}^2 + \rho_{jl}^2 + \rho_{jm}^2) + \rho_{il}^2 \rho_{jm}^2 - \rho_{ij}^l \rho_{jm}^l \rho_{il}^2 \rho_{jm}^2 - \rho_{ij}^l \rho_{jl}^2 \rho_{jm}^2 - \rho_{ij}^l \rho_{jl}^2 \rho_{jm}^2 - \rho_{il}^2 \rho_{ij}^l \rho_{jm}^2 \right) - \rho_{mi}^l \rho_{mj}^l \rho_{ml}^l,
\]
where \( \rho_{ij}^l \) are the elements of the matrix \( p^{-1} \).

**Proof.** Denote the vector of all sample and population correlations from a \((p+1)\)-variate normal distribution (in lexicographic order) by

\[
r = (r_{01}, r_{02}, \ldots, r_{p-1}, p),
\]

\[
\rho = (\rho_{01}, \rho_{01}, \ldots, \rho_{p-1}, p).
\]

The vector \( r \) as a function of sample moments has an asymptotic multivariate normal distribution, i.e.,

\[
\sqrt{n} (r - \rho) \sim N(0, \Phi_\infty).
\]

The asymptotic variance matrix \( \Phi_\infty \) was first obtained by Pearson and Filon (1898). In our notation \( \Phi_\infty = (\text{cov}_\infty [r_{ij}, r_{lm}]) \), where \( \text{Cov}_\infty (r_{ij}, r_{lm}) \) is given by (1), (2), and (4). It is well known (Anderson, 1958, p. 77) that if \( T = (T_1, \ldots, T_k) \)' has an asymptotic \( k \)-variate normal distribution given by

\[
\sqrt{n} (T - \theta) \sim N(0, \Psi),
\]

and if \( f_1, \ldots, f_q \) (\( q \leq k \)) are functions of \( \theta \) having first and second derivatives in a neighborhood of \( \theta \), then \( f(T) = [f_1(T), \ldots, f_q(T)] \) has an asymptotic \( q \)-variate normal distribution given by

\[
\sqrt{n} [f(T) - f(\theta)] \sim N(0, A' \Psi A),
\]

where \( A = (a_{ij}) \) and \( a_{ij} = \partial f_1(t)/\partial \theta_j \) evaluated at \( t = \theta \). An application of this theorem with \( T = r, \theta = \rho, f_1(r) = R(0), \ldots, f_q(p) = R(0, \ldots, k) \) and the fact that
\[ \frac{\partial R(U)}{\partial r_{ij}} = 2R(U)r_{ij} \text{ if } i,j \in U, \quad \frac{\partial R(U)}{\partial r_{ij}} = 0 \text{ if } i,j \not\in U, \]
yields the result.

Remark. Note that the theorem above remains true if \( d \) is replaced with a vector of determinants of arbitrary non-singular matrices whose elements are correlation coefficients.

IV. JOINT DISTRIBUTIONS OF CORRELATION COEFFICIENTS

We now use Theorem 1 to obtain the asymptotic joint distributions of a vector of squared multiple correlations (Section A), partial correlations (Section B), partial correlations (Section C), and partial-multiple correlations (Section D).

A. Squared Multiple Correlations

Denote the vector of sample and population squared multiple correlations, arranged in lexicographic order, by

\[ \hat{\rho} = (r_{01}^2, r_{02}^2, \ldots, r_{0k}^2, r_{0(12)}^2, \ldots, r_{0(12\ldots k)}^2), \]
\[ \hat{\rho} = (\rho_{01}^2, \rho_{02}^2, \ldots, \rho_{0k}^2, \rho_{0(12)}^2, \ldots, \rho_{0(12\ldots k)}^2), \]
where \( r_{0}(s_1, \ldots, s_m) \) denotes the sample multiple correlation between \( X_0 \) and \( X_{s_1}, \ldots, X_{s_m} \). Recall that \( X_i, i = 1, \ldots, k \) is itself a vector.

Theorem 2. The asymptotic distribution of \( \hat{\rho} \) is given by

\[ \sqrt{n}(\hat{\rho} - \hat{\rho}) \sim N(0, \Gamma_\infty), \]
where the covariance matrix of the limiting distribution has elements of the form
\[ \text{Cov}_\omega(r^2_0(\alpha), r^2_0(\alpha)) = \rho^2_0(\alpha) [1 - \rho^2_0(\alpha)]^2, \]

\[ \text{Cov}_\omega(r^2_0(\alpha), r^2_0(\alpha)) = \frac{1}{P(\alpha)P(\beta)} \text{Cov}(R(\alpha), R(\beta)) \]

\[ + \frac{P(0, \alpha)P(0, \beta)}{P(\alpha)^2P(\beta)^2} \text{Cov}(R(0, \alpha), R(0, \beta)) \]

\[ - \frac{P(0, \alpha)}{P(\alpha)^2P(\beta)} \text{Cov}(R(0, \alpha), R(\beta)) \]

\[ - \frac{P(0, \beta)}{P(\alpha)P(\beta)^2} \text{Cov}(R(\alpha), R(0, \beta)), \]

where \( \alpha \) and \( \beta \) denote sets of subscripts. The relevant covariance terms are given in Theorem 1.

**Proof.** The proof follows from a direct application of Olkin and Siotani (1976, Theorem 2.1) after writing

\[ r^2_0(\alpha) = 1 - \frac{R(0, \alpha)}{R(\alpha)}. \]

**B. Partial Correlations**

Denote the vector of sample and population partial correlations between \( X_0 \) and \( X_1 \) for fixed \( X_2, X_3, \ldots, X_k \) (arranged in lexicographic order) by

\[ r^* = (r_{01}(2), r_{01}(3), \ldots, r_{01}(23), \ldots, r_{01}(2, \ldots, k)) \]

\[ \rho^* = (\rho_{01}(2), \rho_{01}(3), \ldots, \rho_{01}(23), \ldots, \rho_{01}(2, \ldots, k)). \]

**Theorem 3.** The asymptotic distribution of \( r^* \) is

\[ \sqrt{n} (r^* - \rho^*) \sim N(0, \Delta^\omega), \]

where the covariance matrix of the limiting distribution has elements of the form

\[ \text{Cov}(r_{01}(\alpha), r_{01}(\alpha)) = (1 - \rho^2_{01}(\alpha))^2, \]

\[ \text{Cov}(r_{01}(\alpha), r_{01}(\beta)) = \frac{1}{\sqrt{P(0, \alpha)P(0, \beta)P(1, \alpha)P(1, \beta)}} \times \]
\[ \text{Cov}(C_{01}(\alpha), C_{01}(\beta)) = \frac{\Gamma_{01}(\alpha)}{2P(0,\alpha)} \text{Cov}(R(0,\alpha), C_{01}(\beta)) \]

\[ - \frac{\Gamma_{01}(\beta)}{2P(0,\beta)} \text{Cov}(R(0,\beta), C_{01}(\alpha)) \]

\[ - \frac{\Gamma_{01}(\alpha)}{2P(1,\alpha)} \text{Cov}(R(1,\alpha), C_{01}(\beta)) \]

\[ - \frac{\Gamma_{01}(\beta)}{2P(1,\beta)} \text{Cov}(R(1,\beta), C_{01}(\alpha)) \]

\[ + \frac{\Gamma_{01}(\alpha)\Gamma_{01}(\beta)}{4} \frac{\text{Cov}(R(0,\alpha), R(0,\beta))}{P(0,\alpha)P(0,\beta)} \]

\[ + \frac{\text{Cov}(R(1,\alpha), R(0,\beta))}{P(1,\alpha)P(0,\beta)} \]

\[ + \frac{\text{Cov}(R(0,\alpha), R(1,\beta))}{P(0,\alpha)P(1,\beta)} + \frac{\text{Cov}(R(1,\alpha), R(1,\beta))}{P(1,\alpha)P(1,\beta)} , \]

where \( C_{01}(\alpha) \) denotes the cofactor of \( r_{01} \) in the matrix \( R(0,1,\alpha) \), \( \Gamma_{01}(\alpha) \) denotes the cofactor of \( \rho_{01} \) in the matrix \( P(0,1,\alpha) \).

\textbf{Proof.} The result follows by Applying Theorem 2.1 of Olkin and Siotani (1976) in the representation

\[ r_{01}^2 = \frac{R(0,1,\alpha)}{\sqrt{R(0,\alpha)R(1,\alpha)}} , \]

\textbf{C. Partial-Multiple Correlations}

In this model we predict \( X_0 \) with variables \( X_1, \ldots, X_m \), and holding variables \( X_{m+1}, \ldots, X_k \) fixed. To simplify notation, let \( \alpha \) denote the subscripts \( 1, 2, \ldots, m \). Denote the vector of sample and population squared partial-multiple correlations, arranged in lexicographic order, by

\[ \tilde{r}_0(\alpha) = (r_{0(\alpha)\cdot m+1}^2, \ldots, r_{0(\alpha)\cdot k}^2) , \]

\[ \tilde{\rho}_0(\alpha) = (\rho_{0(\alpha)\cdot m+1}^2, \ldots, \rho_{0(\alpha)\cdot k}^2) . \]

\textbf{Theorem 4.} The asymptotic distribution of \( \tilde{r}_0(\alpha) \) is

\[ \sqrt{n} \left( \tilde{r}_0(\alpha) - \tilde{\rho}_0(\alpha) \right) \sim N(0, \Sigma) , \]
where the covariance matrix of the limiting distribution has elements of the form

\[ \text{Cov}(r^2_{0(a), \beta}, r^2_{0(a), \gamma}) = (a, b) \Lambda (a, b)' \]

where

\[ a = \frac{P(a)P(0, \alpha, \beta)}{P(\alpha, \beta)P(0, \alpha)} \left( \frac{1}{P(\alpha)'P(0, \alpha, \beta)} - \frac{1}{P(\alpha, \beta)P(0, \alpha)} \right), \]

\[ b = \frac{P(a)P(0, \alpha, \gamma)}{P(\alpha, \gamma)P(0, \alpha)} \left( \frac{1}{P(\alpha)'P(0, \alpha, \gamma)} - \frac{1}{P(\alpha, \gamma)P(0, \alpha)} \right), \]

and \( \Lambda \) is the matrix of covariances of \( R(\beta), R(0, \alpha, \beta), R(\alpha, \beta), R(0, \beta), R(\gamma), R(0, \alpha, \gamma), R(\alpha, \gamma), R(0, \alpha) \) as given by Theorem 1.

**Proof.** The result follows by an application of Theorem 2.1 of Olkin and Siotani (1976) to

\[ r_{0(\alpha), \beta} = \frac{R(\beta)R(0, \alpha, \beta)}{R(\alpha)R(0, \beta)}. \]

V. SPECIAL RESULTS FOR THE TRIVARIATE NORMAL DISTRIBUTION

In this section we state some results for the trivariate normal distribution. Although these results are special cases of the results in Section IV, the results can be stated explicitly. Assume that the variates \( X_0, X_1, X_2 \) have the trivariate normal distribution with correlation matrix \( \rho \).

The asymptotic joint distribution of correlations with the multiple, squared multiple, and partial correlations are given.

**Theorem 5.** The asymptotic distribution of \((r_{01}, r_{12}, r_{0(12)})\) is given by

\[ \sqrt{n} \left[ (r_{01}, r_{12}, r_{0(12)}) - (\rho_{01}, \rho_{12}, \rho_{0(12)}) \right] \sim N(0, \Psi), \]

where the \( 3 \times 5 \) matrix \( \Psi_n = (\psi_{ij}) \) has elements
\[ \psi_{11} = (1 - \rho_{01}^2) \]
\[ \psi_{22} = (1 - \rho_{12}^2) \]
\[ \psi_{33} = (1 - \rho_{0(12)}^2) \]
\[ \psi_{12} = \text{Cov}_\infty(r_{01}, r_{12}) \]
\[ \psi_{13} = -\frac{1}{\rho_{0(12)}} \left[ \text{Cov}_\infty(r_{01}, r_{01}) \frac{\rho_{12}}{\rho_{11}} + \text{Cov}_\infty(r_{01}, r_{02}) \frac{\rho_{13}}{\rho_{11}} \right. \]
\[ \left. + \text{Cov}_\infty(r_{01}, r_{12}) \frac{\rho_{12} \rho_{13}}{\rho_{11} \rho_{11}} \right] \]
\[ \psi_{23} = -\frac{1}{\rho_{0(12)}} \left[ \text{Cov}_\infty(r_{01}, r_{02}) \frac{\rho_{12}}{\rho_{11}} + \text{Cov}_\infty(r_{01}, r_{02}) \frac{\rho_{13}}{\rho_{11}} \right. \]
\[ \left. + \text{Cov}_\infty(r_{02}, r_{12}) \frac{\rho_{12} \rho_{13}}{\rho_{11} \rho_{11}} \right] \]
\[ p^{-1} = (\rho_{ij}) \text{, and } \text{Cov}_\infty(r_{ij}, r_{ik}) \text{ is given by (1) and (2)} \]

**Theorem 6.** The asymptotic distribution of \( (r_{01}^2, r_{12}^2, r_{0(12)}^2) \) is given by
\[ \sqrt{n} \left[ (r_{01}^2, r_{12}^2, r_{0(12)}^2) - (\rho_{01}^2, \rho_{12}^2, \rho_{0(12)}^2) \right] \sim N(0, \Phi) \],
where
\[ \Phi = (\phi_{ij}) \]
\[ \phi_{11} = 4\rho_{01}^2(1 - \rho_{01}^2) \]
\[ \phi_{22} = 4\rho_{12}^2(1 - \rho_{12}^2) \]
\[ \phi_{33} = 4\rho_{0(12)}^2(1 - \rho_{0(12)}^2) \]
\[ \phi_{12} = 4\rho_{01}^2 \rho_{12} \text{Cov}_\infty(r_{01}, r_{12}) \]
\[ \phi_{13} = -4\rho_{12} \left[ \text{Cov}_\infty(r_{01}, r_{01}) \frac{\rho_{12}}{\rho_{11}} + \text{Cov}_\infty(r_{01}, r_{02}) \frac{\rho_{13}}{\rho_{11}} \right. \]
\[ \left. + \text{Cov}_\infty(r_{02}, r_{12}) \frac{\rho_{12} \rho_{13}}{\rho_{11} \rho_{11}} \right] \]
\[ \phi_{23} = -4\rho_{23} \left[ \text{Cov}_\infty(r_{01}, r_{02}) \frac{\rho_{12}}{\rho_{11}} + \text{Cov}_\infty(r_{02}, r_{02}) \frac{\rho_{13}}{\rho_{11}} \right. \]
\[ \left. + \text{Cov}_\infty(r_{02}, r_{12}) \frac{\rho_{12} \rho_{13}}{\rho_{11} \rho_{11}} \right] \]
\[ p^{-1} = (\rho_{ij}) \text{, and } \text{Cov}(r_{ij}, r_{ik}) \text{ is given by (1) and (2)} \]

**Theorem 7.** The asymptotic distribution of \( (r_{01}, r_{12}, r_{01.2}) \) is given by
\[ \sqrt{n} \left[ (r_{01}, r_{12}, r_{01} \cdot 2) - (\rho_{01}, \rho_{12}, \rho_{01} \cdot 2) \right] \sim N(0, \Omega_{\infty}), \]

where \( \Omega_{\infty} = (\omega_{ij}), \omega_{11} = (1-\rho_{01})^2, \omega_{22} = (1-\rho_{12})^2, \]
\( \omega_{33} = (1-\rho_{01} \cdot 2)^2, \)
\( \omega_{12} = \text{Cov}_{\infty}(r_{01}, r_{02}), \)
\( \omega_{13} = \frac{1}{\sqrt{(1-\rho_{02})(1-\rho_{12})}} \text{Cov}_{\infty}(r_{01}, r_{02}), \)
\[ = \text{Cov}_{\infty}(r_{01}, r_{02}) \frac{(\rho_{01} \rho_{02} - \rho_{12})}{(1-\rho_{02})} \]
\[ + \text{Cov}_{\infty}(r_{01}, r_{12}) \frac{\rho_{01} \rho_{12} \rho_{02}}{(1-\rho_{12})}, \]
\( \omega_{23} = \frac{1}{\sqrt{(1-\rho_{02})(1-\rho_{12})}} \text{Cov}_{\infty}(r_{01}, r_{02}), \)
\[ = \text{Cov}_{\infty}(r_{02}, r_{02}) \frac{(\rho_{01} \rho_{02} - \rho_{12})}{(1-\rho_{02})} \]
\[ + \text{Cov}_{\infty}(r_{02}, r_{12}) \frac{(\rho_{01} \rho_{12} - \rho_{02})}{(1-\rho_{12})}, \]

where \( p^{-1} = (\rho_{ij}) \) and \( \text{Cov}(r_{ij}, r_{ik}) \) is given by (1) and (2).

REFERENCES